O’ZBEKISTON RESPUBLIKASI
OLIY TA`LIM, FAN VA INNOVASIYALAR VAZIRLIGI

NAMANGAN DAVLAT UNIVERSITETI

Matematik analiz kafedrasi

“M A T E M A T I K A N A L I Z”

fanidan

O’ Q U V – U S L U B I Y M A J M U A

Bilim sohasi:	**100 000 - Gumanitar**
Ta’lim sohasi:	**130 000 – Matematika**
Ta’lim yo’nalishi:	**5130100 - Matematika**

III-SEMESTR

Namangan-2023

O‘quv uslubiy majmua 2021 yil O‘zR OO‘MTV tomonidan № BD 5130100-2.01. raqami bilan 2021 yil 25 avgustdagi 744-sonli buyrug‘i bilan tasdiqlangan fan dasturi asosida ishlab chiqilgan.

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O‘quv uslubiy majmua Namangan davlat universiteti Kengashininig 2023 yil "…" _________dagi "..." - son yig‘ilishida ko‘rib chiqilgan va foydalanishga tavsiya etilgan.

MUNDARIJA

So‘z boshi

So‘nggi yillarda mamlakatimizda oliy ta’lim sifatini oshirishga qaratilgan bir qancha chora-tadbirlar amalga oshirilmoqda. Chunki, jahon talablari darajasidagi raqobatbardosh kadrlar tayyorlash maqsadida talabalarga dunyo standartlariga javob beradigan bilim va ko‘nikmalar berish bugungi kunning eng dolzarb masalalaridan biri bo‘lib qolmoqda.

Mazkur o‘quv-uslubiy majmua “Matematik analiz” fani bo‘yicha tayyorlangan bo‘lib, u “5130100-Matematika” yo‘nalishi talabalari uchun mo‘ljallangan va Namangan davlat universiteti “Matematika” kafedrasi o‘qituvchilari tomonidan tayyorlangan. Ushbu majmua mamlakatimizda “Matematik analiz” fanini o‘qitish bo‘yicha uzoq yillardan beri to‘plangan boy tajriba, O`zbekiston milliy universiteti professor-o`qituvchilarining ma’ruzalari va o`quv qo`lanmalari hamda rivojlangan xorijiy davlatlarning yetakchi Oliy ta’lim muassasalarining tajribalaridan foydalangan holda, shuningdek, ularning o‘quv dasturlaridagi asosiy adabiyotlardan foydalangan holda yaratildi.

Matematik analiz fani matematikaning fundamental bo‘limlaridan biri bo‘lib, u matematikaning poydevori hisoblanadi. Matematik analiz kursi davomida ko‘pgina tushuncha va tasdiqlar, shuningdek, ularning tatbiqlari keltiriladi.

Matematik analiz fanining asosiy vazifasi shu fanning tushuncha, tasdiqlar va boshqa matematik ma’lumotlar majmuasi bilan tanishtirishgina bo‘lmasdan, balki talabalarda mantiqiy fikrlash, matematik usullarni amaliy masalalarni yechishga qo‘llash ko‘nikmalarini shakllantirishdan iborat.

Ushbu o‘quv-uslubiy majmuada har bir mavzu bo‘yicha materiallar batartib berilgan. Bunda har bir mavzu bo‘yicha ma’ruza matnlari, mavzuga doir misollar, nazorat savollari, mashqlar, glossariy, amaliy mashg`ulot materiallari, test savollari va keyslar banki keltirilgan.

1-Mavzu. Sonli qatorlar va ularning yaqinlashuvchiligi.
1-Ma'ruza.
REJA:

1^0. Sonli qator tushunchasi.
2^0. Yaqinlashuvchi qatorlarning xossalari.
3^0. Yaqinlashuvchi qatorlarning xossalari.
4^0. Koshi teoremasi

**Tayanch so`z va iboralar:** *Sonli qator, yaqinlashuvchi (uzoqlashuvchi) sonli qator, qatorning yig`indisi, qatorning qoldig`i, yaqinlashuvchi qatorlarning xossalari. qator yaqinlashuvchiligining zaruriy sharti, yaqinlashuvchi qatorlarning xossalari, Koshi teoremasi.*

1^0. Sonli qator tushunchasi. Faraz qilaylik,

$$\{a_n\}: a_1, a_2, a_3, ..., a_n, ...$$

haqiqiy sonlar ketma-ketligi berilgan bo`lsin. Ular yordamida ushbu

$$a_1 + a_2 + a_3 + ... + a_n + ... \qquad (1)$$

ifodani hosil qilamiz. (1) ifoda sonli qator, qisqacha **qator** deyiladi va u $\sum_{n=0}^{\infty} a_n$ kabi belgilanadi **[3, Definition 16, p.95]:**

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$$

Bunda $a_1, a_2, a_3, ..., a_n, ...$ sonlar qatorning hadlari, a_n esa qatorning umumiy hadi (yoki n-hadi) deyiladi.

Quyidagi

$$S_n = a_1 + a_2 + ... + a_n \qquad (n = 1, 2, 3, ...)$$

yig`indi (1) qatorning $n$**-qismiy yig`indisi** deyiladi.

Demak, (1) qator berilganda har doim bu qatorning qismiy yig`indilaridan iborat ushbu $\{S_n\}$:

$$S_1, S_2, S_3, ..., S_n, ...$$

ketma-ketlikni hosil qilish mumkin.

Masalan,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{n(n+1)} + ...$$

qatorning $n-$qismiy yig`indisi

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} =$$

$$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{n} - \frac{1}{n+1}) =$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

bo`lib, ulardan tuzilgan $\{S_n\}$ ketma-ketlik

$$\frac{1}{2}\ ,\ \frac{2}{3}\ ,\ \frac{3}{4}\ , \ldots ,\frac{n}{n+1}\ , \ldots$$

bo`ladi.

1-ta`rif.** **[3, Definition 20, p.95]** Agar $n \to \infty$ da $\{S_n\}$ ketma-ketlik $S$ ga $(S \in R)$ yaqinlashsa, (1) qator **yaqinlashuvchi** deyiladi, $S$ uning **yig`indisi deyiladi:

$$\lim_{n\to\infty} S_n = S\ ,\ S = \sum_{n=1}^{\infty} a_n\ .$$

Agar $\{S_n\}$ ketma-ketlik chekli limitga ega bo`lmasa (limit mavjud bo`lmasa yoki cheksiz bo`lsa), (1) qator **uzoqlashuvchi** deyiladi.

2^0. Yaqinlashuvchi qatorlarning xossalari. Aytaylik, biror

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

qator berilgan bo`lsin.

Ushbu

$$\sum_{n=m+1}^{\infty} a_n = a_{m+1} + a_{m+2} + \ldots \qquad (2)$$

qator (bunda $m-$tayinlangan natural son) (1) **qatorning qoldig`i** deyiladi.

1-xossa. Agar (1) qator yaqinlashuvchi bo`lsa, (2) qator ham yaqinlashuvchi bo`ladi va aksincha; (2) qatorning yaqinlashuvchi bo`lishidan (1) qatorning yaqinlashuvchiligi kelib chiqadi.

◄ (1) qatorning qismiy yig`indisi

$$S_n = a_1 + a_2 + \ldots + a_n\ ,$$

(2) qatorning qismiy yig`indisi

$$M_k^{(m)} = a_{m+1} + a_{m+2} + \ldots + a_{m+k}$$

lar uchun

$$S_{m+n} = S_m + M_k^{(m)}\ , \qquad (3)$$

bo`ladi.

Aytaylik, (1) qator yaqinlashuvchi bo`lsin. Unda $k\to\infty$ da $S_{m+n}$ chekli limitga ega bo`lib, (3) munosabatga ko`ra $k\to\infty$ da $M_k^{(m)}$ ham chekli limitga ega bo`ladi. Demak, (2) qator yaqinlashuvchi.

Aytaylik, (2) qator yaqinlashuvchi bo`lsin. Unda $k\to\infty$ da $M_k^{(m)}$ chekli limitga ega bo`ladi. Yana (3) munosabatga ko`ra $k\to\infty$ da $S_{m+n}$ ham chekli limitga ega bo`ladi. Demak, (1) qator yaqinlashuvchi. ►

2-xossa. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots$$

qator yaqinlashuvchi bo`lib, uning yig`indisi S ga teng bo`lsa, u holda

$$\sum_{n=1}^{\infty} c\cdot a_n = c\cdot a_1 + c\cdot a_2 + \dots + c\cdot a_n + \dots$$

qator ham yaqinlashuvchi va uning yig`indisi $c\cdot S$ ga teng bo`ladi, bunda $c\neq 0$ bo`lgan o`zgarmas son.

3-xossa. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots ,$$

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \dots + b_n + \dots$$

qatorlar yaqinlashuvchi bo`lib, ularning yig`indisi mos ravishda S_1 va S_2 ga teng bo`lsa, u holda

$$\sum_{n=1}^{\infty} (a_n + b_n) = (a_1 + b_1) + (a_2 + b_2) + \dots + (a_n + b_n) + \dots$$

qator ham yaqinlashuvchi va uning yig`indisi $S_1 + S_2$ ga teng bo`ladi.

2) va 3)- xossalarning isboti sonli qatorlar va ularning yaqinlashuvchiligi ta`rifidan bevosita kelib chiqadi.

3^0. Yaqinlashuvchi qatorlarning xossalari. Avvalgi mavzudagi yaqinlashuvchi qatorlarning xossalarini o`rganishni davom ettiramiz.

Aytaylik, biror

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots \qquad (1)$$

qator berilgan bo`lsin.

Ushbu

$$\sum_{n=m+1}^{\infty} a_n = a_{m+1} + a_{m+2} + \dots \qquad (2)$$

qator (bunda $m-$tayinlangan natural son) (1) qatorning qoldig`i deyiladi.

4-xossa (zaruriy shart). [3, Corollary 7, p.95] Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

qator yaqinlashuvchi bo`lsa, $n \to \infty$ da a_n nolga intiladi:

$$\lim_{n\to\infty} a_n = 0 .$$

◄ Aytaylik, $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`lib, uning yig`indisi S ga teng bo`lsin: Ta`rifga binoan

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(a_1 + a_2 + \ldots + a_n) = S .$$

Ravshanki,

$$a_n = S_n - S_{n-1}$$

bo`ladi. Keyingi tenglikdan topamiz:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = S - S = 0 . \ ►$$

Eslatma. Qatorning umumiy hadi a_n ning $n \to \infty$ da nolga intilishidan uning yaqinlashuvchi bo`lishi har doim kelib chiqavermaydi. Masalan, ushbu

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots$$

qatorning umumiy hadi $a_n = \frac{1}{\sqrt{n}}$ bo`lib, u $n \to \infty$ da nolga intiladi. Ammo bu qator uzoqlashuvchi , chunki

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \geq n\frac{1}{\sqrt{n}} = \sqrt{n}$$

ketma-ketlik $n \to \infty$ da $+\infty$ ga intiladi:

$$\lim_{n\to\infty} S_n = \infty .$$

Yuqorida keltirilgan 4)- xossa qator yaqinlashuvchi bo`lishining zaruriy shartini ifodalaydi.

5-xossa. Aytaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

qator berilgan bo`lsin. Bu qatorning hadlarini guruxlab quyidagi

$$(a_1 + a_2 + \ldots + a_{n_1}) + (a_{n_1+1} + a_{n_1+2} + \ldots + a_{n_2}) + \ldots \qquad (4)$$

qatorni hosil qilamiz, bunda

$$n_1 < n_2 < \ldots$$

bo`lib, $\{n_k\}$ ketma-ketlik natural sonlar ketma-ketligi $\{n\}$ ning qismiy ketma-ketligi.

Agar (1) qator yaqinlashuvchi bo`lib, uning yig`indisi S ga teng bo`lsa, u holda (4) qator ham yaqinlashuvchi va yig`indisi S bo`ladi.

◄ (1) qator yaqinlashuvchi bo`lib, yigindisi $S$ ga teng bo`lsin. U holda

$$n \to \infty \quad \text{da} \quad S_n = a_1 + a_2 + \ldots + a_n \to S$$

bo`ladi.

Aytaylik, (4) qatorning qismiy yig`indilaridan iborat ketma-ketlik $\{S_{n_k}\}$ bo`lsin ($\kappa = 1,2,3,\ldots$). Ravshanki, bu ketma-ketlik $\{S_n\}$ ketma-ketlikning qismiy ketma-ketligi bo`ladi. Ma`lum teoremaga ko`ra

$$k \to \infty \ \text{da} \ S_{n_k} \to S$$

bo`ladi. Demak, (4) qator yaqinlashuvchi va uning yig`indisi S ga teng. ►

4⁰. Qatorning yaqinlashuvchiligi. Koshi teoremasi.

Faraz qilaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

qator berilgan bo`lsin. Ma`lumki, bu qatorning yaqinlashuvchiligi ushbu

$$S_n = a_1 + a_2 + \ldots + a_n \qquad (n = 1, 2, 3, \ldots)$$

ketma-ketlikning $n \to \infty$ da chekli limitga ega bo`lishidan iborat.

9-ma`ruzada sonlar ketma-ketligining chekli limitga ega bo`lishi haqida Koshi teoremasi, ya`ni $\{S_n\}$ ketma-ketlikning $n \to \infty$ da chekli limitga ega bo`lishi uchun

$$\forall \varepsilon > 0 \ , \exists n_0 \in N \ , \forall n > n_0 , \ \forall m \in N \ \text{da} \quad |S_{n+m} - S_n| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli ekani keltirilgan edi.

Bu tushuncha va tasdiqdan $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchiligini ifodalaydigan quyidagi teorema kelib chiqadi.

Teorema (Koshi teoremasi). [3, Theorem 6, p.95] $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`lishi uchun $\forall \varepsilon > 0$ son olinganda ham shunday $n_0 \in N$ topilib, $\forall n > n_0$ va $m = 1,2,3,\ldots$ bo`lganda

$$|S_{n+m} - S_n| = |a_{n+1} + a_{n+2} + \ldots . + a_{n+m}| < \varepsilon \qquad (5)$$

tengsizlikning bajarilishi zarur va yetarli.

Eslatma. Agar $\sum_{n=1}^{\infty} a_n$ qator uchun (5) shart bajarilmasa, ya`ni

$$\exists \varepsilon_0 > 0, \quad \forall k \in N, \quad \exists \ n \geq k, \quad \exists m \in N$$

$$|a_{n+1} + a_{n+2} + \ldots . + a_{n+m}| \geq \varepsilon_0 \qquad (6)$$

bo`lsa, u holda $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`ladi.

1-Amaliy mashg`ulot.

1-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n(n-1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n(n+1)}+...$$

qator uchun $S_n=1-\dfrac{1}{1+n}$ bo`lib,

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}(1-\frac{1}{n+1})=1$$

bo`ladi. Demak, berilgan qator yaqinlashuvchi va uning yig`in-disi 1 ga teng:

$$\sum_{n=1}^{\infty}\frac{1}{n(n-1)}=1.$$

2-misol. Quyidagi

$$\sum_{n=1}^{\infty}n=1+2+3+...+n+...$$

qator uzoqlashuvchi bo`ladi, chunki

$$S_n=1+2+3+...+n=\frac{n(n+1)}{2}$$

uchun

$$\lim_{n\to\infty}S_n=+\infty\ .$$

3-misol. Ushbu

$$\sum_{m=1}^{\infty}(-1)^{n+1}=1-1+1-1+...+(-1)^{n+1}+...$$

qator uchun

$$S_n=1-1+1-1+...+(-1)^{n+1}=\begin{cases}0, & \text{agar} \quad n-\text{juft} \quad \text{son}\\ 1, & \text{agar} \quad n-\text{toq} \quad \text{son}\end{cases}$$

bo`lib u $n\to\infty$ da limitga ega emas.
Demak, berilgan qator uzoqlashuvchi.

4-misol. Ushbu

$$\sum_{n=1}^{\infty}aq^{n-1}=a+aq+aq^2+...+aq^{n-1}+... \qquad (a\in R\ ,\ q\in R)$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Odatda, bu geometrik qator deb yuritiladi.

Berilgan qator uchun

$$S_n=a+aq+aq^2+...+aq^{n-1}=\frac{a-aq^n}{1-q} \qquad (q\neq 1)$$

bo`lib, $|q|<1$ bo`lganda

$$\lim_{n\to\infty} S_n = \frac{a}{q-1}$$

bo`ladi. Demak, bu holda geometrik qator yaqinlashuvchi va uning yig`indisi $\frac{a}{1-q}$ ga teng .

Agar $q>1$ bo`lsa,

$$\lim_{n\to\infty} S_n = \infty \ ,$$

$q=1$ bo`lsa,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty$$

bo`lib, bu hollarda berilgan qator uzoqlashuvchi bo`ladi.
$q\le -1$ bo`lganda esa $\{S_n\}$ ketma-ketlik limitga ega emas. Demak, bu holda ham qator uzoqlashuvchi bo`ladi.

Shunday qilib, geometrik qator $|q|<1$ bo`lganda yaqinla-shuvchi, $|q|\ge 1$ bo`lganda uzoqlashuvchi bo`ladi. ►

5-misol. Ushbu

$$\frac{4}{1\cdot 3}+\frac{4}{3\cdot 5}+\frac{4}{5\cdot 7}+\ \dots\ +\frac{4}{(2n-1)(2n+1)}+\ \dots$$

qatorning yaqinlashuvchiligi aniqlansin, yig`indisi topilsin.

◄Bu qatorning umumiy hadini quyidagicha

$$\frac{4}{(2n-1)(2n+1)} = \frac{2}{2n-1} - \frac{2}{2n+1}$$

yozib, uning qismiy yig`indisini topamiz:

$$S_n = \frac{4}{1\cdot 3}+\frac{4}{3\cdot 5}+\frac{4}{5\cdot 7}+\ \dots+\frac{4}{(2n-1)(2n+1)} =$$

$$= 2-\frac{2}{3}+\frac{2}{3}-\frac{2}{5}+\frac{2}{5}-\ \dots\ -\frac{2}{2n-1}+\frac{2}{2n+1} = 2-\frac{2}{2n+1}$$

Ravshanki,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(2-\frac{2}{2n+1}\right) = 2 .$$

Demak, berilgan qator yaqinlashuvchi, uning yig`indisi 2 ga teng.►

6-misol. Ushbu

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ \dots\ +(-1)^{n-1}\frac{1}{2^{n-1}}+\ \dots$$

qator yaqinlashuvchilikka tekshirilsin.

◄Ravshanki, bu qatorning qismiy yig`indisi

$$S_n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ \ldots + (-1)^{n-1}\frac{1}{2^{n-1}}$$

bo`ladi. Uni quyidagicha yozib olamiz:

$$S_n = 1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \ \ldots \ + \left(-\frac{1}{2}\right)^{n-1} = \frac{1}{\frac{2}{3}} + \left[\frac{-\frac{1}{2}}{1-\left(-\frac{1}{2}\right)} - \frac{\left(-\frac{1}{2}\right)^{n-1}}{1-\left(-\frac{1}{2}\right)}\right] =$$

$$= \frac{1}{1+\frac{1}{2}} - \frac{\left(-\frac{1}{2}\right)^{n-1}}{1+\frac{1}{2}} = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^{n-1}.$$

Natijada

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^{n-1}\right) = \frac{2}{3}$$

bo`lib, undan berilgan qatorning yaqinlashuvchi, yig`indisi $S = \frac{2}{3}$ bo`lishi kelib chiqadi.►

1-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{\sin n}{2^n} = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \ldots + \frac{\sin n}{2^n} + \ldots$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Bu qator uchun Koshi teoremasidagi (5) shartning bajarilishini tekshiramiz :

$$\left|\frac{\sin(n+1)}{2^{n+1}} + \frac{\sin(n+2)}{2^{n+2}} + \ldots + \frac{\sin(n+m)}{2^{n+m}}\right| \le$$

$$\le \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^{n+m}} \le \frac{\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = \frac{1}{2^n}$$

Agar $\forall\varepsilon > 0$ songa ko`ra $n_0 = [-\log_2 \varepsilon] + 1$ deb olinsa, u holda $\forall n > n_0$ va $m = 1,2,3,\ldots$ lar uchun

$$\left|\frac{\sin(n+1)}{2^{n+1}} + \frac{\sin(n+2)}{2^{n+2}} + \ldots + \frac{\sin(n+m)}{2^{n+m}}\right| < \varepsilon$$

bo`ladi. Demak, berilgan qator yaqinlashuvchi. ►

2-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots \qquad (7)$$

qator yaqinlashuvchilikka tekshirilsin.

◀ $\varepsilon_0 = \frac{1}{2}$ va ixtiyoriy $k \in N$ uchun $n = k,\quad m = k$ bo`lganda

$$\left|\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+m}\right| = \frac{1}{k+1} + \frac{1}{k+2} + \ldots$$

$$\ldots + \frac{1}{2k} > \frac{1}{2k} \cdot k = \frac{1}{2} = \varepsilon_0$$

bo`ladi.

(6) shartga ko`ra (7) qator uzoqlashuvchi bo`ladi. ▶

Odatda, (7) qator garmonik qator deyiladi. Demak, garmonik qator uzoqlashuvchi qator.

3-misol. Ushbu

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{\ln n}}$$

qator yaqinlashuvchilikka tekshirilsin.

◀Bu qatorning umumiy hadi

$$a_n = \frac{1}{\sqrt[n]{\ln n}}$$

bo`ladi. $n \to \infty$ da a_n ning limitini topamiz:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\ln n}} = \lim_{n\to\infty} \frac{1}{e^{\frac{1}{n}\ln\ln n}} = \lim_{n\to\infty} \frac{1}{e^{\frac{\ln\ln n}{n}}} = 1$$

Qator yaqinlashishining zaruriy sharti bajarilmaydi. Binobarin, qator uzoqlashuvchi bo`ladi.▶

4-misol. Ushbu

$$\sum_{n=0}^{\infty} aq^n = a + aq + aq^2 + \ \ldots + aq^{n-1} + \ \ldots$$

qator yaqinlashuvchilikka tekshirilsin.

◀Odatda bu qatorni geometrik qator deyiladi. Qatorning qismiy yig`indisini topamiz:

$$S_n = a + aq + aq^2 + \ \ldots\ + aq^{n-1} = \frac{aq^n - a}{q-1} = \frac{a}{1-q} - \frac{aq^n}{1-q} \qquad (q \neq 1)$$

Ravshanki,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a}{1-q} - \frac{aq^n}{1-q}\right) = \begin{cases} \frac{a}{1-q}, & \text{agar } |q|<1 \text{ bo`lsa}, \\ +\infty, & \text{agar } q>1 \text{ bo`lsa}, \\ \text{mavjud emas}, & \text{agar } q \le -1 \text{ bo`lsa}, \end{cases}$$

va $q=1$ da $\lim_{n\to\infty} S_n = \infty$ bo`ladi.

Demak, geometrik qator $|q|<1$ bo`lganda yaqinlashuvchi, $|q|\ge 1$ bo`lganda uzoqlashuvchi bo`ladi.►

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Nazorat savollari

1. Sonli qator nima?
2. Sonli qator yig`indisining ta'rifini ayting
3. Yaqinlashuvchi sonli qatorlarning qanday xossalarini bilasiz?
4. Sonli qator yaqinlashuvchiligining zaruriy sharti yetarli bo`ladimi?
5. Sonli qator yaqinlashuvchiligining zaruriy sharti nima uchun yetarli bo`lmaydi?
6. Yaqinlashuvchi sonli qatorlarning qanday xossalarini bilasiz?
7. Koshi teoremasini ayting.

Glossariy

Sonli qator - $\{a_n\}$: $a_1, a_2, a_3, \ldots, a_n, \ldots$ haqiqiy sonlar ketma-ketligi berilgan bo'lsin. Ular yordamida ushbu

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \qquad (1)$$

ifodani hosil qilamiz. (1) ifoda sonli qator, qisqacha qator deyiladi va u $\sum_{n=0}^{\infty} a_n$ kabi belgilanadi:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$$

Sonli qator yig'indisi - Agar $n \to \infty$ da $\{S_n\}$ ketma-ketlik S ga $(S \in R)$ yaqinlashsa, (1) qator yaqinlashuvchi deyiladi, S uning yig`indisi deyiladi:

$$\lim_{n\to\infty} S_n = S \ , \ S = \sum_{n=1}^{\infty} a_n \ .$$

Yaqinlashuvchi sonli qator - Agar $n \to \infty$ da $\{S_n\}$ ketma-ketlik S ga $(S \in R)$ yaqinlashsa, (1) qator yaqinlashuvchi deyiladi, S uning yig`indisi deyiladi:

$$\lim_{n\to\infty} S_n = S \ , \ S = \sum_{n=1}^{\infty} a_n \ .$$

Koshi teoremasi - $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`lishi uchun $\forall \varepsilon > 0$ son olinganda ham shunday $n_0 \in N$ topilib, $\forall n > n_0$ va $m = 1,2,3,...$ bo`lganda

$$\left|S_{n+m} - S_n\right| = \left|a_{n+1} + a_{n+2} + ... + a_{n+m}\right| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Keys banki

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)(n+2)}$$

qator yig'indisini toping.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

2-keys. Masala o`rtaga tashlanadi: Agar

$$\sum_{n=1}^{\infty} a_n \qquad (a_n \geq 0, \ n = 1,2,3,...)$$

qator yaqinlashuvchi bo`lsa, u holda

$$\sum_{n=1}^{\infty} a_n^2$$

qatorning ham yaqinlashuvchi bo`lishi isbotlansin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Test

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	Sonli qatorning yig`ndisini toping $\sum_{n=1}^{\infty}\frac{1}{3^n}$.	$\frac{1}{2}$	2	1
2	Sonli qatorning yig`ndisini toping $\sum_{n=1}^{\infty}\frac{2}{n(n+1)}$.	2	1	-1
3	Agar sonli qator yaqinlashuvchi bo`lsa, uning umumiy hadi qaysi songa intiladi?	0	1	2
4	$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ qator yig'indisini toping.	1	0	-1
5	$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ qator n ta hadi yig'indisini toping.	$\frac{n}{n+1}$	$\frac{2n}{n+1}$	$\frac{2-n}{n+1}$
6	$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ qatorni yaqinlashishga tekshiring.	$\frac{1}{2}$	$1\frac{1}{2}$	$-\frac{1}{2}$
7	$\frac{1}{1\cdot 3}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\dots\ a_n-?$	$a_n=\frac{1}{n(2n+1)}$	$a_n=\frac{1}{n(2n-1)}$	$a_n=\frac{1}{2n(2n+1)}$
8	$\sum_{n=1}^{\infty}a_n$ qator yaqinlashuvchi bo`lishi uchun quyidagi shartning qaysi biri bajarilishi zarur:	$\lim_{n\to\infty}a_n=0$	$a_n<\infty$	$\lim_{n\to\infty}a_n=\infty$
9	$\sum_{n=1}^{\infty}a_n$ qator uzoqlashuvchi, agar $\left(S_n=\sum_{k=1}^{n}a_k\right)$	$\lim_{n\to\infty}S_n=\infty$	$\lim_{n\to\infty}S_n=0$	$\lim_{n\to\infty}S_n=-1$

10	$\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`lishi uchun	$\lim_{n\to\infty} S_n = S < \infty$	$S_n < \infty$	$\lim_{n\to\infty} S_n$ mavjud emas

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ Koshi kriteriyasidan foydalanib yaqinlashishga tekshiring.	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
2	$\sum_{n=1}^{\infty} \frac{n}{\sqrt{1+n^2}}$ Koshi kriteriyasidan foydalanib yaqinlashishga tekshiring.	uzoqlashuvchi	yaqinlashuvchi	Aniqlab bo`lmaydi
3	$\sum_{n=1}^{\infty} \frac{1}{n}$ Koshi kriteriyasidan foydalanib yaqinlashishga tekshiring.	uzoqlashuvchi	yaqinlashuvchi	Aniqlab bo`lmaydi
4	Sonli qator yaqinlashishi uchun Koshi shartini aniqlang.	$\forall\varepsilon>0\ \exists n_0:$ $\forall n>n_0\ \forall m>n_0$ $\Rightarrow \left\lvert\sum_{k=n}^{m} a_k\right\rvert<\varepsilon$	$\forall\varepsilon>0\ \exists n_0:$ $\forall n>n_0$ $\exists m>n \Rightarrow$ $\left\lvert\sum_{k=n}^{m} a_k\right\rvert<\varepsilon$	$\exists\varepsilon>0\ \forall n_0:$ $\exists n>n_0$ $\exists m>n_0 \Rightarrow$ $\left\lvert\sum_{k=n}^{m} a_k\right\rvert\ge\varepsilon$
5	$\sum_{k=1}^{\infty} a_k$ sonli qator Koshi kriteriyasiga ko`ra uzoqlashishi uchun quyidagi shartlardan qaysi biri bajarilishi kerak?	$\exists\varepsilon>0\ \forall n_0:$ $\exists n>n_0\ \exists m>n_0:$ $\left\lvert\sum_{k=n}^{m} a_k\right\rvert\ge\varepsilon$	$\forall\varepsilon>0\ \exists n_0:$ $\forall n>n_0$ $\forall m>n \Rightarrow$ $\left\lvert\sum_{k=n}^{m} a_k\right\rvert<\varepsilon$	$\exists\varepsilon>0\ \forall n_0:$ $\exists n>n_0$ $\exists m>n_0$ $\left\lvert\sum_{k=n}^{m} a_k\right\rvert<\varepsilon$
6	$\sum_{n=1}^{\infty} \frac{1}{1+n}$ qatorni yaqinlashishga tekshiring.	uzoqlashuvchi	yaqinlashuvchi	ham yaqinlashuvchi, ham uzoqlashuvchi

2-Mavzu. Musbat hadli qatorlar.
2-Ma’ruza.
REJA:

1^0. Musbat hadli qatorlar va ularning yaqinlashuvchiligi.
2^0. Musbat hadli qatorlarda taqqoslash teoremalari.
3^0. Koshi alomati.
4^0. Dalamber alomati.
5^0. Integral alomati.
6^0. Raabe alomati.

**Tayanch so`z va iboralar:** *Musbat hadli sonli qator, yaqinlashuvchining zarur va yetarli sharti, taqqoslash teoremalari. Musbat hadli qatorlarning Koshi alomati, Dalamber alomati. Koshining integral alomati, boshlang`ich funksiya, chegaralangan ketma-ketlik, Raabe alomati, Gauss alomati. Koshining integral alomati, boshlang`ich funksiya, chegaralangan ketma-ketlik, Raabe alomati, Gauss alomati.*

1^0. Musbat hadli qatorlar va ularning yaqinlashuvchiligi.

Faraz qilaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

qator berilgan bo`lsin.

Agar bu qatorda $a_n \geq 0 \quad (\forall n \in N)$ bo`lsa, (1) **musbat hadli qator** deyiladi.

Musbat hadli qatorlarda, ularning qismiy yig`indi-laridan iborat $\{S_n\}$ ketma-ketlik o`suvchi ketma-ketlik bo`ladi. Haqiqatan ham,

$$S_{n+1} = a_1 + a_2 + \ldots + a_n + a_{n+1} = S_n + a_{n+1} \geq S_n .$$

1-teorema. **[3, Theorem 7, p.98]** Musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

qatorning yaqinlashuvchi bo`lishi uchun

$$\{S_n\} = \{a_1 + a_2 + \ldots + a_n\} \quad (n = 1, 2, 3, \ldots)$$

ketma-ketlikning yuqoridan chegaralangan bo`lishi zarur va yetarli.

◄ **Zarurligi.** (1) qator yaqinlashuvchi bo`lsin. Unda $n \to \infty$ da $\{S_n\}$ ketma-ketlik chekli limitga ega bo`ladi. Yaqinlashuvchi ketma-ketlikning xossasiga ko`ra $\{S_n\}$ chegaralangan, jumladan yuqoridan chegaralangan bo`ladi.

Yetarliligi. $\{S_n\}$ ketma-ketlik yuqoridan chegaralangan bo`lsin. Unda monoton ketma-ketlikning limiti haqidagi teoremaga ko`ra $\{S_n\}$ ketma-ketlik $n \to \infty$ da chekli limitga ega bo`ladi. Demak, (1) qator yaqinlashuvchi. ►

Eslatma. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$$

musbat hadli qatorda, uning qismiy yig`indilaridan iborat $\{S_n\}$ ketma-ketlik yuqoridan chegaralanmagan bo`lsa, u holda qator uzoqlashuvchi bo`ladi.

2^0.Musbat hadli qatorlarda taqqoslash teoremalari.

Ikkita

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... , \quad \sum_{n=1}^{\infty} b_n = b_1 + b_2 + ... + b_n + ...$$

musbat hadli qatorlar berilgan bo`lsin.

2-teorema. [3, Theorem 8, p.98] Faraz qilaylik $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar uchun $\forall n \in N$ da

$$a_n \le b_n \qquad (2)$$

tengsizlik bajarilsin.

U holda:

1) $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo`ladi,

2) $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`lsa, $\sum_{n=1}^{\infty} b_n$ qator ham uzoqlashuvchi bo`ladi.

◄ $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlarning qismiy yig`indilari mos ravishda

$$S_n = a_1 + a_2 + ... + a_n, \quad S'_n = b_1 + b_2 + ... + b_n$$

bo`lsin. U holda (2) munosabatga ko`ra

$$S_n \le S'_n \qquad (3)$$

bo`ladi.

Aytaylik, $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsin. Unda 1-teoremaga binoan $\{S'_n\}$ ketma-ketlik yuqoridan chegaralangan bo`ladi. Ayni paytda, (3) munosabatni e`tiborga olib, $\{S_n\}$ ketma-ketlikning ham yuqoridan chegaralangan bo`lishini topamiz. Yana 1-teoremaga ko`ra $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`ladi.

Aytaylik, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`lsin. Unda (3) munosabat va eslatmadan foydalanib, $\sum_{n=1}^{\infty} b_n$ qatorning uzoqlashuvchi bo`lishini topamiz. ►

1-misol. Ushbu

$$\sum_{n=1}^{\infty} \sin\frac{\pi}{2^n} = \sin\frac{\pi}{2} + \sin\frac{\pi}{2^2} + \ldots + \sin\frac{\pi}{2^n} + \ldots$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Ravshanki, bu qator hadlari uchun

$$0 < \sin\frac{\pi}{2^n} < \frac{\pi}{2^n} \qquad (n = 1,2,3,\ldots)$$

tengsizlik o`rinli bo`ladi.

Natijada berilgan qatorning har bir hadi yaqinla-shuvchi $\sum_{n=1}^{\infty} \frac{1}{2^n}$ qatorning (geometrik qatorning) mos hadidan kichik. 2-teoremaga muvofiq berilgan qator yaqinlashuvchi bo`ladi. ►

3-teorema. Faraz qilaylik, musbat hadli $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlarning umumiy hadlari uchun

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K \qquad (b_n > 0, \quad n = 1,2,\ldots)$$

bo`lsin. U holda:

1) $K < +\infty$ bo`lib, $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo`ladi.

2) $K > 0$ bo`lib, $\sum_{n=1}^{\infty} b_n$ qator uzoqlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator ham uzoqlashuvchi bo`ladi.

◄ Aytaylik, $K < +\infty$ bo`lib, $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsin.

Ravshanki,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K \Rightarrow \quad \forall\varepsilon > 0 \quad \exists n_0 \in N, \quad \forall n > n_0:$$

$$\left|\frac{a_n}{b_n} - K\right| < \varepsilon \quad \Rightarrow \quad (K-\varepsilon)b_n < a_n < (K+\varepsilon)b_n.$$

Bundan esa, $\sum_{n=1}^{\infty}(K+\varepsilon)b_n$ qator yaqinlashuvchi bo`lgani uchun 2-teoremaga ko`ra $\sum_{n=1}^{\infty}a_n$ qatorning yaqinlashuvchiligi kelib chiqishini topmaiz.

Aytaylik, $K>0$ bo`lib, $\sum_{n=1}^{\infty}b_n$ qator uzoqlashuvchi bo`lsin.

Ravshanki, $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=K$ va $0<K_1<K$ bo`lishidan $\forall n>n_0\in N$ uchun $\frac{a_n}{b_n}>K_1$ ya`ni $b_n<\frac{1}{K_1}a_n$ bo`lishi kelib chiqadi. 2-teoremadan foydalanib $\sum_{n=1}^{\infty}a_n$ qatorning uzoqlashuvchi bo`lishini topamiz. ►

Natija. Musbat hadli $\sum_{n=1}^{\infty}a_n$ va $\sum_{n=1}^{\infty}b_n$ qatorlar uchun

$$\lim_{n\to\infty}\frac{a_n}{b_n}=K\ ,\qquad (0<K<+\infty)$$

bo`lsa, u holda $\sum_{n=1}^{\infty}a_n$ va $\sum_{n=1}^{\infty}b_n$ qatorlar bir vaqtda yoki yaqinlashuvchi bo`ladi yoki uzoqlashuvchi bo`ladi.

2-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Berilgan qator bilan birga uzoqlashuvchiligi ma`lum bo`lgan $\sum_{n=1}^{\infty}\frac{1}{n}$ garmonik qatorni qaraymiz. Bu qatorlarnig umumiy hadlari uchun

$$\lim_{n\to\infty}\frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{n^{1+\frac{1}{n}}}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}}=1$$

bo`ladi. Demak, berilgan qator uzoqlashuvchi. ►

4-teorema. Aytaylik, musbat hadli $\sum_{n=1}^{\infty}a_n$ va $\sum_{n=1}^{\infty}b_n$ qatorlar uchun

$$\frac{a_{n+1}}{a_n}\le\frac{b_{n+1}}{b_n}$$

bo`lsin $(a_n>0,\quad b_n>0,\quad n=1,2,3,...)$

U holda:

1) $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator ham yaqinlashuvchi bo`ladi,

2) $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`lsa, $\sum_{n=1}^{\infty} b_n$ qator ham uzoqlashuvchi bo`ladi.

◀ Faraz qilaylik, $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ qatorlar $(a_n > 0, b_n > 0,\ n = 1,2,3,...)$ uchun

$$\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n} \qquad (n = 1,2,3,....)$$

tengsizliklar bajarilsin. Bu shartdan quyidagi munosabat kelib chiqadi:

$$\frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot ... \cdot \frac{a_n}{a_{n-1}} \le \frac{b_2}{b_1} \cdot \frac{b_3}{b_2} \cdot ... \cdot \frac{b_n}{b_{n-1}}.$$

Keyingi tengsizlikdan topamiz:

$$a_n \le \frac{a_1}{b_1} b_n . \qquad (n = 1,2,3,...)$$

Aytaylik, $\sum_{n=1}^{\infty} b_n$ qator yaqinlashuvchi bo`lsin. Ravshanki, $\sum_{n=1}^{\infty} \frac{a_1}{b_1} b_n$ qator ham yaqinlashuvchi bo`ladi. 2-teoremadan foydalanib, $\sum_{n=1}^{\infty} a_n$ qatorning yaqinlashuvchi bo`lishini topamiz. Xuddi shunga o`xshash $\sum_{n=1}^{\infty} a_n$ qatorning uzoqlashuvchi bo`lishidan $\sum_{n=1}^{\infty} b_n$ qatorning uzoqlashuvchi bo`lishi kelib chiqishi ko`rsatiladi. ▶

Yuqorida keltirilgan teorema va misollardan ko`rinadiki, musbat hadli qatorning yaqinlashuvchiligi yoki uzoqlashuvchiligini bilgan holda, hadlari bu qator hadlari bilan ma`lum munosabatda bo`lgan (taqqoslangan) ikkinchi musbat hadli qatorning yaqinlashuvchiligi yoki uzoqlashuvchiligini aniqlash mumkin bo`lar ekan.

Izoh. Yuqorida keltirilgan teoremalar n ning biror n_0 qiymatidan boshlab bajarilganda ham o`rinli bo`ladi.

Musbat hadli qatorlar mavzusida bayon etilgan taqqoslash teoremalaridan foydalanib, yaqinlashish alomatlarini keltiramiz.

3^0. Koshi alomati. **[3, Corollary 9, p.99]** Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \quad (1)$$

qatorda barcha $n \geq 1$ uchun

$$\sqrt[n]{a_n} \leq q < 1 \quad (2)$$

bo`lsa, (1) qator yaqinlashuvchi bo`ladi;

$$\sqrt[n]{a_n} \geq 1 \quad (3)$$

bo`lsa, (1) qator uzoqlashuvchi bo`ladi.

◀ Aytaylik, (1) qator hadlari uchun

$$\sqrt[n]{a_n} \leq q < 1$$

bo`lsin. Ravshanki, bu tengsizlikdan

$$a_n \leq q^n$$

bo`lishi kelib chiqadi.

Demak, berilgan qatorning har bir hadi yaqinlashuvchi geometrik qatorning mos hadidan katta emas. Unda 50-ma`ruzadagi 2-teoremaga ko`ra (1) qator yaqinlashuvchi bo`ladi.

Aytaylik, (1) qator hadlari uchun

$$\sqrt[n]{a_n} \geq 1, \text{ ya`ni } a_n \geq 1$$

bo`lsin. Bu munosabat berilgan qatorning har bir hadini uzoqlashuvchi

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + \ldots + 1 + \ldots$$

qatorning mos hadidan kichik emasligini ko`rsatadi. Bu holda yana o`sha 2-teoremaga ko`ra (1) qator uzoqlashuvchi bo`ladi. ▶

Ko`pincha Koshi alomatining quyida keltirilgan limit ko`rinishidagi tasdig`idan foydalaniladi.

Faraz qilaylik, musbat hadli (1) qatorda

$$\lim_{n\to\infty} \sqrt[n]{a_n} = k$$

mavjud bo`lsin. U holda :

1) $k < 1$ bo`lganda (1) qator yaqinlashuvchi bo`ladi,
2) $k > 1$ bo`lganda (1) qator uzoqlashuvchi bo`ladi.

1-eslatma. Koshi alomatidagi (2) va (3) tengsizliklar n ning biror n_0 qiymatidan boshlab bajarilganda ham tasdiq o`rinli bo`ladi.

2-eslatma. Koshi alomatining limit ko`rinishidagi ifodasida $k = 1$ bo`lsa, u holda (1) qator yaqinlashuvchi ham, uzoqlashuvchi ham bo`lishi mumkin.

4^0. Dalamber alomati. **[3, Corollary 10, p.100]** Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \quad (1)$$

qatorda barcha $n \geq 1$ uchun

$$\frac{a_{n+1}}{a_n} \leq q < 1 \qquad (a_n > 0\ ,\ n = 1,2...) \qquad (4)$$

bo`lsa, (1) qator yaqinlashuvchi bo`ladi;

$$\frac{a_{n+1}}{a_n} \geq 1 \qquad (a_n > 0\ ,\ n = 1,2...) \qquad (5)$$

bo`lsa, (1) qator uzoqlashuvchi bo`ladi.

◄ Aytaylik, (1) qator hadlari uchun

$$\frac{a_{n+1}}{a_n} \leq q < 1$$

bo`lsin. Bu tengsizlikni quyidagicha

$$\frac{a_{n+1}}{a_n} \leq \frac{q^{n+1}}{q^n} \qquad (q < 1)$$

yozish mumkin.

Ravshanki,

$$\sum_{n=1}^{\infty} q^n \qquad (0 < q < 1)$$

qator (geometrik qator) yaqinlashuvchi. 50-ma`ruzada kelti-rilgan 3-teoremadan foydalanib, berilgan qatorning yaqinlashuvchi bo`lishini topamiz.

(1) qator hadlari uchun

$$\frac{a_{n+1}}{a_n} \geq 1$$

bo`lganda (1) qatorning uzoqlashuvchi bo`lishini aniqlash qiyin emas. ►

Dalamber alomatining quyidagi limit ko`rinishidagi tasdiqidan foydalaniladi.

Faraz qilaylik, musbat hadli (1) qatorda

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = d$$

limit mavjud bo`lsin. U holda :

1) $d < 1$ bo`lganda (1) qator yaqinlashuvchi bo`ladi,
2) $d > 1$ bo`lganda (1) qator uzoqlashuvchi bo`ladi.

3-eslatma. Dalamber alomatidagi (4) va (5) tengsizliklar n ning biror n_0 qiymatidan boshlab bajarilganda ham tasdiq o`rinli bo`ladi.

4-eslatma. Dalamber alomatining limit ko`rinishidagi ifodasida $d = 1$ bo`lsa, u holda (1) qator yaqinlashuvchi ham, uzoqlashuvchi ham bo`lishi mumkin.

Musbat hadli qatorlar mavzusida bayon etilgan taqqoslash teoremalaridan foydalanib, yaqinlashish alomatlarini keltiramiz.

5^0. Koshining integral alomati. **[3, Proposition 2., p.101]** Faraz qilaylik, musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \qquad (1)$$

qator berilgan bo`lsin. Ayni paytda, $[1,+\infty)$ oraliqda berilgan $f(x)$ funksiya quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[1,+\infty)$ da uzluksiz,
2) $f(x)$ funksiya $[1,+\infty)$ da kamayuvchi,
3) $\forall x \in [1,+\infty)$ da $f(x) \geq 0$,
4) $f(n) = a_n \quad (n = 1,2,3,...)$.

Bunda berilgan qator ushbu

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$$

ko`rinishga keladi.

Yuqoridagi shartlardan foydalanib, $n < x < n+1 \ (n \in N)$ bo`lganda

$$f(n) \geq f(x) \geq f(n+1), \text{ ya`ni } a_n \geq f(x) \geq a_{n+1}$$

bo`lishini topamiz. Keyingi tengsizlikni $[n, n+1]$ oraliq bo`yicha integrallash natijasida

$$a_{n+1} \leq \int_n^{n+1} f(x)dx \leq a_n \qquad (2)$$

bo`lishi kelib chiqadi.

Endi berilgan

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$$

qator bilan birga ushbu

$$\sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx \qquad (3)$$

qatorni qaraymiz. Bu qatorning qismiy yig`indisi

$$\sum_{k=1}^{n} \int_k^{k+1} f(x)dx = \int_1^{n+1} f(x)dx$$

bo`ladi.

Aytaylik, $F(x)$ funksiya $[1,+\infty]$ oraliqda $f(x)$ funksiyaning boshlang`ich funksiyasi bo`lsin: $F'(x) = f(x)$.

Uni quyidagicha

$$F(x) = \int_1^x f(t)dt \quad , \qquad F(1) = 0$$

ifodalash mumkin. Natijada

$$\sum_{k=1}^{n} \int_k^{k+1} f(x)dx = F(n+1)$$

bo`ladi.

Agar $n \to \infty$ da $F(n+1)$ chekli songa intilsa, (bu holda (3) qatorning qismiy yig`indisi chekli limitga ega bo`ladi) unda (3) qator yaqinlashuvchi.

Binobarin, $\int_1^n f(x)dx \quad (n = 1,2,3,...)$ ketma-ketlik yuqoridan chegaralangan bo`ladi. (2) munosabatga ko`ra berilgan qatorning qismiy yig`indilaridan iborat ketma-ketlik yuqoridan chegaralangan bo`lib, musbat hadli qatorlarning yaqinlashuvchiligi haqidagi teoremaga muvofiq berilgan $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`ladi.

Agar $n \to \infty$ da $F(n+1) \to \infty$ bo`lsa, berilgan qator uzoqla-shuvchi bo`ladi.

Shunday qilib, quyidagi integral alomatga kelamiz.

Koshining integral alomati. Agar

$$\lim_{x \to +\infty} F(x) = b$$

bo`lib, $b$ chekli son bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo`ladi, $b = \infty$ bo`lsa, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`ladi.

6^0. Raabe alomati. Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \qquad (1)$$

qatorda $n \in N$ ning biror $n_0 (n_0 \ge 1)$ qiymatidan boshlab, $n > n_0$ uchun

$$n(1 - \frac{a_{n+1}}{a_n}) \ge r > 1$$

bo`lsa, (1) qator yaqinlashuvchi bo`ladi,

$$n(1 - \frac{a_{n+1}}{a_n}) \le 1$$

bo`lsa, (1) qator uzoqlashuvchi bo`ladi.

◀Aytaylik, (1) qator hadlari uchun

$$n(1 - \frac{a_{n+1}}{a_n}) \ge r > 1$$

bo`lsin. Bu tengsizlikdan

$$\frac{a_{n+1}}{a_n} \le 1 - \frac{r}{n} \qquad (4)$$

bo`lishi kelib chiqadi.

Endi $r > \alpha > 1$ tengsizlikni qanoatlantiruvchi α sonini olib, uni

$$\alpha = \lim_{n\to\infty}\frac{(1-\frac{1}{n})^{\alpha}-1}{-\frac{1}{n}}$$

kabi ifodalaymiz. Limit xossasiga ko`ra, shunday $n_0' \in N$ topiladiki, barcha $n > n_0'$ lar uchun

$$\frac{(1-\frac{1}{n})^{\alpha}-1}{-\frac{1}{n}} \le r\,,$$

ya`ni

$$(1-\frac{1}{n})^{\alpha} \ge 1-\frac{r}{n} \qquad (5)$$

tengsizlik o`rinli bo`ladi.

(4) va (5) munosabatlardan barcha $n > \bar{n}_0 \quad (\bar{n}_0 = \max\{n_0, n_0'\})$ lar uchun

$$\frac{a_{n+1}}{a_n} \le (1-\frac{1}{n})^{\alpha} = \frac{\frac{1}{n^{\alpha}}}{\frac{1}{(n-1)^{\alpha}}}$$

bo`lishi kelib chiqadi.

Bu tengsizlikni va $\alpha > 1$ bo`lganda $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ qatorning yaqinlashuvchiligini e`tiborga olib, so`ng berilgan $\sum_{n=1}^{\infty} a_n$ qatorning yaqinlashuvchi bo`lishini topamiz.

Endi (1) qatorning hadlari uchun $n > n_0$ bo`lganda

$$n(1-\frac{a_{n+1}}{a_n}) \le 1$$

bo`lsin. Bu tengsizlikni quyidagicha:

$$\frac{a_{n+1}}{a_n} \ge \frac{\frac{1}{n}}{\frac{1}{n-1}}$$

yozish mumkin.

Bu tengsizlikni va $\sum_{n=1}^{\infty}\frac{1}{n}$ qatroning uzoqlashuvchiligi e`tiborga olib, berilgan $\sum_{n=1}^{\infty} a_n$ qatorning uzoqlashuvchi bo`lishini topamiz. ►

Ko`p hollarda Raabe alomatining quyidagi limit ko`rinishidan foydalaniladi:

Faraz qilaylik, musbat hadli (1) qator hadlari uchun

$$\lim_{n\to\infty} n(1-\frac{a_{n+1}}{a_n}) = \rho$$

mavjud bo`lsin. U holda:

1) $\rho > 1$ bo`lganida (1) qator yaqinlashuvchi bo`ladi,
2) $\rho < 1$ bo`lganida (1) qator uzoqlashuvchi bo`ladi.

Gauss alomati. Agar (1) qator uchun

$$\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + \frac{\theta_n}{n^{1+\varepsilon}} \qquad \left(|\theta_n| < c, \quad \varepsilon > 0\right)$$

bo`lsa, u holda
1) $\lambda > 1$ bo`lganda (1) qator yaqinlashuvchi,
2) $\lambda < 1$ bo`lganda (1) qator uzoqlashuvchi,
3) $\lambda = 1, \quad \mu > 1$ bo`lganda (1) qator yaqinlashuvchi,
4) $\lambda = 1, \quad \mu \le 1$ bo`lganda (1) qator uzoqlashuvchi
bo`ladi.

2-Amaliy mashg`ulot.

1-misol. Ushbu

$$\sum_{n=1}^{\infty} \sin\frac{\pi}{2^n} = \sin\frac{\pi}{2} + \sin\frac{\pi}{2^2} + \ldots + \sin\frac{\pi}{2^n} + \ldots$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Ravshanki, bu qator hadlari uchun

$$0 < \sin\frac{\pi}{2^n} < \frac{\pi}{2^n} \qquad (n = 1,2,3,\ldots)$$

tengsizlik o‘rinli bo‘ladi.

Natijada berilgan qatorning har bir hadi yaqinla-shuvchi $\sum_{n=1}^{\infty}\frac{1}{2^n}$ qatorning (geometrik qatorning) mos hadidan kichik. 2-teoremaga muvofiq berilgan qator yaqinlashuvchi bo‘ladi. ►

2-misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Berilgan qator bilan birga uzoqlashuvchiligi ma’lum bo‘lgan $\sum_{n=1}^{\infty}\frac{1}{n}$ garmonik qatorni qaraymiz. Bu qatorlarnig umumiy hadlari uchun

$$\lim_{n\to\infty}\frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{n^{1+\frac{1}{n}}}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}}=1$$

bo‘ladi. Demak, berilgan qator uzoqlashuvchi. ►

3-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{5+3(-1)^n}{2^{n+3}}$$

qator yaqinlashuvchilikka tekshirilsin.

◄Bu qatorning umumiy hadi

$$a_n=\frac{5+3(-1)^n}{2^{n+3}}$$

bo`lib, uning uchun

$$2\le\frac{5+3(-1)^n}{2^{n+3}}\le\frac{8}{2^{n+3}}=\frac{1}{2^n}$$

bo`ladi. Ravshanki,

$$\sum_{n=1}^{\infty}\frac{1}{2^n}$$

qator (maxraji $q=\frac{1}{2}$ ga teng bo`lgan geometrik qator) yaqinlashuvchi. Unda 2-Teoremaga ko`ra berilgan qator yaqinlashuvchi bo`ladi.►

4-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{3^n-n}=\frac{1}{3-1}+\frac{1}{3^2-2}+\ \ldots+\frac{1}{3^n-n}+\ldots$$

qator yaqinlashuvchilikka tekshirilsin.

◄Ma`lumki, quyidagi

$$\sum_{n=1}^{\infty}\frac{1}{3^n}=\frac{1}{3}+\frac{1}{3^2}+\ldots\ +\frac{1}{3^n}+\ \ldots$$

qator yaqinlashuvchi. (maxraji $q=\frac{1}{3}$ bo`lgan geometrik qator bo`lganligi uchun).

Bu qator bilan berilgan qatorga 2-Teoremani qo`llaymiz:

$$\left(a_n=\frac{1}{3^n-n},\ \ b_n=\frac{1}{3^n}\right)$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(\frac{1}{3^n-n}:\frac{1}{3^n}\right)=\lim_{n\to\infty}\frac{3^n}{3^n-n}=\lim_{n\to\infty}\frac{1}{1-\frac{n}{3^n}}=1.$$

Demak, alomatga ko`ra berilgan qator yaqinlashuvchi bo`ladi.►

5-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ \ldots+\frac{1}{\sqrt{n}}+\ldots$$

qator yaqinlashuvchilikka tekshirilsin.

◄Bu qatorning qismiy yig`indisi

$$S_n=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ \ldots+\frac{1}{\sqrt{n}}$$

uchun quyidagi tengsizlik o`rinli bo`ladi:

$$S_n>\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\ \ldots\ +\frac{1}{\sqrt{n}}=n\cdot\frac{1}{\sqrt{n}}=\sqrt{n}$$

Unda

$$\lim_{n\to\infty}S_n=+\infty$$

bo`lganligi sababli, berilgan qator uzoqlashuvchi bo`ladi.►

Mashq. Garmonik qator $\sum_{n=1}^{\infty}\frac{1}{n}$ ning qismiy yiqindisi S_n uchun ushbu

$$0<S_n-\ln(n+1)<1\quad (n\geq 2)$$

tengsizlar o`rinli ekani ko`rsatilsin.

1-misol. Ushbu

$$\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^{n^2}$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Bu qatorning umumiy hadi

$$a_n=\left(\frac{n+1}{n+2}\right)^{n^2}$$

bo`lib, uning uchun

$$\sqrt[n]{a_n}=\left(\frac{n+1}{n+2}\right)^{n}=\frac{(1+\frac{1}{n})^n}{(1+\frac{2}{n})^n}$$

bo`ladi. Ravshanki,

$$\lim_{n\to\infty}\frac{(1+\frac{1}{n})^n}{(1+\frac{2}{n})^n}=\frac{1}{e}.$$

Demak, $k=\frac{1}{e}<1$, berilgan qator yaqinlashuvchi. ►

2-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{n!}{n^n}$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Berilgan qator uchun

$$a_n=\frac{n!}{n^n}\quad,\qquad a_{n+1}=\frac{(n+1)!}{(n+1)^{n+1}}$$

bo`lib,

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)!\cdot n^n}{(n+1)^{n+1}\cdot n!}=\frac{1}{(1+\frac{1}{n})^n}$$

bo`ladi. Ravshanki,

$$\lim_{n\to\infty}=\frac{1}{(1+\frac{1}{n})^n}=\frac{1}{e}.$$

Demak, $d=\frac{1}{e}<1$, berilgan qator yaqinlashuvchi. ►

3-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{n^{n^2}\cdot 2^n}{(n+1)^{n^2}}$$

qator yaqinlashuvchilikka tekshirilsin.

◄Berilgan qatorning umumiy hadi

$$a_n=\frac{n^{n^2}\cdot 2^n}{(n+1)^{n^2}}$$

bo`ladi. Bu qatorga Koshi alomatini qo`llab topamiz:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\left(\frac{n^{n^2}\cdot 2^n}{(1+n)^{n^2}}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^n\cdot 2}{(1+n)^n}=\lim_{n\to\infty}2\left(\frac{n}{1+n}\right)^n=\frac{2}{e}.$$

Ravshanki, $\frac{2}{e}<1$. Demak, berilgan qator Koshi alomatiga ko`ra yaqinlashuvchi bo`ladi.►

4-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{n^5}{2^n+3^n}$$

qator yaqinlashuvchilikka tekshirilsin.

◀Berilgan qator uchun

$$a_n=\frac{n^5}{2^n+3^n},\quad a_{n+1}=\frac{(n+1)^5}{2^{n+1}+3^{n+1}}$$

bo`lib,

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)^5}{2^{n+1}+3^{n+1}}\cdot\frac{2^n+3^n}{n^5}=\left(1+\frac{1}{n}\right)^5\frac{3^n\left(1+\left(\frac{2}{3}\right)^n\right)}{3^{n+1}\left(1+\left(\frac{2}{3}\right)^{n+1}\right)}=\frac{1}{3}\left(1+\frac{1}{n}\right)^5\cdot\frac{1+\left(\frac{2}{3}\right)^n}{1+\left(\frac{2}{3}\right)^{n+1}}$$

bo`ladi. Endi bu nisbatning limitini topamiz:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{3}\left(1+\frac{1}{n}\right)^5\cdot\frac{1+\left(\frac{2}{3}\right)^n}{1+\left(\frac{2}{3}\right)^{n+1}}=\frac{1}{3}<1$$

Demak, berilgan qator Dalamber alomatiga ko`ra yaqinlashuvchi bo`ladi.▶

1-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}=1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\ldots+\frac{1}{n^\alpha}+\ldots\qquad(\alpha>0)$$

qator yaqinlashuvchilikka tekshirilsin.

◀ Agar $f(x)=\frac{1}{x^\alpha}$ $(\alpha>0)$ deyilsa, unda bu funksiya $[1,+\infty)$ oraliqda integral alomatda keltirilgan barcha shartlarni qanoatlantiradi. Bu funksiyaning boshlang`ich funksiyasi

$$F(x)=\int_1^x f(t)dt=\int_1^x\frac{dt}{t^\alpha}=\frac{1}{1-\alpha}\left(\frac{1}{x^{\alpha-1}}-1\right)\qquad(\alpha\neq 1)$$

bo`ladi.

Ravshanki,

$$\lim_{x\to+\infty}F(x)=\lim_{x\to\infty}\frac{1}{1-\alpha}\left(\frac{1}{x^{\alpha-1}}-1\right)=$$

$$=\begin{cases}\frac{1}{\alpha-1}, & \text{agar}\quad \alpha>1\quad \text{bo`lsa},\\ \infty, & \text{agar}\quad \alpha<1\quad \text{bo`lsa},\end{cases}$$

bo`lib, $\alpha=1$ bo`lganda

$$\lim_{x\to+\infty}F(x)=\lim_{x\to\infty}\int_1^x\frac{dt}{t}=\infty$$

bo`ladi.

Demak, integral alomatga ko`ra

$$\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$$

qator $\alpha > 1$ bo`lganda yaqinlashuvchi, $\alpha \le 1$ bo`lganda uzoqlashuvchi bo`ladi. ►

Odatda, $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ qator umumlashgan garmonik qator deyiladi.

2-misol. Ushbu

$$\sum_{n=2}^{\infty} a^{-(1+\frac{1}{2}+...+\frac{1}{n-1})} \qquad (a>0)$$

qator yaqinlashuvchilikka tekshirilsin.

◄ Bu qator uchun

$$n(1-\frac{a_{n+1}}{a_n}) = \left(1-\frac{a^{-(1+\frac{1}{2}+...+\frac{1}{n})}}{a^{-(1+\frac{1}{2}+...+\frac{1}{n-1})}}\right) = n\left(1-a^{-\frac{1}{n}}\right)$$

bo`lib,

$$\lim_{n\to\infty} n(1-\frac{a_{n+1}}{a_n}) = \lim_{n\to\infty}\frac{a^{-\frac{1}{n}}-1}{-\frac{1}{n}} = \ln a$$

bo`ladi. Agar $\ln a > 1$, ya`ni $a > e$ bo`lsa, berilgan qator yaqinlashuvchi bo`ladi.

Agar $\ln a < 1$, ya`ni $a < e$ bo`lsa, berilgan qator uzoqlashuvchi bo`ladi

Agar $a = e$ bo`lsa, Raabe alomati berilgan qatorning yaqinlashuvchiligi yoki uzoqlashuvchiligi haqida xulosa qilolmaydi.►

Mashqlar.

Raabe va Gauss alomatlaridan foydalanib quyidagi qatorlar yaqinlashishga tekshirilsin:

1. $\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\ \ ...\ (2n-1)}{2\cdot 4\cdot 6\ ...\ (2n)}\right]^{p}\frac{1}{n^{q}}$ 2. $\sum_{n=1}^{\infty}\frac{\ln 2\cdot\ln 3...\ln(n+1)}{\ln(2+a)\cdot\ln(3+a)...\ln(n+1+a)},\ \ a>0$

3. $\sum_{n=1}^{\infty}\frac{p(p+1)...(p+n-1)}{n!}\cdot\frac{1}{n^{q}}$.

Foydalanish uchun adabiyotlar

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Nazorat savollari

1. Qanday qatorni musbat hadli qator deyiladi?
2. Qator yaqinlashuvchi bo`lishining zaruriy va yetarli sharti qanday?
3. Musbat hadli qatorlarda qanday taqqoslash teoremalarini bilasiz?
4. Koshi alomatini ayting.
5. Dalamber alomatini ayting.
6. Integral alomatini ayting.
7. Raabe alomatini ayting.
8. Gauss alomatini ayting.

Glossariy

Musbat hadli qator - $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$ (1)

qator berilgan bo'lsin. Agar bu qatorda $a_n \geq 0 \quad (\forall n \in N)$ bo'lsa, (1) musbat hadli qator deyiladi.

Qator yaqinlashuvchi bo`lishining zaruriy va yetarli sharti - Musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$$

qatorning yaqinlashuvchi bo`lishi uchun

$$\{S_n\} = \{a_1 + a_2 + ... + a_n\} \quad (n = 1, 2, 3, ...)$$

ketma-ketlikning yuqoridan chegaralangan bo`lishi zarur va yetarli.

Koshi alomati. Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \qquad (1)$$

qatorda barcha $n \geq 1$ uchun

$$\sqrt[n]{a_n} \leq q < 1$$

bo`lsa, (1) qator yaqinlashuvchi bo`ladi;

$$\sqrt[n]{a_n} \geq 1$$

bo`lsa, (1) qator uzoqlashuvchi bo`ladi.

Dalamber alomati. Agar musbat hadli

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \qquad (1)$$

qatorda barcha $n \geq 1$ uchun

$$\frac{a_{n+1}}{a_n} \leq q < 1 \qquad (a_n > 0 \, , \, n = 1,2...)$$

bo`lsa, (1) qator yaqinlashuvchi bo`ladi;

$$\frac{a_{n+1}}{a_n} \geq 1 \qquad (a_n > 0 \, , \, n = 1,2...)$$

bo`lsa, (1) qator uzoqlashuvchi bo`ladi.

Keys banki

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=2}^{\infty} \frac{\ln^3 n}{n^2}$$

qatorning yaqinlashuvchi bo`lishi isbotlansin.

2-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=3}^{\infty} \frac{1}{(\ln)^{\ln \ln n}}$$

qatorning yaqinlashuvchi ekani isbotlansin.

3-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=1}^{\infty}\left(\frac{x}{a_n}\right)^n \qquad (x>0)$$

qator yaqinlashuvchilikka tekshirilsin, bunda $\{a_n\}$ ketma-ketlikning har bir hadi musbat bo`lib, $\lim\limits_{n\to\infty} a_n = a \quad (a\in R)$.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Test

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$\sum\limits_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}$ qatorni yaqinlashuvchilikka tekshirilsin.	uzoqlashuvchi	yaqinlashuvchi	Aniqlab bo`lmaydi
2	$\sum\limits_{n=1}^{\infty}\sin\frac{\pi}{2^n}$ qatorni yaqinlashuvchilikka tekshirilsin.	yaqinlashuvchi	uzoqlashuvchi	Aniqlab bo`lmaydi
3	$\sum\limits_{n=1}^{\infty}a_n$ - musbat hadli qator berilgan bo`lib, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=q$ bo`lsa, u holda	$q<1$ qator yaqinlashuvchi	$q>1$ qator yaqinlashuvchi	$q=1$ qator yaqinlashuvchi
4	$\sum\limits_{n=3}^{\infty}\frac{1}{(\ln)^{\ln\ln n}}$ qatorni yaqinlashuvchilikka tekshirilsin.	yaqinlashuvchi	uzoqlashuvchi	Aniqlab bo`lmaydi
5	$\sum\limits_{n=2}^{\infty}\frac{\ln^3 n}{n^2}$ q qatorni yaqinlashuvchilikka tekshirilsin.	yaqinlashuvchi	uzoqlashuvchi	Aniqlab bo`lmaydi
6	$\sum\limits_{n=1}^{\infty}a_n$ - musbat hadli qator	$q<1$ qator yaqinlashuvchi	$q>1$ qator yaqinlashuvchi	$q=1$ qator yaqinlashuvchi

	berilgan bo`lib, $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=q$ bo`lsa, u holda			
7	$\sum\limits_{n=1}^{\infty}\frac{1}{n\sqrt{n+1}}$ qatorni yaqinlashishga tekshiring.	yaqinlashuvchi	uzoqlashuvchi	Aniqlab bo`lmaydi
8	$\sum\limits_{n=1}^{\infty}\frac{1}{1+n}$ qatorni yaqinlashishga tekshiring.	uzoqlashuvchi	yaqinlashuvchi	ham yaqinlashuvchi, ham uzoqlashuvchi
	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	Qatorni yaqinlashishga tekshiring: $\sum\limits_{n=1}^{\infty}\frac{n^2}{\left(n+\frac{1}{n}\right)^{n^2}}$	yaqinlashuvchi	uzoqlashuvchi	Aniqlab bo`lmaydi
2	Qatorni yaqinlashishga tekshiring: $\sum\limits_{n=1}^{\infty}\left(\frac{an}{n+2}\right)^n, \quad a>0$	$0<\alpha<1$ da yaqinlashuvchi, $\alpha\geq 1$ da uzoqlashuvchi	$0<\alpha<1$ da uzoqlashuvchi $\alpha\geq 1$ da yaqinlashuvchi	$\alpha>0$ da yaqinlashuvchi
3	Qatorni yaqinlashishga tekshiring: $\sum\limits_{n=2}^{\infty}\frac{1}{(\ln n)^n}$	yaqinlashuvchi	uzoqlashuvchi	Aniqlab bo`lmaydi
4	Qatorni yaqinlashishga tekshiring: $\sum\limits_{n=1}^{\infty}3^{n+1}\left(\frac{n+2}{n+3}\right)^{n^2}$	uzoqlashuvchi	yaqinlashuvchi	Aniqlab bo`lmaydi
5	Qatorni yaqinlashishga tekshiring: $\sum\limits_{n=1}^{\infty}\frac{n!a^n}{n^n}, \quad a\neq e, \quad a>0$	$a<e$ da yaqinlashuvchi, $a>e$ da uzoqlashuvchi	$a<e$ da uzoqlashuvchi, $a>e$ da yaqinlashuvchi	Ixtiyoriy a da yaqinlashuvchi

3-Mavzu. Absolyut va shartli yaqinlashuvchi qatorlar

3-Ma’ruza.
REJA:

1^0. Absolyut va shartli yaqinlashuvchi qatorlar tushunchasi.
2^0. Absolyut yaqinlashuvchi qatorlarning xossalari.

Tayanch so`z va iboralar: *Absolyut yaqinlashuvchi sonli qator, shartli yaqinlashuvchi sonli qator, Koshi teoremasi, Dalamber alomati, Koshi alomati, absolyut yaqinlashuvchi qatorlarning xossalari.*

1^0. Absolyut va shartli yaqinlashuvchi qatorlar tushunchasi.

Faraz qilaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \qquad (1)$$

qator berilgan bo`lsin. Bu qatorning har bir hadi ixtiyoriy ishorali haqiqiy sonlardan iborat. (Odatda, bunday qator **ixtiyoriy hadli qator** deyiladi.)

(1) qator hadlarining absolyut qiymatlaridan ushbu

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_n| + ... + |a_n| + ... \qquad (2)$$

qatorni tuzamiz.

1-teoema. Agar (2) qator yaqinlashuvchi bo`lsa, u holda (1) qator ham yaqinlashuvchi bo`ladi.

◄ Aytaylik, (2) qator yaqinlashuvchi bo`lsin. Unda qator yaqinlashuvchiligi haqidagi Koshi teoremasiga ko`ra

$$\forall \varepsilon > 0, \quad \exists n_0 \in N, \quad \forall n > n_0 \quad m = 1,2,3... \text{ da}$$

$$|a_{n+1}| + |a_{n+2}| + ... + |a_{n+m}| < \varepsilon$$

bo`ladi. Ravshanki,

$$|a_{n+1} + a_{n+2} + ... + a_{n+m}| \le |a_{n+1}| + |a_{n+2}| + ... + |a_{n+m}| .$$

Keyingi ikki munosabatdan

$$\forall \varepsilon > 0, \quad \exists n_0 \in N, \forall n > n_0, \quad m = 1,2,3,... \text{ da}$$

$$|a_{n+1} + a_{n+2} + ... + a_{n+m}| < \varepsilon$$

bo`lishi kelib chiqadi. Koshi teoremasiga muvofiq (1) qator yaqinlashuvchi bo`ladi. ►

1-ta`rif.** **[3, Definition 21, p.97]** Agar $\sum_{n=1}^{\infty}|a_n|$ qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty}a_n$ qator **absolyut yaqinlashuvchi qator deyiladi.

Masalan, ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}(-1)^{n-1}=1-\frac{1}{2^{\alpha}}+\frac{1}{3^{\alpha}}-\frac{1}{4^{\alpha}}+...+\frac{(-1)^{n-1}}{n^{\alpha}}+...$$

qator $\alpha>1$ bo`lganda absolyut yaqinlashuvchi qator bo`ladi, chunki

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^{\alpha}}(-1)^{n-1}\right|=1+\frac{1}{2^{\alpha}}+\frac{1}{3^{\alpha}}-\frac{1}{4^{\alpha}}+...+\frac{1}{n^{\alpha}}+...$$

umumlashgan garmonik qator $\alpha>1$ bo`lganda yaqinlashuvchi.

2-ta`rif**. Agar $\sum_{n=1}^{\infty}a_n$ qator yaqinlashuvchi bo`lib, $\sum_{n=1}^{\infty}|a_n|$ qator uzoqlashuvchi bo`lsa, $\sum_{n=1}^{\infty}a_n$ qator **shartli yaqinlashuvchi qator deyiladi.

Misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n-1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{(-1)^{n-1}}{n}+...$$

qator shartli yaqinlashuvchi qator bo`ladi.

◄ Ravshanki, berilgan qatorning qismiy yig`indisi

$$S_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{(-1)^{n-1}}{n} \tag{3}$$

bo`ladi.

Ma`lumki, $\ln(1+x)$ funksiyaning Makloren formulasiga ko`ra yoyilmasi

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...+\frac{(-1)^{n-1}x^n}{n}+R_{n+1}(x),$$

bo`lib, $0\le x\le 1$ bo`lganda

$$|R_{n+1}(x)|<\frac{1}{n+1}$$

bo`lar edi. (qaralsin [1], 6-bob, 7-§)

Xususan, $x=1$ bo`lganda

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{(-1)^{n-1}}{n}+R_{n+1}(1)$$

$$|R_{n+1}(1)| < \frac{1}{n+1} \qquad (4)$$

bo`ladi.

(3) va (4) munosabatlardan

$$\ln 2 = S_n + R_{n+1}(1)$$

va undan

$$|S_n - \ln 2| < \frac{1}{n+1}$$

bo`lishi kelib chiqadi.

Demak, $n \to \infty$ da $S_n \to \ln 2$. Bu esa qaralayotgan qatorning yaqinlashuvchi ekanini bildiradi.

Ayni paytda, berilgan qator hadlarining absolyut qiymat-laridan tuzilgan qator

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$$

garmonik qator bo`lib, uning uzoqlashuvchiligi ma`lum. Demak, berilgan qator shartli yaqinlashuvchi qator.►

2–teorema (Koshi teoremasi). **[3, Theorem 6, p.95]** Ixtiyoriy hadli (1) qatorning yaqinlashuvchi bo`lishi uchun

$$\forall \;\; \varepsilon > 0, \;\; \exists \;\; n_0 \in N, \;\; \forall \;\; n > n_0, \;\;\; m = 1, \; 2, \; 3, ...:$$

$$|S_{n+m} - S_n| = |a_{n+1} + a_{n+2} + ... + a_{n+m}| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Endi

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + ... + |a_n| + ...$$

qatorning musbat hadli qator ekanini e`tiborga olib, $\sum_{n=1}^{\infty} a_n$ qatorning absolyut yaqinlashuvchiligini ifodalovchi alomatlarni keltiramiz. Ularning isboti 51-ma`ruzada bayon etilgan alomatlardan kelib chiqadi.

Dalamber alomati. Faraz qilaylik ,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... \qquad (a_n \neq 0, \quad n = 1,2,...)$$

qator hadlari uchun

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = d$$

limit mavjud bo`lsin. U holda:

1) $d<1$ bo`lganda, $\sum_{n=1}^{\infty} a_n$ qator absolyut yaqinlashuvchi bo`ladi,

2) $d>1$ bo`lganda, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`ladi.

Koshi alomati. Faraz qilaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

qator hadlari uchun

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = K$$

limit mavjud bo`lsin. U holda:

1) $K<1$ bo`lganda, $\sum_{n=1}^{\infty} a_n$ qator absolyut yaqinlashuvchi bo`ladi.

2) $K>1$ bo`lganda, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo`ladi.

***2⁰. Absolyut yaqinlashuvchi qatorlarning xossalari*.**

Absolyut yaqinlashuvchi qatorlarning xossalarini keltiramiz.

1-Xossa. [3, p.95] Agar qator absolyut yaqinlashuvchi bo`lsa, u holda bu qator yaqinlashuvchi bo`ladi.

◄ Bu xossaning isboti 1-teoremadan kelib chiqadi.►

2-Xossa. Agar

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

qator absolyut yaqinlashuvchi bo`lib, $\{b_n\}$ sonlar ketma-ketligi chegaralangan bo`lsa, u holda

$$\sum_{n=1}^{\infty} a_n b_n = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n + \ldots \qquad (5)$$

qator absolyut yaqinlashuvchi bo`ladi.

◄ Shartga ko`ra $\{b_n\}$ sonlar ketma-ketligi chegaralangan. Demak,

$$\exists M>0, \ \forall n \in N \text{ da } |b_n| \le M \qquad (6)$$

bo`ladi.

(1) qator absolyut yaqinlashuvchi. Unda Koshi teoremasiga ko`ra $\forall \varepsilon > 0$ son olinganda ham $\frac{\varepsilon}{M}$ ga ko`ra shunday $n_0 \in N$ topiladiki, $\forall n > n_0$ va $m = 1,2,3,\ldots$ bo`lganda

$$|a_{n+1}|+|a_{n+2}|+...+|a_{n+m}|<\frac{\varepsilon}{M} \qquad (7)$$

bo`ladi.

(6) va (7) munosabatlardan foydalanib topamiz:

$$|a_{n+1}b_{n+1}|+|a_{n+2}b_{n+2}|+...+|a_{n+m}b_{n+m}|\le$$

$$\le M\left(|a_{n+1}|+|a_{n+2}|+...+|a_{n+m}|\right)<\varepsilon.$$

Yana Koshi teoremasidan foydalanib, $\sum_{n=1}^{\infty}a_nb_n$ qatorning absolyut yaqinlashuvchi ekanini topamiz. ►

3-Xossa. Faraz qilaylik,

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+...+a_n+... \qquad (1)$$

qator hadlarining o`rinlarini almashtirish natijasida ushbu

$$\sum_{j=1}^{\infty}a'_j=a'_1+a'_2+...+a'_j+... \qquad (8)$$

qator hosil qilingan bo`lsin.

Ravshanki, (8) qatorning har bir a'_j hadi $(j=1,2,...)$ (1) qatorning tayin bir a_{k_j} hadining aynan o`zidir, ya`ni $\forall j\in N,\ \exists k_j\in N, a_{k_j}=a'_j$ bo`ladi.

Agar (1) qator absolyut yaqinlashuvchi bo`lib,uning yig`indisi S ga teng bo`lsa, u holda bu qator hadlarining o`rinlarini ixtiyoriy ravishda almashtirishdan hosil bo`lgan (8) qator absolyut yaqinlashuvchi va uning yig`indisi ham S ga teng bo`ladi.

◄ Aytaylik, (1) qator absolyut yaqinlashuvchi bo`lib, uning yig`indisi S ga teng bo`lsin.

(8) qator hadlarining absolyut qiymatlaridan tuzilgan $\sum_{j=1}^{\infty}|a'_j|$ qatorning qismiy yig`indisini σ'_n bilan belgilaylik:

$$\sigma'_n=\sum_{j=1}^{n}|a'_j|. \qquad (a'_j=a_{k_j})$$

Agar $n'=\max\limits_{1\le j\le n}k_j$ deyilsa, unda $n'\ge n$ va $\forall n\in N$ bo`lganda

$$\sigma'_n\le\sum_{k=1}^{n'}|a_k|$$

bo`ladi.

(1) qator absolyut yaqinlashuvchi bo`lgani sababli uning qismiy yig`indilari ketma-ketligi yuqoridan chegaralan-gandir. Binobarin, σ'_n yig`indi ham yuqoridan chegaralangan bo`ladi. Unda musbat hadli qatorning yaqinlashuvchiligi haqidagi teoremaga ko`ra $\sum_{j=1}^{\infty}|a'_j|$ qator va ayni paytda $\sum_{j=1}^{\infty}a'_j$ qator ham yaqinlashuvchi bo`ladi. Demak, $\sum_{j=1}^{\infty}a'_j$ qator absolyut yaqinlashuvchi. Uning yig`indisini S' deylik.

Endi berilgan $\sum_{n=1}^{\infty}a_n$ qator hadlarining o`rinlarini ixtiyoriy ravishda almashtirishdan hosil bo`lgan

$$\sum_{n=1}^{\infty}a'_k = a'_1 + a'_2 + \ldots + a'_k + \ldots$$

qator yig`indisini $S$ ga teng ekanini isbotlaymiz. Buning uchun $\forall\varepsilon>0$ ga ko`ra shunday $\overline{n}\in N$ topilib, $\forall n>\overline{n}$ da

$$\left|\sum_{k=1}^{n}a'_k - S\right| < \varepsilon \qquad (9)$$

bo`lishini ko`rsatish yetarli bo`ladi.

Ixtiyoriy musbat ε sonni tayinlab olamiz. Modomiki, $\sum_{k=1}^{\infty}a_k$ qator absolyut yaqinlashuvchi ekan, unda Koshi teoremasiga binoan olingan $\varepsilon>0$ songa ko`ra shunday n_0 nomer topiladiki,

$$\sum_{k-n_0+1}^{n_0+m}|a_k| < \frac{\varepsilon}{2} \qquad (m=1,2,3,\ldots) \qquad (10)$$

shuningdek, qatorning yaqinlashish ta`rifiga ko`ra

$$\left|\sum_{k=1}^{n_0}a_k - S\right| < \frac{\varepsilon}{2} \qquad (11)$$

bo`ladi.

Yuqoridagi natural son $\overline{n}$ ni shunday katta qilib olamizki, $\sum_{n=1}^{\infty}a'_k$ qatorning $\overline{n}$ dan katta bo`lgan $n$ nomerli ixtiyoriy qismiy yig`indisi

$$S'_n = \sum_{k=1}^{n}a'_k \text{ da } \sum_{k=1}^{\infty}a_k$$

qatorning barcha dastlabki n_0 ta hadi qatnashsin.

Ravshanki,

$$\sum_{k=1}^{n}a'_k - S = \left(\sum_{k=1}^{n}a'_k - \sum_{k=1}^{n_0}a_k\right) + \left(\sum_{k=1}^{n_0}a_k - S\right).$$

Keyingi munosabatdan va (11) tengsizlikni e`tiborga olib topamiz.

$$\left|\sum_{k=1}^{n} a'_k - S\right| \le \left|\sum_{k=1}^{n} a'_k - \sum_{k=1}^{n_0} a_k\right| + \left|\sum_{k=1}^{n_0} a_k - S\right| < \left|\sum_{k=1}^{n} a'_k - \sum_{k=1}^{n_0} a_k\right| + \frac{\varepsilon}{2} \qquad (12)$$

Ma`lumki, $n > \bar{n}$ bo`lganda $\sum_{k=1}^{\infty} a'_k$ qatorda $\sum_{k=1}^{\infty} a_k$ qatorning barcha dastlabki n_0 ta hadi qatnashadi. Binobarin,

$$\sum_{k=1}^{n} a'_k - \sum_{k=1}^{n_0} a_k$$

ayirma $\sum_{k=1}^{\infty} a_k$ qatorning, har bir hadining nomeri n_0 dan katta bo`lgan $n - n_0$ ta hadining yig`indisidan iborat.

Endi natural m sonni shunday katta qilib olamizki, bunda $n_0 + m$ son yuqorida aytilgan barcha $n - n_0$ ta hadlarning nomerlaridan katta bo`lsin.

Unda

$$\left|\sum_{k=1}^{n} a'_k - \sum_{k=1}^{n_0} a_k\right| \le \sum_{k=n_0+1}^{n_0+m} |a_k| \qquad (13)$$

bo`ladi.

(12), (13) va (10) munosabatlardan foydalanib, (9) tengsizlikning, ya`ni

$$\left|\sum_{k=1}^{n} a'_k - S\right| < \varepsilon$$

tengsizlikning bajarilishini topamiz. ►

3-Amaliy mashg`ulot

1–misol. Koshi teoremasidan foydalanib, ushbu

$$\sum_{n=1}^{\infty} \frac{\sin x}{n(n+1)}$$

qatorning yaqinlashuvchi bo`lishi ko`rsatilsin.

◄Ravshanki, bu qator uchun

$$|S_{n+m} - S_n| = \left|\sum_{k=m+1}^{n+m} \frac{\sin kx}{k(k+1)}\right| \le \sum_{k=n+1}^{n+m} \frac{1}{k(k+1)} = \frac{1}{n+1} - \frac{1}{n+m} < \frac{1}{n+1}$$

bo`ladi. Demak,

$$|a_{n+1} + a_{n+2} + \ldots + a_{n+m}| < \frac{1}{n+1}$$

va

$$\lim_{n\to\infty}\frac{1}{n+1}=0.$$

Bundan esa berilgan qatorning yaqinlashuvchi bo`lishi kelib chiqadi.►

Mashqlar

Aytaylik,

$$\sum_{n=1}^{\infty}a_n \qquad (*)$$

ixtiyoriy hadli qator bo`lib,

$$u_n=\begin{cases}a_n, a_n\ge 0,\\ 0,\ \ a_n<0\end{cases} \qquad v_n=\begin{cases}-a_n,\ a_n<0,\\ 0,\ \ \ a_n\ge 0\end{cases}$$

bo`lsin.

1. Agar (*) qator absolyut yaqinlashuvchi bo`lsa, u holda $\sum_{n=1}^{\infty}u_n$, $\sum_{n=1}^{\infty}v_n$ qatorlar yaqinlashuvchi va

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}u_n-\sum_{n=1}^{\infty}v_n,\quad \sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}u_n+\sum_{n=1}^{\infty}v_n$$

bo`lishi isbotlansin.

2. Agar (*) qator shartli yaqinlashuvchi bo`lsa, u holda $\sum_{n=1}^{\infty}u_n$, $\sum_{n=1}^{\infty}v_n$ qatorlarning uzoqlashuvchi bo`lishi isbotlansin.

Quyidagi qatorlar absolyut va shartli yaqinlashishga tekshirilsin.

3. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{p+\frac{1}{n}}}$. 4. $\sum_{n=2}^{\infty}\ln\left(1+\frac{(-1)^n}{n^p}\right)$. 5. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n\sin^{2n}x}{n}$.

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Nazorat savollari

1. Absolyut yaqinlashuvchi qatorning ta'rifini ayting.
2. Shartli yaqinlashuvchi qator deb qanday qatorga aytiladi?
3. Koshi teoremasini ayting.
4. Absolyut yaqinlashuvchi qatorlarning qanday xossalarini bilasiz?

Glossariy

Absolyut yaqinlashuvchi qator-qator hadlarining absolyut qiymatlaridan tuzilgan ushbu $\sum_{n=1}^{\infty}|a_n| = |a_1| + |a_n| + \ldots + |a_n| + \ldots$ qator yaqinlashuvchi bo'lsa, $\sum_{n=1}^{\infty} a_n$ qator absolyut yaqinlashuvchi qator deyiladi.

Shartli yaqinlashuvchi qator -$\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi bo'lib, $\sum_{n=1}^{\infty}|a_n|$ qator uzoqlashuvchi bo'lsa, $\sum_{n=1}^{\infty} a_n$ qator shartli yaqinlashuvchi qator deyiladi.

Dalamber alomati-$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (a_n \neq 0, \quad n = 1,2,\ldots)$ qator hadlari uchun

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = d$$

limit mavjud bo'lsin. U holda:

1) $d < 1$ bo'lganda, $\sum_{n=1}^{\infty} a_n$ qator absolyut yaqinlashuvchi bo'ladi,
2) $d > 1$ bo'lganda, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo'ladi.

Koshi alomati-Faraz qilaylik,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

qator hadlari uchun

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = K$$

limit mavjud bo'lsin. U holda:

1) $K < 1$ bo'lganda, $\sum_{n=1}^{\infty} a_n$ qator absolyut yaqinlashuvchi bo'ladi.
2) $K > 1$ bo'lganda, $\sum_{n=1}^{\infty} a_n$ qator uzoqlashuvchi bo'ladi.

Keys banki

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n-1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{(-1)^{n-1}}{n}+...$$

qator shartli yaqinlashuvchi qator bo‘ladi.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Test

	Test topshirig`i	To’g`ri javob	Muqobil javob	Muqobil javob
1	Quyidagilardan qaysi biri Dalamber alomatini ifodalaydi:	$l=\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}$	$l=\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}$	$l=\lim\limits_{n\to\infty}\sqrt{\frac{a_{n+1}}{a_n}}$
2	Ko’rsatilgan qatorlardan qaysi biri yaqinlashuvchi?	$\sum_{n=1}^{\infty}\frac{4}{n^4}$	$\sum_{n=1}^{\infty}(-1)^n$	$\sum_{n=1}^{\infty}\frac{n+4}{n+2}$
3	Ko’rsatilgan qatorlardan qaysi biri uzoqlashuvchi?	$\sum_{n=1}^{\infty}\frac{1}{3n}$	$\sum_{n=1}^{\infty}\frac{5}{n^2}$	$\sum_{n=1}^{\infty}\frac{1}{7^n}$
4	Tasdiqni davom ettiring: Agar sonli qator $\sum_{n=1}^{\infty}\lvert a_n\rvert$ yaqinlashsa, u holda $\sum_{n=1}^{\infty}a_n$ …	ham yaqinlashadi	uzoqlashadi	uzoqlashishi ham yaqinlashishi ham bo`lishi mumkin
5	α ning qanday qiymatlarida $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{\alpha}}$ sonli qator absolyut yaqinlashuvchi bo’ladi?	$\alpha>1$	$\alpha=1$	$\alpha\le 1$

4-Mavzu. Ixtiyoriy hadli qatorlarda yaqinlashish alomatlari.
4-Ma'ruza.
REJA:

1^0. Leybnits alomati.

2^0. Dirixle va Abel' alomatlari.

3^0. Mavzuga doir misollar.

Tayanch so`z va iboralar: *Ishorasi almashinuvchi qatorlar, Leybnits alomati, Dirixle va Abel' alomatlari.*

1^0. Leybnits alomati. **[3, p.100]** Ushbu

$$\sum_{n=1}^{\infty}(-1)^{n-1}c_n = c_1 - c_2 + c_3 - c_4 + \ldots + (-1)^{n-1}c_n + \ldots \qquad (1)$$

qatorni qaraymiz, bunda $c_n > 0 \quad (n = 1,2,3,\ldots)$.

Odatda, bunday qator hadlarining **ishoralari navbat bilan o`zgarib keladigan qator** deyiladi.

Ravshanki, (1) qator ixtiyoriy hadli qatorning bitta holidir.

Masalan, ushbu

$$\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1}\frac{1}{n} + \ldots$$

qator hadlarining ishoralari navbat bilan o`zgarib keladigan qator bo`ladi.

1-teorema (Leybnits alomati). Agar hadlarining ishoralari navbat bilan o`zgarib keladigan (1) qatorda:

1) $c_{n+1} < c_n$, $\quad (n = 1,2,3,\ldots)$

2) $\lim\limits_{n\to\infty} c_n = 0$

bo`lsa, u holda (1) qator yaqinlashuvchi bo`ladi.

◄ (1) qatorning dastlabki $2m$ ta $(m \in N)$ hadidan iborat qismiy yig`indisi

$$S_{2m} = c_1 - c_2 + c_3 - c_4 + \ldots + c_{2m-1} - c_{2m}$$

ni olaylik. Unda $S_{2(m+1)}$ uchun

$$S_{2(m+1)} = S_{2m} + (c_{2m+1} - c_{2m+2})$$

bo`lib, $c_{2m+2} < c_{2m+1}$ bo`lganligi sababli (bunda $c_{2m+1} - c_{2m+2} > 0$ bo`ladi)

$$S_{2(m+1)} > S_{2m} \qquad (m = 1,2,3,\ldots)$$

bo`ladi. Demak, $\{S_{2m}\}$ ketma-ketlik o`suvchi.

Endi S_{2m} yig`indini quyidagicha yozamiz:

$$S_{2m} = c_1 - (c_2 - c_3) - (c_4 - c_5) - ... - (c_{2m-2} - c_{2m-1}) - c_{2m}.$$

Bu tenglikning o`ng tomonidagi ifodada qatnashgan qavs ichidagi ayirmalarning, shuningdek $c_{2m}$ ning musbat bo`lishini e`tiborga olib,

$$S_{2m} < c_1$$

bo`lishini topamiz. Demak, $\{S_{2m}\}$ ketma-ketlik yuqoridan chegaralangan.

Monoton ketma-ketlikning limiti haqidagi teoremaga ko`ra

$$\lim_{m\to\infty} S_{2m} = S \qquad (S - \text{chekli son}) \qquad (2)$$

mavjud.

Endi (1) qatorning dastlabki $2m-1$ ta $(m \in N)$ sondagi hadidan iborat ushbu

$$S_{2m-1} = c_1 - c_2 + c_3 - c_4 + ... + c_{2m-1}$$

qismiy yig`indisini olaylik. Ravshanki,

$$S_{2m-1} = S_{2m} + c_{2m}.$$

Teoremaning $n \to \infty$ da $c_n \to 0$ bo`lishi sharti hamda (2) munosabatdan foydalanib topamiz:

$$\lim_{m\to\infty} S_{2m-1} = \lim_{m\to\infty}(S_{2m} + c_{2m}) = S.$$

Shunday qilib, berilgan (1) qatorning qismiy yig`indilaridan iborat ketma-ketlik chekli limitga ega ekani ko`rsatildi. Demak, (1) qator yaqinlashuvchi. ►

Masalan,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{(-1)^{n-1}}{n} + ... \qquad (3)$$

qator hadlari keltirilgan teoremaning barcha shartlarini qanoatlantiradi. Teoremaga ko`ra (3) qator yaqinlashuvchi bo`ladi ((3) qatorning yaqinlashuvi va yig`indisi $\ln 2$ ga teng bo`lishi ko`rsatilgan edi).

2^0. Dirixle va Abel' alomatlari. Faraz qilaylik,

$$a_1\ ,\ a_2\ ,\ a_3\ ,... a_n\ ,...,$$

$$b_1\ , b_2\ ,\ b_3\ ,\ ...\ , b_n\ ,...$$

ixtiyoriy haqiqiy sonlar ketma-ketliklari bo`lib,

$$S_n = a_1 + a_2 + ... + a_n$$

bo`lsin. U holda $\forall n \in N$, $\forall m \in N$ uchun

$$\sum_{k=n}^{n+m} a_k b_k = \sum_{k=n}^{n+m-1} S_k (b_k - b_{k-1}) + S_{n+m} \cdot b_{n+m} - S_{n-1} b_n \qquad (4)$$

munosabat o`rinli bo`ladi.
Odatda, (4) munosabat Abel' ayniyati deyiladi.

◄ Ravshanki, $a_k = S_k - S_{k-1}$ bo`ladi. Unda

$$\sum_{k=n}^{n+m} a_k b_k$$

yig`indi ushbu ko`rinishga

$$\sum_{k=n}^{n+m} a_k b_k = \sum_{k=n}^{n+m} S_k b_k - \sum_{k=n}^{n+m} S_{k-1} b_k \tag{5}$$

keladi. Bu tenglikning o`ng tomonidagi birinchi qo`shiluvchini quyidagicha:

$$\sum_{k=n}^{n+m} S_k b_k = \sum_{k=n}^{n+m-1} S_k b_k + S_{n+m} b_{n+m},$$

ikkinchi qo`shiluvchini esa quyidagicha

$$\sum_{k=n}^{n+m} S_{k-1} b_k = \sum_{k=n-1}^{n+m-1} S_k b_{k+1} = S_{n-1} b_n + \sum_{k=n}^{n+m-1} S_k b_{k+1}$$

yozib olamiz.

Natijada (5) tenglik quyidagicha:

$$\sum_{k=n}^{n+m} S_k b_k = \sum_{k=n}^{n+m-1} S_k b_k + S_{n+m} b_{n+m} - \sum_{k=n}^{n+m-1} S_k b_{k+1} - S_{n-1} b_n =$$

$$= \sum_{k=n}^{n+m-1} S_k (b_k - b_{k+1}) - S_{n+m} b_{n+m} - S_{n-1} b_n$$

bo`ladi. ►

Dirixle alomati quyidagi teorema orqali ifodalanadi.

2-teorema (Dirixle alomati). Aytaylik,

$$\sum_{k=1}^{\infty} a_k b_k = a_1 b_1 + a_2 b_2 + \ldots + a_k b_k + \ldots \tag{6}$$

qator berilgan bo`lsin. Agar:

1) $\{b_k\}$ ketma-ketlik kamayuvchi va u cheksiz kichik miqdor,

2) $\sum_{k=1}^{\infty} a_k$ qatorning qismiy yig`indilari ketma-ketligi chegaralangan bo`lsa,

(6) qator yaqinlashuvchi bo`ladi.

◄ Agar $\sum_{k=1}^{\infty} a_k$ qatorning qismiy yig`indisini

$$S_n = a_1 + a_2 + \ldots + a_n$$

desak, unda teoremaning shartiga ko`ra, shunday $M>0$ son topiladiki, barcha $n\in N$ uchun

$$|S_n|\le M \qquad (7)$$

bo`ladi.

Shartga ko`ra $\{b_k\}$ ketma-ketlik kamayuvchi va u cheksiz kichik miqdor. Unda $\forall\varepsilon>0$ ga ko`ra shunday $n_0\in N$ topiladiki, $\forall n>n_0$ da

$$0\le b_n<\frac{\varepsilon}{2M} \qquad (8)$$

bo`ladi.

Endi

$$\sum_{k=n}^{n+m} a_k b_k$$

yig`indiga Abel' ayniyatini qo`llaymiz:

$$\sum_{k=n}^{n+m} a_k b_k = \sum_{k=n}^{n+m-1} S_k(b_k - b_{k+1}) + S_{n+m}b_{n+m} - S_{n-1}b_n .$$

Natijada

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| \le \sum_{k=n}^{n+m-1} |S_k(b_k - b_{k+1})| + |S_{n+m}b_{n+m}| + |S_{n-1}b_n| =$$

$$= \sum_{k=n}^{n+m-1} |S_k|\cdot(b_k - b_{k+1}) + |S_{n+m}|b_{n+m} + |S_{n-1}|\cdot b_n$$

bo`ladi.

(7) tengsizlikdan foydalanib topamiz:

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| \le M\left[\sum_{k=n}^{n+m-1}(b_k - b_{k+1}) + b_{n+m}\right] + M\cdot b_n .$$

Agar

$$\sum_{k=n}^{n+m-1}(b_k - b_{k+1}) + b_{n+m} = (b_n - b_{n+1}) + (b_{n+1} - b_{n+2}) + \ldots + (b_{n+m-1} - b_{n+m}) + b_{n+m} = b_n$$

bo`lishini e`tiborga olsak, unda

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| \le 2M\cdot b_n$$

bo`lib, (8) munosabatga ko`ra

$$\left|\sum_{k=n}^{n+m} a_k b_k\right| < \varepsilon$$

bo`ladi. Bundan Koshi teoremasiga ko`ra $\sum_{k=n}^{n+m} a_k b_k$ qatorning yaqinlashuvchiligi kelib chiqadi. ►

Quyida Abel' alomatini isbotsiz keltiramiz.

3–teorema (Abel'). Agar (6) qatorda:

1. $\sum_{n=1}^{\infty} a_n$ qator yaqinlashuvchi,
2. $\{b_n\}$ – chegaralangan, monoton ketma-ketlik bo`lsa,

u holda $\sum_{n=1}^{\infty} a_n b_n$ qator yaqinlashuvchi bo`ladi.

4-Amaliy mashg`ulot.

1-Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{\cos kx}{k} = \frac{\cos x}{1} + \frac{\cos 2x}{2} + \ldots + \frac{\cos kx}{k} + \ldots$$

qator yaqinlashuvchilikka tekshirilsin, bunda x – tayinlangan haqiqiy son.

◄ Agar $x = 2\pi$ bo`lsa, berilgan qator

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = \sum_{k=1}^{\infty} \frac{\cos 2\pi \cdot k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

garmonik qator bo`lib, u uzoqlashuvchi bo`ladi.

Aytaylik, $x \neq 2\pi$ bo`lsin. Berilgan qatorda

$$a_k = \cos kx \quad , b_k = \frac{1}{k}$$

belgilashlarni bajaramiz.

Ravshanki, $\{b_k\} = \left\{\frac{1}{k}\right\}$ ketma-ketlik kamayuvchi va cheksiz kichik miqdor bo`ladi $(k \to \infty$ da $\frac{1}{k} \to 0)$.

Endi $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \cos kx$ qatorning qismiy yig`indisi S_n ni topamiz:

$$S_n = \sum_{k=1}^{n} \cos kx = \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} 2\sin\frac{x}{2}\cos kx =$$

$$= \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} \left[\sin(k+\frac{1}{2})x - \sin(k-\frac{1}{2})x\right] = \frac{\sin(n+\frac{1}{2})x - \sin\frac{x}{2}}{2\sin\frac{x}{2}}.$$

Keyingi munosabatdan, 2π ga karrali bo`lmagan x lar uchun

$$|S_n| \le \frac{1}{\left|\sin\frac{x}{2}\right|}$$

bo`lishi kelib chiqadi. Demak, $\{S_n\}$ ketma-ketlik chegaralangan. Unda berilgan qator 2-teoremaga ko`ra yaqinlashuvchi bo`ladi.►

2–Misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

qatorning ixtiyoriy $x \in R$ da yaqinlashuvchi bo`lishi ko`rsatilsin.

◄Agar $x = k\pi, \ k \in Z$ bo`lsa, unda qatorning yaqinlashuvchi bo`lishi ravshan.

Aytaylik, $x \ne k\pi, \ k \in Z$ bo`lsin. Berilgan qatorda

$$\frac{1}{n} = a_n, \quad \sin nx = b_n \qquad (n \ge 1)$$

deyilsa, unda berilgan qator

$$\sum_{n=1}^{\infty} a_n b_n$$

ko`rinishdagi qatorga keladi. Bu qator uchun Dirixle teoremasining shartlarining bajarilishini ko`rsatamiz. Haqiqatdan ham,

$$a_n = \frac{1}{n}$$

bo`lganligidan, uning monotonligi hamda $\mathbf{n \to \infty}$ da $\mathbf{a_n \to 0}$ bo`lishini topamiz.

Endi uchinchi shartning bajarilishini ko`rsatamiz:

$$\left|\sum_{k=1}^{n} b_k\right| = \left|\sum_{k=1}^{n} \sin kx\right| = \frac{1}{2\left|\sin\frac{x}{2}\right|}\left|\sum_{k=1}^{n} 2\sin\frac{x}{2}\cdot\sin kx\right| = \frac{1}{2\left|\sin\frac{x}{2}\right|}\left|\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x\right| \le \frac{1}{\left|\sin\frac{x}{2}\right|}$$

Demak, Dirixle alomatiga ko`ra berilgan qator yaqinlashuvchi bo`ladi.►

Mashqlar

1.Ushbu

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \ldots\right) =$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) = 0$$

munosabatda $2n$ xatolik topilsin.

2.Ushbu

$$\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{5}}{\sqrt{n}\ln(n+1)}\left(1+\frac{1}{n}\right)^{-\pi}$$

qator yaqinlashuvchilikka tekshirilsin.

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Nazorat savollari

1. Qanday qatorni ishorasi almashinuvchi qator deyiladi?
2. Leybnits alomatini ayting.
3. Dirixe alomatini ayting.
4. Abel' alomatini ayting.

Glossariy

Ishorasi almashinuvchi qator –bu ushbu

$$\sum_{n=1}^{\infty}(-1)^{n-1}c_n = c_1 - c_2 + c_3 - c_4 + \ldots + (-1)^{n-1}c_n + \ldots \qquad (1)$$

ko`rinishdagi qatordir, bunda $c_n > 0 \quad (n = 1,2,3,\ldots)$.

Leybnits alomati – Agar (1) qatorda:

1) $c_{n+1} < c_n$, $\quad (n = 1,2,3,\ldots)$ $\quad$ 2) $\lim\limits_{n\to\infty} c_n = 0$

bo`lsa, u holda (1) qator yaqinlashuvchi bo`ladi.

Dirixle va Abel' alomatlari – ushbu

$$\sum_{k=1}^{\infty} a_k b_k = a_1 b_1 + a_2 b_2 + \ldots + a_k b_k + \ldots$$

ko`rinishdagi qatorlar uchun tadbiq qilinadi.

Keys banki

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$$

qatorni yaqinlashuvchiligini isbotlang.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Test

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	Ko'rsatilgan qatorlardan qaysi biri absolyut yaqinlashuvchi?	$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$	$\sum_{n=1}^{\infty}(-1)^n$	$\sum_{n=1}^{\infty}\sin n$
2	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{p+\frac{1}{n}}}$ qator absolyut va shartli yaqinlashishga tekshirilsin.	$p>1$ da absolyut va $0<p\le 1$ da shartli yaqinlashadi.	$p\ge 1$ da absolyut va $0<p<1$ da shartli yaqinlashadi.	$p>0$ da absolyut va $1<p\le 0$ da shartli yaqinlashadi
3	$\sum_{n=1}^{\infty}\frac{(-1)^{n-2}}{n^p}$ qatorni absolyut va shartli yaqinlashishga tekshirilsin.	$p>1$ da absolyut yaqinlashuvchi, $0<p\le 1$ da shartli yaqinlashuvchi	$0<p\le 1$ da absolyut yaqinlashuvchi, $p>1$ da shartli yaqinlashuvchi	$p>0$ da absolyut yaqinlashuvchi
4	$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[n^2]{n}}$ qatorni absolyut va shartli yaqinlashishga tekshirilsin.	uzoqlashuvchi	yaqinlashuvchi	uzoqlashishi ham yaqinlashishi ham bo`lishi mumkin
5	$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+.....+\frac{(-1)^{n-1}}{3^{n-1}}$ +.... qatorning yig`indisi topilsin.	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{36}$

5-Mavzu. Cheksiz ko`paytmalar va ularning xossalari.
5-Ma'ruza.
REJA:

1^0. Cheksiz ko`paytma tushunchasi.

2^0. Yaqinlashuvchi cheksiz ko`paytmaning xossalari.

3^0. Cheksiz ko`paytmalar bilan qatorlar orasidagi bog`lanish.

**Tayanch so`z va iboralar:** *Cheksiz ko`paytma, yaqinlashuvchi cheksiz ko`paytma, yaqinlashuvchi cheksiz ko`paytmaning xossalari: cheksiz ko`paytmaning qoldig`i va uning limiti, zaruruy shart, cheksiz ko`paytmaning yaqinlashuvchi bo`lishining zaruriy va yetarli shartlari.*

1^0. Cheksiz ko`paytma tushunchasi. Faraz qilaylik, biror

$$\{c_n\}: c_1, c_2, c_3, ..., c_n, ...$$

haqiqiy sonlar ketma-ketligi berilgan bo`lsin. Ular yordamida ushbu

$$c_1 \cdot c_2 \cdot c_3 \cdot ... \cdot c_n ... \tag{1}$$

ifodani tuzamiz.

(1) ifoda **cheksiz ko`paytma** deyiladi va u $\prod_{n=1}^{\infty} c_n$ kabi belgilanadi:

$$\prod_{n=1}^{\infty} c_n = c_1 \cdot c_2 \cdot c_3 \cdot ... \cdot c_n ...$$

Bunda $c_1, c_2, ..., c_n ...$ sonlar cheksiz ko`paytmaning hadlari, $c_n$ esa ko`paytmaning umumiy yoki n-hadi deyiladi.

Quyidagi

$$P_n = c_1 \cdot c_2 ... c_n \quad (n = 1,2,3,...)$$

ko`paytma, (1) cheksiz ko`paytmaning n**-qismiy ko`paytmasi** deyiladi.

Demak, (1) cheksiz ko`paytma berilganda har doim uning qismiy ko`paytmalaridan iborat ushbu $\{P_n\}$:

$$P_1, P_2, P_3, ..., P_n, ...$$

ketma-ketlikni hosil qilish mumkin.

Masalan,

$$\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) ... \left(1 - \frac{1}{n^2}\right) ... \qquad (n = 2,3,...)$$

cheksiz ko`paytmaning $n$-qismiy ko`paytmasi

$$P_n=\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=$$

$$=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\ldots\frac{n-1}{n}\cdot\frac{n+1}{n}=\frac{1}{2}\cdot\frac{n+1}{n}\quad (n=2,3,4,\ldots)$$

bo`lib, ulardan tuzilgan $\{P_n\}$ ketma-ketlik

$$\frac{3}{4},\frac{2}{3},\frac{5}{8},\frac{3}{5},\ldots,\frac{1}{2}\cdot\frac{n+1}{n},\ldots$$

bo`ladi.

1-ta`rif.** **[3, Ex.8, p.148]** Agar $n\to\infty$ da $\{P_n\}$ ketma-ketlik noldan farqli chekli $P$ songa intilsa (yaqinlashsa), (1) **cheksiz ko`paytma yaqinlashuvchi deyiladi, P esa uning qiymati deyiladi:

$$\lim_{n\to\infty}P_n=P,\qquad P=\prod_{n=1}^{\infty}P_n.$$

Agar $\{P_n\}$ ketma-ketlik limitga ega bo`lmasa (yoki uning limiti 0 bo`lsa), (1) **cheksiz ko`paytma uzoqlashuvchi** deyiladi.

Masalan, yuqorida keltirilgan

$$\prod_{n=2}^{\infty}(1-\frac{1}{n^2})$$

cheksiz ko`paytma uchun

$$\lim_{n\to\infty}P_n=\lim_{n\to\infty}\frac{1}{2}\cdot\frac{n+1}{n}=\frac{1}{2}$$

bo`ladi. Demak, $\prod_{n=2}^{\infty}(1-\frac{1}{n^2})$ cheksiz ko`paytma yaqinlashuvchi va uning qiymati $\frac{1}{2}$ ga teng.

2^0. Yaqinlashuvchi cheksiz ko`paytmaning xossalari. Aytaylik, biror

$$\prod_{n=1}^{\infty}c_n=c_1\cdot c_2\cdot c_3\cdot\ldots\cdot c_n\cdot\ldots \tag{1}$$

cheksiz ko`paytma berilgan bo`lsin.

Ushbu

$$\prod_{n=m+1}^{\infty}c_n=c_{m+1}\cdot c_{m+2}\cdot\ldots \tag{2}$$

cheksiz ko`paytma (bunda $m$-tayinlangan natural son) (1) cheksiz ko`paytmaning qoldig`i deyiladi.

1-Xossa. Agar (1) cheksiz ko`paytma yaqinlashuvchi bo`lsa, (2) cheksiz ko`paytma ham yaqinlashuvchi bo`ladi va aksincha.

◄ (1) cheksiz ko`paytmaning qismiy ko`paytmasi

$$P_n = c_1 \cdot c_2 ... c_n \ ,$$

(2) cheksiz ko`paytmaning qismiy ko`paytmasi

$$Q_k^{(m)} = c_{m+1} \cdot c_{m+2} \cdot ... \cdot c_{m+k}$$

lar uchun

$$P_n = P_m \cdot Q_k^{(m)} \ ,$$

(bunda, $n = m + k$) bo`ladi. Bu munosabatdan, $n \to \infty$ da $P_n$ ning chekli limitga ega bo`lishidan $k \to \infty$ da $Q_k^{(m)}$ ning ham chekli limitga ega bo`lishi, shuningdek, $k \to \infty$ da $Q_k^{(m)}$ ning chekli limitga ega bo`lishidan, $n \to \infty$ da P_n ning ham chekli limitga ega bo`lishi kelib chiqadi. ►

2-Xossa. Agar (1) cheksiz ko`paytma yaqinlashuvchi bo`lsa, u holda

$$\lim_{m \to \infty} (c_{m+1} \cdot c_{m+2} \cdot ... \cdot c_{m+k} ...) = 1$$

bo`ladi.

◄ Aytaylik, (1) cheksiz ko`paytma yaqinlashuvchi bo`lib, uning qiymati P bo`lsin. Unda

$$P_m \cdot c_{m+1} \cdot c_{m+2} \cdot ... \cdot c_{m+k} ... = P$$

bo`lib, undan $m \to \infty$ da

$$c_{m+1} \cdot c_{m+2} \cdot ... \cdot c_{m+k} ... = \frac{P}{P_m} \to \frac{P}{P} = 1$$

bo`lishi kelib chiqadi. ►

3-Xossa (zaruruy shart). Agar (1) cheksiz ko`paytma yaqinlashuvchi bo`lsa, u holda

$$\lim_{n \to \infty} c_n = 1$$

bo`ladi.

◄ Aytaylik, (1) cheksiz ko`paytma yaqinlashuvchi bo`lib, uning qiymati P bo`lsin:

$$\lim_{n \to \infty} P_n = \lim_{n \to \infty} (c_1 \cdot c_2 \cdot ... \cdot c_n) = P.$$

Unda $P_n = P_{n-1} \cdot c_n$, ya`ni $c_n = \dfrac{P_n}{P_{n-1}}$ bo`lib,

$$\lim_{n \to \infty} c_n = \lim_{n \to \infty} \frac{P_n}{P_{n-1}} = \frac{P}{P} = 1$$

bo`ladi. ►

Yuqorida keltirilgan xossalardan quyidagi xulosalarni chiqarish mumkin:

Cheksiz ko`paytmalarning yaqinlashishida, ularning dastlabki chekli sondagi hadlarining ta`siri bo`lmaydi.

Agar cheksiz ko`paytma yaqinlashuvchi bo`lsa, unda $n \to \infty$ da $c_n \to 1$ bo`lganligi sababli, uning biror hadidan boshlab keyingi hadlarini musbat deb olish mumkin bo`ladi.

Bu xossalar yaqinlashuvchi cheksiz ko`paytmalarda ularning hadlarini musbat deb olish imkonini beradi.

3⁰. Cheksiz ko`paytmalar bilan qatorlar orasidagi bog`lanish. Faraz qilaylik,

$$\prod_{n=1}^{\infty} c_n = \tilde{n}_1 \cdot \tilde{n}_2 \cdot \ldots \cdot c_n \ldots \quad (c_n > 0, \quad n = 1, 2, \ldots)$$

cheksiz ko`paytma berilgan bo`lsin. Bu cheksiz ko`paytma hadlarining logarifmlaridan ushbu

$$\sum_{n=1}^{\infty} \ln c_n = \ln c_1 + \ln c_2 + \ldots + \ln c_n + \ldots \qquad (3)$$

qatorni hosil qilamiz.

1-teorema. **[3, Exercise 7, p.148]** (1) cheksiz ko`paytmaning yaqinlashuvchi bo`lishi uchun (3) qatorning yaqinlashuvchi bo`lishi zarur va yetarli.

◀ **Zarurligi.** Aytaylik, (1) cheksiz ko`paytma yaqinlashuvchi bo`lsin:

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} (c_1 \cdot c_2 \cdot \ldots \cdot c_n) = P . \qquad (P - \text{chekli son})$$

Unda (3) qatorning qismiy yig`indisi uchun

$$S_n = \sum_{k=1}^{n} \ln c_k = \ln c_1 + \ln c_2 + \ldots + \ln c_n = \ln(c_1 \cdot c_2 \cdot \ldots \cdot c_n)$$

bo`lib,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(c_1 \cdot c_2 \cdot \ldots \cdot c_n) = \ln P$$

bo`ladi. Demak, (3) qator yaqinlashuvchi.

Yetarliligi. Aytaylik, (3) qator yaqinlashuvchi bo`lsin:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \ln c_k = \lim_{n\to\infty} \ln(c_1 \cdot c_2 \cdot \ldots \cdot c_n) = S .$$

Unda (1) cheksiz ko`paytmaning qismiy ko`paytmasi uchun

$$P_n = c_1 \cdot c_2 \cdot \ldots \cdot c_n = e^{\ln(c_1 \cdot c_2 \cdot \ldots \cdot c_n)}$$

bo`lib,

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} e^{\ln(c_1\cdot c_2\cdot ...c_n)} = e^s$$

bo`ladi. Demak, (1) cheksiz ko`paytma yaqinlashuvchi. ►

Ko`pincha, $\prod_{n=1}^{\infty} c_n$ cheksiz ko`paytmani o`rganishda, uning umumiy hadi c_n ni quyidagicha

$$c_n = 1 + a_n$$

ifodalash qulay bo`ladi. U holda (1) cheksiz ko`paytma

$$\prod_{n=1}^{\infty}(1+a_n) \ ,$$

cheksiz qator esa

$$\sum_{n=1}^{\infty}\ln(1+a_n)$$

ko`rinishlarga ega bo`ladi.

Faraz qilaylik, n ning $(n\in N)$ etarlicha katta qiymatlarida

$$a_n > 0 \ \ (\text{yoki } a_n < 0)$$

bo`lsin.

2-teorema. Ushbu

$$\prod_{n=1}^{\infty}(1+a_n)$$

cheksiz ko`paytmaning yaqinlashuvchi bo`lishi uchun

$$\sum_{n=1}^{\infty}a_n$$

qatorning yaqinlashuvchi bo`lishi zarur va yetarli.

◄ Ravshanki, $\prod_{n=1}^{\infty}(1+a_n)$ cheksiz ko`paytma va $\sum_{n=1}^{\infty}a_n$ qatorni yaqinlashuvchi bo`lishi uchun avvalo

$$\lim_{n\to\infty} a_n = 0$$

bo`lishi kerak. Shu munosabat bajarilsin.

Keltirilgan teoremaning isboti

$$\lim_{n\to\infty}\frac{\ln(1+a_n)}{a_n} = 1$$

munosabat hamda 1-teoremaning qo`llanishidan kelib chiqadi. ►

Masalan, bu teoremadan foydalanib, ushbu

$$\prod_{n=1}^{\infty}(1+\frac{1}{n^{\alpha}}) = (1+1)\cdot(1+\frac{1}{2^{\alpha}})\cdot(1+\frac{1}{3^{\alpha}})...(1+\frac{1}{n^{\alpha}})...$$

cheksiz ko`paytmaning $\alpha > 1$ bo`lganda yaqinlashuvchi bo`lishini (chunki, $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ $(\alpha > 1)$ yaqinlashuvchi),

$$\prod_{n=1}^{\infty}(1+\frac{1}{n}) = (1+1)\cdot(1+\frac{1}{2})\cdot(1+\frac{1}{3})...(1+\frac{1}{n})...$$

cheksiz ko`paytmaning esa uzoqlashuvchi bo`lishini (chunki, $\sum_{n=1}^{\infty}\frac{1}{n}$ uzoqlashuvchi) topamiz.

Mashqlar

1. Quyidagi

$$\prod_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{1+\frac{1}{n}} = e^{c}$$

tenglik isbotlansin, bunda

$$c = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$$

(Eyler o`zgarmasi).

Foydalanish uchun adabiyotlar

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Nazorat savollari

1. Cheksiz ko`paytma deb nimaga aytiladi?
2. Yaqinlashuvchi cheksiz ko`paytma deb nimaga aytiladi?
3. Yaqinlashuvchi cheksiz ko`paytmaning qanday xossalarini bilasiz?
4. Cheksiz ko`paytmalar bilan qatorlar orasida qanday bog`lanishlar bor?

Glossariy

Cheksiz ko`paytma – bu ushbu

$$\prod_{n=1}^{\infty} c_n = c_1 \cdot c_2 \cdot c_3 \cdot \ldots \cdot c_n \ldots \qquad (1)$$

ko`rinishdagi ifodadir, bunda $c_n > 0 \quad (n = 1,2,3,\ldots)$.

Cheksiz ko`paytmaning** $n$**-qismiy ko`paytmasi – bu

$$P_n = c_1 \cdot c_2 \ldots c_n \qquad (n = 1,2,3,\ldots)$$

**Yaqinlashuvchi cheksiz ko`paytma** – Agar $n \to \infty$ da $\{P_n\}$ ketma-ketlik noldan farqli chekli $P$ songa intilsa (yaqinlashsa), (1) cheksiz ko`paytma yaqinlashuvchi deyiladi.

Cheksiz ko`paytma yaqinlashuvchiligining zaruriy sharti – $\lim_{n\to\infty} c_n = 1$.

Cheksiz ko`paytmalar bilan qatorlar orasidagi bog`lanish –

$$\sum_{n=1}^{\infty} \ln c_n = \ln c_1 + \ln c_2 + \ldots + \ln c_n + \ldots \qquad (2)$$

(1) cheksiz ko`paytmaning yaqinlashuvchi bo`lishi uchun (2) qatorning yaqinlashuvchi bo`lishi zarur va yetarli.

Keys banki

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\prod_{n=1}^{\infty} \left(1 + x^{2n}\right) \qquad \left(|x| < 1\right)$$

cheksiz ko`paytmaning yaqinlashuvchiligi ko`rsatilsin va qiymati topilsin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Test

	Test topshirig`i	To’g`ri javob	Muqobil javob	Muqobil javob
1	$\prod_{n=1}^{\infty} c_n$ cheksiz ko`paytma yaqinlashuvchiligining zaruriy sharti bu	$\lim_{n\to\infty} c_n = 1$	$\lim_{n\to\infty} c_n = 0$	$\lim_{n\to\infty} c_n = \infty$
2	$\prod_{n=1}^{\infty}(1+a_n)$ cheksiz ko`paytmaning yaqinlashuvchi bo`lishi uchun	$\sum_{n=1}^{\infty} a_n$ qatorning yaqinlashuvchi bo`lishi zarur va yetarli.	$\sum_{n=1}^{\infty} \ln a_n$ qatorning yaqinlashuvchi bo`lishi zarur va yetarli.	$\sum_{n=1}^{\infty} a_n^{-1}$ qator yaqinlashuvchi bo`lishi zarur va yetarli.
3	$\prod_{n=1}^{\infty} c_n \quad (c_n > 0,\ n = 1, 2, ...)$ yaqinlashuvchi bo`lishi uchun	$\sum_{n=1}^{\infty} \ln c_n$ yaqinlashuvchi bo`lishi zarur va yetarli.	$\sum_{n=1}^{\infty} c_n$ yaqinlashuvchi bo`lishi zarur va yetarli.	$\sum_{n=1}^{\infty} (c_n + 1)$ yaqinlashuvchi bo`lishi zarur va yetarli.
4	$\prod_{n=1}^{\infty} \frac{n}{n^2+1}$ cheksiz ko`paytma yaqinlashuvchi bo`ladimi?	uzoqlashuvchi	yaqinlashuvchi	aniqlab bo`lmaydi
5	$\prod_{n=1}^{\infty} \frac{n^2-4}{n^2-1}$ cheksiz ko`paytmaning qiymatini toping.	$\frac{1}{4}$	$\frac{1}{2}$	0

6-Mavzu. Funksional ketma-ketliklar va ularning tekis yaqinlashuvchiligi
6-Ma’ruza.
REJA:

1^0. Funksional ketma-ketlik va limit funksiya tushunchalari.
2^0. Funksional ketma-ketlikning tekis yaqinlashuvchiligi.
3^0. Koshi teoremasi.
4^0. Tekis yaqinlashuvchi funksional ketma-ketlikning xossalari.

**Tayanch so`z va iboralar:** *Funksional ketma-ketlik, yaqinlashuvchi funksional ketma-ketlik, limit funksiya, funksional ketma-ketlikning yaqinlashish to`plami, tekis yaqinlashuvchi funksional ketma-ketlik, funksional ketma-ketlik tekis yaqilashishining zaruriy va yetarli sharti. Koshi teoremasi, notekis yaqinlashuvchi funksional ketma-ketliklar, limit funksiyaning uzluksizligi, integral ostida limitga o`tish.*

1^0. Funksional ketma-ketlik va limit funksiya tushunchalari. Aytaylik, har bir natural n songa $E\subset R$ to`plamda aniqlangan bitta $f_n(x)$ funksiyani mos qo`yuvchi qoida berilgan bo`lsin. Bu qoidaga ko`ra

$$f_1(x),\ f_2(x)\ ,\ ...,\ f_n(x),... \qquad (1)$$

to`plam hosil bo`ladi. Uni **funksional ketma-ketlik** deyiladi. E to`plam (1) funksional ketma-ketlikning **aniqlanish to`plami** deyiladi.

Odatda, (1) funksional ketma-ketlik, uning n-hadi yordamida $\{f_n(x)\}$ yoki $f_n(x)$ kabi belgilanadi. Masalan,

$$f_n(x)=\frac{n+1}{n+x^2}\ :\frac{2}{1+x^2},\frac{3}{2+x^2},...,\frac{n+1}{n+x^2},...;$$

$$f_n(x)=\sin\frac{\sqrt{x}}{n}\ :\sin\frac{\sqrt{x}}{1},\sin\frac{\sqrt{x}}{2},...,\sin\frac{\sqrt{x}}{n},...$$

lar funksional ketma-ketliklar bo`ladi va ularning aniqlanish to`plami mos ravishda

$$E=R\ ,\ E=[0,+\infty)$$

bo`ladi. Ravshanki, $x$ o`zgaruvchining biror tayinlangan $x=x_0\in E$ qiymatida ushbu

$$\{f_n(x_0)\}:\ f_1(x_0), f_2(x_0),..., f_n(x_0),...$$

sonlar ketma-ketligiga ega bo`lamiz.

**1-ta`rif.** **[4, Definition 1, p.363]** Agar $\{f_n(x_0)\}$ sonli ketma-ketlik yaqinlashuvchi (uzoqlashuvchi) bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik $x=x_0$

nuqtada **yaqinlashuvchi (uzoqlashuvchi)** deyiladi. x_0 nuqta esa bu funksional ketma-ketlikning **yaqinlashish (uzoqlashish) nuqtasi** deyiladi.

2-ta`rif**. $\{f_n(x)\}$ funksional ketma-ketlikning barcha yaqinlashish nuqtalarida iborat $E_0 \subset E$ to`plam, $\{f_n(x)\}$ funksional ketma-ketlikning **yaqinlashish to`plami deyiladi.

Masalan, ushbu

$$f_n(x) = x^n : x, x^2, x^3, ..., x^n, ...$$

funksional ketma-ketlik aniqlashish to`plami $E = R$ bo`lib, u $\forall x \in (-1,1]$ nuqtada yaqinlashuvchi, $x \in R \setminus (-1,1]$ da uzoqlashuvchi bo`ladi. Demak, ketma-ketlikning yaqinlashish to`plami $E_0 = (-1,1]$ bo`ladi.

Faraz qilaylik, $\{f_n(x)\}$ funksional ketma-ketlikning yaqinlashish to`plami $E_0 (E_0 \subset R)$ bo`lsin. Ravshanki, bu holda har bir $x \in E_0$ da

$$f_1(x), f_2(x), ..., f_n(x), ...$$

ketma-ketlik yaqinlashuvchi, ya`ni

$$\lim_{n\to\infty} f_n(x)$$

mavjud bo`ladi. Endi har bir $x \in E$ ga $\lim\limits_{n\to\infty} f_n(x)$ ni mos qo`ysak, ushbu

$$f : x \to \lim_{n\to\infty} f_n(x)$$

funksiya hosil bo`ladi. Bu $f(x)$ funksiya $\{f_n(x)\}$ funksional ketma-ketlikning limit funksiyasi deyiladi:

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E_0).$$

Bu munosabat quyidagini anglatadi: ixtiyoriy $\varepsilon > 0$ son va har bir $x \in E_0$ uchun shunday natural $n_0 = n_0(\varepsilon, x)$ son topiladiki, ixtiyoriy $n > n_0$ da

$$|f_n(x) - f(x)| < \varepsilon ,$$

ya`ni

$$\forall \varepsilon > 0 \ , \ \exists n_\circ = n_0(\varepsilon, x) \in N \ , \ \forall n > n_0 \ : |f_n(x) - f(x)| < \varepsilon$$

bo`ladi.

2^0^. Funksional ketma-ketlikning tekis yaqinlashuvchiligi. Faraz qilaylik, $\{f_n(x)\}$:

$$f_1(x) \ , \ f_2(x), ..., f_n(x), ...$$

funksional ketma-ketlik $E_\circ$ to`plamda yaqinlashuvchi (ya`ni yaqinlashish to`plami $E_0$) bo`lib, uning limit funksiyasi $f(x)$ bo`lsin:

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Ma`lumki, bu munosabat

$$\forall\varepsilon>0\ ,\ \exists n_0=n_0(\varepsilon,x)\in N,\ \forall n>n_0:\ |f_n(x)-f(x)|<\varepsilon$$

bo`lishini anglatadi. Shuni ta`kidlash lozimki, yuqoridagi natural n_0 son ixtiyoriy olingan $\varepsilon>0$ son bilan birga qaralayotgan $x\in E_0$ nuqtaga ham bog`liq bo`ladi (chunki, $x\in E_0$ ning turli qiymatlarida ularga mos ketma-ketlik, umuman aytganda turlicha bo`ladi).

**3-ta`rif**. **[4, Definition 9, p.367]** Agar $\forall\varepsilon>0$ son olinganda ham shu $\varepsilon>0$ gagina bog`liq bo`lgan natural $n_0=n_0(\varepsilon)$ son topilsaki, $\forall n>n_0$ va ixtiyoriy $x\in E_0$ da

$$|f_n(x)-f(x)|<\varepsilon$$

tengsizlik bajarilsa, ya`ni

$$\forall\varepsilon>0\ ,\ \exists n_0=n_0(\varepsilon)\in N\ ,\ \forall n>n_0,\ \ \forall x\in E_0:\ |f_n(x)-f(x)|<\varepsilon$$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik $E_0$ to`plamda $f(x)$ ga **tekis yaqinlashadi** (funksional ketma-ketlik E_0 to`plamda tekis yaqinlashuvchi) deyiladi.

Shunday qilib, $\{f_n(x)\}$ funksional ketma-ketlik E_0 to`plamda $f(x)$ limit funksiyaga ega bo`lsa, uning shu limit funksiyasiga yaqinalishish ikki xil bo`lar ekan:

1) $\forall\varepsilon>0\ ,\ \exists n_0=n_0(\varepsilon,x)\in N\ ,\forall n>n_0:\ |f_n(x)-f(x)|<\varepsilon$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik E_0 da $f(x)$ ga yaqinlashadi (**oddiy yaqinlashadi**). Bu holda

$$f_n(x)\to f(x)\qquad (x\in E_0)$$

kabi belgilanadi.

2) $\forall\varepsilon>0\ ,\ \exists n_0=n_0(\varepsilon)\in N\ ,\ \ \forall n>n_0,\ \ \forall x\in E_0\ :\ |f_n(x)-f(x)|<\varepsilon$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik E_0 da $f(x)$ ga tekis yaqinlashadi. Bu holda

$$f_n(x)\rightrightarrows f(x)\qquad (x\in E_0)$$

kabi belgilanadi.

Ravshanki, $\{f_n(x)\}$ funksional ketma-ketlik E_0 to`plamda $f(x)$ funksiyaga tekis yaqinlashsa u shu to`plamda $f(x)$ ga yaqinlashadi:

$$f_n(x)\rightrightarrows f(x)\ \Rightarrow\ f_n(x)\to f(x)\qquad (x\in E_0).$$

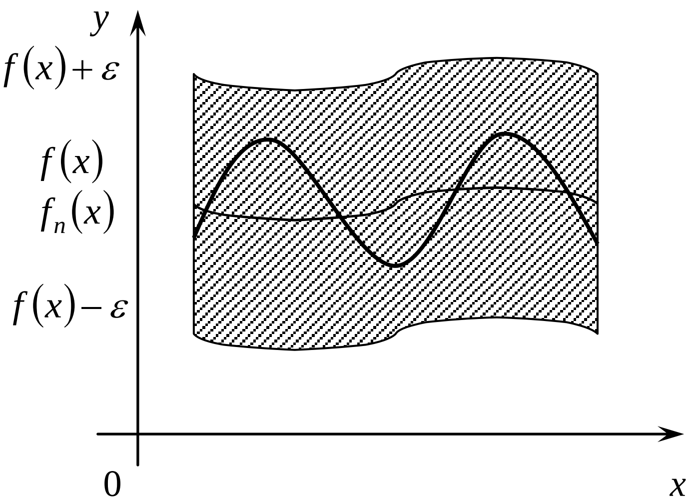
Aytaylik,

$$f_n(x) \rightrightarrows f(x) \qquad (x \in E_0)$$

bo`lsin. Bu holda $\forall n > n_0$ va $\forall x \in E_0$ da

$$|f_n(x) - f(x)| < \varepsilon, \text{ ya`ni } f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$$

bo`ladi. Bu esa $\{f_n(x)\}$ funksional ketama-ketlikning biror hadidan boshlab, keyingi barcha hadlari $f(x)$ funksiyaning "$\varepsilon$-oralig`i" da butunlay joylashishini bildiradi (1-chizma).



1-chizma

Faraz qilaylik, $\{f_n(x)\}$ funksional ketma-ketlik E_0 to`plamda $f(x)$ limit funksiyaga ega bo`lsin.

1-teorema. $\{f_n(x)\}$ funksional ketma-ketlik E_0 to`plamda $f(x)$ funksiyaga tekis yaqilashishi uchun

$$\lim_{n\to\infty} \sup_{x\in E_0} |f_n(x) - f(x)| = 0$$

bo`lishi zarur va yetarli.

◀ **Zarurligi**. Aytaylik,

$$f_n(x) \rightrightarrows f(x) \qquad (x \in E_0)$$

bo`lsin. Ta`rifga binoan

$$\forall \varepsilon > 0, \ \exists n_0 = n_0(\varepsilon) \in N, \ \forall n > n_0, \forall x \in E_0 \ : |f_n(x) - f(x)| < \varepsilon$$

bo`ladi. Bu tengsizlikdan

$$\sup_{x\in E_0} |f_n(x) - f(x)| \le \varepsilon$$

bo`lib, undan

$$\lim_{n\to\infty} \sup_{x\in E_0} |f_n(x) - f(x)| = 0$$

bo`lishi kelib chiqadi.

Yetarliligi. Aytaylik

$$\lim_{n\to\infty}\sup_{x\in E_0}|f_n(x)-f(x)|=0$$

bo`lsin. Limit ta`rifga ko`ra

$$\forall\varepsilon>0,\ \exists n_0\in N\ \ \forall n>n_0,:\ \sup_{x\in E_0}|f_n(x)-f(x)|<\varepsilon$$

bo`ladi. Ravshanki

$$|f_n(x)-f(x)|\le\sup_{x\in E_0}|f_n(x)-f(x)|.$$

U holda $\forall x\in E_0$ uchun

$$|f_n(x)-f(x)|<\varepsilon$$

bo`ladi. Bundan

$$f_n(x)\rightrightarrows f(x)\qquad (x\in E_0)$$

bo`lishi kelib chiqadi.►

3⁰. Koshi teoremasi. Funksional ketma-ketlikning limit funksiyaga ega bo`lishi va unga tekis yag`inlashishini ifodalovchi teoremani keltiramiz:

1-teorema (Koshi teoremasi). [4, Theorem, p.370] $\{f_n(x)\}$ funksional ketma-ketlik E to`plamda limit funksiyaga ega bo`lishi va unga tekis yaqinlashishi uchun $\forall\varepsilon>0$ son olinganda ham shunday $n_0=n_{0(\varepsilon)}\in N$ topilib, $\forall n>n_0$, $\forall p\in N$ va $\forall x\in E$ da

$$|f_{n+p}(x)-f_n(x)|<\varepsilon,$$

ya`ni

$$\forall\varepsilon>0\ ,\exists n_0=n_0(\varepsilon)\in N,\ \forall n>n_0,\ \forall p\in N \text{ va } \forall x\in E \text{ da}$$

$$|f_{n+p}(x)-f_n(x)|<\varepsilon \qquad (1)$$

bo`lishi zarur va yetarli.

◄ **Zarurligi.** Aytaylik, E to`plamda $\{f_n(x)\}$ funksional ketma-ketlik limit funksiya $f(x)$ ga ega bo`lib, unga tekis yaqinlashsin:

$$f_n(x)\rightrightarrows f(x).\qquad (x\in E_0)$$

Tekis yaqinlashish ta`rifiga ko`ra

$$\forall\varepsilon>0,\ \exists n_0=n_0(\varepsilon)\in N,\ \forall k>n_0,\ \forall x\in E:|f_k(x)-f(x)|<\frac{\varepsilon}{2}$$

bo`ladi. Xususan, $k=n,\ \ n>n_0$ va $k=n+p,\ \ p\in N$ da

$$\left|f_n(x)-f(x)\right|<\frac{\varepsilon}{2},\quad \left|f_{n+p}(x)-f(x)\right|<\frac{\varepsilon}{2}$$

tengsizliklar bajarilib, ulardan

$$\left|f_{n+p}(x)-f_n(x)\right|=\left|f_{n+p}(x)-f(x)-(f_n(x)-f(x))\right|\le$$

$$\le\left|f_{n+p}(x)-f(x)\right|+\left|f_n(x)-(x)\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

bo`lishi kelib chiqadi. Demak, (1) shart o`rinli.

Yetarliligi. $\{f_n(x)\}$ funksional ketma-ketlik uchun (1) shart bajarilsin. Uni quyidagicha yozamiz:

$\forall\varepsilon>0,\ \exists n_0=n_0(\varepsilon)\in N,\ \forall n>n_0,\ \forall p\in N, \forall x\in E$ da

$$\left|f_{n+p}(x)-f_n(x)\right|<\frac{\varepsilon}{2} \tag{2}$$

bo`ladi.

Ravshanki, tayin $x_0\in E$ da $\{f_n(x_0)\}$ sonlar ketma-ketligi uchun (2) shartning bajarilishidan uning fundamental ketma-ketlik ekanligi kelib chiqadi. Koshi teoremasiga ko`ra $\{f_n(x_0)\}$ yaqinlashuvchi bo`ladi. Binobarin, chekli

$$\lim_{n\to\infty} f_n(x_0) \tag{3}$$

limit mavjud.

Modomiki, har bir $x\in E$ da (3) limit mavjud bo`lar ekan, unda avval ayganimizdek, $E$ to`plamda aniqlangan

$$x\to\lim_{n\to\infty} f_n(x)\qquad (x\in E)$$

funksiya hosil bo`ladi Uni $f(x)$ bilan belgilaymiz. Bu funksiya $\{f_n(x)\}$ funksional ketma-ketlikning limit funksiyasi bo`ladi:

$$f_n(x)\to f(x)\qquad (x\in E).$$

Endi (2) tengsizlikda, n va x larni tayinlab $(n>n_0,\, x\in E)$ $p\to\infty$da limitga o`tamiz. Natijada

$$\left|f(x)-f_n(x)\right|\le\frac{\varepsilon}{2}<\varepsilon$$

hosil bo`ladi. Bu

$$f_n(x)\rightrightarrows f(x)\qquad (x\in E_0)$$

bo`lishini bildiradi. ►

1-misol. Ushbu

$$f_n(x)=\frac{\ln nx}{\sqrt{nx}}$$

funksional ketma-ketlik $E=(0,1)$ to`plamda tekis yaqinlashuvchilikka tekshirilsin.

◀Agar ixtiyoriy $k\in N$ uchun

$$n=k\ ,\ p=k=n\ ,\ x^*=\frac{1}{k}=\frac{1}{n}$$

deyilsa,

$$\left|f_{n+p}(x)-f(x)\right|=\left|f_{2n}\left(\frac{1}{n}\right)-f_n\left(\frac{1}{n}\right)\right|=\left|\frac{\ln 2}{\sqrt{2}}-\ln 1\right|=\frac{\ln 2}{\sqrt{2}}=\varepsilon_0$$

bo`ladi. Demak,

$$\exists\varepsilon_0=\frac{\ln 2}{\sqrt{2}}\ \forall k\in N,\exists n\geq k,\exists p\in N,\ \exists x^*=\frac{1}{n}\in(0,1):\left|f_{n+p}(x^*)-f_n(x^*)\right|\geq\varepsilon_0.$$

Bu esa yuqoridagi teoremaning shartini bajarilmasligini ko`rsatadi. Demak, berilgan funksional ketma-ketlik $E=(0,1)$ da tekis yaqinlashuvchi emas.▶

Aytaylik, $\{f_n(x)\}$ funksional ketma-ketlik E to`plamda yaqinlashuvchi bo`lib, $f(x)$ funksiya uning limit funksiyasi bo`lsin:

$$f_n(x)\to f(x)\qquad (x\in E).$$

Agar

$$\exists\varepsilon_0>0\ ,\ \forall k\in N\ ,\ \exists n>k\ ,\ \exists x^*\in E:\left|f_n(x^*)-f(x^*)\right|\geq\varepsilon_0$$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik $E$ to`plamda $f(x)$ funksiyaga **notekis yaqinlashadi** deyiladi.

2-misol. Ushbu

$$f_n(x)=n\sin\frac{1}{nx}$$

funksional ketma-ketlik $E=(0,1)$ da tekis yaqinlashishiga tekshirilsin.

◀ Ravshanki,

$$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}n\sin\frac{1}{nx}=\lim_{n\to\infty}\frac{\sin\frac{1}{nx}}{\frac{1}{nx}\cdot x}=\frac{1}{x}.$$

Demak, berilgan funksional ketma-ketlikning limit funksiyasi $f(x)=\frac{1}{x}$ bo`ladi.

Aytaylik, $x^*=\frac{1}{n}$ bo`lsin. Unda

$$\left|f_n(x^*)-f(x^*)\right|=\left|n\sin 1-n\right|\geq 1-\sin 1=\varepsilon_0$$

munosabat ixtiyoriy $n\in N$ da o`rinli bo`ladi.

Demak, $f_n(x)=n\sin\frac{1}{nx}$ funksional ketma-ketlik limit funksiya $f(x)=\frac{1}{x}$ ga $E=(0,1)$ da tekis yaqinlashmaydi.►

4^0. Tekis yaqinlashuvchi funksional ketma-ketlikning xossalari. Tekis yaqinlashuvchi funksiyaonal ketma-ketliklar qator xossalarga ega. Bu xossalarni keltiramiz.

Aytaylik. $\{f_n(x)\}$:

$$f_1(x)\ ,\ f_2(x),...,\ f_n(x),...$$

funksional ketma-ketlik $E\subset R$ to`plamda yaqinlashuvchi bo`lib, $f(x)$ uning limit funksiyasi bo`lsin:

$$f_n(x)\to f(x) \qquad (x\in E).$$

1-xossa. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,3,...)$ hadi E to`plamda uzluksiz bo`lib,

$$f_n(x)\rightrightarrows f(x) \qquad (x\in E)$$

bo`lsa, limit funksiya $f(x)$ shu $E$ to`plamda uzluksiz bo`ladi.

Demak, bu holda

$$\lim_{t\to x}\left(\lim_{n\to\infty} f_n(t)\right)=\lim_{n\to\infty}\left(\lim_{t\to x} f_n(t)\right)$$

munosabat o`rinli bo`ladi.

2-xossa. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,3,...)$ hadi $E=[a,b]$ da uzluksiz bo`lib,

$$f_n(x)\rightrightarrows f(x) \qquad (x\in[a,b])$$

bo`lsa,

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b f(x)dx$$

bo`ladi.

Demak, bu holda

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b\left(\lim_{n\to\infty} f_n(x)\right)dx$$

munosabat o`rinli bo`ladi.

3-xossa. Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $(n=1,2,3,...)$ hadi $E=[a,b]$ da uzluksiz $f_n'(x)$ $(n=1,2,3,...)$ hosilalarga ega bo`lib,

$$f_n'(x)\rightrightarrows \varphi(x) \qquad (x\in[a,b])$$

bo`lsa,

$$\varphi(x) = f'(x)$$

bo`ladi.

Shu kabi xossalarga keyinroq o`rganiladigan tekis yaqinlashuvchi funksional qatorlar ham ega bo`ladi. Ayni paytda, ular bir mulohaza asosida isbotlanadi Mazkur xossalarning isbotini funksional qatorlarga nisbatan keltiramiz.

6-Amaliy mashg`ulot.

1-misol. Ushbu

$$f_n(x) = n\sin\frac{\sqrt{x}}{n}$$

funksional ketma-ketlikning limit funksiyasi topilsin.

◀ Berilgan funksional ketma-ketlik $E = [0,+\infty)$ da aniqlangan. Uning limit funksiyasi

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n\sin\frac{\sqrt{x}}{n} = \lim_{n\to\infty}\frac{\sin\frac{\sqrt{x}}{n}}{\frac{\sqrt{x}}{n}}\cdot\sqrt{x} = \sqrt{x}$$

bo`ladi. Demak, funksional ketma-ketlik $E = [0,+\infty)$ da yaqinlashuvchi va

$$\lim_{n\to\infty} n\sin\frac{\sqrt{x}}{n} = \sqrt{x}\,.$$ ▶

2-misol. Ushbu

$$f_n(x) = x^n$$

funksional ketma-ketlikning limit funksiyasi topilsin.

◀ Bu funksional ketma-ketlik $E = R$ da aniqlangan. Ravshanki

$\forall x \in (1,+\infty)$ da $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^n = +\infty$,

$\forall x \in (-1,1)$ da $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^n = 0$,

$x = 1$ da $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} 1 = 1$,

$\forall x \in (-\infty,-1)$ da $\lim\limits_{n\to\infty} f_n(x)$ mavjud emas.

Demak, berilgan funksional ketma-ketlik $E_0 = (-1,1]$ yaqinla-shuvchi bo`lib, uning limit funksiyasi

$$f(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & \text{agar} \quad -1 < x < 1 \quad \text{bo`lsa,} \\ 1, & \text{agar} \quad x = 1 \quad \text{bo`lsa} \end{cases}$$

bo`ladi.▶

3-misol. Ushbu

$$f_n(x)=n^2\left(\sqrt[n]{x}-\sqrt[n+1]{x}\right) \qquad (x>0)$$

funksional ketma-ketlikning limit funksiyasi topilsin.

◀Berilgan funksional ketma-ketlikning limit funksiyasi quyidagicha topiladi:

$$f(x)=\lim_{n\to\infty}(f_n(x))=\lim_{n\to\infty}n^2\left(\sqrt[n]{x}-\sqrt[n+1]{x}\right)=\lim_{n\to\infty}n^2\left(x^{\frac{1}{n}}-x^{\frac{1}{n+1}}\right)=$$

$$=\lim_{n\to\infty}n^2x^{\frac{1}{n+1}}\left(x^{\frac{1}{n}-\frac{1}{n+1}}-1\right)=\lim_{n\to\infty}\frac{n^2}{n^2+n}x^{\frac{1}{n+1}}\cdot\frac{x^{\frac{1}{n^2+n}}-1}{\frac{1}{n^2+n}}=\ln x. \ ▶$$

4-misol. Ushbu

$$f_n(x)=\frac{\sin nx}{n}$$

funksional ketma-ketlikning R da tekis yaqinlashuvchiligi ko`rsatilsin.

◀Ravshanki,

$$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{\sin nx}{n}=0.$$

Demak, limit funksiya $f(x)=0$.

Agar $\forall\varepsilon>0$ son olinganda $n_0=\left[\frac{1}{\varepsilon}\right]$ deyilsa, unda $\forall n>n_0$ va $\forall x\in R$ uchun

$$|f_n(x)-f(x)|=\left|\frac{\sin nx}{n}-0\right|=\left|\frac{\sin nx}{n}\right|\le\frac{1}{n}\le\frac{1}{n_0+1}<\varepsilon$$

bo`lishini topamiz. Demak ta`rifga binoan

$$\frac{\sin nx}{n}\rightrightarrows 0$$

bo`ladi.▶

5-misol. Ushbu

$$f_n(x)=\sqrt{x^2+\frac{1}{n^2}}$$

funksional ketama-ketlikning $E_0=R$ da tekis yaqinlashuvchiligi ko`rsatilsin.

◀Berilgan funksional ketma-ketlikning limit funksiyasi

$$f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\sqrt{x^2+\frac{1}{n^2}}=|x| \qquad (x\in R)$$

bo`ladi. endi

$$\sup_x |f_n(x) - f(x)|$$

ni topamiz:

$$\sup_{x\in R}\left|\sqrt{x^2+\frac{1}{n^2}}-|x|\right| = \sup_{x\in R}\left|\frac{\frac{1}{n^2}}{\left(\sqrt{x^2+\frac{1}{n^2}}+|x|\right)}\right| = \sup_{x\in R}\frac{1}{n^2}\cdot\frac{1}{\left(\sqrt{x^2+\frac{1}{n^2}}+|x|\right)} = \frac{1}{n}.$$

Demak,

$$\lim_{n\to\infty}\sup_{x\in R}\left|\sqrt{x^2+\frac{1}{n^2}}-|x|\right| = \lim_{n\to\infty}\frac{1}{n} = 0$$

bo`lib,

$$\sqrt{x^2+\frac{1}{n^2}} \rightrightarrows |x| \qquad (x\in R)$$

bo`ladi.►

Eslatma. Agar $\{f_n(x)\}$ funksional ketma-ketligi uchun $E\subset R$ to`plamda

$$\lim_{n\to\infty}\sup_{x\in E}|f_n(x)-f(x)|\neq 0$$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik E da tekis yaniqlashishi shart emas.

Mashqlar. Berilgan to‘plamlarda funksional ketma-ketliklarni tekis yaqinlashishga tekshiring

1. $f_n(x)=\frac{x}{n}\ln\frac{x}{n};0<x<1.$

2. $f_n(x)=x^n;0\le x\le\frac{1}{4}.$

3. $f_n(x)=e^{-(x-n)^2};-1<x<1.$

4. $f_n(x)=x^n-x^{n+1};0\le x\le 1.$

5. $f_n(x)=e^{n\cdot(x-1)};0<x<1.$

6. $f_n(x)=x^n;0\le x\le 1.$

7. $f_n(x)=x\,arctg\,nx,0<x<+\infty.$

8. $f_n(x)=x^n-x^{2n};0\le x\le 1.$

9. $f_n(x)=arctg\,nx;0<x<+\infty.$

10. $f_n(x)=\frac{1}{x+n};0<x<+\infty.$

11. $f_n(x)=\sin\frac{x}{n};-\infty<x<+\infty.$

12. $f_n(x)=\frac{nx}{1+n+x};0\le x\le 1.$

13. $f_n(x)=\frac{\sin nx}{n};-\infty<x<+\infty.$

14. $f_n(x)=\frac{x^n}{1+x^n};0\le x\le 1-\varepsilon,\varepsilon>0.$

15. $f_n(x)=n\left(\sqrt{x+\frac{1}{n}}-\sqrt{x}\right);0<x<+\infty.$

16. $f_n(x)=\frac{x^n}{1+x^n};1-\varepsilon\le x\le 1+\varepsilon,\varepsilon>0.$

17. $f_n(x)=\sqrt{x^2+\frac{1}{n^2}};-\infty<x<+\infty.$

18. $f_n(x)=\frac{x^n}{1+x^n};2\le x<+\infty.$

19. $f_n(x)=\dfrac{2nx}{1+n^2x^2};1<x<+\infty.$ **20.** $f_n(x)=\dfrac{2nx}{1+n^2x^2};0\le x\le 1.$

21. $f_n(x)=\dfrac{n+x}{n+x+\sqrt{nx}},a)0\le x<+\infty,b)0\le x\le 1.$

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NAZORAT SAVOLLARI

1. Yaqinlashuvchi funksional ketma-ketlik deb nimaga aytiladi?
2. Limit funksiya nima?
3. Funksional ketma-ketlikning yaqinlashish to`plamini tushuntiring.
4. Tekis yaqinlashuvchi funksional ketma-ketlikning ta'rifini ayting.
5. Funksional ketma-ketlik tekis yaqilashishining zarur va yetarli shartini ayting.
6. Koshi teoremasini ayting.
7. Notekis yaqinlashuvchi funksional ketma-ketlik deb nimaga aytiladi?
8. Tekis yaqinlashuvchi funksional ketma-ketlikning qanday xossalarini bilasiz?

GLOSSARIY

$\{f_n(x)\}$ **funksional ketma-ketlik** $x=x_0$ **nuqtada yaqinlashuvchi (uzoqlashuvchi)** - Agar $\{f_n(x_0)\}$ sonli ketma-ketlik yaqinlashuvchi (uzoqlashuvchi) bo'lsa

$\{f_n(x)\}$ **funksional ketma-ketlikning yaqinlashish to'plami** - $\{f_n(x)\}$ funksional ketma-ketlikning barcha yaqinlashish nuqtalarida iborat $E_0\subset E$ to'plam

funksional ketma-ketlik E_0 to‘plamda $f(x)$ ga tekis yaqinlashadi (funksional ketma-ketlik E_0 to‘plamda tekis yaqinlashuvchi) - Agar $\forall\varepsilon>0$ son olinganda ham shu $\varepsilon>0$ gagina bog‘liq bo‘lgan natural $n_0=n_0(\varepsilon)$ son topilsaki, $\forall n>n_0$ va ixtiyoriy $x\in E_0$ da

$$|f_n(x)-f(x)|<\varepsilon$$

tengsizlik bajarilsa, ya’ni

$$\forall\varepsilon>0\ ,\ \exists n_0=n_0(\varepsilon)\in N\ ,\ \forall n>n_0,\ \ \forall x\in E_0:\ |f_n(x)-f(x)|<\varepsilon$$

bo‘lsa, $\{f_n(x)\}$ deyiladi.

$\{f_n(x)\}$ **funksional ketma-ketlik E_0 da $f(x)$ ga yaqinlashadi (oddiy yaqinlashadi)** - $\forall\varepsilon>0$, $\exists n_0=n_0(\varepsilon,x)\in N$,$\forall n>n_0$: $|f_n(x)-f(x)|<\varepsilon$ bo‘lsa.
Belgilanishi: $f_n(x)\to f(x)$ $\quad (x\in E_0)$

$\{f_n(x)\}$ **funksional ketma-ketlik E_0 da $f(x)$ ga tekis yaqinlashadi** - $\forall\varepsilon>0$, $\exists n_0=n_0(\varepsilon)\in N$, $\forall n>n_0$, $\forall x\in E_0$: $|f_n(x)-f(x)|<\varepsilon$ bo‘lsa.

Belgilanishi: $f_n(x)\rightrightarrows f(x)$ $\quad (x\in E_0)$

Koshi teoremasi - $\{f_n(x)\}$ funksional ketma-ketlik E to`plamda limit funksiyaga ega bo`lishi va unga tekis yaqinlashishi uchun $\forall\varepsilon>0$ son olinganda ham shunday $n_0=n_{0(\varepsilon)}\in N$ topilib, $\forall n>n_0$, $\forall p\in N$ va $\forall x\in E$ da

$$|f_{n+p}(x)-f_n(x)|<\varepsilon,$$

ya`ni

$$\forall\varepsilon>0\ ,\exists n_0=n_0(\varepsilon)\in N,\ \forall n>n_0,\ \forall p\in N \text{ va } \forall x\in E \text{ da}$$

$$|f_{n+p}(x)-f_n(x)|<\varepsilon$$

bo`lishi zarur va yetarli.

Notekis yaqinlashuvchi funksional ketma-ketlik - Agar

$$\exists\varepsilon_0>0\ ,\ \forall k\in N\ \ ,\ \exists n>k\ ,\ \exists x^*\in E:\ |f_n(x^*)-f(x^*)|\ge\varepsilon_0$$

bo`lsa, $\{f_n(x)\}$ funksional ketma-ketlik $E$ to`plamda $f(x)$ funksiyaga **notekis yaqinlashadi** deyiladi.

Limit funksiyaning uzluksizligi - Agar $\{f_n(x)\}$ funksional ketma-ketlikning har bir $f_n(x)$ $\quad (n=1,2,3,...)$ hadi E to`plamda uzluksiz bo`lib,

$$f_n(x)\rightrightarrows f(x)\qquad (x\in E)$$

bo`lsa, limit funksiya $f(x)$ shu $E$ to`plamda uzluksiz bo`ladi.

KEYS BANKI

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$f_n(x)=n\left(\sqrt{x+\frac{1}{n}}-\sqrt{x}\right)$$

funksional ketma-ketlikni $E=(0,+\infty)$ da tekis yaqinlashuvchilikka tekshirilsin.

2-keys. Masala o`rtaga tashlanadi: Aytaylik, $f(x)$ funksiya $(a,b)$da uzluksiz $f'(x)$ hosilaga ega bo`lib,

$$f_n(x)=n\left(f\left(x+\frac{1}{n}\right)-f(x)\right)$$

bo`lsin. Bu funksional ketma-ketlikning $[a_i,b_i]\subset(a,b)$ da $f'(x)$ ga tekis yaqinlashishi isbotlansin

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

TEST

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$f_n(x)=\frac{\sin nx}{n}$ funksional ketma-ketlik uchun $n\to\infty$ da limit funksiyani toping.	$f(x)\equiv 0$	x^3	$f(x)\equiv 2$
2	α ning qanday qiymatlarida $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^\alpha}$ sonli qator absolyut yaqinlashuvchi bo'ladi?	$\alpha>1$	$\alpha=1$	$\alpha\le 1$
3	α ning qanday qiymatlarida $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n^\alpha}$ sonli qator absolyut yaqinlashuvchi bo'ladi?	$\alpha>1$	$\alpha=2$	$\alpha\le 1$
4	Qanday α da $\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ sonli qator uzoqlashuvchi?	$\alpha\le 1$	$\alpha\ge 2$	$\alpha\le 1$
5	Qanday α da $\sum_{n=1}^{\infty}\frac{1}{4n^\alpha}$ sonli qator uzoqlashuvchi?	$\alpha\le 1$	$\alpha\ge 4$	$\alpha\le 1$

6	Ko'rsatilgan qatorlardan qaysi biri absolyut yaqinlashuvchi?	$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$	$\sum_{n=1}^{\infty}(-1)^n$	$\sum_{n=1}^{\infty}\sin n$
7	Ko'rsatilgan qatorlardan qaysi biri absolyut yaqinlashuvchi?	$\sum_{n=1}^{\infty}(-1)^n\frac{1}{25n^2}$	$\sum_{n=1}^{\infty}(-1)^{3n}$	$\sum_{n=1}^{\infty}\sin n$
8	$f_n(x)=\frac{\sin nx}{n}$ funksional ketma-ketlik uchun $n\to\infty$ da limit funksiyani toping.	$f(x)\equiv 0$	x^3	$f(x)\equiv 2$
9	$f_n(x)=\frac{\sin 2nx}{2n}$ funksional ketma-ketlik uchun $n\to\infty$ da limit funksiyani toping.	$f(x)\equiv 0$	x^2	$f(x)\equiv 2$
10	$f_n(x)=x^n,\ n\in N$ funksional ketma-ketlik $x=1$ nuqtada qanday songa intiladi?	1	4	∞

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$f_n(x)=x^n,\ n\in N$ funksional ketma-ketlik $x=1/2$ nuqtada qanday songa intiladi?	0	1	∞
2	Ko'rsatilgan funksional ketma-ketliklardan qaysilari tekis yaqinlashadi?	$f_n(x)=\frac{\sin nx}{n^2},\ n\in N$	$f_n(x)=\sin nx, n\in N$	$f_n(x)=\frac{n^2}{\sin nx},\ n\in N$
3	Ko'rsatilgan funksional ketma-ketliklardan qaysilari tekis yaqinlashadi?	$f_n(x)=\frac{2\sin nx}{n^2},\ n\in N$	$f_n(x)=\sin 3nx, n\in N$	$f_n(x)=\frac{n^2}{\sin nx},\ n\in N$
4	$f_n(x)=\frac{1}{x^2+n}$, $\lim\limits_{n\to\infty}f_n(x)=f(x)$ bo'lsa $f(x)-?$	$f(x)\equiv 0$	limit funksya yo'q	$f(x)\equiv 1$
5	$f_n(x)=\frac{1}{x^2+n^2}$, $\lim\limits_{n\to\infty}f_n(x)=f(x)$ bo'lsa $f(x)-?$	$f(x)\equiv 0$	limit funksya cheksiz ko'p	$f(x)\equiv 1$
6	$f_n(x)=x^n,\ n\in N$ funksional ketma-ketlikning yaqinlashish sohasini toping.	$(-1,1]$	$(-2,3]$	$(-\infty,+\infty)$
7	$\{f_n(x)\}$ funktsional ketma-ketlik $n\to\infty$ da X to`plamda $f(x)$ funksiyaga tekis yaqinlashadi, agar	$\lim\limits_{n\to\infty}\sup\limits_{x\in X}\lvert f_n(x)-f(x)\rvert=0$	$\sup\limits_{x\in X}\lim\limits_{n\to\infty}\lvert f_n(x)-f(x)\rvert=0$	$\forall\varepsilon>0\ \exists n_0(\varepsilon)$ $\forall n>n_0\Rightarrow$ $\lvert f_n(x)-f(x)\rvert<\varepsilon$

7-Mavzu. Funksional qatorlar va ularning tekis yaqinlashuvchanligi
7-Ma'ruza.
REJA:

1^0. Funksional qator va uning yig'indisi.
2^0. Funksional qatorning tekis yaqinlashuvchiligi.
3^0. Veyershtrass alomati.
4^0. Dirixle alomati.
5^0. Abel' alomati.

**Tayanch so`z va iboralar:** *Funksional qator, nuqtada yaqinlashuvchi (uzoqlashuvchi) funksional qator, absolyut yaqinlashuvchi funksional qator, funksional qator yig`indisi, tekis yaqinlashuvchi funksional qator, zaruriy va yetarli shart, Koshi teoremasi. Veyershtrass, Dirixle, Abel' alomatlari.*

***1^0. Funksional qator va uning yig`indisi.*** Faraz qilaylik, $E \subset R$ to`plamda aniqlangan

$$u_1(x), u_2(x), \ldots, u_n(x), \ldots$$

funksional ketma-ketlik berilgan bo`lsin. Bu ketma-ketlik hadlari yordamida tuzilgan quyidagi

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

ifoda **funksional qator** deyiladi va $\sum_{n=1}^{\infty} u_n(x)$ kabi belgilanadi:

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \,. \qquad (1)$$

Bunda E funksional qatorning **aniqlanish to`plami** deyiladi. Masalan,

1) $\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots + x^{n-1} + \cdots,$

2) $\sum_{n=1}^{\infty} ne^{nx} = e^x + 2e^{2x} + 3e^{3x} + \cdots + ne^{nx} + \cdots$

funksional qatorlar bo`lib, ularning aniqlanish to`plami $E = (-\infty, +\infty)$ bo`ladi. (1) funksional qator hadlaridan ushbu

$$
\begin{aligned}
&S_1(x)=u_1(x),\\
&S_2(x)=u_1(x)+u_2(x),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&S_n(x)=u_1(x)+u_2(x)+\cdots+u_n(x)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\qquad (2)
$$

yig`indilarni tuzamiz. Ular (1) funksional qatorning qismiy yig`indilari deyiladi. Demak, (1) funksional qator berilgan holda har doim bu qatorning (2) qismiy yig`indilaridan iborat $\{S_n(x)\}$:

$$S_1(x), S_2(x), \ldots, S_n(x), \ldots$$

funksional ketma-ketlik hosil bo`ladi. Ravshanki, $x = x_0 \in E$ nuqtada $\{S_n(x_0)\}$ sonlar ketma-ketligi bo`ladi.

1-ta`rif.** **[4, Definition 1, p.372]** Agar $\{S_n(x_0)\}$ yaqinlashuvchi (uzoqlashuvchi) bo`lsa, $\sum_{n=1}^{\infty} u_n(x)$ funksional qator $x = x_0$ nuqtada **yaqinlashuvchi (uzoqlashuvchi) deyiladi, x_0 nuqta funksional qatorning **yaqinlashish (uzoqlashish) nuqtasi** deyiladi.

**2-ta`rif.** $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning barcha yaqinlashish nuqtalaridan iborat $E_0 \subset E$ to`plam, $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning **yaqinlashish to`plami** deyiladi. Bu holda $\sum_{n=1}^{\infty} u_n(x)$ funksional qator $E_0$ to`plamda yaqinlashuvchi ham deb yuritiladi.

Agar E_0 to`plamda ushbu

$$\sum_{n=1}^{\infty} |u_n(x)| = |u_1(x)| + |u_2(x)| + \cdots + |u_n(x)| + \cdots$$

qator yaqinlashuvchi bo`lsa, $\sum_{n=1}^{\infty} u_n(x)$ funksional qator E_0 da absolyut yaqinlashuvchi deyiladi.

**3-ta`rif.** **[4, Definition 3, p.372]** $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning qismiy yig`indilaridan iborat $\{S_n(x)\}$ ketma-ketlikning limit funksiyasi $S(x)$:

$$S_n(x) \to S(x) \qquad (x \in E_0)$$

$\sum_{n=1}^{\infty} u_n(x)$ **funksional qator yig`indisi** deyiladi.

$$\sum_{n=1}^{\infty} u_n(x) = S(x) \qquad (x \in E_0) \text{ kabi yoziladi.}$$

1-misol. Ushbu

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots + x^{n-1} + \cdots$$

funksional qatorning yaqinlashish to`plami va yig`indisi topilsin.

◄ Berilgan funksional qatorning aniqlanish to`plami $E = R$ bo`ladi. Qatorning qismiy yig`indisini topamiz:

$$S_n(x) = 1 + x + x^2 + \cdots + x^{n-1} = \begin{cases} \dfrac{1-x^n}{1-x}, & \text{agar } õ \neq 1 \\ n \quad, & \text{agar } õ = 1. \end{cases}$$

Ravshanki, $n \to \infty$ da $S_n(x)$ ning limiti x ga bog`liq bo`ladi:

a) $x \in (-1,1)$ da

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\left(\frac{1}{1-x} - \frac{x^n}{1-x}\right) = \frac{1}{1-x};$$

б) $x \in [1,+\infty)$ da

$$\lim_{n\to\infty} S_n(x) = \infty;$$

в) $x \in (-\infty,-1]$ da $\lim_{n\to\infty} S_n(x)$ mavjud emas.

Demak, berilgan funksional qatorning yaqinlashish to`plami $E_0 = (-1,1)$ bo`lib, yig`indisi

$$S(x) = \frac{1}{1-x}$$

bo`ladi. ►

2-misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{x^n}{1+x^{2n}}$$

funksional qatorning yaqinlashish to`plami topilsin.

◄ Sonli qatorlar nazariyasidagi Dalamber alomatidan foydalanib topamiz:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{1+x^{2n+2}} : \frac{x^n}{1+x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x(1+x^{2n})}{1+x^{2n+2}}\right|;$$

a) $x\in(-1,1)$ da

$$\lim_{n\to\infty}\left|\frac{x(1+x^{2n})}{1+x^{2n+2}}\right| = |x|.$$

Bu holda berilgan funksional qator $(-1,1)$ da yaqinlashuvchi bo\`ladi.

b) $x\in(-\infty,-1)\cup(1,+\infty)$ da

$$\lim_{n\to\infty}\left|\frac{x\left(1+x^{2n}\right)}{1+x^{2n+2}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{x^{2n+1}}+\frac{1}{x}}{\frac{1}{x^{2n+2}}+1}\right| = \left|\frac{1}{x}\right|$$

bo\`lib, funksional qator $x\in(-\infty,-1)\cup(1,+\infty)$ da yaqinlashuvchi bo\`ladi.

v) $x=\pm1$ da berilgan funksional qator mos ravishda ushbu

$$\sum_{n=1}^{\infty}\frac{1}{2}\ ,\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{2}$$

sonli qatorga aylanadi va ular uzoqlashuvchi bo\`ladi.

Shunday qilib, qaralayotan funksional qatorning yaqinlashish to\`plami

$$E_0 = R\setminus\{-1,1\} = (-\infty,-1)\cup(-1,1)\cup(1,+\infty)$$

bo\`ladi. ►

2^0. Funksional qatorning tekis yaqinlashuvchiligi. Aytaylik,

$$\sum_{n=1}^{\infty}u_n(x) = u_1(x)+u_2(x)+\ \ldots\ +u_n(x)+\ldots$$

funksional qator E_0 to\`plamda yaqinlashuvchi (ya\`ni qatorning yaqinlashish to\`plami $E_0$) bo\`lib, yig\`indisi $S(x)$ bo\`lsin:

$$S_n(x)\to S(x) \qquad (x\in E_0) \tag{3}$$

bunda, $S_n(x)=u_1(x)+u_2(x)+\cdots+u_n(x)$. (3) munosabat

$$\forall\varepsilon>0,\ \forall x\in E_0,\exists n_0=n_0(\varepsilon,x)\in N,\ \forall n>n_0 : |S_n(x)-S(x)|<\varepsilon$$

bo\`lishini anglatadi.

**4-ta\`rif.** Agar $E_0$ to\`plamda

$$S_n(x)\underset{\to}{\to}S(x), \quad (x\in E_0)$$

ya`ni

$$\forall \varepsilon > 0 \, , \exists n_0 = n_0(\varepsilon) \in N \, , \forall n > n_0 \, , \ \forall x \in E_0 : |S_n(x) - S(x)| < \varepsilon$$

bo`lsa, $\sum_{n=1}^{\infty} u_n(x)$ funksional qator $E_0$ to`plamda **tekis yaqinlashuvchi** deyiladi.

Agar

$$r_n(x) = S(x) - S_n(x) \, ,$$

deyilsa, funksional qatorning E_0 to`plamda tekis yaqinlashuvchiligini quyidagicha

$$r_n(x) \rightrightarrows 0 \qquad (x \in E_0),$$

ya`ni

$$\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in N \, , \forall n > n_0 \ \ \forall x \in E_0 \ : |r_n(x)| < \varepsilon$$

ko`rinishda ta`riflash mumkin bo`ladi.

Shunday qilib

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

funksional qator, uning qismiy yig`indisi

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$$

va yig`indisi $S(x)$ uchun

$$S_n(x) \to S(x) \qquad\qquad (x \in E_0)$$

bo`lsa, funksional qator E_0 da yaqinlashuvchi,

$$S_n(x) \rightrightarrows S(x) \qquad\qquad (x \in E_0)$$

bo`lsa, funksional qator $E_0$ da tekis yaqinlashuvchi bo`ladi.

1-teorema. $\sum_{n=1}^{\infty} u_n(x)$ funksional qator E_0 da qator yig`indisi $S(x)$ funksiyaga tekis yaqinlashishi uchun

$$\lim_{n\to\infty} \sup_{x \in E_0} |S_n(x) - S(x)| = 0 \, ,$$

ya`ni

$$\lim_{n\to\infty} \sup_{x \in E_0} |r_n(x)| = 0$$

bo`lishi zarur va yetarli.

◄ Bu teoremaning isboti ravshan, (qaralsin 65-ma`ruza,1-teorema.) ►

3-misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n+1)}$$

funksional qatorning $[0,+\infty)$ da tekis yaqinlashuvchi bo`lishi isbotlansin.

◄ Berilgan funksional qatorning qismiy yig`indisini hisoblab, so`ng yig`indisini topamiz:

$$S_n(x)=\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}+\cdots+\frac{1}{(x+n)(x+n+1)}=$$
$$=\left(\frac{1}{x+1}-\frac{1}{x+2}\right)+\left(\frac{1}{x+2}-\frac{1}{x+3}\right)+\cdots+\left(\frac{1}{x+n}-\frac{1}{x+n+1}\right)=$$
$$=\frac{1}{x+1}-\frac{1}{x+n+1},$$

$$\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}\left(\frac{1}{x+1}-\frac{1}{x+n+1}\right)=\frac{1}{x+1}.$$

Demak,

$$S(x)=\frac{1}{x+1}.$$

Unda

$$S_n(x)-S(x)=\frac{1}{x+1}-\frac{1}{x+n+1}-\frac{1}{x+1}=-\frac{1}{x+n+1}$$

bo`lib,

$$\sup_{x\in[0,+\infty)}|S_n(x)-S(x)|=\frac{1}{n+1}$$

bo`ladi. Keyingi tenglikdan

$$\lim_{n\to\infty}\sup_{x\in[0,+\infty)}|S_n(x)-S(x)|=0$$

bo`lishi kelib chiqadi. 1-teoremaga ko`ra berilgan funksional qator $[0,+\infty)$ da tekis yaqinlashuvchi. ►

Eslatma. Agar

$$\lim_{n\to\infty}\sup_{x\in E_0}|S_n(x)-S(x)|\neq 0$$

bo`lsa, $\sum_{n=1}^{\infty}u_n(x)$ funksional qator $E_0$ da tekis yaqinlashuvchi bo`lisht shart emas: Masalan,

$$\sum_{n=1}^{\infty}x^{n-1}=1+x+x^2+\cdots+x^{n-1}+\cdots$$

funksional qatorning $(-1,1)$ da yaqinlashuvchi, yig`indisi

$$S(x)=\frac{1}{1-x}$$

bo`lishini ko`rgan edik. Bu funksional qator uchun

$$\lim_{n\to\infty}\sup_{-1<x<1}|S_n(x)-S(x)|=\lim_{n\to\infty}\sup_{-1<x<1}\left|\frac{x^n}{1-x}\right|=+\infty$$

bo`ladi. Demak, funksional qator $(-1,1)$ da tekis yaqinlashuvchi emas.

Faraz qilaylik,

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$$

funksional qator $E\subset R$ to`plamda berilgan bo`lsin.

2-teorema (Koshi). [4, Theorem 1, p.373] $\sum_{n=1}^{\infty}u_n(x)$ funksional qator E to`plamda tekis yaqinlashuvchi bo`lishi uchun

$$\forall\varepsilon>0,\ \exists n_0=n_0(\varepsilon)\in N,\ \forall n>n_0,\ \forall p\in N,\ \forall x\in E \text{ da}$$

$$|S_{n+p}(x)-S_n(x)|=|u_{n+1}(x)+u_{n+2}(x)+\cdots+u_{n+p}(x)|<\varepsilon$$

bo`lishi zarur va yetarli.

Bu teoremaning isboti [3] dagi 65-ma`ruzadagi 2-teoremadan kelib chiqadi.

3^0. Veyershtrass alomati.

Teorema (Veyershtrass alomati). [4, Corollary 2, p.375] Aytaylik, E to`plamda

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \qquad (1)$$

funksional qator berilgan bo`lib,

1) $\forall n\in N$, $\forall x\in E$ da $|u_n(x)|\le C_n$,

2) $\sum_{n=1}^{\infty}C_n=C_1+C_2+\ldots+C_n+\ldots$ sonli qator yaqinlashuvchi bo`lsin. U holda (1) funksional qator $E$ to`plamda tekis yaqinlashuvchi bo`ladi.

◄ 1) – shartga ko`ra $\forall n>n_0$, $\forall p\in N$ va $\forall x\in E$ uchun

$$|u_{n+1}(x)+u_{n+2}(x)+\cdots+u_{n+p}(x)|\le|u_{n+1}(x)|+|u_{n+2}(x)|+\cdots+|u_{n+p}(x)|\le c_{n+1}+$$
$$+c_{n+2}+\cdots+c_{n+p}$$

bo`lib, 2) – shartda, ya`ni $\sum_{n=1}^{\infty}c_n$ qatorning yaqinlashuvchiligidan Koshi teoremasiga binoan

$$\forall\varepsilon>0,\ \exists n_0=n_0(\varepsilon)\in N,\ \forall n>n_0,\ \forall p\in N \text{ da}$$

$$c_{n+1}+c_{n+2}+\cdots+c_{n+p}<\varepsilon$$

bo`ladi. Demak,

$$\left|u_{n+1}(x)+u_{n+2}(x)+\cdots+u_{n+p}(x)\right|<\varepsilon .$$

Avvalgi mavzudagi 2-teoremaga ko`ra $\sum_{n=1}^{\infty}u_n(x)$ funksional qator $E$ to`plamda tekis yaqinlashuvchi bo`ladi.►

1-misol. Ushbu

$$\sum_{n=1}^{\infty}\frac{x\sin x}{\sqrt{1+n^2}\left(1+nx^2\right)}$$

funksional qator tekis yaqinlashishga tekshirilsin.

◄ Berilgan qatorning aniqlanish to`plami $E=(-\infty,+\infty)$ bo`lib, uning umumiy hadi

$$u_n(x)=\frac{x\sin x}{\sqrt{1+n^2}\left(1+nx^2\right)} \qquad (n=1,2,3,\ldots)$$

bo`ladi. Ravshanki,

$$\left|u_n(x)\right|=\left|\frac{x\sin x}{\sqrt{1+n^2}\left(1+nx^2\right)}\right|\le\frac{|x|}{\sqrt{1+n^2}\left(1+nx^2\right)}.$$

Endi $\forall x\in(-\infty,+\infty)$ uchun

$$\frac{|x|}{1+nx^2}\le\frac{1}{2\sqrt{n}}$$

bo`lishini e`tiborga olib topamiz:

$$\frac{|x|}{\sqrt{1+n^2}\left(1+nx^2\right)}\le\frac{1}{2\sqrt{n\left(1+n^2\right)}}\le\frac{1}{2n^{3/2}}.$$

Demak, berilgan funksional qatorning hadlari uchun

$$\left|u_n(x)\right|\le\frac{1}{2n^{3/2}}$$

bo`ladi. Ma`lumki, $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ qator yaqinlashuvchi. Binobarin, Veyershtrass alomatiga ko`ra berilgan funksional qator $(-\infty,+\infty)$ da tekis yaqinlashuvchi bo`ladi. ►

4⁰. Dirixle alomati.

Funksional qatorlarning tekis yaqinlashishini ifodalovchi keyingi alomatlarini isbotsiz keltiramiz.

Teorema (Dirixle alomati). [4, Proposition 3, p.377] Aytaylik, $E \subset R$ to`plamda aniqlangan $u_n(x)$ va $v_n(x)$ $(n = 1\,2,3,\ldots)$ funksiyalar quyidagi shartlarni bajarsin:

1) $\forall x \in E$ da $\{u_n(x)\}$ ketma-ketlik monoton;

2) $\{u_n(x)\}$ funksional ketma-ketlik E da 0 ga tekis yaqinlashuvchi:

$$u_n(x) \rightrightarrows 0 \quad (x \in E);$$

3) shunday $C \in R$ mavjudki, $\forall n \in N$, $\forall x \in E$ da

$$|v_1(x) + v_2(x) + \cdots + v_n(x)| = \left|\sum_{k=1}^{n} v_k(x)\right| \leq C.$$

U holda

$$\sum_{n=1}^{\infty} u_n(x) \cdot v_n(x)$$

funksional qator E to`plamda tekis yaqinlashuvchi bo`ladi.

2-misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{\sin x \cdot \sin nx}{\sqrt{n+x}}$$

funksional qator $E = [0, +\infty)$ da tekis yaqinlashuvchiligi isbotlansin.

◄ Aytaylik,

$$u_n(x) = \frac{1}{\sqrt{n+x}}, \quad v_n(x) = \sin x \cdot \sin nx$$

bo`lsin. Bu funksiyalar uchun Dirixle alomatidagi uchta shart bajariladi. Haqiqatdan ham,

1) $\forall x \in E$ da $u_n(x) = \dfrac{1}{\sqrt{n+x}}$ uchun

$$\frac{1}{\sqrt{n+x}} - \frac{1}{\sqrt{n+1+x}} = \frac{\sqrt{n+1+x} - \sqrt{n+x}}{\sqrt{n+x} \cdot \sqrt{n+1+x}} =$$

$$= \frac{1}{\sqrt{(n+x)(n+1+x)} \cdot \left(\sqrt{n+1+x} + \sqrt{n+x}\right)} > 0$$

bo`lganligidan uning kamayuvchiligi kelib chiqadi;

2) Ravshanki,

$$u_n(x) = \frac{1}{\sqrt{n+x}} \leq \frac{1}{\sqrt{n}}, \; n \to \infty \text{ da } \frac{1}{\sqrt{n}} \to 0.$$

Demak,

$$u_n(x) \rightrightarrows 0 \quad (x \in E);$$

3) bu holda

$$\left|\sum_{k=1}^{n} v_k(x)\right| = \left|\sum_{k=1}^{n} \sin x \sin kx\right| = 2\left|\cos\frac{x}{2}\right|\left|\sin\frac{nx}{2} \cdot \sin\frac{n+1}{2}x\right| \le 2$$

bo`ladi.

Dirixle alomatiga ko`ra berilgan funksional qator $E = [0, +\infty)$ da tekis yaqinlashuvchi.►

5⁰. Abel' alomati.

Teorema (Abel' alomati). Aytaylik, $E \subset R$ to`plamda aniqlangan $u_n(x)$ va $v_n(x)$ $(n = 1\ 2, 3, \ldots)$ funksiyalar quyidagi shartlarni bajarsin:

1) $\forall x \in E$ da $\{u_n(x)\}$ ketma-ketlik monoton;

2) shunday $C \in R$ topiladiki, $\forall n \in E$, $\forall x \in E$ da

$$|u_n(x)| \le C;$$

3) $\sum_{n=1}^{\infty} v_n(x)$ funksional qator E to`plamda tekis yaqinlashuvchi. U holda

$$\sum_{n=1}^{\infty} u_n(x) \cdot v_n(x)$$

funksional qator E to`plamda tekis yaqinlashuvchi bo`ladi.

3-misol. Ushbu

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

funksional qatorning $E = [0, 1]$ da tekis yaqinlashuvchi ekanligi isbotlansin.

◄ Aytaylik,

$$u_n(x) = x^n \;, \quad v_n(x) = \frac{(-1)^{n+1}}{n} \qquad (x \in [0, 1])$$

bo`lsin. Bu funksiyalar uchun Abel' alomatidagi uchta shart bajariladi (bu ravshan). Unda Abel' alomatiga ko`ra berilgan funksional qator $[0, 1]$ da tekis yaqinlashuvchi bo`ladi.►

7-Amaliy mashg`ulot

Mashq. 1. Agar

$$\sum_{n=1}^{\infty} u_n(x) \qquad (x \in E)$$

funksional qator E to`plamda tekis yaqinlashuvchi bo`lsa, $\{u_n(x)\}$ funksional ketma-ketlikni E to`plamda 0 ga tekis yaqinlashishi isbotlansin.

2. Ushbu $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+x^2}\text{arctg}nx \quad (x\in R)$ funksional qator R da tekis yaqinlashuvchi ekanligi isbotlansin.

Mashqlar. Berilgan to‘plamlarda qatorning tekis yaqinlashishini Veyershtrass alomatiga ko‘ra ko‘rsating.

1 $\sum_{n=1}^{\infty} arctg\frac{2x}{x^2+n^3};|x|<+\infty.$

2 $\sum_{n=1}^{\infty} x^2e^{-nx};0\le x<+\infty$

3 $\sum_{n=1}^{\infty}\ln\left(1+\frac{x^2}{n\ln^2 n}\right);|x|<3.$

4 $\sum_{n=1}^{\infty}\frac{\sin nx}{n\sqrt{n}};|x|<+\infty.$

5 $\sum_{n=1}^{\infty}\frac{\cos nx}{n^2};|x|<+\infty.$

6 $\sum_{n=1}^{\infty}\frac{\sin nx}{\sqrt[3]{n^4+x^4}};|x|<+\infty.$

7 $\sum_{n=1}^{\infty}\frac{x^n}{n!};|x|<2.$

8 $\sum_{n=1}^{\infty}\frac{n^2}{\sqrt{n!}}\left(x^n+x^{-n}\right);\frac{1}{2}\le x\le 2.$

9 $\sum_{n=1}^{\infty}\frac{nx}{1+n^5x^2};|x|<+\infty.$

10 $\sum_{n=1}^{\infty}\frac{x}{1+n^4x^2};0\le x<+\infty.$

11 $\sum_{n=1}^{\infty}\frac{(-1)^n}{x+2^n};-2<x<+\infty.$

12 $\sum_{n=1}^{\infty}\frac{1}{x^2+n^2};-\infty<x<+\infty.$

13 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{x+\sqrt{n^3}};0\le x<+\infty.$

14 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{\sqrt{n^3}};0\le x\le 1.$

15 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n};-1<x<1.$

16 $\sum_{n=1}^{\infty}\frac{1}{(x+2n-1)\cdot(x+2n+1)};0\le x<+\infty.$

17 $\sum_{n=1}^{\infty}\sin\frac{1}{\sqrt{n}}\, arctg\frac{2x}{x^2+n^2};-\infty<x<+\infty.$

18 $\sum_{n=1}^{\infty}\ln\left(1+\frac{nx^2}{2+n^3x^2}\right);-\infty<x<+\infty.$

19 $\sum_{n=1}^{\infty}\frac{(x-1)^n}{(3n+1)\cdot 3^n};-1\le x<3.$

20 $\sum_{n=1}^{\infty}\frac{n\ln(1+nx)}{x^n},2<x<+\infty.$

21 $\sum_{n=1}^{\infty}\ln\left[1+\frac{x}{n\cdot\ln^2(n+1)}\right];0\le x\le 2.$

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NAZORAT SAVOLLARI

1. Yaqinlashuvchi funksional qatorning ta'rifini ayting.
2. Absolyut yaqinlashuvchi funksional qatorning ta'rifini ayting.
3. Funksional qator yig`indisi nima?
4. Tekis yaqinlashuvchi funksional qatorning ta'rifini ayting.
5. Funksional qator tekis yaqilashishining zarur va yetarli sharti qanday?
6. Koshi teoremasini ayting.
7. Veyershtrass alomatini ayting.
8. Dirixle alomatini ayting.
9. Abel' alomatini ayting.

GLOSSARIY

$\sum_{n=1}^{\infty} u_n(x)$ **funksional qator** $x = x_0$ **nuqtada yaqinlashuvchi (uzoqlashuvchi) deyiladi** - Agar $\{S_n(x_0)\}$ yaqinlashuvchi (uzoqlashuvchi) bo'lsa.

$\sum_{n=1}^{\infty} u_n(x)$ **funksional qatorning yaqinlashish to'plami** - $\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning barcha yaqinlashish nuqtalaridan iborat $E_0 \subset E$ to'plam.

$\sum_{n=1}^{\infty} u_n(x)$ **funksional qator** E_0 **da absolyut yaqinlashuvchi** deyiladi - Agar E_0 to'plamda ushbu

$$\sum_{n=1}^{\infty} |u_n(x)| = |u_1(x)| + |u_2(x)| + \cdots + |u_n(x)| + \cdots$$

qator yaqinlashuvchi bo'lsa.

$\sum_{n=1}^{\infty} u_n(x)$ funksional qatorning qismiy yig'indilaridan iborat $\{S_n(x)\}$ ketma-ketlikning limit funksiyasi $S(x)$:

$$S_n(x) \to S(x) \qquad (x \in E_0)$$

$\sum_{n=1}^{\infty} u_n(x)$ funksional qator yig'indisi deyiladi.

Veyershtrass alomati - Aytaylik, E to`plamda

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \qquad (1)$$

funksional qator berilgan bo`lib,

1) $\forall n \in N$, $\forall x \in E$ da $|u_n(x)| \le C_n$,

2) $\sum_{n=1}^{\infty} C_n = C_1 + C_2 + \ldots + C_n + \ldots$ sonli qator yaqinlashuvchi bo`lsin. U holda (1) funksional qator $E$ to`plamda tekis yaqinlashuvchi bo`ladi.

Dirixle alomati - Aytaylik, $E \subset R$ to`plamda aniqlangan $u_n(x)$ va $v_n(x)$ $(n = 1\ 2, 3, \ldots)$ funksiyalar quyidagi shartlarni bajarsin:

1) $\forall x \in E$ da $\{u_n(x)\}$ ketma-ketlik monoton;

2) $\{u_n(x)\}$ funksional ketma-ketlik E da 0 ga tekis yaqinlashuvchi:

$$u_n(x) \rightrightarrows 0 \quad (x \in E);$$

3) shunday $C \in R$ mavjudki, $\forall n \in N$, $\forall x \in E$ da

$$|v_1(x) + v_2(x) + \cdots + v_n(x)| = \left|\sum_{k=1}^{n} v_k(x)\right| \le C.$$

U holda

$$\sum_{n=1}^{\infty} u_n(x) \cdot v_n(x)$$

funksional qator E to`plamda tekis yaqinlashuvchi bo`ladi.

Abel' alomati - Aytaylik, $E \subset R$ to`plamda aniqlangan $u_n(x)$ va $v_n(x)$ $(n = 1\ 2, 3, \ldots)$ funksiyalar quyidagi shartlarni bajarsin:

1) $\forall x \in E$ da $\{u_n(x)\}$ ketma-ketlik monoton;

2) shunday $C \in R$ topiladiki, $\forall n \in E$, $\forall x \in E$ da

$$|u_n(x)| \le C;$$

3) $\sum_{n=1}^{\infty} v_n(x)$ funksional qator E to`plamda tekis yaqinlashuvchi. U holda

$$\sum_{n=1}^{\infty} u_n(x) \cdot v_n(x)$$

funksional qator E to`plamda tekis yaqinlashuvchi bo`ladi.

KEYS BANKI

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$$

funksional qatorning $E=[0,1]$ da tekis yaqinlashuvchi ekanligi isbotlansin.

2-keys. Masala o`rtaga tashlanadi: Ushbu $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+x^2}\operatorname{arctg} nx \quad (x\in R)$ funksional qator R da tekis yaqinlashuvchi ekanligi isbotlansin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

TEST

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$\sum_{n=1}^{\infty}x^{n-1}$ funksional qatorning yaqinlashish sohasini toping.	$(-1,1)$	$(-2,3]$	$(-\infty,+\infty)$
2	$\sum_{n=1}^{\infty}\frac{n}{x^n}$ funksional qatorning yaqinlashish sohasini toping.	$\lvert x\rvert>1$ da absolyut yaqinlashuvchi	$\lvert x\rvert<1$ da absolyut yaqinlashuvchi	uzoqlashuvchi
3	$\sum_{n=1}^{\infty}ne^{-nx}$ funksional qatorning yaqinlashish sohasini toping.	$\lvert x\rvert>0$ da absolyut yaqinlashuvchi	$\lvert x\rvert>1$ da absolyut yaqinlashuvchi	$\lvert x\rvert>1/2$ da absolyut yaqinlashuvchi
4	$f_n(x)=\frac{1}{x^2+n}$, $\lim_{n\to\infty}f_n(x)=f(x)$ bo'lsa $f(x)-?$	$f(x)\equiv 0$	limit funksya yo'q	$f(x)\equiv 1$
5	$f_n(x)=x^n,\ n\in N$ funksional ketma-ketlikning yaqinlashish sohasini toping.	$(-1,1]$	$(-2,3]$	$(-\infty,+\infty)$
6	Qaysi tasdiqlar to`g`ri? 1. $x\in(-\infty;+\infty)$ da $\mathrm{Sin}\frac{x}{n}\rightrightarrows 0$. 2. $x\in(-\infty;+\infty)$ da $\frac{Sinx}{n}\rightrightarrows 0$.	2	1	Ikkalasi noto`g`ri
7	Qanday shartlar bajarilganda $\sum_{k=1}^{\infty}u_k(x)$ qator X to`plamda tekis yaqinlashadi?	1,3	1,2	2,3

	1. $\forall\varepsilon>0\ \exists n_0:$ $\exists n>n_0\ \forall x\in X\ \Rightarrow\left\|\sum_{k=n}^{\infty}u_k(x)\right\|<\varepsilon$. **2.** $\forall x\in X\ \ \forall\varepsilon>0\ \exists n_0:$ $\forall n>n_0\Rightarrow\left\|\sum_{k=n}^{\infty}u_k(x)\right\|<\varepsilon$. **3.** $\forall\varepsilon>0\ \exists n_0:\ \forall n>n_0$ $\forall m>n_0\ \forall x\in X\ \Rightarrow\left\|\sum_{k=n}^{m}u_k(x)\right\|<\varepsilon$			

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$\sum_{n=1}^{\infty}\frac{x^n}{n^2}$ funksional qatorni $-1\le x\le 1$ da yaqinlashishga tekshiring..	tekis yaqinlashuvchi	notekis yaqinlashuvchi	uzoqlashuvchi
2	$\sum_{n=1}^{\infty}\frac{x}{((n-1)x+1)(nx+1)}$ funksional qatorni $0<x<+\infty$ da yaqinlashishga tekshiring.	notekis yaqinlashuvchi	uzoqlashuvchi	tekis yaqinlashuvchi
3	$\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ funksional qatorni $\varepsilon\le x\le 2\pi-\varepsilon$ da tekis yaqinlashishga tekshiring.	tekis yaqinlashuvchi	notekis yaqinlashuvchi	uzoqlashuvchi
4	$\sum_{n=1}^{\infty}\frac{(-1)^n}{x+n}$ funksional qatorni $0<x<+\infty$ da tekis yaqinlashishga tekshiring.	tekis yaqinlashuvchi	notekis yaqinlashuvchi	uzoqlashuvchi
5	$\sum_{n=1}^{\infty}2^n\sin\frac{1}{3^n x}$ funksional qatorni $0<x<+\infty$ da tekis yaqinlashishga tekshiring.	notekis yaqinlashuvchi	tekis yaqinlashuvchi	uzoqlashuvchi

8-Mavzu. Tekis yaqinlashuvchi funksional qatorlarning xossalari
8-Ma’ruza.
REJA:

1^0. Funksional qator yig‘indisining uzluksizligi.
2^0. Funksional qatorlarni hadlab integrallash.
3^0. Funksional qatorlarni hadlab differensiallash.

Tayanch so`z va iboralar: *Funksional qator yig‘indisining uzluksizligi, funksional qatorlarni hadlab integrallash, funksional qatorlarni hadlab differensiallash.*

***1^0. Funksional qator yig`indisining uzluksizligi.*** Faraz qilaylik, $E \subset R$ to`plamda

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots \qquad (1)$$

funksional qator berilgan bo`lib, uning yig`indisi $S(x)$ bo`lsin.

1-teorema. Aytaylik, (1) qator ushbu shartlarni bajarsin:

1) qatorning har bir $u_n(x) \quad (n = 1, 2, 3, \dots)$ hadi E to`plamda uzluksiz,

2) $\sum_{n=1}^{\infty} u_n(x)$ qator E da tekis yaqinlashuvchi. U holda funksional qatorning yig`indisi $S(x)$ funksiya $E$ to`plamda uzluksiz bo`ladi.

◀ Aytaylik, $x_0 \in E$,

$$S_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$$

bo`lsin. Teoremaning 2) – shartiga ko`ra

$$S_n(x) \rightrightarrows S(x) \quad (x \in E)$$

bo`ladi. Ta`rifga binoan

$$\forall \varepsilon > 0, \ \exists n_0 = n_0(\varepsilon) \in N, \ \forall n > n_0 \text{ va } \forall x \in E \text{ da}$$

$$|S_n(x) - S(x)| < \frac{\varepsilon}{3} \qquad (2)$$

jumladan

$$|S_n(x_0) - S(x_0)| < \frac{\varepsilon}{3} \qquad (3)$$

tengsizliklar bajariladi.

Ravshanki, (2) va (3) tengsizliklar n ning n_0 dan katta biror tayin n_1 qiymatida ham o`rinli bo`ladi:

$$|S_{n_1}(x) - S(x)| < \frac{\varepsilon}{3}, \qquad (2')$$

$$\left|S_{n_1}(x_0)-S(x_0)\right|<\frac{\varepsilon}{3}. \qquad (3')$$

Teoremaning 1) shartidan va chekli sondagi uzluksiz funksiyalar yig`indisi yana uzluksiz bo`lishidan

$$S_{n_1}(x)=u_1(x)+u_2(x)+\cdots+u_{n_1}(x)$$

funksiyaning E to`plamda uzluksiz ekanligi kelib chiqadi. Demak, $S_{n_1}(x)$ funksiya $x=x_0$ da uzluksiz. Unda, ta`rifga binoan

$\forall\varepsilon>0$, $\exists\delta=\delta(\varepsilon)>0$, $|x-x_0|<\delta$ tengsizlikni qanoatlantiruvchi barcha $x\in E$ da

$$\left|S_{n_1}(x)-S_{n_1}(x_0)\right|<\frac{\varepsilon}{3} \qquad (4)$$

bo`ladi.

Yuqoridagi ($2'$), ($3'$) va (4) tengsizliklardan foydalanib topamiz:

$$|S(x)-S(x_0)|=\left|\left(S(x)-S_{n_1}(x)\right)+\left(S_{n_1}(x)-S_{n_1}(x_0)\right)+\left(S_{n_1}(x_0)-S(x_0)\right)\right|\le\left|S(x)-S_{n_1}(x)\right|+$$

$$+\left|S_{n_1}(x)-S_{n_1}(x_0)\right|+\left|S_{n_1}(x_0)-S(x_0)\right|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Bu esa $S(x)$ funksiyaning x_0 nuqtada uzluksiz bo`lishini bildiradi. Modomiyki, $x_0$ nuqta $E$ to`plamning ixtiyoriy nuqtasi ekan, $S(x)$ funksiya E to`plamda uzluksiz bo`ladi.►

Yuqorida keltirilgan teoremaning shartlari bajarilganda uning tasdig`ini quyidagicha

$$\lim_{x\to x_0}\sum_{n=1}^{\infty}u_n(x)=\sum_{n=1}^{\infty}\left(\lim_{x\to x_0}u_n(x)\right)$$

ifodalash mumkin.

2⁰. Funksional qatorlarni hadlab integrallash. Faraz qilaylik, $[a,b]$ segmentda

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \qquad (5)$$

funksional qator berilgan bo`lsin.

2-teorema. [4, Corollary 4, p.387] Aytaylik, (5) qator quyidagi shartlarni bajarsin:

1) qatorning har bir $u_n(x)$ $(n=1,2,3,\ldots)$ hadi $[a,b]$ segmentda uzluksiz,

2) $\sum_{n=1}^{\infty}u_n(x)$ qator $[a,b]$ segmentda tekis yaqinlashuvchi,

3) $\sum_{n=1}^{\infty}u_n(x)=S(x)$.

U holda

$$\sum_{n=1}^{\infty}\int_a^x u_n(t)dt=\int_a^x u_1(t)dt+\int_a^x u_2(t)dt+\cdots$$

qator $[a,b]$ da yaqinlashuvchi va

$$\sum_{n=1}^{\infty}\int_a^x u_n(t)dt=\int_a^x S(t)dt \qquad (x\in[a,b])$$

bo`ladi.

◀ Berilgan funksional qatorning qismiy yig`indisi

$$S_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)$$

ni olamiz. Unda teoremaning 2) – va 3) – shartlariga ko`ra

$$S_n(x)\rightrightarrows S(x) \qquad (x\in[a,b])$$

bo`ladi. Tekis yaqinlashish ta`rifiga binoan
$\forall\varepsilon>0$, $\exists n_0=n_0(\varepsilon)\in N$, $\forall n>n_0$ va $\forall t\in[a,b]$ da

$$|S_n(t)-S(t)|<\frac{\varepsilon}{b-a}$$

tengsizlik bajariladi.

Teoremaning 1) – shartidan hamda yuqorida isbot etilgan 1-teoremadan foydalanib

$$\int_a^x u_n(t)dt \quad (n=1,2,3,\ldots), \qquad \int_a^x S(t)dt$$

integrallarning mavjudligini topamiz.

Ushbu

$$\sum_{n=1}^{\infty}\int_a^x u_n(t)dt=\int_a^x u_1(t)dt+\int_a^x u_2(t)dt+\cdots+\int_a^x u_n(t)dt+\cdots$$

funksional qatorni qaraymiz. Bu qatorning qismiy yig`indisi

$$\sigma_n(x)=\sum_{k=1}^{n}\int_a^x u_k(t)dt \qquad (x\in[a,b])$$

bo`lsin. Ravshanki,

$$\sum_{k=1}^{n}\int_a^x u_k(t)dt=\int_a^x\left(\sum_{k=1}^{n}u_k(t)\right)dt.$$

Demak,

$$\sigma_n(x)=\int_a^x S_n(t)dt.$$

Endi

$$\sum_{n=1}^{\infty}\int_a^x u_n(t)dt$$

funksional qatorning $[a,b]$ da tekis yaqinlashuvchiligini ko`rsatamiz. Quyidagi

$$\left|\sigma_n(x)-\int_a^x S(t)dt\right|$$

ayirma uchun

$$\left|\sigma_n(x)-\int_a^x S(t)dt\right|=\left|\int_a^x S_n(t)dt-\int_a^x S(t)dt\right|\le\int_a^x\left|S_n(t)-S(t)\right|dt<\frac{\varepsilon}{b-a}\int_a^x dt=\frac{\varepsilon}{b-a}\cdot(x-a)<\varepsilon$$

bo`ladi. Demak,

$$\sigma_n(x)\rightrightarrows\int_a^x S(t)dt \qquad (x\in[a,b]).$$

Bu esa

$$\sum_{n=1}^{\infty}\int_a^x u_n(t)dt$$

funksional qatorni $[a,b]$ da tekis yaqinlashuvchiligi va

$$\sum_{n=1}^{\infty}\int_a^x u_n(t)dt=\int_a^x S(t)dt$$

bo`lishini bildiradi.►

Keltirilgan teoremaning shartlari bajarilganda teoremaning tasdig`ini quyidagicha

$$\sum_{k=1}^{\infty}\int_a^x u_k(t)dt=\int_a^x\left(\sum_{k=1}^{\infty}u_k(t)\right)dt$$

ifodalash mumkin.

3^0. Funksional qatorlarni hadlab differensiallash. Faraz qilaylik, $[a,b]$ segmentda

$$\sum_{n=1}^{\infty}u_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \qquad (6)$$

funksional qator berilgan bo`lsin.

3-teorema. **[4, Corollary 5, p.389]** Aytaylik, (6) funksional qator quyidagi shartlarni bajarsin:

1) qatorning har bir $u_n(x)$ $(n=1,2,3,\ldots)$ hadi $[a,b]$ segmentda uzluksiz $u_n'(x)$ $(n=1,2,3,\ldots)$ hosilaga ega,

2) Ushbu

$$\sum_{n=1}^{\infty}u_n'(x)=u_1'(x)+u_2'(x)+\ldots+u_n'(x)+\ldots$$

funksional qator $[a,b]$ da tekis yaqinlashuvchi,

3) $x_0\in[a,b]$ nuqta mavjudki,

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x_0) + u_2(x_0) + \ldots + u_n(x_0) + \ldots$$

qator yaqinlashuvchi. U holda

a) $\sum_{n=1}^{\infty} u_n(x)$ funksional qator $[a,b]$ da tekis yaqinlashuvchi,

b) bu qatorning yig`indisi

$$S(x) = \sum_{n=1}^{\infty} u_n(x)$$

$[a,b]$ da uzluksiz $S'(x)$ hosilaga ega,

v) $S'(x) = \sum_{n=1}^{\infty} u_n{}'(x)$

bo`ladi.

◀ Ushbu $\sum_{n=1}^{\infty} u_n{}'(x)$

qatorning yig`indisini $\sigma(x)$ bilan belgilaylik:

$$\sigma(x) = \sum_{n=1}^{\infty} u_n{}'(x). \tag{7}$$

Bu qator tekis yaqinlashuvchi va har bir hadi $[a,b]$ da uzluksiz. Yuqorida keltirilgan 2 – teoremaga ko`ra (7) ni hadlab integrallash mumkin:

$$\int_{x_0}^{x} \sigma(x)dx = \sum_{n=1}^{\infty} \int_{x_0}^{x} u_n{}'(x)dx,$$

bunda $x_0 \in [a,b]$, $x \in [a,b]$. Ayni paytda,

$\sum_{n=1}^{\infty} \int_{x_0}^{x} u_n{}'(x)$ funksional qator $[a,b]$ da tekis yaqinlashuvchi.

Ravshanki,

$$\int_{x_0}^{x} u_n{}'(x)dx = u_n(x) - u_n(x_0).$$

Demak, $\sum_{n=1}^{\infty} (u_n(x) - u_n(x_0))$ qator $[a,b]$ tekis yaqinlashuvchi.

Shartga ko`ra

$$\sum_{n=1}^{\infty} u_n(x_0)$$

qator yaqinlashuvchi (uni $[a,b]$ da tekis yaqinlashuvchi deb qarash mumkin).

Shunday qilib

$$\sum_{n=1}^{\infty} (u_n(x) - u_n(x_0)), \quad \sum_{n=1}^{\infty} u_n(x_0)$$

qatorlar $[a,b]$ da tekis yaqinlashuvchi bo`ladi. Bundan esa bu qatorlarning yig`indisi bo`lgan

$$\sum_{n=1}^{\infty} u_n(x)$$

funksional qatorning $[a,b]$ da tekis yaqinlashuvchiligi kelib chiqadi. Shuni e`tiborga olib topamiz:

$$\int_{x_0}^{x} \sigma(x)dx = \sum_{n=1}^{\infty}(u_n(x) - u_n(x_0)) = \sum_{n=1}^{\infty} u_n(x) - \sum_{n=1}^{\infty} u_n(x_0) = S(x) - S(x_0).$$

$\sigma(x)$ funksiya, har bir hadi uzluksiz, o`zi tekis yaqinlashuvchi

$$\sum_{n=1}^{\infty} u_n{}'(x)$$

qatorning yig`indisi bo`lgani uchun 1-teoremaga ko`ra, $[a,b]$ da uzluksiz bo`ladi.

Unda keyingi tenglikdan

$$\sigma(x) = (S(x) - S(x_0))' = S'(x)$$

bo`lishi kelib chiqadi.

Demak,

$$\sum_{n=1}^{\infty} u_n(x)$$

qator yig`indisi uzluksiz $S'(x)$ hosilaga ega va

$$S'(x) = \sum_{n=1}^{\infty} u_n'(x)$$

bo`ladi.►

Bu keltirilgan teoremaning shartlari bajarilganda uning tasdiqini quyidagicha yozish mumkin:

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty}\left(\frac{d}{dx} u_n(x)\right).$$

1-misol. Ushbu

$$\sum_{n=1}^{\infty} \ln\frac{(n+1)(n+x)}{n(n+1+x)} \qquad (0 \le x \le +\infty)$$

funksional qatorning yig`indisi topilsin.

◄ Ma`lumki,

$$\sum_{n=1}^{\infty} \frac{1}{(n+x)(n+1+x)}$$

funksional qator $[0,+\infty)$ da tekis yaqinlashuvchi bo`lib, uning yig`indisi

$$S(x) = \frac{1}{1+x}$$

ga teng (qaralsin, 66-ma`ruza):

$$\frac{1}{1+x}=\sum_{n=1}^{\infty}\frac{1}{(n+x)(n+1+x)}.$$

Ravshanki, bu qatorning har bir hadi $[0,+\infty)$ da uzluksiz. Demak, uni 2 – teoremaga ko\`ra hadlab integrallash mumkin:

$$\int_0^x\frac{dt}{1+t}=\sum_{n=1}^{\infty}\int_0^x\frac{dt}{(n+t)(n+1+t)}.$$

Aniq integrallarni hisoblaymiz:

$$\int_0^x\frac{dt}{1+t}=\ln(1+t)\Big|_0^x=\ln(1+x),$$

$$\int_0^x\frac{1}{(n+t)(n+1+t)}dt=\int_0^x\left(\frac{1}{n+t}-\frac{1}{n+1+t}\right)dt=$$

$$=\ln(n+t)\Big|_0^x-\ln(n+1+t)\Big|_0^x=\ln\frac{(n+1)(n+x)}{n(n+1+x)}.$$

Demak, $\sum_{n=1}^{\infty}\ln\frac{(n+1)(n+x)}{n(n+1+x)}=\ln(1+x).$ ►

8-Amaliy mashg\`ulot

Mashqlar.

1. Ushbu

$$\sum_{n=1}^{\infty}nx^n \qquad (-1<x<1)$$

funksional qatorning yig\`indisi topilsin.

2. Ushbu

$$S(x)=\sum_{n=1}^{\infty}\frac{1}{n^2+x^2} \qquad (x\in R)$$

funksiya funksional qatorni hadlab differensiallash bilan topilsin.

Berilgan to'plamlarda qatorni tekis yaqinlashishga tekshiring.

1 $\sum_{n=1}^{\infty}\frac{nx}{(1+x)(1+2x)\ldots(1+nx)};\ 0\le x\le 1.$

2 $\sum_{n=1}^{\infty}\frac{nx}{(1+x)(1+2x)\ldots(1+nx)};1\le x<+\infty.$

3 $\sum_{n=1}^{\infty}\frac{1}{(x+n)(x+n+1)};0<x<+\infty.$

4 $\sum_{n=1}^{\infty}\frac{x}{[(n-1)x+1](nx+1)};0<x<+\infty.$

5 $\sum_{n=1}^{\infty}\left(\frac{x^n}{n}-\frac{x^{n+1}}{n+1}\right);-1\le x\le 1.$

6 $\sum_{n=0}^{\infty}(1-x)x^n;0\le x\le 1.$

7 $\sum_{n=0}^{\infty}\frac{x^n}{n!};0<x<+\infty.$

8 $\sum_{n=1}^{\infty}\frac{x^n}{n^3};-1\le x\le 1.$

9 $\sum_{n=1}^{\infty}\frac{\sin nx}{n};\frac{\pi}{2}\le x\le\frac{3\pi}{2}.$

10 $\sum_{n=1}^{\infty}\frac{\sin nx}{n};0\le x\le 2\pi.$

11 $\sum_{n=1}^{\infty} 2^n \cdot \sin\frac{1}{3^n x}; 0 < x < +\infty.$

12 $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}; 0 < x < +\infty.$

13 $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+\sin x}; 0 \le x \le 2\pi.$

14 $\sum_{n=1}^{\infty} \frac{\cos\frac{2n\pi}{3}}{\sqrt{n^2+x^2}}; -\infty < x < +\infty.$

15 $\sum_{n=1}^{\infty} e^{-ntgx}, 0 < x < \frac{\pi}{2}.$

16 $\sum_{n=1}^{\infty} \ln^2\left(1+\frac{x}{1+n^2x^2}\right); 0 \le x < +\infty.$

17 $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n\cdot(x^n+1)}; 1 \le x < +\infty.$

18 $\sum_{n=1}^{\infty} \frac{n\sqrt{x}}{1+n^3x^3}; 0 \le x < +\infty.$

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NAZORAT SAVOLLARI

1. Funksional qator yig‘indisining uzluksizligi haqidagi teoremani ayting.
2. Funksional qatorlarni hadlab integrallash haqidagi teoremani ayting.
3. Funksional qatorlarni hadlab differensiallash haqidagi teoremani ayting.

GLOSSARIY

Funksional qator yig‘indisining uzluksizligi.
Faraz qilaylik, $E \subset R$ to‘plamda

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \quad (1)$$

funksional qator berilgan bo‘lib, uning yig‘indisi $S(x)$ bo‘lsin.

Aytaylik, (1) qator ushbu shartlarni bajarsin:

1) qatorning har bir $u_n(x)$ $(n = 1, 2, 3, \ldots)$ hadi E to‘plamda uzluksiz,

2) $\sum_{n=1}^{\infty} u_n(x)$ qator E da tekis yaqinlashuvchi.

U holda funksional qatorning yig‘indisi $S(x)$ funksiya E to‘plamda uzluksiz bo‘ladi.

Funksional qatorlarni hadlab integrallash.

Faraz qilaylik, $[a,b]$ segmentda

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \tag{2}$$

funksional qator berilgan bo‘lsin.

Aytaylik, (2) qator quyidagi shartlarni bajarsin:

1) qatorning har bir $u_n(x) \quad (n = 1,2,3,\ldots)$ hadi $[a,b]$ segmentda uzluksiz,

2) $\sum_{n=1}^{\infty} u_n(x)$ qator $[a,b]$ segmentda tekis yaqinlashuvchi,

3) $\sum_{n=1}^{\infty} u_n(x) = S(x)$.

U holda

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)dt = \int_a^x u_1(t)dt + \int_a^x u_2(t)dt + \cdots$$

qator $[a,b]$ da yaqinlashuvchi va

$$\sum_{n=1}^{\infty} \int_a^x u_n(t)dt = \int_a^x S(t)dt \qquad (x \in [a,b])$$

bo‘ladi.

Funksional qatorlarni hadlab differensiallash.

Faraz qilaylik, $[a,b]$ segmentda

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \tag{3}$$

funksional qator berilgan bo‘lsin.

Aytaylik, (3) funksional qator quyidagi shartlarni bajarsin:

1) qatorning har bir $u_n(x) \quad (n = 1,2,3,\ldots)$ hadi $[a,b]$ segmentda uzluksiz $u_n'(x) \quad (n = 1,2,3,\ldots)$ hosilaga ega,

2) Ushbu

$$\sum_{n=1}^{\infty} u_n'(x) = u_1'(x) + u_2'(x) + \ldots + u_n'(x) + \ldots$$

funksional qator $[a,b]$ da tekis yaqinlashuvchi,

3) $x_0 \in [a,b]$ nuqta mavjudki,

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x_0) + u_2(x_0) + \ldots + u_n(x_0) + \ldots$$

qator yaqinlashuvchi. U holda

a) $\sum_{n=1}^{\infty} u_n(x)$ funksional qator $[a, b]$ da tekis yaqinlashuvchi,

b) bu qatorning yig‘indisi

$$S(x) = \sum_{n=1}^{\infty} u_n(x)$$

$[a, b]$ da uzluksiz $S'(x)$ hosilaga ega,

v) $S'(x) = \sum_{n=1}^{\infty} u_n{}'(x)$

bo‘ladi.

KEYS BANKI

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=1}^{\infty} \ln \frac{(n+1)(n+x)}{n(n+1+x)} \qquad (0 \le x \le +\infty)$$

funksional qatorning yig‘indisi topilsin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

TEST

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$\sum_{n=1}^{\infty} u_n(x)$ qatorni (a,b) da hadlab differensiallash haqidagi teoremaga qaysi shartlar kiradi? 1. $\sum_{n=1}^{\infty} u_n(x)$ -yaqinlashuvchi 2. $\sum_{n=1}^{\infty} u_n^{'}(x)$ - yaqinlashuvchi 3. $\sum_{n=1}^{\infty} u_n(x)$ -tekis yaqinlashuvchi 4. $\sum_{n=1}^{\infty} u_n^{'}(x)$ -tekis yaqinlashuvchi	1 va 4	2 va 3	3 va 4
2	Qaysi shartlarning bajarilishi $\lim_{x\to a}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} u_n(a)$ tenglikning bajarilishini ta`minlaydi? 1. $u_n(x)$ lar a nuqtada uzluksiz. 2. $\sum_{n=1}^{\infty} u_n(x)$ qator $\|x-a\|<\delta_0$ da yaqinlashadi. 3. $\sum_{n=1}^{\infty} u_n(x)$ qator $\|x-a\|<\delta_0$ da tekis yaqinlashadi.	1 va 3	1 va 2	faqat 1
3	Qaysi shartlarning bajarilishi $\int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right]dx = \sum_{n=1}^{\infty}\left[\int_a^b u_n(x)dx\right]$ tenglikning bajarilishini ta`minlaydi? 1. $\int_a^b u_n(x)dx$ lar barcha n lar uchun mavjud; 2. $\sum_{n=1}^{\infty} u_n(x)$ qator [a,b] da yaqinlashadi; 3. $\sum_{n=1}^{\infty} u_n(x)$ qator [a,b] da tekis yaqinlashadi.	1 va 3	1 va 2	3 yetarli
4	$\sum_{n=0}^{\infty} \frac{n!x^n}{n^n}$ yaqinlashish sohasi topilsin	$\|x\| < e$	$\|x\| \le e$	$\|x\| < 1$
5	$f_n(x) = \frac{\sin nx}{n}$ funksional ketma-ketlik uchun $n \to \infty$ da limit funksiyani toping.	$f(x) \equiv 0$	x^3	$f(x) \equiv 2$

9-Mavzu. Darajali qatorlar, ularning yaqinlashish radiusi va yaqinlashish intervallari

9-Ma'ruza.

REJA:

1^0. Darajali qator tushunchasi. Abel' teoremasi.

2^0. Darajali qatorning yaqinlashish radiusi va yaqinlashish intervali.

3^0. Darajali qatorning yaqinlashish radiusini topish.

Tayanch so`z va iboralar: *Darajali qator, Abel' teoremasi, darajali qatorning yaqinlashish radiusi va yaqinlashish intervali, Koshi-Adamar formulasi.*

1^0. Darajali qator tushunchasi*. *Abel' teoremasi. Har bir hadi

$$u_n(t) = a_n(t - t_0)^n \qquad (t_0 \in R;\ n = 0,1,2...)$$

funksiyadan iborat bo`lgan ushbu

$$\sum_{n=0}^{\infty} a_n(t - t_0)^n = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + ... \tag{1}$$

funksional qator **darajali qator** deyiladi, bunda

$$a_0, a_1, ..., a_n, ...$$

haqiqiy sonlar darajali **qatorning koeffitsientlari** deyiladi.

(1) da $t - t_0 = x$ deyilsa, u quyidagi

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ... \qquad (x \in R) \tag{2}$$

ko`rinishga keladi va biz shu ko`rinishdagi darajali qatorlarni o`rganamiz.

Ravshanki, (2) qatorning qismiy yig`indisi

$$S_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

ko`phaddan iborat. Ayni paytda, $x = 0$ da $S_n(0) = a_0$ bo`ladi. Demak, har qanday (2) ko`rinishdagi darajali qator $x = 0$ nuqtada yaqinlashuvchi bo`ladi.

1-teorema (Abel'). [4, Proposition 2, p.375] Agar

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

darajali qator $x = x_0 \neq 0$ nuqtada yaqinlashuvchi bo`lsa, ushbu

$$|x| < |x_0|$$

tengsizlikni qanoatlantiruvchi barcha x larda darajali qator yaqinlashuvchi (absolyut yaqinlashuvchi) bo`ladi.

◀Aytaylik, $x = x_0 \neq 0$ da

$$\sum_{n=0}^{\infty} a_n x_0^n$$

qator yaqinlashuvchi bo`lsin. Qator yaqinlashishining zaruriy shartiga ko`ra

$$\lim_{n\to\infty} a_n x_0^n = 0$$

bo`ladi. Demak, $\{a_n x_0^n\}$ ketma-ketlik chegaralangan:

$$\exists M > 0,\ \forall n \in N \text{ da } \left|a_n x_0^n\right| \leq M .$$

Ravshanki,

$$\left|a_n x^n\right| = \left|a_n x_0^n\right| \left|\frac{x}{x_0}\right|^n \leq M \cdot \left|\frac{x}{x_0}\right|^n \qquad (3)$$

va $|x| < |x_0|$ da $\left|\frac{x}{x_0}\right| = q < 1$ bo`ladi. Demak $\sum_{n=0}^{\infty} \left|\frac{x}{x_0}\right|^n = \sum_{n=0}^{\infty} q^n$ geometrik qator yaqinlashuvchi. Unda ushbu

$$\sum_{n=0}^{\infty} M \left|\frac{x}{x_0}\right|^n$$

qator ham yaqinlashuvchi bo`ladi. (3) munosabatni e`tiborga olib, so`ng solishtirish teoremasidan foydalanib

$$\sum_{n=0}^{\infty} a_n x^n$$

darajali qatorning yaqinlashishini (absolyut yaqinlashi-shini) topamiz.▶

Natija. Agar

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

darajali qator $x = x_1$ nuqtada uzoqlashuvchi (ushbu

$$\sum_{n=0}^{\infty} a_n x_1^n = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n + \ldots$$

sonli qator uzoqlashuvchi) bo`lsa, quyidagi

$$|x| > |x_1|$$

tengsizlikni qanoatlantiruvchi barcha x larda $\sum_{n=0}^{\infty} a_n x^n$ qator uzoqlashuvchi bo`ladi.

◀Teskarisini faraz qilaylik, $\sum_{n=0}^{\infty} a_n x^n$ qator $|x| > |x_1|$ tengsizlikni qanoatlantiruvchi biror $x = x^*$ nuqtada $\left(|x^*| > |x_1|\right)$ yaqinlashuvchi bo`lsin. U holda Abel' teoremasiga ko`ra $|x| < |x^*|$ tengsizlikning qanotalantiruvchi barcha x larda yaqinla-shuvchi, jumladan x_1 nuqtada ham yaqinlashuvchi bo`lib qoladi. Bu esa shartga ziddir.▶

Abel' teoremasi va uning natijasi darajali qator-larning yaqinlashish (uzoqlashish) to`plamining struktura-sini (tuzilishini) aniqlab beradi.

2^0. Darajali qatorning yaqinlashish radiusi va yaqinlashish intervali. Faraz qilaylik,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

darajali qator berilgan bo`lsin. Bu qatorning yaqinlashish yoki uzoqlashish nuqtalari haqida quyidagi uch hol bo`lishi mumkin:

1) barcha musbat sonlar qatorning yaqinlashish nuqtalari bo`ladi;
2) barcha musbat sonlar qatorning uzoqlashish nuqtalari bo`ladi;
3) shunday musbat sonlar borki, ular qatorning yaqinlashish nuqtalari bo`ladi, shunday musbat sonlar borki, ular qatorning uzoqlashish nuqtalari bo`ladi.

Birinchi holda, Abel' teoremasiga ko`ra darajali qator barcha $x \in R$ da yaqinlashuvchi bo`lib, darajali qatorning yaqinlashish to`plami $E = (-\infty, +\infty)$ bo`ladi. Bunday qatorga ushbu

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!} x^2 + \dots + \frac{1}{n!} x^n + \dots$$

darajali qator misol bo`ladi.

Ikkinchi holda, Abel' teoremasining natijasiga ko`ra darajali qator barcha $x \in R \setminus \{0\}$ da uzoqlashuvchi bo`lib, uning yaqinlashish to`plami $E = \{0\}$ bo`ladi. Bunday qatorga ushbu

$$\sum_{n=1}^{\infty} n! x^n = x + 2! x^2 + 3! x^3 + \dots + n! x^n + \dots$$

darajali qator misol bo`laoladi.

Endi uchinchi holni qaraymiz. Bu holga ushbu

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$$

darajali qator misol bo`ladi. Bu darajali qator barcha $x \in (0,1)$ da yaqinlashuvchi va demak, Abel' teoremasiga ko`ra qator $(-1,1)$ da yaqinlashadi, barcha $x \in [1,+\infty)$ da qator uzoqlashuvchi va demak, Abel' teoremasining natijasiga ko`ra qator $(-\infty,-1] \cup [1,+\infty)$ da uzoqlashadi. Demak, darajali qatorning yaqinlashish to`plami $E = (-1,1)$ bo`ladi.

Aytaylik,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

darajali qator r_1 nuqtada $(r_1 > 0)$ yaqinlashuvchi, R_1 nuqtada $(R_1 > 0)$ nuqtada esa uzoqlashuvchi bo`lsin. Ravshanki,

$$r_1 < R_1$$

bo`ladi.

Agar $\sum_{n=0}^{\infty} a_n x^n$ darajali qator

$$\frac{r_1 + R_1}{2}$$

nuqtada yaqinlashuvchi bo`lsa,

$$r_2 = \frac{r_1 + R_1}{2}, \qquad R_2 = R_1$$

deb, uzoqlashuvchi bo`lsa,

$$r_2 = r_1, \quad R_2 = \frac{r_1 + R_1}{2}$$

deb r_2 va R_2 nuqtalarni olamiz. Ravshanki,

$$r_1 \le r_2, \; R_1 \ge R_2 \;\text{ va }\; R_2 - r_2 = \frac{R_1 - r_1}{2}$$

bo`ladi. Bu munosabatdagi $r_2$ va $R_2$ sonlarga ko`ra r_3 va R_3 sonlarni yuqoridagiga o`xshash aniqlaymiz:

Agar $\sum_{n=0}^{\infty} a_n x^n$ darajali qator

$$\frac{r_1 + R_1}{2}$$

nuqtada yaqinlashuvchi bo`lsa,

$$r_3 = \frac{r_2 + R_2}{2}, \quad R_3 = R_2$$

deb, uzoqlashuvchi bo`lsa,

$$r_3 = r_2, \quad R_3 = \frac{r_2 + R_2}{2}$$

deb r_3 va R_3 nuqtalarni olamiz. Bunda

$$r_2 \le r_3, \ R_2 \ge R_3 \ \text{va} \ R_3 - r_3 = \frac{R_1 - r_1}{2^2}$$

bo`ladi.

Bu jarayonni davom ettiraborish natijasida $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish nuqtalaridan iborat $\{r_n\}$, uzoqlashish nuqtalaridan iborat $\{R_n\}$ ketma-ketliklar hosil bo`ladi. Bunda

$$r_1 \le r_2 \le ... \le r_n \le ..., \qquad R_1 \ge R_2 \ge ... \ge R_n \ge ...,$$

va $n \to \infty$ da

$$R_n - r_n = \frac{R_1 - r_1}{2^{n-1}} \to 0$$

bo`ladi. Unda [3], 3-bob, 8-§ da keltirilgan teoremaga ko`ra $\lim\limits_{n\to\infty} r_n$ va $\lim\limits_{n\to\infty} R_n$ limitlar mavjud va

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} R_n$$

bo`ladi. Uni r bilan belgilaymiz:

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} R_n = r.$$

Endi x o`zgaruvchining $|x| < r$ tengsizlikni qanoatlan-tiruvchi ixtiyoriy qiymatini olaylik. Unda

$$\lim_{n\to\infty} r_n = r$$

bo`lishidan, shunday $n_0 \in N$ topiladiki,

$$|x| < r_{n_0} < r$$

bo`ladi. Binobarin, berilgan darajali qator $r_{n_0}$ nuqtada, demak qaralayotgan $x$ nuqtada yaqinlashuvchi bo`ladi.

x o`zgaruvchining $|x| > r$ tenglikni qanoatlantiruvchi ixtiyoriy qiymatini olaylik. Unda

$$\lim_{n\to\infty} R_n = r$$

bo`lishidan, shunday $n_1 \in N$ topiladiki,

$$|x| > R_{n_1} > r$$

bo`ladi. Binobarin, berilgan darajali qator $R_{n_1}$ nuqtada, demak qaralayotgan $x$ nuqtada uzoqlashuvchi bo`ladi.

Demak, 3)-holda $\sum_{n=0}^{\infty} a_n x^n$ darajali qator uchun shunday musbat r soni mavjud bo`ladiki, $|x| < r$, ya`ni $\forall x \in (-r, r)$ da qator yaqinlashuvchi, $|x| > r$, ya`ni $\forall x \in (-\infty, -r) \cup (r, +\infty)$ da qator uzoqlashuvchi bo`ladi. $x = \pm r$ nuqtalarda $\sum_{n=0}^{\infty} a_n x^n$ darajali qator yaqinlashuvchi ham bo`lishi mumkin, uzoqlashuvchi ham bo`lishi mumkin.

1-ta`rif. Yuqorida keltirilgan r son $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning **yaqinlashish radiusi**, $(-r, r)$ interval esa darajali qatorning **yaqinlashish intervali** deyiladi.

Eslatma. 1)-holda darajali qatorning yaqinlashish radiusi $r = +\infty$ deb, 2)-holda darajali qatorning yaqinlashish radiusi $r = 0$ deb olinadi.

3^0. Darajali qatorning yaqinlashish radiusini topish. Biror

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

darajali qatorni qaraylik. Bu qator koeffitsientlaridan tuzilgan $\{a_n\}$ $(n = 0,1,2,\dots)$ ketma-ketlik uchun

1) $\forall n \geq 0$ da $a_n \neq 0$,

2) $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ mavjud bo`lsin. U holda $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

bo`ladi.

◀Aytaylik, darajali qator uchun

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = L \qquad (a_n \neq 0,\ n = 0,1,2,3\dots)$$

bo`lsin. qaralayotgan $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorda $x$ ni parametr hisoblab, Dalamber alomatiga ko`ra uni yaqinlashishga tekshiramiz:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{x+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot|x| = |x|\lim_{n\to\infty}\left|\frac{1}{\frac{a_n}{a_{n+1}}}\right| = |x|\cdot\frac{1}{L}$$

Demak,

$$\frac{|x|}{L}<1,\ \text{ya`ni}\ |x|<L$$

bo`lganda qator yaqinlashuvchi bo`ladi,

$$\frac{|x|}{L}>1,\ \text{ya`ni}\ |x|>L$$

bo`lganda darajali qator uzoqlashuvchi bo`ladi.

Bundan $\sum_{n=0}^{\infty}a_n x^n$ darajali qatorning yaqinlashish radiusi

$$r = L = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad (4)$$

bo`lishi kelib chiqadi.►

1-misol. Ushbu

$$\sum_{n=0}^{\infty}\frac{n^n}{e^n n!}x^n \qquad (0!=1)$$

darajali qatorning yaqinlashish radiusi topilsin.

◄ Bu qator uchun

$$a_n = \frac{n^n}{e^n n!}\quad , \qquad a_{n+1} = \frac{(n+1)^{n+1}}{e^{n+1}(n+1)!}$$

bo`ladi. Ravshanki,

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n^n}{e^n\cdot n!}\cdot\frac{e^{n+1}(n+1)!}{(n+1)^{n+1}}\right| = \lim_{n\to\infty}\frac{e}{\left(1+\frac{1}{n}\right)^n} = 1.$$

Demak, berilgan darajali qatorning yaqinlashish radiusi $r=1$ bo`ladi. ►

Ixtiyoriy darajali qatorning yaqinlashish radiusini aniqlab beradigan teoremani isbotsiz keltiramiz.

2-teorema (Koshi-Adamar). **[4, Theorem 2, p.376]** Ushbu

$$\sum_{n=0}^{\infty}a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

darajali qatorning yaqinlashish radiusi

$$r = \frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}} \qquad (5)$$

bo`ladi.[1]

Eslatma. Agar

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} = +\infty$$

bo`lsa, $\sum\limits_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi $r = 0$ deb,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} = 0$$

bo`lsa, $\sum\limits_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi $r = +\infty$ deb olinadi.

2-misol. Ushbu

$$\sum_{n=0}^{\infty} 2^n x^{5n}$$

darajali qatorning yaqinlashish radiusi topilsin.

◀ Avvalo

$$2x^5 = t$$

deb olamiz. Natijada berilgan qator quyidagi

$$\sum_{n=0}^{\infty} t^n = 1 + t + t^2 + ... + t^n + ...$$

ko`rinishga keladi. Bu qatorning yaqinlashish radiusi (5) formulaga ko`ra

$$r = \frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{1}} = 1$$

bo`ladi. Demak, $|t| < 1$ da qator yaqinlashuvchi, $|t| > 1$ da uzoqlashuvchi. Unda

$$\left|2x^5\right| < 1,$$

ya`ni $|x| < \frac{1}{\sqrt[5]{2}}$ da berilgan qator yaqinlashuvchi,

$$\left|2x^5\right| > 1,$$

ya`ni $|x|>\frac{1}{\sqrt[5]{2}}$ da uzoqlashuvchi bo`ladi. Berilgan darajali qatorning yaqinlashish radiusi $r=\frac{1}{\sqrt[5]{2}}$ bo`ladi. ►

3-misol. Ushbu

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n-1}\sqrt{n}}x^n$$

darajali qatorning yaqinlashish to`plami topilsin.

◄ Ravshanki,

$$a_n=\frac{(-1)^n}{3^{n-1}\sqrt{n}},\qquad a_{n+1}=\frac{(-1)^{n+1}}{3^n\sqrt{n+1}}.$$

Berilgan darajali qatorning yaqinlashish radiusini (4) formulaga ko`ra topamiz:

$$r=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n}{3^{n-1}\sqrt{n}}\cdot\frac{3^n\sqrt{n+1}}{(-1)^{n+1}}\right|=\lim_{n\to}3\sqrt{\frac{n+1}{n}}=3.$$

Darajali qator $x=-3$ nuqtada ushbu $\sum_{n=1}^{\infty}\frac{3}{\sqrt{n}}$ sonli qatorga aylanadi va bu sonli qator uzoqlashuvchi bo`ladi. $x=3$ nuqtada esa quyidagi $\sum_{n=1}^{\infty}3\frac{(-1)^n}{\sqrt{n}}$ sonli qator hosil bo`ladi va bu qator Leybnits teoremasiga ko`ra yaqinlashuvchi bo`ladi. Demak, berilgan darajali qatorning yaqinlashish to`plami $E=(-3,3]$ dan iborat. ►

9-Amaliy mashg`ulot.

1. Agar

$$\sum_{n=0}^{\infty}a_nx^n$$

darajali qatorning yaqinlashish radiusi $r>0$ bo`lsa, ushbu

$$\sum_{n=0}^{\infty}(n^2+1)a_nx^n,\qquad \sum_{n=0}^{\infty}\frac{1}{n+3}a_nx^n$$

darajali qatorning yaqinlashish radiuslari ham r ga teng bo`lishi isbotlansin.

2. Ushbu

$$\sum_{n=0}^{\infty}\frac{(n!)}{(2n)!}x^n$$

darajali qatorning yaqinlashish intervali topilsin.

3. Funksional qatorlarning yaqinlashish radiusini toping.

1 $\sum_{n=1}^{\infty}\frac{(-1)^n}{(x+n)^{-1/5}}$;	**2** $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cdot\left(\frac{1-x}{1+x}\right)^n$;
3 $\sum_{n=1}^{\infty}\frac{n}{n+1}\cdot\frac{1}{(3x^2+4x+2)^n}$.	**4** $\sum_{n=1}^{\infty}\frac{n+1}{3^n}\cdot(x^2-4x+6)^n$.
5 $\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}$.	**6** $\sum_{n=1}^{\infty}\frac{n+3}{n+1}\cdot\frac{1}{(27x^2+12x+2)^n}$.
7 $\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$.	**8** $\sum_{n=1}^{\infty}\frac{n\cdot 2^n}{n+1}\cdot\frac{1}{(3x^2+8x+6)^n}$.
9 $\sum_{n=1}^{\infty}\frac{1}{n+3}\left(\frac{1+x}{1-x}\right)^n$.	**10** $\sum_{n=1}^{\infty}\frac{(x^2-6x+12)^n}{4^n\cdot(n^2+1)}$.
11 $\sum_{n=1}^{\infty}\frac{1}{\left(\sqrt[3]{n^2}+\sqrt{n}+1\right)^{2x+1}}$.	**12** $\sum_{n=1}^{\infty}\frac{(-1)^n}{(x+n)^3}$.
13 $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{x+n}}$.	**14** $\sum_{n=1}^{\infty}\frac{(x^2-5x+11)^n}{5^n\cdot(n^2+5)}$.
15 $\sum_{n=1}^{\infty}\frac{(n+x)^n}{n^n}$.	**16** $\sum_{n=1}^{\infty}\frac{1}{n(n+x)}$.
17 $\sum_{n=1}^{\infty}\frac{(-1)^n}{(x+n)^2}$.	**18** $\sum_{n=1}^{\infty}\frac{1+x^n}{1-x^n}$.
19 $\sum_{n=1}^{\infty}\frac{n+1}{x\cdot n^x}$.	**20** $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^{x^2-1}}$.

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NAZORAT SAVOLLARI

1. Darajali qatorning ko`rinishi qanday?
2. Abel' teoremasini ayting.
3. Darajali qatorning yaqinlashish radiusi nima?
4. Darajali qatorning yaqinlashish intervali nima?
5. Darajali qatorning yaqinlashish radiusi qanday hisoblanadi?

GLOSSARIY

Darajali qator - Har bir hadi

$$u_n(t)=a_n(t-t_0)^n \qquad (t_0\in R;\ n=0,1,2...)$$

funksiyadan iborat bo‘lgan ushbu

$$\sum_{n=0}^{\infty}a_n(t-t_0)^n=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\dots \tag{1}$$

funksional qator darajali qator deyiladi, bunda

$$a_0, a_1, ..., a_n, ...$$

haqiqiy sonlar darajali qatorning koeffitsientlari deyiladi.

(1) da $t-t_0=x$ deyilsa, u quyidagi

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\dots+a_nx^n+\dots \qquad (x\in R) \tag{2}$$

ko‘rinishga keladi va biz shu ko‘rinishdagi darajali qatorlarni o‘rganamiz.

Ravshanki, (2) qatorning qismiy yiђindisi

$$S_n(x)=a_0+a_1x+a_2x^2+\dots+a_nx^n$$

ko‘phaddan iborat. Ayni paytda, $x=0$ da $S_n(0)=a_0$ bo‘ladi. Demak, har qanday (2) ko‘rinishdagi darajali qator $x=0$ nuqtada yaqinlashuvchi bo‘ladi.

Darajali qatorning yaqinlashish radiusi va yaqinlashish intervali - $\sum_{n=0}^{\infty}a_nx^n$ darajali qatorning yaqinlashish nuqtalaridan iborat $\{r_n\}$, uzoqlashish nuqtalaridan iborat $\{R_n\}$ ketma-ketliklar bo‘lsin. Bunda

$$r_1\le r_2\le\dots\le r_n\le\dots, \qquad R_1\ge R_2\ge\dots\ge R_n\ge\dots,$$

va $n\to\infty$ da

$$R_n-r_n=\frac{R_1-r_1}{2^{n-1}}\to 0$$

bo‘ladi.

Darajali qatorning yaqinlashish intervali - Yuqorida keltirilgan r son $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi, $(-r, r)$ interval esa darajali qatorning yaqinlashish intervali deyiladi.

KEYS BANKI

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n-1}\sqrt{n}} x^n$$

darajali qatorning yaqinlashish to‘plami topilsin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

TEST

	Test topshirig`i	To’g`ri javob	Muqobil javob	Muqobil javob
1	$f_n(x) = x^n, \ n \in N$ funksional ketma-ketlikning yaqinlashish sohasini toping.	$(-1, 1]$	$(-2, 3]$	$(-\infty, +\infty)$
2	$\sum_{n=1}^{\infty} x^{n-1}$ funksional qatorning yaqinlashish sohasini toping.	$(-1, 1)$	$(-2, 3]$	$(-\infty, +\infty)$
3	$\sum_{n=1}^{\infty} x^{n-1}$ darajali qatorning yaqinlashish radiusini toping.	$R = 1$	$R = 2$	$R = 4$
4	$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ darajali qatorning yaqinlashish sohasini toping.	$[-1, 1]$	$(-2, 3]$	$(-\infty, +\infty)$
5	$\sum_{n=0}^{\infty} \frac{n! x^n}{n^n}$ qatorning yaqinlashish sohasi topilsin.	$\|x\| < e$	$\|x\| \le 1$	$\|x\| \le e$
6	$\sum_{n=1}^{\infty} \frac{x^n}{n}$ qatorning yaqinlashish to`plami topilsin?	-1≤x<1	-1<x<1	|x|≤1

10-Mavzu. Darajali qatorning tekis yaqinlashishi.
10-Ma’ruza.
REJA:

1^0 .Darajali qatorning tekis yaqinlashishi.

2^0. Darajali qatorning xossalari.

3^0. Funksiyaning Teylor qatori.

4^0. Funksiyani Teylor qatoriga yoyish.

**Tayanch so`z va iboralar:** *Darajali qatorning tekis yaqinlashishi, darajali qator yig`indisining uzluksizligi, darajali qatorni hadlab integrallash va differensiallash. Teylor qatori, Teylor teoremasi, ko`rsatkichli va giperbolik funksiyalarning Teylor qatorlari, trigonometrik funksiyalarning Teylor qatorlari, logarifmik funksiyaning Teylor qatori, darajali funksiyaning Teylor qatori.*

1^0. Darajali qatorning tekis yaqinlashishi. Aytaylik, ushbu

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \qquad (1)$$

darajali qatorning yaqinlashish radiusi $r > 0$ bo`lsin.

1-teorema. [4, Theorem 2, p.376] (1) darajali qator $[\alpha, \beta] \subset (-r, r)$ da tekis yaqinlashuvchi bo`ladi, bunda $\alpha \in R,\ \beta \in R$.

◄ Ravshanki, (1) darajali qator $(-r, r)$ da absolyut yaqinlashuvchi bo`ladi.

Aytaylik, $\alpha \in (0, r)$ bo`lsin. Unda $\forall n \geq 0$ va $\forall x \in [-\alpha, \alpha]$ da

$$\left| a_n x^n \right| \leq \left| a_n \alpha^n \right|$$

bo`lganligi uchun, Veyershtrass alomatiga ko`ra (1) qator $[-\alpha, \alpha]$ da tekis yaqinlashuvchi bo`ladi. ►

Demak, $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yaqinlashish radiusi $r > 0$ bo`lsa, yuqorida keltirilgan teoremaga ko`ra bu qator $[-c, c] \subset (-r, r)$ da $(c > 0)$ tekis yaqinlashuvchi bo`ladi. Bunda $c$ sonni $r$ songa har qancha yaqin qilib olish mumkin bo`lsada, qator $(-r, r)$ da tekis yaqinlashmasdan qolishi mumkin. Masalan, ushbu

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots$$

darajali qatorning yaqinlashish radiusi $r = 1$, biroq qator $(-1, 1)$ da tekis yaqinlashuvchi emas.

2^0. Darajali qatorning xossalari. Ma`lumki, darajali qatorlar funksional qatorlarning xususiy holi. Binobarin, ular tekis yaqinlashuvchi funksional qatorlarning xossalari kabi xossalarga ega.

2-teorema. Agar

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

darajali qatorning yaqinlashish radiusi $r > 0$ bo`lib, yig`indisi

$$S(x) = \sum_{n=0}^{\infty} a_n x^n$$

bo`lsa, $S(x)$ funksiya $(-r, r)$ da uzluksiz bo`ladi.

◀ Ravshanki, qaralayotgan darajali qator $(-r, r)$ da yaqinlashuvchi bo`ladi.

Aytaylik, $x_0 \in (-r, r)$ bo`lsin. Ushbu

$$|x_0| < c < r$$

tengsizlikni qanoatlantiruvchi c sonini olaylik. Unda darajali qator $[-c, c]$ da tekis yaqinlashuvchi bo`ladi. Tekis yaqinlashuvchi funksional qatorning xossasiga ko`ra $\sum_{n=0}^{\infty} a_n x^n$ darajali qatorning yig`indisi $S(x)$ funksiya $[-c, c]$ da uzluksiz, jumladan x_0 nuqtada uzluksiz. ▶

3-teorema. Aytaylik, darajali qatorning yaqinlashish radiusi $r > 0$ bo`lib, yig`indisi $S(x)$ bo`lsin:

$$S(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Bu qatorni $(-r, r)$ ga tegishli bo`lgan ixtiyoriy $[a, b]$ bo`yicha $([a, b] \subset (-r, r))$ hadlab integrallash mumkin:

$$\int_a^b S(x)dx = \sum_{n=0}^{\infty} \left(\int_a^b a_n x^n dx \right).$$

Xususan, $\forall x \in (-r, r)$ uchun

$$\int_0^x S(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \qquad (2)$$

bo`ladi.

◀ Ravshanki, darajali qator $[a, b]$ da $([a, b] \subset (-r, r))$ tekis yaqinlashuvchi bo`ladi. Tekis yaqinlashuvchi funksional qatorning xossasiga ko`ra uni hadlab integrallash mumkin. Ayni paytda, (2) qatorning yaqinlashish radiusi r ga teng bo`ladi. Haqiqatan ham Koshi-Adamar teoremasiga ko`ra

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left| \frac{a_n}{n+1} \right|} = \overline{\lim_{n\to\infty}} \frac{\sqrt[n]{|a_n|}}{\sqrt[n]{n+1}} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = r$$

bo`ladi. ▶

Natija. Aytaylik, $\sum_{n=0}^{\infty} a_n x^n$ darajali qator berilgan bo`lib, uning yaqinlashish radiusi $r > 0$ bo`lsin. Bu qatorni $[0, x]$ bo`yicha $(\forall x \in (-r, r))$ ixtiyoriy marta hadlab integrallash mumkin. Integrallash natijasida hosil bo`lgan darajali qatorning yaqinlashish radiusi ham r ga teng bo`ladi.

3-teorema. **[4, Proposition 3, p.390]** Faraz qilaylik,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

darajali qatorning yaqinlashish radiusi $r > 0$, yig`indisi $S(x)$ bo`lsin:

$$\sum_{n=0}^{\infty} a_n x^n = S(x).$$

U holda $S(x)$ funksiya $(-r, r)$ da uzluksiz $S'(x)$ hosilaga ega va

$$S'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{3}$$

bo`ladi, bunda (3) qatorning yaqinlashish radiusi ham r ga teng.

◀ Berilgan darajali qator $[-c, c]$ da $(0 < c < r)$ tekis yaqinlashuvchi bo`ladi. Tekis yaqinlashuvchi funksional qatorning xossasiga ko`ra darajali qatorni hadlab differensiallash mumkin. Demak, $\forall x \in (-r, r)$ da

$$S'(x) = \sum_{n=0}^{\infty} \left(a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Bu darajali qatorning yaqinlashish radiusi ham r ga teng bo`lishi quyidagi munosabatdan kelib chiqadi:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|n a_n|} = \overline{\lim_{n\to\infty}} \left(\sqrt[n]{n} \cdot \sqrt[n]{|a_n|}\right) = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}\,. \blacktriangleright$$

Natija. Aytaylik, $\sum_{n=0}^{\infty} a_n x^n$ darajali qator berilgan bo`lib, uning yaqinlashish radiusi $r > 0$ bo`lsin. Bu qatorni $(-r, r)$ da ixtiyoriy marta hadlab differentsiallash mumkin. Differentsiallash natijasida hosil bo`lgan darajali qatorning yaqinlashish radiusi ham $r$ ga teng bo`ladi.

4-teorema. Aytaylik,

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

darajali qatorning yaqinlashish radiusi $r > 0$, yig`indisi $S(x)$ bo`lsin:

$$\sum_{n=0}^{\infty} a_n x^n = S(x)\quad . \tag{4}$$

U holda $\forall n \ge 0$ da

$$a_n = \frac{S^{(n)}(0)}{n!}$$

bo`ladi.

◀(4) munosabatda $x=0$ deb topamiz:

$$a_0 = S(0).$$

(4) qatorni hadlab differentsiallaymiz:

$$S'(x) = \sum_{n=0}^{\infty}(a_n x^n)' = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

Bu tenglikda $x=0$ deyilsa

$$a_1 = S'(0)$$

bo`lishi kelib chiqadi. Shu jarayonni davom ettiraborib

$$a_n = \frac{S^{(n)}(0)}{n!} \qquad (n = 2,3,\ldots)$$

bo`lishini topamiz. ▶

3^0. Funksiyaning Teylor qatori. [3, Definition 6, p.225] Aytaylik, $f(x)$ funksiya $x_0 \in R$ nuqtaning biror

$$U_\delta(x_0) = \{x \in R : x_0 - \delta < x < x_0 + \delta; \delta > 0\}$$

atrofida istalgan tartibdagi hosilaga ega bo`lsin. Bu hol $f(x)$ funksiyaning Teylor formulasini yozish imkonini beradi:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_n(x),$$

bunda $r_n(x)$-qoldiq had.

Modomiki, $f(x)$ funksiya $U_\delta(x_0)$ da istalgan tartibdagi hosilaga ega ekan, unda

$$f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots \qquad (1)$$

darajali qatorni qarash mumkin bo`ladi.

(1) darajali qatorning koeffitsientlari sonlar bo`lib, ular $f(x)$ funksiya va uning hosilalarining x_0 nuqtadagi qiymatlari orqali ifodalangan.

(1) darajali qator $f(x)$ funksiyaning Teylor qatori deyiladi.

Xususan, $x_0 = 0$ bo`lganda (1) darajali qator ushbu

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots = \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

ko`rinishga keladi.

Faraz qilaylik, $f(x)$ funksiya biror $(-r, r)$ da $(r>0)$ istalgan tartibdagi hosilaga ega bo`lib, uning $x_0 = 0$ nuqtadagi Teylor qatori

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots \qquad (2)$$

bo`lsin. Bu qatorning qoldiq hadini $r_n(x)$ deylik:

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+r_n(x).$$

1-teorema. [3, p.226] (2) darajali qator $(-r,r)$ da $f(x)$ ga yaqinlashishi uchun ushbu

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots\frac{f^{(n)}(0)^n}{n!}+r_n(x)$$

Teylor formulasida, $\forall x\in(-r,r)$ uchun

$$\lim_{n\to\infty} r_n(x)=0$$

bo`lishi zarur va yetarli.

◄ **Zarurligi**. Aytaylik, (2) darajali qator $(-r,r)$ da yaqinlashuvchi, yi\indisi $f(x)$ bo`lsin. Ta`rifga binoan

$$\lim_{n\to\infty} S_n(x)=f(x), \qquad (x\in(-r,r))$$

bo`ladi, bunda

$$S_n(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots\frac{f^{(n)}(0)^n}{n!}.$$

Ravshanki, $\forall x\in(-r,r)$ da $\lim_{n\to\infty} S_n(x)=f(x)$ bo`lishidan

$$\lim_{n\to\infty}[f(x)-S_n(x)]=\lim_{n\to\infty} r_n(x)=0$$

bo`lishi kelib chiqadi.

Yetarliligi. Aytaylik, $\forall x\in(-r,r)$ da $\lim_{n\to\infty} r_n(x)=0$ bo`lsin. U holda

$$\lim_{n\to\infty}[f(x)-S_n(x)]=\lim_{n\to\infty} r_n(x)=0$$

bo`lib, undan

$$\lim_{n\to\infty} S_n(x)=f(x)$$

bo`lishi kelib chiqadi. Demak,

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots\frac{f^{(n)}(0)^n}{n!}+\ldots$$

bo`ladi. ►

Odatda, bu munosabat o`rinli bo`lsa, $f(x)$ funksiya Teylor qatoriga yoyilgan deyiladi.

4^0. Funksiyani Teylor qatoriga yoyish. Faraz qilaylik, $f(x)$ funksiya biror $(-r,r)$ da istalgan tartibdagi hosilalarga ega bo`lsin.

2-teorema (Teylor). Agar $\exists M>0,\ \forall x\in(-r,\ r),\ \forall n\geq 0$ da

$$\left|f^{(n)}(x)\right|\leq M$$

bo`lsa, $f(x)$ funksiya $(-r,r)$ da Teylor qatoriga yoyiladi:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots\frac{f^{(n)}_{(0)}}{n!}x^n+\ldots \qquad (3)$$

◀ Ma`lumki, $f(x)$ funksiyaning Lagranj ko`rinishidagi qoldiq hadli Teylor formulasi quyidagicha bo`ladi:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots\frac{f^{(n)}_{(0)}}{n!}x^n+r_n(x),$$

bunda,

$$r_n(x)=\frac{f^{(n)}(\theta x)}{(n+1)!}x^{n+1}\ . \qquad (0<\theta<1).$$

Teoremaning shartidan foydalanib topamiz:

$$|r_n(x)|=\left|\frac{f^{(n)}(\theta x)}{(n+1)!}x^{n+1}\right|\leq M\cdot\frac{r^{n+1}}{(n+1)!}\ . \qquad (x\in(-r,\ r)).$$

Ravshanki,

$$\lim_{n\to\infty}\frac{r^{n+1}}{(n+1)!}=0.$$

Demak, $\forall x\in(-r,r)$ da

$$\lim_{n\to\infty}r_n(x)=0$$

bo`lib, undan qaralayotgan $f(x)$ funksiyaning Teylor qatoriga yoyilishi kelib chiqadi. ▶

10-Amaliy mashg`ulot.

1-misol. Ushbu

$$\sum_{n=1}^{\infty}nx^n=x+2x^2+3x^3+\ldots+nx^n+\ldots$$

darajali qator yig`indisi topilsin.

◀Ma`lumki,

$$\sum_{n=1}^{\infty}x^n$$

darajali qator $(-1,1)$ da yaqinlashuvchi va uning yig`indisi $\frac{x}{1-x}$ ga teng:

$$\sum_{n=1}^{\infty}x^n=\frac{x}{1-x}.$$

Bu qatorni hadlab differentsiallab topamiz:

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{x}{1-x}\right),$$

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Keyingi tenglikning har ikki tomonini x ga ko`paytirsak, unda

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

bo`lishi kelib chiqadi. ►

2-misol. Ushbu

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \ln(1+x)$$

tenglikning to`g`riligi isbotlansin.

◄Ravshanki,

$$\sum_{n=0}^{\infty} x^n$$

darajali qator $(-1,1)$ da yaqinlashuvchi bo`lib, uning yig`indisi $\frac{1}{1-x}$ ga teng:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Bu tenglikda x ni $-x$ ga almashtirsak, natijada

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

tenglik hosil bo`ladi. Uni $[0,x]$ bo`yicha $(0<x<1)$ integrallab topamiz:

$$\int_0^x \sum_{n=0}^{\infty} (-1)^n t^n dt = \int_0^x \frac{dt}{1+t},$$

$$\sum_{n=0}^{\infty} (-1)^n \int_0^x t^n dt = \ln(1+t)\Big|_0^x$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \ln(1+x).$$

3-misol. Ushbu

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

darajali qator yig`indisi topilsin va undan foydalanib

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{\pi}{4}$$

bo`lishi ko`rsatilsin.

◀Ma`lumki,

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x} \qquad (-1<x<1).$$

Bu tenglikda x ni $-x^2$ ga almashtiramiz. Natijada $\sum_{n=0}^{\infty}(-1)^n x^{2n}=\frac{1}{1+x^2}$ hosil bo`ladi. Uni $[0,x]$ bo`yicha $(0<x<1)$ integrallab topamiz:

$$\int_0^x\left(\sum_{n=0}^{\infty}(-1)^n t^{2n}\right)dt=\int_0^x\frac{dt}{1+t^2},$$

$$\sum_{n=0}^{\infty}(-1)^n\int_0^x t^{2n}dt=arctgt\Big|_0^x,$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=arctgx.$$

Keyingi tenglikda $x=1$ deylik. Unda tenglikning chap tomoni

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

sonli qatorga aylanib, u Leybnits teoremasiga ko`ra, yaqin-lashuvchi bo`ladi. Demak,

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\mathrm{arctg}1=\frac{\pi}{4}.\ ▶$$

Mashqlar

1. Hadlab differensiallash bilan ushbu

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+...$$

darajali qatorning yig`indisi topilsin.

2. Hadlab integrallash bilan ushbu

$$x-4x^2+9x^3-16x^4+...$$

darajali qatorning yig`indisi topilsin.

Elementar funksiyalarni Teylor qatoriga yoyish.*[3, p.229]*

a) Ko`rsatkichli va giperbolik funksiyalarni Teylor qatorlarini topamiz. Aytaylik,

$$f(x)=e^x$$

bo`lsin. Ravshanki, $f(0)=1,\ f^{(n)}(0)=1$ $(n\in N)$ bo`lib, $\forall x\in(-\alpha,\alpha)$ da $(\alpha>0)$

$$0<f(x)<e^{\alpha},\ \ 0<f^{(n)}(x)<e^{\alpha}$$

bo`ladi. Binobarin, 2-teoremaga ko`ra $f(x)=e^x$ funksiya $(-\alpha,\alpha)$ da Teylor qatoriga yoyiladi va (3) formulada foydalanib topamiz:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots\ (0!=1). \qquad (4)$$

$\alpha>0$ ixtiyoriy musbat son. Demak, (4) darajali qatorning yaqinlashish radiusi $r=+\infty$ bo`ladi.

(4) munosabatda x ni $-x$ ga almashtirib topamiz:

$$e^{-x}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}=1-\frac{x}{1!}+\frac{x^2}{2!}-\ldots+(-1)^n\cdot\frac{x^n}{n!}+\ldots$$

Ma`lumki giperbolik sinus hamda giperbolik kosinus funksiyalari quyidagicha

$$shx=\frac{e^x-e^{-x}}{2},\ \ chx=\frac{e^x+e^{-x}}{2}$$

ta`riflanar edi.

Yuqoridagi

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots,$$

$$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\ldots+(-1)^n\frac{x^n}{n!}+\ldots$$

formulalardan foydalanib topamiz:

$$shx=\frac{x}{1!}+\frac{x^3}{3!}+\ldots+\frac{x^{2n+1}}{(2n+1)!}+\ldots=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!},$$

$$chx=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}.$$

Bu $shx,\ chx$ funksiyalarining Teylor qatorlari bo`lib, ular ifodalangan darajali qatorlarning yaqinlashish radiuslari $r=+\infty$ bo`ladi.

b) Trigonometrik funksiyalarning Teylor qatorlarini topamiz. Aytaylik, $f(x)=\sin x$ bo`lsin. Ravshanki, $\forall x\in R,\ \forall n\in N$ da

$$|f(x)|\le 1,\ \ |f^{(n)}(x)|\le 1$$

bo`lib, $f(0),\ f'(0)=1,\ f^{(2n)}(0)=0,\ f^{(2n+1)}(0)=(-1)^n$ $(n\in N)$ bo`ladi. Demak, 2-teoremaga ko`ra $f(x)=\sin x$ funksiya Teylor qatoriga yoyiladi va (3) formulaga binoan

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\ldots \qquad (5)$$

bo`ladi.

Aytaylik,

$$f(x)=\cos x$$

bo`lsin. Bu funksiya uchun $\forall x\in \boldsymbol{R},\ \forall n\in N$ da

$$|f(x)|\le 1,\ |f^{(n)}(x)|\le 1$$

bo`lib,

$$f(0)=1,\ f'(0)=0,\ f^{(2n)}(0)=(-1)^n,\ f^{(2n+1)}(0)=0 \quad (n\in N)$$

bo`ladi. Unda 2–teoremaga ko`ra $f(x)=\cos x$ funksiya Teylor qatoriga yoyiladi va (3) formulaga binoan

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\ldots \qquad (6)$$

bo`ladi.

(5) va (6) darajali qatorlarning yaqinlashish radiusi $r=+\infty$ bo`ladi.

v) Logarifmik funksiyaning Teylor qatorini topamiz. Aytaylik,

$$f(x)=\ln(1+x)$$

bo`lsin. Ma`lumki,

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \qquad (n\in N)$$

bo`lib,

$$\frac{f^{(n)}(0)}{n!}=\frac{(-1)^{n-1}}{n}$$

bo`ladi. Bu funksiyaning Teylor formulasi

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+r_n(x) \qquad (7)$$

ko`rinishga ega.

$f(x)=\ln(1+x)$ funksiyani Teylor qatoriga yoyishda 1-teoremadan foydalanmiz. Buning uchun (7) formulada $r_n(x)$ ning 0 ga intilishini ko`rsatish etarli bo`ladi.

Aytaylik, $x\in\boldsymbol{[0,1]}$ bo`lsin. Bu holda Lagranj ko`rinishida yozilgan

$$r_n(x)=\frac{(-1)^n x^{n+1}}{(n+1)(1+\theta x)^{n+1}} \qquad (0<\theta<1)$$

qoldiq had uchun

$$|r_n(x)|\le\frac{1}{n+1}$$

bo`ladi va

$$\lim_{n\to\infty} r_n(x)=0$$

tenglik bajariladi.

Aytaylik, $x\in[-\alpha,0]$ bo\`lsin, bunda $0<\alpha<1$.
Bu holda Koshi ko\`rinishida yozilgan

$$r_n(x)=\frac{(-1)^n(1-\theta_1)^n\cdot x^{n+1}}{(1+\theta_1 x)^{n+1}} \qquad (0<\theta_1<1)$$

qoldiq had uchun

$$|r_n(x)|\le\frac{\alpha^{n+1}}{1-\alpha}$$

bo\`lib,

$$\lim_{n\to\infty} r_n(x)=0$$

bo\`ladi.
Demak, $\forall x\in(-1,1]$

$$\lim_{n\to\infty} r_n(x)=0.$$

Unda 1-teoremaga ko\`ra

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots \qquad (8)$$

bo\`ladi.

(8) darajali qatorning yaqinlashish radiusi $r=1$ ga teng.
Agar yuqoridagi $\ln(1+x)$ ning yoyilmasida x ni $-x$ ga almashtirilsa, unda

$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}=-x-\frac{x^2}{2}-\frac{x^3}{3}-\ldots-\frac{x^n}{n}-\ldots$$

formula kelib chiqadi.

g) Darajali funksiyaning Teylor qatorini topamiz.

Aytaylik,

$$f(x)=(1+x)^{\alpha} \qquad (\alpha\in R)$$

bo\`lsin. Ma\`lumki,

$$f^{(n)}(x)=\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)(1+x)^{\alpha-n} \qquad (n\in N)$$

bo\`lib,

$$f^{(n)}(0)=\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)$$

bo\`ladi. Bu funksiyaning Teylor formulasi ushbu

$$(1+x)^{\alpha}=1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+\ldots+\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n+r_n(x)$$

ko\`rinishga ega.

Endi $n\to\infty$ da $r_n(x)\to 0$ bo\`lishini ko\`rsatamiz.

Ma`lumki, Teylor formulasidagi qoldiq hadning Koshi ko`rinishi quyidagicha

$$r_n(x)=\frac{(\alpha-1)(\alpha-2)...[(\alpha-1)-(n-1)]}{n!}x^n\alpha\cdot x(1+\theta x)^{\alpha-1}\left(\frac{1-\theta}{1+\theta x}\right)^n$$

$(0<\theta<1)$ bo`lar edi.

Aytaylik, $x\in(-1,1)$ bo`lsin. Bu holda:

1) $\lim\limits_{n\to\infty}\frac{1}{n!}(\alpha-1)(\alpha-2)...[(\alpha-1)-(n-1)]x^n=0$ bo`ladi,

chunki, limit ishorasi ostidagi ifoda yaqinlashuvchi ushbu

$$1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}x^n$$

qatorning umumiy hadi;

2) $|\alpha\cdot x|(1-|x|)^{\alpha-1}<\alpha\cdot x(1+\theta x)^{\alpha-1}<|\alpha\cdot x|(1+|x|)^{\alpha-1}$;

3) $\left|\frac{1-\theta}{1+\theta x}\right|^n\leq\left|\frac{1-\theta}{1+\theta x}\right|<1$

bo`ladi. Bu munosabatlardan foydalanib, $\forall x\in(-1,1)$ da

$$\lim_{n\to\infty}r_n(x)=0$$

bo`lishini topamiz. 1-teoremaga ko`ra

$$(1+x)^{\alpha}=1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+...+\frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}x^n+... \qquad (9)$$

bo`ladi.

Bu darajali qatorning yaqinlashish radiusi $\alpha\neq 0,\ \alpha\notin N$ bo`lganda 1 ga teng: $r=1$.

(9) munosabatda $\alpha=-1$ deb olinsa, unda ushbu

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n=1-x+x^2-x^3+x^4-...+(-1)^n x^n+...$$

formula hosil bo`ladi. Bu formulada x ni $-x$ ga almashtirib topamiz:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^n x^n=1+x+x^2+...+x^n+...$$

1-misol. Ushbu

$$f(x)=\ln\frac{1+x}{1-x}$$

funksiya Teylor qatoriga yoyilsin.

◀Ma`lumki,

$$\ln\frac{1+x}{1-x}=\ln(1+x)-\ln(1-x)$$

bo`ladi.

Biz yuqorida

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n-1}\frac{x^n}{n}+...$$

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-...\frac{x^n}{n}-...$$

bo`lishini ko`rgan edik. Bu munosabatlardan foydalanib topamiz:

$$\ln(1+x)-\ln(1-x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n-1}\frac{x^n}{n}+...$$

$$-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-...-\frac{x^n}{n}-...\right)=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+...+\frac{2x^{2n-1}}{2n-1}+...$$

Demak,

$$\ln\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+...+\frac{x^{2n-1}}{2n-1}+...\right). \tag{10}$$

(10) darajali qatorning yaqinlashish radiusi $r=1$ bo`lib, yaqinlashish to`plamsi $(-1,1)$ bo`ladi.▶

2-misol. Ushbu

$$f(x)=\int_0^x\frac{\sin t}{t}dt$$

funksiya Teylor qatoriga yoyilsin.

◀Ma`lumki,

$$\sin t=t-\frac{t^3}{3!}+\frac{t^5}{5!}-...+(-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!}+....$$

Unda

$$\frac{\sin t}{t}=1-\frac{t^2}{3!}+\frac{t^4}{5!}-...+(-1)^{n-1}\frac{t^{2n-2}}{(2n-1)!}+...$$

bo`ladi. Bu darajali qatorni hadlab integrallab topamiz:

$$\int_0^x\frac{\sin t}{t}dt=\int_0^x\left(1-\frac{t^2}{3!}+\frac{t^4}{5!}-...+(-1)^{n-1}\frac{t^{2n-2}}{(2n-1)!}+...\right)dt=$$

$$=x-\frac{x^3}{3!\cdot3}+\frac{x^5}{5!\cdot5}-...+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!\cdot(2n-1)}+...$$

Keyingi darajali qatorning yaqinlashish radiusi $r=+\infty$ bo`ladi.▶

3-misol. Ushbu

$$f(x)=\frac{2x-1}{x^2+x-6}$$

funksiya Teylor qatoriga yoyilsin va bu qatorning yaqinlashish radiusi topilsin.

◀ Avvalo $f(x)$ funksiyani quyidagicha yozib olamiz:

$$f(x)=\frac{2x-1}{x^2+x-6}=\frac{1}{x+2}+\frac{1}{x-3}=\frac{1}{2\left(1+\frac{1}{2}x\right)}-\frac{1}{3\left(1-\frac{1}{3}x\right)}$$

Ma`lumki,

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n\cdot x^n,$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n.$$

Bu formulalardan foydalanib topamiz:

$$\frac{1}{2\left(1+\frac{1}{2}x\right)}=\sum_{n=0}^{\infty}\frac{1}{2}(-1)^n\cdot\left(\frac{1}{2}x\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n, \qquad (r=2)$$

$$\frac{1}{3\left(1-\frac{1}{3}x\right)}=\sum_{n=0}^{\infty}\frac{1}{3}\left(\frac{1}{3}x\right)^n=\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n \qquad (r=3)$$

Demak,

$$\frac{2x-1}{x^2+x-6}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n-\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n=\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2^{n+1}}-\frac{1}{3^{n+1}}\right)x^n$$

bo`ladi.

Bu darajali qatorning yaqinlashish radiusi $r=2$ bo`ladi. ▶

Mashqlar

1. Ushbu

$$f(x)=\sin^3 x,\ f(x)=\ln(1-x^2), \qquad f(x)=\frac{1}{1+x+x^2}$$

funksiyalar Teylor qatoriga yoyilsin.

2. Ushbu

$$\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!} \qquad (x\in R)$$

qatorning yig`indisi topilsin.

3. Berilgan nuqta atrofida funksiyani Teylor qatoriga yoying va ushbu qatorlarning yaqinlashish sohalarini toping.

1 $f(x)=\sqrt{x}; x_0=4.$ **2** $f(x)=e^x; x_0=-2.$

3 $f(x)=\cos x; x_0=\frac{\pi}{4}.$

4 $f(x)=\frac{1}{x}; x_0=-2.$

5 $f(x)=\frac{1}{\sqrt{x^2-4x+8}}; x_0=2.$

6 $f(x)=\frac{1}{x^2-5x+6}; x_0=1.$

7 $f(x)=\cos^4 x; x_0=-\frac{\pi}{2}.$

8 $f(x)=\sin^4 x; x_0=\frac{\pi}{2}.$

9 $f(x)=\frac{1}{x-1}; x_0=3.$

10 $f(x)=e^{2x}; x_0=-1.$

11 $f(x)=\sin x; x_0=\frac{\pi}{4}.$

12 $f(x)=\sqrt{x}; x_0=3.$

13 $f(x)=\sin^2 x; x_0=-\frac{\pi}{2}.$

14 $f(x)=\cos^2 x; x_0=\frac{\pi}{2}.$

15 $f(x)=\frac{1}{x^2-5x+6}; x_0=-1.$

16 $f(x)=\sqrt[3]{x}; x_0=8.$

17 $f(x)=\sin^3 x; x_0=\frac{\pi}{4}.$

18 $f(x)=\cos^3 x; x_0=\frac{\pi}{4}.$

19 $f(x)=\frac{1}{1-x}; x_0=-1.$

20 $f(x)=\frac{1}{1-2x-x^2}; x_0=2.$

21 $f(x)=\cos^4 x; x_0=\frac{\pi}{4}.$

Foydalanish uchun adabiyotlar

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NAZORAT SAVOLLARI

1. Darajali qatorning tekis yaqinlashishi nima?
2. Darajali qator yig`indisining uzluksizligi haqidagi teoremani ayting.
3. Qanday shartlarda darajali qatorni hadlab integrallash mumkin?
4. Qanday shartlarda darajali qatorni hadlab differensiallash mumkin?
5. Qanday qatorni Teylor qatori deyiladi?
6. Teylor teoremasini ayting.
7. Elementar funksiyalarni Teylor qatiriga yoyilmasi qanday?

GLOSSARIY

Darajali qatorning tekis yaqinlashishi - darajali qator $[\alpha,\beta]\subset(-r,r)$ da tekis yaqinlashuvchi bo`ladi, bunda $\alpha\in R,\ \beta\in R,\ r>0$-yaqinlashish radiusi.

Darajali qator yig`indisining uzluksizligi -

$$S(x)=\sum_{n=0}^{\infty}a_nx^n$$

$S(x)$ funksiya $(-r,r)$ da uzluksiz bo`ladi, $r>0$-yaqinlashish radiusi.

Darajali qatorni hadlab integrallash - $S(x)=\sum_{n=0}^{\infty}a_nx^n$ qatorni $(-r,r)$ ga tegishli bo`lgan ixtiyoriy $[a,b]$ bo`yicha $([a,b]\subset(-r,r))$ hadlab integrallash mumkin:

$$\int_a^b S(x)dx=\sum_{n=0}^{\infty}\left(\int_a^b a_nx^ndx\right).$$

Darajali qatorni hadlab differensiallash - Faraz qilaylik,

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$$

darajali qatorning yaqinlashish radiusi $r>0$, yig`indisi $S(x)$ bo`lsin:

$$\sum_{n=0}^{\infty}a_nx^n=S(x).$$

U holda $S(x)$ funksiya $(-r,r)$ da uzluksiz $S'(x)$ hosilaga ega va

$$S'(x)=\sum_{n=0}^{\infty}na_nx^{n-1}$$

bo`ladi, bunda (3) qatorning yaqinlashish radiusi ham r ga teng.

Funksiyaning Teylor qatori - Aytaylik, $f(x)$ funksiya $x_0\in R$ nuqtaning biror

$$U_\delta(x_0)=\{x\in R: x_0-\delta<x<x_0+\delta;\delta>0\}$$

atrofida istalgan tartibdagi hosilaga ega bo‘lsin. Bu hol $f(x)$ funksiyaning Teylor formulasini yozish imkonini beradi:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x),$$

bunda $r_n(x)$-qoldiq had.

Modomiki, $f(x)$ funksiya $U_\delta(x_0)$ da istalgan tartibdagi hosilaga ega ekan, unda

$$f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots \qquad (1)$$

darajali qatorni qarash mumkin bo‘ladi.

(1) darajali qatorning koeffitsientlari sonlar bo‘lib, ular $f(x)$ funksiya va uning hosilalarining x_0 nuqtadagi qiymatlari orqali ifodalangan.

(1) darajali qator $f(x)$ funksiyaning Teylor qatori deyiladi.

Xususan, $x_0 = 0$ bo‘lganda (1) darajali qator ushbu

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots=\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

ko‘rinishga keladi.

Faraz qilaylik, $f(x)$ funksiya biror $(-r,r)$ da $(r>0)$ istalgan tartibdagi hosilaga ega bo‘lib, uning $x_0 = 0$ nuqtadagi Teylor qatori

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots$$

bo‘lsin. Bu qatorning qoldiq hadini $r_n(x)$ deymiz:

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+r_n(x).$$

KEYS BANKI

1-keys. Masala o`rtaga tashlanadi: Hadlab differensiallash bilan ushbu

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$$

darajali qatorning yig`indisi topilsin.

2-keys. Masala o`rtaga tashlanadi: Hadlab integrallash bilan ushbu

$$x-4x^2+9x^3-16x^4+\ldots$$

darajali qatorning yig`indisi topilsin.

3-keys. Masala o`rtaga tashlanadi: **.** Ushbu

$$f(x)=\int_0^x\frac{\sin t}{t}dt$$

funksiya Teylor qatoriga yoyilsin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

TEST

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}x^n$ darajali qatorning yaqinlashish oralig`ini toping.	$\|x\|<4$	$\|x\|\le 4$	$\|x\|>4$
2	$\sum_{n=1}^{\infty}\frac{n!}{a^{n^2}}x^n$ $(a>1)$ darajali qatorning yaqinlashish radiusini toping.	$R=+\infty$	$R=0$	$R=1$
3	$\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\right)^p\left(\frac{x-1}{2}\right)^n$ darajali qatorning yaqinlashish radiusini toping.	$R=2$	$R=1$	$R=3$
4	$\sum_{n=1}^{\infty}\frac{x^n}{a^{\sqrt{n}}}$ $(a>0)$ darajali qatorning yaqinlashish oralig`ini toping.	$\|x\|<1$	$\|x\|>1$	$\|x\|>2$
5	$\frac{1}{(1-x)^2}$ funksiyani darajali qatorga yoying.	$\sum_{n=1}^{\infty}nx^{n-1}$ $(\|x\|<1)$	$\sum_{n=1}^{\infty}(n-1)x^n$ $(\|x\|<1)$	$\sum_{n=1}^{\infty}n^2x^{n-1}$ $(\|x\|<1)$

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$ qator qaysi funksiyaning yoyilmasi bo`ladi?	e^x	cosx	ln(1+x)
2	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ qator qaysi funksiyaning yoyilmasi bo`ladi?	cosx	sinx	e^x
3	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ qator qaysi funksiyaning yoyilmasi bo`ladi?	sinx	cosx	e^x
4	$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ qator qaysi funksiyaning yoyilmasi bo`ladi?	ln(1+x)	e^x	sinx
5	f(x)=arctgx funksiya qaysi yoyilmaga ega?	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n}$
6	f(x)=ln$\frac{1+x}{1-x}$ funksiya qaysi yoyilmaga ega?	$2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$

11-Mavzu. Uzluksiz funksiyani ko`phad bilan yaqinlashtirish. Veyershtrass teoremasi.
11-Ma'ruza
REJA:

1^0. Bernshteyn ko`phadi.

2^0. Muhim lemma.

3^0. Uzluksiz funksiyani ko`phad bilan yaqinlashtirish.

**Tayanch so`z va iboralar:** *Bernshteyn ko`phadi, muhim lemma, N'yuton-binomi formulasi, uzluksiz funksiyani ko`phad bilan yaqinlashtirish: Bernshteyn teoremasi, Veyershtrass teoremasi.*

***1^0. Bernshteyn ko`phadi***. Aytaylik, $f(x)$ funksiya $[0,1]$ segmentda berilgan bo`lsin.

Ushbu

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k} = f(0)\cdot(1-x)^n + f\left(\frac{1}{n}\right) C_n^1 x(1-x)^{n-1} + \ldots + f(1)x^n$$

ko`phad $f(x)$ funksiyaning Bernshteyn ko`phadi deyiladi va $B_n(f;x)$ kabi belgilanadi:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}.$$

Bunda

$$C_n^k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

Demak, Bernshteyn ko`phadi $n$-darajali ko`phad bo`lib, uning koeffitsientlari $f(x)$ funksiyaning

$$\frac{k}{n} \qquad (k = 0,1,2,\ldots,n)$$

nuqtalardagi qiymatlari orqali ifodalanadi.
Masalan,

$$B_1(f;x) = f(0) + [f(1) - f(0)]x,$$

$$B_2(f;x) = f(0) + \left[2f\left(\frac{1}{2}\right) - 2f(0)\right]x + \left[f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right]x^2$$

bo`ladi.

2^0. Muhim lemma. Ushbu

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1, \qquad (1)$$

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 C_n^k x^k (1-x)^{n-k} = \frac{x(1-x)}{n} \qquad (0 \le x \le 1) \qquad (2)$$

ayniyatlar o`rinli.

◀N'yuton-binomi formulasi

$$\sum_{k=0}^{n} C_n^k a^k b^{n-k} = (a+b)^n$$

da $a = x$, $b = 1-x$ deyilsa, u holda

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1$$

bo`lishi kelib chiqadi.

(2) ayniyatni isbotlash uchun quyidagi

$$\sum_{k=0}^{n} \frac{k}{n} C_n^k x^k (1-x)^{n-k}, \quad \sum_{k=0}^{n} \frac{k^2}{n^2} C_n^k x^k (1-x)^{n-k}$$

yig`indilarni hisoblaymiz.

Bu yig`indini hisoblashda yuqoridagi keltirilgan C_n^k ning ifodasi va N'yuton binomi formulasidan foyda-lanamiz:

$$\sum_{k=0}^{n} \frac{k}{n} C_n^k x^k (1-x)^{n-k} = \sum_{k=1}^{n} \frac{k}{n} C_n^k x^k (1-x)^{n-k} = \sum_{k=1}^{n} C_{n-1}^{k-1} x^k (1-x)^{n-k} =$$

$$= x \cdot \sum_{k=1}^{n} C_{n-1}^{k-1} x^{k-1} (1-x)^{n-1-(k-1)} = x[x + (1-x)]^{n-1} = x.$$

Demak,

$$\sum_{k=0}^{n} \frac{k}{n} C_n^k x^k (1-x)^{n-k} = x. \qquad (3)$$

Endi

$$\sum_{k=0}^{n} \frac{k^2}{n^2} C_n^k x^k (1-x)^{n-k}$$

yig`indini hisoblaymiz:

$$\sum_{k=0}^{n}\frac{k^2}{n^2}C_n^k x^k(1-x)^{n-k}=\sum_{k=1}^{n}\frac{k}{n}C_{n-1}^{k-1}x^k(1-x)^{n-k}=\sum_{k=1}^{n}\frac{n-1}{n}\cdot\frac{k-1}{n-1}C_{n-1}^{k-1}x^k(1-x)^{n-k}+$$

$$+\sum_{k=1}^{n}\frac{1}{n}C_{n-1}^{k-1}x^k(1-x)^{n-k}=\frac{n-1}{n}x^2\sum_{k=2}^{n}C_{n-2}^{k-2}x^{k-2}(1-x)^{n-2-(k-2)}+$$

$$+\frac{1}{n}x\sum_{k=1}^{n}C_{n-1}^{k-1}x^{k-1}(1-x)^{n-1-(k-1)}=\frac{n-1}{n}x^2[x+(1-x)]^{n-2}+\frac{1}{n}x[x+(1-x)]^{n-1}=$$

$$=x^2+\frac{x(1-x)}{n}.$$

Demak,

$$\sum_{k=0}^{n}\frac{k^2}{n^2}C_n^k x^k(1-x)^{n-k}=x^2+\frac{x(1-x)}{n}. \qquad (4)$$

Yuqoridagi (1), (3) va (4) munosabatlardan foydalanib topamiz:

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2C_n^k x^k(1-x)^{n-k}=\sum_{k=0}^{n}\frac{k^2}{n^2}C_n^k x^k(1-x)^{n-k}-2x\sum_{k=0}^{n}\frac{k}{n}C_n^k(1-x)^{n-k}+$$

$$+x^2\sum_{k=0}^{n}C_n^k x^k(1-x)^{n-k}=x^2+\frac{x(1-x)}{n}-2x\cdot x+x^2=\frac{x(1-x)}{n}. \ ►$$

Natija. $\forall x\in[0,1]$, $\forall n\in N$ uchun

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2C_n^k x^k(1-x)^{n-k}\le\frac{1}{4n} \qquad (5)$$

tengsizlik o`rinli bo`ladi.

◄Ravshanki, $\forall x\in[0,1]$ uchun

$$x(1-x)\le\frac{1}{4}$$

bo`ladi. Bu tengsizlik va (2) munosabatdan (5) tengsizlikning o`rinli bo`lishi kelib chiqadi.►

***3^0. Uzluksiz funksiyani ko`phad bilan yaqinlashtirish*.**

1-teorema. (Bernshteyn). Agar $f(x)$ funksiya **[0,1]** segmentda uzluksiz bo`lsa, u holda

$$\lim_{n\to\infty}\max_{0\le x\le 1}|f(x)-B_n(f,x)|=0$$

bo`ladi, bunda

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)C_n^k x^k(1-x)^{n-k}. \qquad (6)$$

◀(1) va (6) munosabatlardan foydalanib topamiz:

$$B_n(f;x)-f(x)=\sum_{k=0}^{n}\left[f\left(\frac{k}{n}\right)-f(x)\right]C_n^k x^k(1-x)^{n-k}.$$

Kantor teoremasiga ko`ra qaralayotgan $f(x)$ funksiya $[0,1]$ da tekis uzluksiz bo`ladi. Unda ta`rifga binoan

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x',x''\in[0,1] \text{ uchun } |x'-x''|<\delta$$

bo`lganda

$$|f(x'')-f(x')|<\frac{\varepsilon}{2}$$

tengsizlik bajariladi.

Ma`lumki,

$$B_n(f;x)-f(x)$$

ayirmani ifodalovchi yig`indida $n+1$ ta had bo`lib, ular k ning $0,1,2,...,n$ qiymatlarida yuzaga keladi. Bu k ning ushbu

$$\left|\frac{k}{n}-x\right|<\delta \qquad (x\in[0,1])$$

tengsizlikni qanoatlantiradigan qiymatlari to`plamini $E_n(k)$ bilan,

$$\left|\frac{k}{n}-x\right|\ge\delta \qquad (x\in[0,1])$$

tengsizlikni qanoatlantiradigan qiymatlari to`plamini $F_n(k)$ bilan belgilaylik.

Ravshanki,

$$E_n(k)\cup F_n(k)=\{0,1,2,...,n\}$$

bo`ladi. Shuni e`tiborga olib, yuqoridagi yig`indini ikki qismga ajratamiz:

$$\sum_{k=0}^{n}\left[f\left(\frac{k}{n}\right)-f(x)\right]C_n^k x^k(1-x)^{n-k}=\sum_{E_n(k)}\left[f\left(\frac{k}{n}\right)-f(x)\right]C_n^k x^k(1-x)^{n-k}+$$

$$+\sum_{F_n(k)}\left[f\left(\frac{k}{n}\right)-f(x)\right]C_n^k x^k(1-x)^{n-k}.$$

Endi bu yig`indilarni baholaymiz. $f(x)$ funksiyaning $[0,1]$ da tekis uzluksizligidan hamda lemmadan foydalanib topamiz:

$$\left|\sum_{E_n(k)}\left[f\left(\frac{k}{n}\right)-f(x)\right]C_n^k x^k(1-x)^{n-k}\right| \le \sum_{E_n(k)}\left|f\left(\frac{k}{n}\right)-f(x)\right|C_n^k x^k(1-x)^{n-k} =$$

$$= \sum_{k:\left|\frac{k}{n}-x\right|<\delta}\left|f\left(\frac{k}{n}\right)-f(x)\right|C_n^k x^k(1-x)^{n-k} < \frac{\varepsilon}{2}\sum_{E_n(k)}C_n^k x^k(1-x)^{n-k} \le$$

$$\le \frac{\varepsilon}{2}\sum_{k=0}^{n}C_n^k x^k(1-x)^{n-k} = \frac{\varepsilon}{2}.$$

Ravshanki, $f(x)$ funksiya **[0,1]** da chegaralangan. Unda $\max\limits_{0\le x\le 1}|f(x)| = M \in R$ bo`ladi. Shuni e`tiborga olib topamiz:

$$\left|\sum_{F_n(k)}\left[f\left(\frac{k}{n}\right)-f(x)\right]C_n^k x^k(1-x)^{n-k}\right| \le \sum_{k:\left|\frac{k}{n}-x\right|\ge\delta}\left|f\left(\frac{k}{n}\right)-f(x)\right|C_n^k x^k(1-x)^{n-k} \le$$

$$\le 2M\sum_{k:\left|\frac{k}{n}-x\right|\ge\delta}C_n^k x^k(1-x)^{n-k}.$$

Agar

$$\left|\frac{k}{n}-x\right|\ge\delta \quad \Rightarrow \left(\frac{k}{n}-x\right)^2\cdot\frac{1}{\delta^2}\ge 1$$

bo`lishini hisobga olsak, unda lemmaga ko`ra

$$\sum_{k:\left|\frac{k}{n}-x\right|\ge\delta}C_n^k x^k(1-x)^{n-k} \le \frac{1}{\delta^2}\sum_{k:\left|\frac{k}{n}-x\right|\ge\delta}\left(\frac{k}{n}-x\right)^2 C_n^k x^k(1-x)^{n-k} \le$$

$$\le \frac{1}{\delta^2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 C_n^k x^k(1-x)^{n-k} \le \frac{1}{4n\delta^2}$$

bo`ladi.

Shunday qilib,

$$|B_n(f;x)-f(x)| \le \left|\sum_{E_n(k)}\left[f\left(\frac{k}{n}\right)-f(x)\right]C_n^k x^k(1-x)^{n-k}\right| +$$

$$+\left|\sum_{F_n(k)}\left[f\left(\frac{k}{n}\right)-f(x)\right]C_n^k x^k(1-x)^{n-k}\right| < \frac{\varepsilon}{2}+\frac{M}{2n\delta^2}$$

bo`ladi. Agar $n > \dfrac{M}{2\delta^2\varepsilon}$ deyilsa, u holda

$$\frac{M}{2\delta^2 n} < \frac{1}{2}\varepsilon$$

bo`lib,

$$|B_n(f;x) - f(x)| < \varepsilon$$

bo`ladi. Bu munosabatdan esa

$$\lim_{n\to\infty} \max_{0\le x\le 1} |B_n(f;x) - f(x)| = 0$$

bo`lishi kelib chiqadi.►

Bu teoremadan $n \to \infty$ da

$$B_n(f;x) \rightrightarrows f(x) \qquad (0 \le x \le 1)$$

bo`lishini topamiz. Demak, **[0,1]** da uzluksiz bo`lgan $f(x)$ funksiya $B_n(f;x)$ ko`phad bilan yaqinlashtirildi:

$$f(x) \approx \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k} . \qquad (0 \le x \le 1)$$

Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda uzluksiz bo`lsin. Ma`lumki, ushbu

$$t = \frac{1}{b-a}x - \frac{a}{b-a}$$

chiziqli almashtirish $[a,b]$ segmentni **[0,1]** segmentga almashtiradi. Bu almashtirishdan foydalanib ushbu

$$\varphi(t) = f(a + (b-a)t) \qquad (0 \le t \le 1) \tag{7}$$

funksiyani hosil qilamiz. Ravshanki, $\varphi(t)$ funksiya **[0,1]** da uzluksiz bo`ladi. Yuqoridagi teoremadan foydalanib topamiz:

$$\lim_{n\to\infty} \max_{0\le t\le 1} |B_n(\varphi;t) - \varphi(t)| = 0, \tag{8}$$

bunda

$$B_n(\varphi;t) = \sum_{k=0}^{n} \varphi\left(\frac{k}{n}\right) C_n^k t^k (1-t)^{n-k} .$$

(7) va (8) munosabatlardan

$$\lim_{n\to\infty} \max_{a\le x\le b} \left| B_n\left(f; \frac{x-a}{b-a}\right) - f(x) \right| = 0$$

bo`lishi kelib chiqadi, bunda

$$B_n\left(f;\frac{x-a}{b-a}\right)=\sum_{k=0}^{n} f\left(a+\frac{b-a}{n}k\right)C_n^k\left(\frac{x-a}{b-a}\right)^k\left(1-\frac{x-a}{b-a}\right)^{n-k}=$$

$$=\sum_{k=o}^{n} f\left(a+\frac{b-a}{n}k\right)C_n^k\,\frac{(x-a)^k(b-x)^{n-k}}{(b-a)^n}.$$

Shunday qilib quyidagi teoremaga kelamiz.

2-teorema (Veyershtrass). Agar $f(x)$ funksiya $[a,b]$ segmentda uzluksiz bo`lsa,

$$\lim_{n\to\infty}\max_{a\le x\le b}\left|B_n\left(f;\frac{x-a}{b-a}\right)-f(x)\right|=0$$

bo`ladi.

Mashqlar

1. Agar $f(x)$ funksiya $[0;1]$ segmentda uzluksiz bo`lsa, $\forall x\in[0;1]$ da

$$\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k f\left(\frac{k}{n}\right)C_n^k x^k(1-x)^{n-k}=0$$

bo`lishi isbotlansin.

2. Agar $f(x)$ funksiya $[0;1]$ segmentda uzluksiz bo`lsa,

$$\sup_{0\le x\le 1}\left|B_n(f;x)-f(x)\right|\le\frac{3}{2}\omega\left(\frac{1}{\sqrt{n}}\right)$$

bo`lishi isbotlansin, bunda $\omega(\delta)$ – $f(x)$ funksiyaning uzluk-siz moduli.

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NAZORAT SAVOLLARI

1. Bernshteyn ko`phadi nima?
2. Muhim lemmani yozing. U qaysi teoremada ishlatiladi?
3. Bernshteyn teoremasini ayting.
4. Veyershtrass teoremasini ayting. Ma’nosini tushuntiring.

GLOSSARIY

$f(x)$ **funksiyaning Bernshteyn ko`phadi** – bu ushbu

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k} = f(0)\cdot(1-x)^n + f\left(\frac{1}{n}\right) C_n^1 x(1-x)^{n-1} + \ldots + f(1)x^n$$

ko`rinishdagi ko`phad. $B_n(f;x)$ kabi belgilanadi:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}.$$

Bunda

$$C_n^k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

**Uzluksiz funksiyani ko`phad bilan yaqinlashtirish** - Agar $f(x)$ funksiya $[0,1]$ segmentda uzluksiz bo`lsa, u holda

$$\lim_{n\to\infty} \max_{0\le x\le 1} \left| f(x) - B_n(f,x) \right| = 0$$

bo`ladi, bunda

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}.$$

Veyershtrass teoremasi - Agar $f(x)$ funksiya $[a,b]$ segmentda uzluksiz bo`lsa,

$$\lim_{n\to\infty} \max_{a\le x\le b} \left| B_n\left(f;\frac{x-a}{b-a}\right) - f(x) \right| = 0$$

bo`ladi.

Keys banki

1-keys. Masala o`rtaga tashlanadi: Agar $f(x)$ funksiya $[0;1]$ segmentda uzluksiz bo`lsa, $\forall x \in [0;1]$ da

$$\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k f\left(\frac{k}{n}\right)C_n^k x^k(1-x)^{n-k}=0$$

bo`lishi isbotlansin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

TEST

	Test topshirig`i	To'g`ri javob	Muqobil javob	Muqobil javob
1	$f(x)=x$ funksiya uchun [0,1] segmentda Bershteyn ko`phadini tuzing.	x	x^2	$1-x$
2	$f(x)=x^2$ funksiya uchun [0,1] segmentda Bershteyn ko`phadini tuzing.	$x^2+\frac{x(1-x)}{n}$	$x^2+\frac{x(1+x)}{n}$	$x^2-\frac{x(1+x)}{n}$
3	Bernshteyn teoremasi. Agar $f(x)$ funksiya [0,1] segmentda uzluksiz bo`lsa, u holda …	$\lim_{n\to\infty}\max_{0\le x\le 1}\lvert f(x)-B_n(f,x)\rvert=0$	$\lim_{n\to\infty}\sup_{0\le x\le 1}\lvert f(x)-B_n(f,x)\rvert=0$	$\lim_{n\to\infty}\min_{0\le x\le 1}\lvert f(x)-B_n(f,x)\rvert=0$
4	Quyidagilardan qaysi biri to`g`ri?	$\sum_{k=0}^{n}C_n^k x^k(1-x)^{n-k}=1$	$\sum_{k=0}^{n}C_n^k x^k(1-x)^{n-k}=2$	$\sum_{k=0}^{n}C_n^k x(1-x)^{n-k}=1$

12-Mavzu: Ikki o`zgaruvchili funksiyaning bir o`zgaruvchisi bo`yicha yaqinlashishi
12-Ma`ruza
REJA:

1^0. Limit funksiya.
2^0. Limit funksiyaga tekis yaqinlashish.
3^0. Mavzuga doir misollar.

Tayanch so`z va iboralar: *Limit funksiya, tekis yaqinlashish, tekis yaqinlashishning zaruriy va yetarli sharti, limit funksiyaning uzluksizligi.*

1^0. Limit funksiya. **[4, Definition 5,6,7, p.366-367]** Faraz qilaylik, $f(x,y)$ funksiya R^2 fazodagi

$$M=\{(x,y)\in R^2 : a\le x\le b,\ y\in E\subset R\}$$

to`plamda berilgan va $y_0\in R$ nuqta $E$ to`plamning limit nuqtasi bo`lsin.

Ravshanki, har bir tayin $x\in[a,b]$da $f(x,y)$ funksiya y o`zgaruvchining funksiyasiga aylanadi. Aytaylik, bu funksiya $\forall x\in[a,b]$da

$$\lim_{y\to y_0} f(x,y)$$

limitga ega bo`lsin.

Har bir $x\in[a,b]$ ga $f(x,y)$ funksiyaning $y\to y_0$ dagi limitini mos qo`yish natijasida

$$\varphi:\quad x\to \lim_{y\to y_0} f(x,y)$$

funksiya hosil bo`ladi.

Odatda, bu funksiya $f(x,y)$ funksiyaning $y\to y_0$ dagi limit funksiyasi deyiladi:

$$\lim_{y\to y_0} f(x,y)=\varphi(x),\qquad (x\in[a,b]) \tag{1}$$

(1) munosabat quyidagicha tushuniladi:
$\forall\varepsilon>0$ son olinganda ham, shunday $\delta=\delta(\varepsilon,x)>0$ son topiladiki, $0<|y-y_0|<\delta$ tengsizlikni qanoatlantiruvchi $\forall y\in E$ uchun

$$|f(x,y)-\varphi(x)|<\varepsilon,\qquad (x\in[a,b])$$

bo`ladi.

Ta`rif. **[4, Definition 9, p.367]** Endi $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2;\ x\in[a,b],\ y\in E\subset R\}$$

to`plamda berilgan va $\infty$ «nuqta» $E$ to`plamning limit nuqtasi bo`lsin.

Agar $\forall\varepsilon>0$ son olinganda ham, shunday $\delta=\delta(\varepsilon,x)>0$ son topilsaki, $|y|>\delta$ tengsizlikni qanoatlantiruvchi $\forall y\in E$ uchun

$$|f(x,y)-\varphi(x)|<\varepsilon$$

tengsizlik bajarilsa, $\varphi(x)$ funksiya $f(x,y)$ ning $y\to\infty$ dagi limit funksiyasi deyiladi.

1-misol. Ushbu

$$f(x,y)=xy$$

funksiyani

$$M=\{(x,y)\in R^2:\ 0\le x\le 1,\ y\in[0,1]\}$$

to`plamda qaraylik. Bu funksiyaning $y\to y_0=1$ dagi limit funksiyasi $\varphi(x)=x$ bo`lishi ko`rsatilsin.

◀Ixtiyoriy $\varepsilon>0$ songa ko`ra, har bir $x\in[0,1]$ uchun $\delta=\varepsilon$ deb olinsa, unda $|y-y_0|=|y-1|<\delta$ tengsizlikni qanoatlan-tiruvchi $y\in[0,1]$ uchun

$$|f(x,y)-\varphi(x)|=|xy-x|=|x||y-1|\le|y-1|<\delta=\varepsilon$$

bo`ladi. Demak,

$$\lim_{y\to 1} xy=x.\ ▶$$

2-misol. Ushbu

$$f(x,y)=x^y,\qquad (0^0=0)$$

funksiyani

$$M=\{(x,y)\in R^2:\ 0\le x\le 1,\ y\in[0,1]\}$$

to`plamda qaraymiz. Bu funksiyaning $y\to y_0=0$ dagi limit funksiyasi topilsin.

◀Aytaylik, $x=0$ bo`lsin. Bu holda $\forall y\in[0,1]$ uchun

$$f(0,y)=0$$

bo`lib, $y\to 0$ da $f(0,y)\to 0$ bo`ladi.

Aytaylik, $x\ne 0$ bo`lsin. Bu holda $y\to 0$ da

$$f(x,y)=x^y \to x^0=1$$

bo`ladi. Haqiqatan ham, ixtiyoriy $\varepsilon>0$ songa ko`ra $\delta=\log_x(1-\varepsilon)$ deyilsa $(x>0)$, unda $|y-y_0|=|y-0|=|y|<\delta$ tengsizlikni qanoatlantiruvchi $y\in[0,1]$ uchun

$$|f(x,y)-\varphi(x)|=|x^y-1|=1-x^y<1-x^{\log_x(1-\varepsilon)}=1-(1-\varepsilon)=\varepsilon$$

bo`ladi. Demak, $y\to 0$ da $f(x,y)=x^y$ funksiyaning limit funksiyasi

$$\varphi(x)=\begin{cases}1, & agar\ \ x\in(0,1]\ bo`lsa,\\ 0, & agar\ \ x=0\ \ bo`lsa\end{cases}$$

bo`ladi. ►

2^0. Limit funksiyaga tekis yaqinlashish.

Faraz qilaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2: a\le x\le b,\ y\in E\subset R\}$$

to`plamda berilgan bo`lib, $y_0\in R$ nuqta esa E to`plamning limit nuqtasi bo`lsin. Bu funksiya har bir tayinlangan $x\in[a,b]$ da y o`zgaruvchining funksiyasi sifatida $y\to y_0$ da $\varphi(x)$ limit funksiyaga ega bo`lsin:

$$\lim_{y\to y_0} f(x,y)=\varphi(x).$$

$f(x,y)$ funksiyaning $\varphi(x)$ ga intilishi xarakteri olingan x ga bog`liq, chunki $x$ ning turli qiymatlarida $f(x,y)$ funksiya, umuman aytganda $y$ o`zgaruvchining tur-licha funksiyalari bo`ladi. Bu vaziyat

$$\lim_{y\to y_0} f(x,y)=\varphi(x), \qquad (x\in[a,b])$$

tushunchasidagi ixtiyoriy $\varepsilon>0$ songa ko`ra, topiladigan $\delta>0$ sonning qaralayotgan $x$ ga bog`liq yoki bog`liq emasligida namoyon bo`ladi.

Yuqorida keltirilgan misollarning birinchisida $\delta=\varepsilon$ bo`lib, u faqat $\varepsilon>0$ gagina bog`liq, ikkinchisida esa $\delta=\log_x(1-\varepsilon)$ bo`lib, u olingan $\varepsilon>0$ bilan birga qaralayotgan $x$ ga ham bog`liq ekanini ko`ramiz.

1-ta`rif. Agar $\forall\varepsilon>0$ son olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ son topilsaki, $|y-y_0|<\delta$ tengsizlikni qanoatlantiruvchi $y\in E$, $\forall x\in[a,b]$ uchun

$$|f(x,y)-\varphi(x)|<\varepsilon$$

tengsizlik bajarilsa, ya`ni

$$\forall\varepsilon>0,\ \exists\delta=\delta(\varepsilon)>0,\ |y-y_0|<\delta,\ y\in E,\ \forall x\in[a,b]:$$

$$|f(x,y)-\varphi(x)|<\varepsilon$$

bo`lsa, $f(x,y)$ funksiya $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashadi deyiladi.

3-misol. Ushbu

$$f(x,y)=x\sin y$$

funksiyani

$$M=\{(x,y)\in R^2:\ 0\le x\le 1,\ y\in[0,\pi]\}$$

to`plamda qaraylik. Bu funksiyaning $y\to y_0=\frac{\pi}{3}$ da limit funksiyasi topilib, unga $[0,1]$ da tekis yaqinlashishi ko`rsatilsin.

◀Ravshanki,

$$\lim_{y\to\frac{\pi}{3}} x\sin y=\frac{\sqrt{3}}{2}x.$$

Demak,

$$\varphi(x)=\frac{\sqrt{3}}{2}x.$$

Agar $\forall\varepsilon>0$ ga ko`ra $\delta=\varepsilon$ deyilsa, u holda $\left|y-\frac{\pi}{3}\right|<\delta$ tengsizlikni qanoatlantiruvchi $y\in[0,\pi]$ va $\forall x\in[0,1]$ uchun

$$|f(x,y)-\varphi(x)|=\left|x\sin y-\frac{\sqrt{3}}{2}x\right|=|x|\left|\sin y-\frac{\sqrt{3}}{2}\right|=|x|\left|\sin y-\sin\frac{\pi}{3}\right|\le\left|y-\frac{\pi}{3}\right|<\delta=\varepsilon$$

bo`ladi. Ta`rifga binoan, $y\to\frac{\pi}{3}$ da $f(x,y)=x\sin y$ funksiya $\varphi(x)=\frac{\sqrt{3}}{2}x$ limit funksiyaga $[0,1]$ da tekis yaqinlashadi. ►

Eslatma. Aytaylik,

$$\lim_{y\to y_0} f(x,y)=\varphi(x)$$

bo`lsin.

Agar $\forall\delta>0$ son olinganda shunday $\varepsilon_0>0,\ x_0\in[a,b]$ va $|y-y_0|<\delta$ tengsizlikni qanoatlantiruvchi $y_1\in E$ topilsaki,

$$|f(x_0,y_1)-\varphi(x_0)|\ge\varepsilon_0$$

bo`lsa, $f(x,y)$ funksiyaning $y \to y_0$ da limit funksiya $\varphi(x)$ ga tekis yaqin-lashmaydi deyiladi.

Masalan,

$$f(x,y)=x^y, \qquad (0^0=0)$$

funksiyaning $y \to 0$ dagi limit funksiya

$$\varphi(x)=\begin{cases}1, & \text{agar} \quad x\in(0,1] \quad \text{bo`lsa,}\\ 0, & \text{agar} \quad x=0 \quad \text{bo`lsa}\end{cases}$$

ga tekis yaqinlashmaydi bo`ladi. Haqiqatan ham, $\forall \delta > 0$ son olinganda, $\varepsilon_0=\frac{1}{4}$, $0<y_1<\delta$ tengsizlikni qanoatlantiruvchi y_1 va $x_0=2^{-\frac{1}{y_1}}$ deb olinsa, u holda

$$|f(x_0,y_1)-\varphi(x_0)|=1-x_0^{y_1}=1-\frac{1}{2}=\frac{1}{2}>\varepsilon_0$$

bo`ladi.

Faraz qilaylik, $f(x,y)$ funksiya R^2 fazodagi

$$M=\{(x,y)\in R^2 : a\le x\le b,\ y\in E\subset R\}$$

to`plamda berilgan va $y_0 \in R$ nuqta $E$ to`plamning limit nuqtasi bo`lsin.

Agar E to`plam $y_0$ ga intiluvchi $\{y_n\}$ ketma-ketlikdan iborat bo`lsa, $f(x,y)$ funksiyani $[a,b]$ da aniqlangan

$$f_n(x)=f(x,y_n)$$

funktsional ketma-ketlik sifatida qarash mumkin. Masalan,

$$f(x,y)=\frac{2xy}{x^2+y^2}$$

funksiyani $M_0=\left\{(x,y)\in R^2 :\ x\in[0,1],\ y\in E=\left\{1,\frac{1}{2},\frac{1}{3},...\right\}\right\}$ to`plamda qarasak, u quyidagi

$$f_n(x)=\frac{2xn}{1+n^2x^2}$$

funktsional ketma-ketlikka aylanadi.

1-teorema. Agar $y \to y_0$ da $f(x,y)$ funksiya $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashsa, u holda E to`plamdagi y_0 ga intiluvchi har bir $\{y_n\}$ ketma-ketlikda ($y_n \subset E$)

$$f_n(x) = f(x, y_n)$$

funktsional ketma-ketlik ham $[a,b]$ da $\varphi(x)$ ga tekis yaqinlashadi.

◀Aytaylik, $f(x,y)$ funksiya $y \to y_0$ da $\varphi(x)$ funksiyaga $[a,b]$ da tekis yaqinlashsin. Unda ta`rifga binoan

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0,\ 0 < |y - y_0| < \delta$$

tengsizlikni qanoatlantiruvchi ixtiyoriy

$$y \in E,\ \forall x \in [a,b]:\ |f(x,y) - \varphi(x)| < \varepsilon$$

bo`ladi.

Modomiki, $\{y_n\}$ ketma-ketlik y_0 ga intilar ekan, unda

$$\forall \delta > 0,\ \exists n_0 \in N,\ \forall n > n_0:\ |y_n - y_0| < \delta$$

tengsizlik bajariladi. Demak,

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ |y_n - y_0| < \delta,\ y_n \in E,\ \forall x \in [a,b]:\ |f(x,y_n) - \varphi(x)| < \varepsilon$$

ya`ni,

$$|f_n(x) - \varphi(x)| < \varepsilon$$

bo`ladi. Bu esa $f_n(x)$ funktsional ketma-ketlikning $[a,b]$ da $\varphi(x)$ funksiyaga tekis yaqinlashishini bildiradi.▶

Endi $f(x,y)$ funksiyaning limit funksiyaga ega bo`lish va unga tekis yaqinlashishi haqidagi teoremani keltiramiz.

2-teorema. $f(x,y)$ funksiya $y \to y_0$ da limit funksiya $\varphi(x)$ ga ega bo`lishi va unga tekis yaqinlashishi uchun $\forall \varepsilon > 0$ olinganda ham $x$ ga bog`liq bo`lmagan shunday $\exists \delta = \delta(\varepsilon) > 0$ topilib, $|y - y_0| < \delta$, $|y' - y_0| < \delta$ tengsizliklarni qanoatlantiruvchi ixtiyoriy $y,\ y' \in E$ hamda $\forall x \in [a,b]$ da

$$|f(x,y) - f(x,y')| < \varepsilon \qquad (2)$$

tengsizlikning bajarilishi zarur va yetarli.

◀ **Zarurligi.** Aytaylik, $f(x,y)$ funksiya $y \to y_0$ da limit funksiya $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashsin. Unda ta`rifga binoan

$$\forall\varepsilon>0,\ \exists\delta>0,\ |y-y_0|<\delta,\ \forall\ y\in E,\ \forall x\in[a,b]:\ |f(x,y)-\varphi(x)|<\frac{\varepsilon}{2} \quad (3)$$

jumladan, $|y'-y_0|<\delta,\ y'\in E$ uchun ham

$$|f(x,y')-\varphi(x)|<\frac{\varepsilon}{2} \quad (4)$$

bo`ladi.

(3) va (4) munosabatlardan

$$|f(x,y)-f(x,y')|<\varepsilon$$

bo`lishi kelib chiqadi.

Yetarliligi. Aytaylik, (2) shart bajarilsin. Modomiki, har bir tayinlangan $x\in[a,b]$ va $|y-y_0|<\delta$, $|y'-y_0|<\delta$, $y\in E$, $y'\in E$ da

$$|f(x,y)-f(x,y')|<\varepsilon$$

tengsizlik bajarilar ekan, unda Koshi teoremasiga ko`ra $y\to y_0$ da $f(x,y)$ funksiya limitga ega bo`ladi. Uni $\varphi(x)$ bilan belgilaylik:

$$\lim_{y\to y_0} f(x,y)=\varphi(x).$$

Endi y o`zgaruvchining $|y-y_0|<\delta$ tengsizlikni qanoat-lantiradigan qiymatida tayinlab, (2) tengsizlikda $y'\to y_0$ da limitga o`tib, topamiz:

$$|f(x,y)-\varphi(x)|\le\varepsilon.$$

Bu esa $y\to y_0$ da $f(x,y)$ funksiya $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashishini bildiradi. ►

3-teorema. $f(x,y)$ funksiya uchun quyidagi shartlar bajarilsin:

1) har bir tayin $y\in E$ da $f(x,y)$ funksiya $[a,b]$ da x o`zgaruvchining funksiyasi sifatida uzluksiz;

2) $y\to y_0$ da $f(x,y)$ funksiya $[a,b]$ da $\varphi(x)$ ga tekis yaqinlashsin.

U holda $\varphi(x)$ funksiya $[a,b]$ da uzluksiz bo`ladi.

◄ E to`plamda y_0 ga intiluvchi ixtiyoriy $\{y_n\}$ ketma-ketlik olib $(y_n\in E, n=1,2,..., y_n\to y_0)$ $[a,b]$ segmentda aniqlangan ushbu

$$f_n(x)=f(x,y_n)$$

funktsional ketma-ketlikni hosil qilamiz. Teoremaning shartlariga ko`ra:

1) $f_n(x)$ funktsional ketma-ketlikning har bir hadi $[a,b]$ da uzluksiz bo`ladi;

2) mazkur ma`ruzadagi 2-teoremaga binoan $y \to y_0$ da $f_n(x)$ funktsional ketma-ketlik $\varphi(x)$ funksiyaga $[a,b]$ da tekis yaqinlashadi.

Unda $\varphi(x)$ funksiya $[a,b]$ segmentda uzluksiz bo`ladi (qaralsin, 65-ma`ruza). ►

12-amaliy mashg'ulot

1-misol. Ushbu

$$f(x,y) = xy$$

funksiyani

$$M = \{(x,y) \in R^2 : \ 0 \le x \le 1, \ y \in [0,1]\}$$

to`plamda qaraylik. Bu funksiyaning $y \to y_0 = 1$ dagi limit funksiyasi $\varphi(x) = x$ bo`lishi ko`rsatilsin.

◄Ixtiyoriy $\varepsilon > 0$ songa ko`ra, har bir $x \in [0,1]$ uchun $\delta = \varepsilon$ deb olinsa, unda $|y - y_0| = |y - 1| < \delta$ tengsizlikni qanoatlan-tiruvchi $y \in [0,1]$ uchun

$$|f(x,y) - \varphi(x)| = |xy - x| = |x||y - 1| \le |y - 1| < \delta = \varepsilon$$

bo`ladi. Demak,

$$\lim_{y \to 1} xy = x. \ ►$$

2-misol. Ushbu

$$f(x,y) = x^y, \qquad (0^0 = 0)$$

funksiyani

$$M = \{(x,y) \in R^2 : \ 0 \le x \le 1, \ y \in [0,1]\}$$

to`plamda qaraymiz. Bu funksiyaning $y \to y_0 = 0$ dagi limit funksiyasi topilsin.

◄Aytaylik, $x = 0$ bo`lsin. Bu holda $\forall y \in [0,1]$ uchun

$$f(0,y) = 0$$

bo`lib, $y \to 0$ da $f(0,y) \to 0$ bo`ladi.

Aytaylik, $x \ne 0$ bo`lsin. Bu holda $y \to 0$ da

$$f(x,y) = x^y \to x^0 = 1$$

bo`ladi. Haqiqatan ham, ixtiyoriy $\varepsilon > 0$ songa ko`ra $\delta = \log_x(1-\varepsilon)$ deyilsa $(x > 0)$, unda $|y - y_0| = |y - 0| = |y| < \delta$ tengsizlikni qanoatlantiruvchi $y \in [0,1]$ uchun

$$|f(x,y) - \varphi(x)| = |x^y - 1| = 1 - x^y < 1 - x^{\log_x(1-\varepsilon)} = 1 - (1 - \varepsilon) = \varepsilon$$

bo`ladi. Demak, $y \to 0$ da $f(x,y) = x^y$ funksiyaning limit funksiyasi

$$\varphi(x)=\begin{cases}1, & \text{агар } x\in(0,1] \text{ бўлса,}\\ 0, & \text{агар } x=0 \text{ бўлса}\end{cases}$$

bo`ladi. ►

3-misol. Ushbu $f(x,y)=x\sin y$ funksiyani

$$M=\{(x,y)\in R^2:\ 0\le x\le 1,\ y\in[0,\pi]\}$$

to`plamda qaraylik. Bu funksiyaning $y\to y_0=\dfrac{\pi}{3}$ da limit funksiyasi topilib, unga $[0,1]$ da tekis yaqinlashishi ko`rsatilsin.

◄Ravshanki,

$$\lim_{y\to\frac{\pi}{3}} x\sin y=\frac{\sqrt{3}}{2}x.$$

Demak, $\varphi(x)=\dfrac{\sqrt{3}}{2}x$.

Agar $\forall\varepsilon>0$ ga ko`ra $\delta=\varepsilon$ deyilsa, u holda $\left|y-\dfrac{\pi}{3}\right|<\delta$ tengsizlikni qanoatlantiruvchi $y\in[0,\pi]$ va $\forall x\in[0,1]$ uchun

$$|f(x,y)-\varphi(x)|=\left|x\sin y-\frac{\sqrt{3}}{2}x\right|=|x|\left|\sin y-\frac{\sqrt{3}}{2}\right|=|x|\left|\sin y-\sin\frac{\pi}{3}\right|\le\left|y-\frac{\pi}{3}\right|<\delta=\varepsilon$$

bo`ladi. Ta`rifga binoan, $y\to\dfrac{\pi}{3}$ da $f(x,y)=x\sin y$ funksiya $\varphi(x)=\dfrac{\sqrt{3}}{2}x$ limit funksiyaga $[0,1]$ da tekis yaqinlashadi. ►

Eslatma. Aytaylik,

$$\lim_{y\to y_0} f(x,y)=\varphi(x)$$

bo`lsin. Agar $\forall\delta>0$ son olinganda shunday $\varepsilon_0>0,\ x_0\in[a,b]$ va $|y-y_0|<\delta$ tengsizlikni qanoatlantiruvchi $y_1\in E$ topilsaki,

$$|f(x_0,y_1)-\varphi(x_0)|\ge\varepsilon_0$$

bo`lsa, $f(x,y)$ funksiyaning $y\to y_0$ da limit funksiya $\varphi(x)$ ga tekis yaqinlashmaydi deyiladi.

4-misol. $f(x,y)=x^y$, $(0^0=0)$ funksiyaning $y\to 0$ dagi limit funksiya

$$\varphi(x)=\begin{cases}1, & \text{agar } x\in(0,1] \text{ bo`lsa,}\\ 0, & \text{agar } x=0 \text{ bo`lsa}\end{cases}$$

ga tekis yaqinlashmaydi bo`ladi. Haqiqatan ham, $\forall\delta > 0$ son olinganda, $\varepsilon_0 = \frac{1}{4}$, $0 < y_1 < \delta$ tengsizlikni qanoatlantiruvchi y_1 va $x_0 = 2^{-\frac{1}{y_1}}$ deb olinsa, u holda

$$|f(x_0, y_1) - \varphi(x_0)| = 1 - x_0^{y_1} = 1 - \frac{1}{2} = \frac{1}{2} > \varepsilon_0$$

bo`ladi.

Faraz qilaylik, $f(x, y)$ funksiya R^2 fazodagi

$$M = \{(x, y) \in R^2 : a \le x \le b,\ y \in E \subset R\}$$

to`plamda berilgan va $y_0 \in R$ nuqta $E$ to`plamning limit nuqtasi bo`lsin.

Agar E to`plam $y_0$ ga intiluvchi $\{y_n\}$ ketma-ketlikdan iborat bo`lsa, $f(x, y)$ funksiyani $[a, b]$ da aniqlangan

$$f_n(x) = f(x, y_n)$$

funktsional ketma-ketlik sifatida qarash mumkin. Masalan,

$$f(x, y) = \frac{2xy}{x^2 + y^2}$$

funksiyani $M_0 = \left\{(x, y) \in R^2 :\ x \in [0,1],\ y \in E = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}\right\}$ to`plamda qarasak, u quyidagi

$$f_n(x) = \frac{2xn}{1 + n^2x^2}$$

funktsional ketma-ketlikka aylanadi.

1. Ushbu

$$f(x, y) = n\left(\sqrt{x + \frac{1}{n}} - \sqrt{x}\right)$$

funksiyani $M = \{(x, n) \in R^2 :\ x \in (0, \infty),\ n \in N\}$ to`plamda qaraylik. Bu funksiyaning $n \to \infty$ da limit funksiyasi topilsin.

2. Ushbu

$$f(x, y) = \frac{1}{y}\left(1 - x^{\frac{1}{y}}\right)x^{\frac{1}{y}}$$

funksiyani $M = \left\{(x, y) \in R^2 :\ x \in \left[\frac{1}{2}, 1\right],\ y \in (0,1]\right\}$ to`plamda qaraylik. Bu funksiya uchun

$$\lim_{y\to+0} f(x,y)=0$$

bo`lishi isbotlansin.

3. Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : x\in[a,b],\ y\in E\subset R\}$$

to`plamda berilgan va $y_0\in R$ esa $E$ ning limit nuqtasi. $y\to y_0$ da $f(x,y)$ funksiyaning $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashishi uchun $E$ to`plamdagi y_0 ga intiluvchan ixtiyoriy $\{y_n\}$ ketma-ketlikda

$$f_n(x)=f(x,y_n)$$

funktsional ketma-ketlikning $[a,b]$ da $\varphi(x)$ ga tekis yaqinlashishi zarur va etarli ekani isbotlansin.

Foydalanish uchun adabiyotlar

1. **Tao T.** *Analysis 1, 2.* Hindustan Book Agency, India, 2014.
2. **Aksoy A. G., Khamsi M. A.** *A problem book in real analysis.* Springer, 2010.
3. **Zorich V.A.** *Mathematical Analysis I.* Springer, 2004, Germany.
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5. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, I, II q.* T. "Voris-nashriyot", 2010.
6. **Shoimqulov B. A., Tuychiyev T. T., Djumaboyev D. X.** *Matematik analizdan mustaqil ishlar.* T. "O'zbekiston faylasuflari milliy jamiyati", 2008.

Nazorat savollari

1. Limit funksiya.
2. Tekis yaqinlashish.
3. Tekis yaqinlashishning zaruriy va yetarli sharti.
4. Limit funksiyaning uzluksizligi haqida teorema.

GLOSSARY

Limit funksiya - Har bir $x\in[a,b]$ ga $f(x,y)$ funksiyaning $y\to y_0$ dagi limitini mos qo`yish natijasida

$$\varphi:\quad x\to\lim_{y\to y_0} f(x,y)$$

hosil bo`luvchi $\varphi(x)$ funksiya.

Tekis yaqinlash - Agar $\forall\varepsilon>0$ son olinganda ham, shunday $\delta=\delta(\varepsilon)>0$ son topilsaki, $|y-y_0|<\delta$ tengsizlikni qanoatlantiruvchi $y\in E$, $\forall x\in[a,b]$ uchun

$$|f(x,y)-\varphi(x)|<\varepsilon$$

tengsizlik bajarilsa, ya`ni

$$\forall\varepsilon>0,\ \exists\delta=\delta(\varepsilon)>0,\ |y-y_0|<\delta,\ y\in E,\ \forall x\in[a,b]:$$

$$|f(x,y)-\varphi(x)|<\varepsilon$$

bo`lsa, $f(x,y)$ funksiya $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashadi deyiladi.

KEYS BANKI

1-keys. Masala o`rtaga tashlanadi: Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : x\in[a,b],\ y\in E\subset R\}$$

to‘plamda berilgan va $y_0\in R$ esa E ning limit nuqtasi. $y\to y_0$ da $f(x,y)$ funksiyaning $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashishi uchun E to‘plamdagi $y_0\in R$ ga intiluvchan ixtiyoriy $\{y_n\}$ ketma-ketlikda

$$f_n(x)=f(x,y_n)$$

funksional ketma-ketlikning $[a,b]$ da $\varphi(x)$ ga tekis yaqinlashishi zarur va yetarli ekani isbotlansin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

13-Mavzu: Parametrga bog\`liq integrallar
13-Ma'ruza
REJA:

1^0. Parametrga bog'liq integral tushunchasi: $J(y)=\int_a^b f(x,y)dx$.

2^0. $J(y)$ funksiyaning limiti.

3^0. $J(y)$ funksiyaning uzluksizligi.

4^0. $J(y)$ funksiyani differensiallash.

5^0. $J(y)$ funksiyani integrallash.

Tayanch so'z va iboralar: *parametrga bog'liq integral, $J(y)$ funksiyaning limiti, $J(y)$ funksiyaning uzluksizligi, $J(y)$ funksiyaning differensiali, $J(y)$ funksiyaning integrali.*

1^0. Parametrga bog\`liq integral tushunchasi. Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : a\le x\le b,\ y\in E\subset R\}$$

to\`plamda berilgan bo\`lsin. Bu funksiya har bir tayinlangan $y\in E$ da x o\`zgaruv-chining funksiyasi sifatida $[a,b]$da integrallanuvchi, ya\`ni

$$\int_a^b f(x,y)dx$$

mavjud deylik. Qaralayotgan integralning qiymati tayin-langan y ga bog\`liq bo\`la-di:

$$J(y)=\int_a^b f(x,y)dx. \qquad (1)$$

Masalan, $y\neq 0$ bo\`lganda

$$\int_0^1 e^{yx}dx=\frac{e^{yx}}{y}\bigg|_0^1=\frac{e^y-1}{y},$$

$y=0$ bo\`lganda

$$\int_0^1 e^{0x}dx=1$$

bo\`ladi. Demak,

$$J(y)=\int_0^1 e^{yx}dx=\begin{cases}\dfrac{e^y-1}{y}, & agar\ \ y\neq 0 \text{ bo\`lsa},\\ 1, & agar\ \ y=0\ \ bo\`lsa.\end{cases}$$

Odatda (1) integral parametrga bog`liq integral, y esa parametr deyiladi.

Ravshanki, $J(y)$ funksiya (parametrga bog`liq integral) berilgan $f(x,y)$ funksiya orqali aniqlanib, unga bog`liq bo`ladi.

Parametrga bog`liq integral mavzusida $f(x,y)$ funksiya-ning funktsional xossalariga ko`ra $J(y)$ funksiyaning funktsional xossalari (limiti, uzluksizligi, differentsiallanuvchiligi, integrallanishi) o`rganiladi.

2^0. $J(y)$ funksiyaning limiti. Aytaylik, $f(x,y)$ funksiya

$$M = \{(x,y) \in R^2 : a \le x \le b,\ y \in E \subset R\}$$

to`plamda berilgan bo`lib, $y_0 \in R$ esa E to`plamning limit nuqtasi bo`lsin. Bu funksiya uchun har bir tayin $y \in E$ da

$$J(y) = \int_a^b f(x,y)dx$$

mavjud bo`lsin.

1-teorema. Faraz qilaylik, $f(x,y)$ funksiya quyidagi shartlarni bajarsin:

1) har bir tayin $y \in E$ da $f(x,y)$ funksiya x o`zgaruvchi-ning funksiyasi sifatida $[a,b]$ da uzluksiz;

2) $y \to y_0$ da $f(x,y)$ funksiya limit funksiya $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashsin.

U holda $y \to y_0$ da $J(y)$ funksiya limitga ega bo`lib,

$$\lim_{y \to y_0} J(y) = \int_a^b \varphi(x)dx \tag{2}$$

bo`ladi.

◄ Keltirilgan teoremaning shartlarini bajarilishi-dan, 74-ma`ruzadagi 3-teoremaga ko`ra, limit funksiya $\varphi(x)$ ning $[a,b]$ da uzluksiz bo`lishi kelib chiqadi. Demak,

$$\int_a^b \varphi(x)dx$$

integral mavjud.

Ayni paytda, $y \to y_0$ da $f(x,y)$ funksiyaning $[a,b]$ da $\varphi(x)$ funksiyaga tekis yaqinlashuvchi bo`lishidan, ta`rifga binoan,

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ |y - y_0| < \delta,\ \forall y \in E,\ \forall x \in [a,b] : |f(x,y) - \varphi(x)| < \frac{\varepsilon}{b-a}$$

bo`lishini topamiz. Ushbu

$$\left|J(y)-\int_a^b \varphi(x)dx\right|$$

ayirmani qaraylik.

Ravshanki, $|y - y_0| < \delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy $y \in E$ uchun

$$\left|J(y)-\int_a^b \varphi(x)dx\right| = \left|\int_a^b f(x,y)dx - \int_a^b \varphi(x)dx\right| \le \int_a^b |f(x,y)-\varphi(x)|dx < \frac{\varepsilon}{b-a}\int_a^b dx = \varepsilon$$

bo`ladi.

Keyingi munosabatdan

$$\lim_{y\to y_0} J(y) = \int_a^b \varphi(x)dx$$

bo`lishi kelib chiqadi. ►

(2) munosabatni quyidagicha

$$\lim_{y\to y_0}\int_a^b f(x,y)dx = \int_a^b \left[\lim_{y\to y_0} f(x,y)\right]dx$$

ham yozish mumkin. Bu integral belgisi ostida limitga o`tish qoidasini ifodalaydi.

3^0. $J(y)$ funksiyaning uzluksizligi. $J(y)$ funksiyaning uzluksizligini quyidagi teorema ifodalaydi.

2-teorema. **[4, Proposition 1, p.408]** Agar $f(x,y)$ funksiya

$$M_0 = \{(x,y)\in R^2 : a \le x \le b,\ c \le y \le d\}$$

to`plamda uzluksiz bo`lsa, $J(y)$ funksiya $[c,d]$ da uzluksiz bo`ladi.

◄ Ixtiyoriy $y_0 \in [c,d]$ va $y_0 + \Delta y \in [c,d]$ nuqtalarni olib, $J(y)$ funksiyaning orttirmasini topamiz:

$$\Delta J(y_0) = J(y_0+\Delta y) - J(y_0) = \int_a^b [f(x,y_0+\Delta y) - f(x,y_0)]dx.$$

$f(x,y)$ funksiya M_0 to`plamda tekis uzluksiz. Unda $\forall\varepsilon > 0$ uchun shunday $\delta > 0$ topiladiki, $|\Delta y| < \delta$ bo`lganda, $\forall x \in [a,b]$ uchun

$$|f(x,y_0+\Delta y) - f(x,y_0)| < \varepsilon$$

bo`ladi. Demak, $|\Delta y| < \delta$ bo`lganda

$$|\Delta J(y_0)| = \left|\int_a^b [f(x,y_0+\Delta y) - f(x,y_0)]dx\right| < \varepsilon(b-a)$$

bo`ladi. Keyingi munosabatdan

$$\lim_{\Delta y\to 0}\Delta J(y_0)=0$$

bo`lishi kelib chiqadi. Bu esa $J(y)$ funksiyani ixtiyoriy $y_0$ nuqtada, binobarin $[c,d]$ da uzluksiz bo`lishini bildiradi.►

4^0. $J(y)$ funksiyani differentsiallash. Aytaylik, $f(x,y)$ funksiya M_0 to`plamda berilgan bo`lsin.

3-teorema. **[4, Proposition 2, p.409]** Faraz qilaylik, $f(x,y)$ funksiya quyidagi shartlarni bajarsin:

1) har bir tayin $y\in[c,d]$ da $f(x,y)$ funksiya $[a,b]$ da x o`zgaruvchining funksiyasi sifatida uzluksiz;

2) $f(x,y)$ funksiya M_0 to`plamda $f'_y(x,y)$ xususiy hosilaga ega va $f'_y(x,y)$ funksiya M_0 da uzluksiz.

U holda $J(y)$ funksiya $[c,d]$ da hosilaga ega va

$$J'(y)=\int_a^b f'_y(x,y)dx \tag{3}$$

bo`ladi.

◄ $y\in[c,d]$, $y+\Delta y\in[c,d]$ nuqtalarni olib, topamiz:

$$\frac{J(y+\Delta y)-J(y)}{\Delta y}=\int_a^b\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}dx.$$

Lagranj teoremasiga ko`ra

$$\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=f'_y(x,y+\theta\Delta y)\qquad(0<\theta<1)$$

bo`lib,

$$\frac{J(y+\Delta y)-J(y)}{\Delta y}=\int_a^b f'(x,y+\theta\Delta y)dx\qquad(0<\theta<1) \tag{4}$$

bo`ladi.

$f'_y(x,y)$ funksiya M_0 to`plamda tekis uzluksiz bo`lganligi sababli

$$\forall\varepsilon>0,\ \exists\delta>0,\ |\Delta y|<\delta,\ \forall x\in[a,b]:\left|f'_y(x,y+\theta\Delta y)-f'_y(x,y)\right|<\frac{\varepsilon}{b-a}$$

tengsizlik bajariladi. (4) munosobatdan foydalanib

$$\left|\frac{J(y+\Delta y)-f(y)}{\Delta y}-\int_a^b f'_y(x,y)dx\right|\le\int_a^b\left|f'_y(x,y+\theta\Delta y)-f'_y(x,y)\right|dx<\varepsilon$$

bo`lishini topamiz. Demak,

$$\lim_{\Delta y\to 0}\frac{J(y+\Delta y)-J(y)}{\Delta y}=\int_a^b f'_y(x,y)dx\,.$$

Bu esa

$$J'(y)=\int_a^b f'_y(x,y)dx$$

ekanini bildiradi. ►

(3) munosabatni quyidagicha

$$\frac{d}{dy}\int_a^b f(x,y)dx=\int_a^b\frac{d}{dy}f(x,y)dx$$

ham yozish mumkin. Bu differentsiallash amalini integral belgisi ostiga o`tkazish qoidasini ifodalaydi.

5⁰. $J(y)$ funksiyani integrallash. Faraz qilaylik, $f(x,y)$ funksiya M_0 to`plamda berilgan va uzluksiz bo`lsin. U holda 2-teoremaga ko`ra

$$J(y)=\int_a^b f(x,y)dx$$

funksiya $[c,d]$ da uzluksiz bo`ladi. Binobarin, bu funksiya $[c,d]$ da integrallanuvchi, ya`ni

$$\int_c^d J(y)dy$$

mavjud bo`ladi.

4-teorema. **[4, Proposition 3, p.4013]** Agar $f(x,y)$ funksiya M_0 to`plamda uzluksiz bo`lsa, u holda

$$\int_c^d J(y)dy=\int_c^d\left[\int_a^b f(x,y)dx\right]dy=\int_a^b\left[\int_c^d f(x,y)dy\right]dx$$

bo`ladi.

◄ $\forall t\in[c,d]$ nuqtani olib, ushbu

$$F(t)=\int_c^t\left[\int_a^b f(x,y)dx\right]dy,\;\; \hat{O}(t)=\int_a^b\left[\int_c^t f(x,y)dy\right]dx$$

funksiyalarni qaraymiz.

Ravshanki,

$$F'(t)=\left(\int_c^t\left[\int_a^b f(x,y)dx\right]dy\right)' = \int_a^b f(x,t)dx\ ,$$

$$\Phi'(t)=\left(\int_a^b\left[\int_c^t f(x,y)dy\right]dx\right)' = \int_a^b\left[\int_c^t f(x,y)dy\right]_t^1 dx=\int_a^b f(x,t)dx\ .$$

Demak,

$$F'(t)=\Phi'(t)=\int_a^b f(x,t)dx$$

bo`lib, undan

$$F(t)=\Phi(t)+C \qquad (C=const)$$

bo`lishi kelib chiqadi.

Agar $t=c$ deyilsa,

$$F(c)=\Phi(c)=0$$

bo`ladi va keyingi tenglikdan $C=0$ bo`lishini topamiz.

Demak,

$$F(t)=\Phi(t).$$

Xususan, $t=d$ bo`lganda $F(d)=\Phi(d)$ bo`lib,

$$\int_c^d\left[\int_a^b f(x,y)dx\right]dy=\int_a^b\left[\int_c^d f(x,y)dy\right]dx$$

bo`ladi. ►

13-Amaliy mashg’ulotlar

Parametirga bog’liq xosmas integrallar

Parametirga bog’liq xosmas integrallarni hisoblang:

3713.

a) $\lim\limits_{\alpha\to 0}\int_{\alpha}^{1+2}\frac{dx}{1+x^2+\alpha^2}=\int_0^1\frac{dx}{1+x^2}=arctgx|_0^1=\frac{\pi}{4}-0=\frac{\pi}{4}.$

$$\lim_{\alpha\to 0}\left(\frac{1}{1+x^2+\alpha^2}-\frac{1}{1+x^2}\right)=\lim_{\alpha\to 0}\frac{1+x^2-1-x^2-\alpha^2}{(1+x^2)(1+x^2+\alpha^2)}=$$

$$\lim_{\alpha\to 0}\frac{-\alpha^2}{(1+x^2)(1+x^2+\alpha^2)}=\frac{0}{(1+x^2)}=o$$

3713.

b)

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(\sqrt{x^2+a^2}-\sqrt{x^2}\right)=\int_{-1}^{1}\sqrt{x^2}dx=$$

$$=\int_{-1}^{1}|x|dx=-\int_{-1}^{0}xdx+\int_{0}^{1}xdx=$$

$$-\frac{x^2}{2}\Big|_{-1}^{0}+\frac{x^2}{2}\Big|_{0}^{1}=-\left(0-\frac{1}{2}\right)+\left(\frac{1}{2}+0\right)=$$

$$=\frac{1}{2}+\frac{1}{2}=1$$

g) $$\lim_{n\to\infty}\int_0^1\frac{dx}{1+(1+\frac{x}{n})^n}=\lim_{n\to\infty}\frac{dx}{1+(1+\frac{n}{x})^{\frac{n}{x}x}}=\int_0^1\frac{dx}{1+e^x}=$$

$$\int_0^1\frac{1+e^x-e^x}{1+e^x}dx=x\Big|_0^1-ln|1+e^x|\Big|_0^1=1-$$

$$-\ln(e+1)+ln2=1-ln\frac{e+1}{2}$$

$$\lim_{n\to\infty}\left(\frac{1}{1+(1+\frac{x}{n})^n}-\frac{1}{1+e^x}\right.$$

$$=\lim_{n\to\infty}\frac{1+e^x-1-\left(1+\frac{x}{n}\right)^n}{(1+(1+\frac{x}{n})(1+e^x}$$

$$=\frac{e^x-(1+\frac{x}{n})^{\frac{n}{x}x}}{(1+e^x)(1+\left(1+\frac{x}{n}\right)^{\frac{n}{x}n}}=$$

$$\frac{e^x-e^x}{(1+e^x)^2}=0.$$

$$\int_{a+\alpha}^{b+\alpha}\frac{1}{x}x\cos 2xdx=\frac{\sin\alpha(b+\alpha)}{b+\alpha}-$$

$$-\frac{\sin\alpha(a+\alpha)}{a+\alpha}+\frac{1}{2}\sin\alpha x|_{a+\alpha}^{b+\alpha}=\frac{\sin\alpha(b+\alpha)}{b+\alpha}-$$

$$-\frac{\sin\alpha(a+\alpha)}{a+\alpha}+\frac{\sin\alpha(b+\alpha)}{\alpha}-\frac{\sin\alpha(a+\alpha)}{\alpha}=$$

$$\sin\alpha(b+\alpha)\left(\frac{1}{b+\alpha}+\frac{1}{2}\right)-\sin\alpha(a+\alpha)\left(\frac{1}{a+\alpha}+\frac{1}{2}\right)$$

Mashqlar

1. Ushbu

$$f(x,y)=\frac{1}{y}\left(1-x^{\frac{1}{y}}\right)x^{\frac{1}{y}}$$

funksiyani $\left\{(x,y)\in R^2 : x\in\left[\frac{1}{2},1\right],\ y\in(0,1]\right\}$ to`plamda qaraylik. Bu funksiya uchun

$$\lim_{y\to+0}\int_{\frac{1}{2}}^{1}f(x,y)dx\neq\int_{\frac{1}{2}}^{1}\left(\lim_{y\to+0}f(x,y)\right)dx$$

bo`lishi isbotlansin.

2. Agar $f(x)$ funksiya $[0,1]$ segmentda uzluksiz hosilaga ega bo`lsa,

$$J(y)=\int_0^1 f(x)\,sign\sin(xy)dx \qquad (y>0)$$

funksiyaning hosilasi topilsin.

3. Agar

$$J_0(y)=\frac{1}{\pi}\int_0^{\pi}\cos(y\cos x)dx \qquad (y\in R)$$

bo`lsa, u quyidagi $yJ_0''(y)+J_0'(y)+yJ_0(y)=0$ tenglamani qanoatlantirishi ko`rsatilsin.

Foydalanish uchun adabiyotlar

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6. **Shoimqulov B. A., Tuychiyev T. T., Djumaboyev D. X.** *Matematik analizdan mustaqil ishlar.* T. "O'zbekiston faylasuflari milliy jamiyati", 2008.

Nazorat savollari

5. Parametrga bog'liq integral deb nimaga aytiladi?
6. $J(y)$ funksiyaning limiti va uzluksizligini tariflang.
7. $J(y)$ funksiyaning differensiallash deganda nimani tushunasiz?
8. $J(y)$ funksiyani integrallash.

GLOSSARY

Parametrga bog`liq integral - Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : a\le x\le b,\ y\in E\subset R\}$$

to`plamda berilgan bo`lsin. Bu funksiya har bir tayinlangan $y\in E$ da x o`zgaruvchining funksiyasi sifatida $[a,b]$da integrallanuvchi, ya`ni

$$\int_a^b f(x,y)dx$$

mavjud deylik. Qaralayotgan integralning qiymati tayin-langan y ga bog`liq bo`ladi:

$$J(y)=\int_a^b f(x,y)dx. \qquad (1)$$

Odatda (1) ko'rinishidagi integral parametrga bog'liq integral deyiladi.

$J(y)$ **funksiyaning limiti -** Faraz qilaylik, $f(x,y)$ funksiya quyidagi shartlarni bajarsin:

1) har bir tayin $y\in E$ da $f(x,y)$ funksiya x o`zgaruvchi-ning funksiyasi sifatida $[a,b]$ da uzluksiz;

2) $y\to y_0$ da $f(x,y)$ funksiya limit funksiya $\varphi(x)$ ga $[a,b]$ da tekis yaqinlashsin.

U holda $y\to y_0$ da $J(y)$ funksiya limitga esa bo`lib,

$$\lim_{y\to y_0} J(y) = \int_a^b \varphi(x)dx \qquad (2)$$

bo`ladi.

$J(y)$ **funksiyaning uzluksizligi -** Agar $f(x,y)$ funksiya

$$M_0 = \{(x,y)\in R^2 : a\le x\le b,\ c\le y\le d\}$$

to`plamda uzluksiz bo`lsa, $J(y)$ funksiya $[c,d]$ da uzluksiz bo`ladi.

KEYS BANKI

1-keys. Ushbu

$$f(x,y) = \frac{1}{y}\left(1 - x^{\frac{1}{y}}\right)x^{\frac{1}{y}}$$

funksiyani

$$\left\{(x,y)\in R^2 : x\in\left[\frac{1}{2},1\right],\ y\in(0,1]\right\}$$

to‘plamda qaraylik. Bu funksiya uchun

$$\lim_{y\to+0}\int_{\frac{1}{2}}^{1} f(x,y)dx \ne \int_{\frac{1}{2}}^{1}\left(\lim_{y\to+0} f(x,y)\right)dx$$

bo‘lishi isbotlansin.

2-keys. Agar $f(x)$ funksiya $[0,1]$ segmentda uzluksiz hosilaga ega bo`lsa,

$$J(y) = \int_0^1 f(x)\,sign\sin(xy)dx \qquad (y>0)$$

funksiyaning hosilasi topilsin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

14-Mavzu: Chegaralari o`zgaruvchi parametrga bog`liq integrallar

14-ma'ruza

REJA:

1^0. $J_1(y)$ funksiyaning uzluksizligi.
2^0. $J_1(y)$ funksiyani differensiallash.

Tayanch so'z va iboralar: *parametrga bog`liq integral, $J_1(y)$ funksiyaning uzluksizligi, $J_1(y)$ funkisyani differensiallash.*

Faraz qilaylik, $f(x,y)$ funksiya

$$M_0 = \{(x,y) \in R^2 : a \le x \le b,\ c \le y \le \mathrm{d}\}$$

to`plamda berilgan va har bir tayin $y \in [c,d]$ da $f(x,y)$ funksiya $x$ o`zgaruvchining funksiyasi sifatida $[a,b]$ da integrallanuvchi bo`lsin. $\alpha(y)$ va $\beta(y)$ funksiyalarning har biri $[c,d]$ da berilgan va $\forall y \in [c,d]$ uchun

$$a \le \alpha(y) \le \beta(y) \le b \tag{1}$$

tengsizliklar bajarilsin.

Ushbu

$$\int_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

integral, ravshanki, y o`zgaruvchiga bog`liq bo`ladi:

$$J_1(y) = \int_{\alpha(y)}^{\beta(y)} f(x,y)dx\,. \tag{2}$$

(2) integral chegaralari ham parametrga bog`liq integral deyiladi.

1^0. $J_1(y)$ funksiyaning uzluksizligi. $J_1(y)$ funksiyaning uzluksizligini quyidagi teorema ifodalaydi:

1-teorema. Faraz qilaylik, $f(x,y)$ funksiya M_0 to`plamda uzluksiz bo`lib, $\alpha(y)$ va $\beta(y)$ funksiyalar esa $[c,d]$ segmentda uzluksiz bo`lsin. U holda

$$J_1(y) = \int_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya $[c,d]$ da uzluksiz bo`ladi.

◀Ixtiyoriy $y_0 \in [c,d]$ nuqtani olaylik. Integralning ma`lum xossalaridan foydalanib topamiz:

$$\begin{aligned} J_1(y) &= \int_{\alpha(y)}^{\beta(y)} f(x,y)dx = \int_{\alpha(y)}^{\alpha(y_0)} f(x,y)dx + \int_{\alpha(y_0)}^{\beta(y_0)} f(x,y)dx + \int_{\beta(y_0)}^{\beta(y)} f(x,y)dx = \\ &= \int_{\alpha(y_0)}^{\beta(y_0)} f(x,y)dx + \int_{\beta(y_0)}^{\beta(y)} f(x,y)dx - \int_{\alpha(y_0)}^{\alpha(y)} f(x,y)dx\,. \end{aligned} \tag{3}$$

Ravshanki,

$$\int_{\alpha(y_0)}^{\beta(y_0)} f(x,y)dx$$

integral chegarasi o`zgarmas bo`lgan parametrga bog`liq integral. Bu funksiya 75-ma`ruzada keltirilgan 2-teoremaga muvofiq y o`zgaruvchining uzluksiz funksiyasi bo`ladi. Demak,

$$y \to y_0 \text{ da } \int_{\alpha(y_0)}^{\beta(y_0)} f(x,y)dx \to \int_{\alpha(y_0)}^{\beta(y_0)} f(x,y_0)dx = J_1(y_0) \qquad (4)$$

bo`ladi.

$f(x,y)$ funksiya M_0 to`plamda uzluksiz bo`lganligi sababli y shu to`plamda chegaralangan bo`ladi:

$$|f(x,y)| \le C \qquad (C = const).$$

SHartga ko`ra $\alpha(y)$ va $\beta(y)$ funksiyalar $[c,d]$ segmentda uzluksiz.

Demak,

$$y \to y_0 \text{ da } \alpha(y) \to \alpha(y_0),$$
$$y \to y_0 \text{ da } \beta(y) \to \beta(y_0).$$

Endi

$$\left| \int_{\beta(y_0)}^{\beta(y)} f(x,y)dx \right| \le C|\beta(y) - \beta(y_0)|,$$

$$\left| \int_{\alpha(y_0)}^{\alpha(y)} f(x,y)dx \right| \le C|\alpha(y) - \alpha(y_0)|,$$

munosabatlardan

$$y \to y_0 \text{ da } \int_{\beta(y_0)}^{\beta(y)} f(x,y)dx \to 0, \qquad (5)$$

$$y \to y_0 \text{ da } \int_{\alpha(y_0)}^{\beta(y)} f(x,y)dx \to 0$$

bo`lishini topamiz.

(3) tenglikda, $y \to y_0$ da limitga o`tish va unda (4) va (5) munosabatlarni hisobga olish natijasida

$$y \to y_0 \text{ da } J_1(y) \to J_1(y_0)$$

bo`lishi kelib chiqadi. Demak, $J_1(y)$ funksiya $[c,d]$ da uzluksiz.►

2^0. $J_1(y)$ funksiyani differensiallash. Faraz qilaylik, $f(x,y)$ funksiya

$$M_0 = \{(x,y) \in R^2 : a \le x \le b,\ c \le y \le d\}$$

to`plamda, $\alpha(y)$ va $\beta(y)$ funksiyalar esa $[c,d]$ segmentda berilgan bo`lib, $\alpha(y)$, $\beta(y)$ funksiyalar (1) shartni bajarsin, ya`ni $\forall y\in[c,d]$ uchun

$$a\le\alpha(y)\le\beta(y)\le b$$

bo`lsin.

2-teorema. **[4, Proposition 2', p.411]** Aytaylik, $f(x,y)$, $\alpha(y)$ va $\beta(y)$ funksiyalar quyidagi shartlarni bajarsin:

1) $f(x,y)$ funksiya M_0 to`plamda uzluksiz;
2) $f(x,y)$ funksiya M_0 to`plamda uzluksiz $f'_y(x,y)$ xususiy hosilaga ega;
3) $\alpha(y)$ va $\beta(y)$ funksiyalar $[c,d]$ da $\alpha'(y)$ va $\beta'(y)$ hosilalarga ega.

U holda

$$J_1(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya $[c,d]$ segmentda $J'_1(y)$ hosilaga ega bo`lib,

$$J'_1(y)=\int_{\alpha(y)}^{\beta(y)} f'_y(x,y)dx+\beta'(y)f(\beta(y),y)-\alpha'(y)f(\alpha(y),y)$$

bo`ladi.

◄ $y_0\in[c,d]$, $y_0+\Delta y\in[c,d]$ nuqtalarni olib, topamiz:

$$\frac{J_1(y_0+\Delta y)-J_1(y_0)}{\Delta y}=\frac{1}{\Delta y}\left[\int_{\alpha(y_0+\Delta y)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx-\int_{\alpha(y_0)}^{\beta(y_0)} f(x,y_0)dx\right].$$

Agar

$$\int_{\alpha(y_0+\Delta y)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx=\int_{\alpha(y_0)}^{\beta(y_0)} f(x,y_0+\Delta y)dx+$$
$$+\int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx-\int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx$$

bo`lishini e`tiborga olsak, unda

$$\frac{J_1(y_0+\Delta y)-J_1(y_0)}{\Delta y}=\int_{\alpha(y_0)}^{\beta(y_0)}\frac{[f(x,y_0+\Delta y)-f(x,y_0)]}{\Delta y}dx+$$
$$+\frac{1}{\Delta y}\int_{\beta(y_0)}^{\beta(y_0+\Delta y)} f(x,y_0+\Delta y)dx-\frac{1}{\Delta y}\int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)} f(x,y_0+\Delta y)dx \qquad (6)$$

bo`lishi kelib chiqadi.

Avvalgi ma`ruzadagi 1- teoremaga ko`ra

$$\lim_{\Delta y\to 0}\int_{\alpha(y_0)}^{\beta(y_0)}\frac{[f(x,y_0+\Delta y)-f(x,y_0)]}{\Delta y}dx=$$
$$=\int_{\alpha(y_0)}^{\beta(y_0)}\lim_{\Delta y\to 0}\frac{[f(x,y_0+\Delta y)-f(x,y_0)]}{\Delta y}dx=\int_{\alpha(y_0)}^{\beta(y_0)}f'_y(x,y_0)dx \quad (7)$$

bo`ladi.

O`rta qiymat haqidagi teoremadan foydalanib, topamiz:

$$\int_{\beta(y_0)}^{\beta(y_0+\Delta y)}f(x,y_0+\Delta y)dx=f(x',y_0+\Delta y)[\beta(y_0+\Delta y)-\beta(y_0)],$$
$$\int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)}f(x,y_0+\Delta y)dx=f(x'',y_0+\Delta y)[\alpha(y_0+\Delta y)-\alpha(y_0)].$$

Bunda x' nuqta $\beta(y_0)$, $\beta(y_0+\Delta y)$ nuqtalar orasida, x'' esa $\alpha(y_0)$, $\alpha(y_0+\Delta y)$ nuqtalar orasida joylashgan. $y\to y_0$ da limitga o`tishi bilan quyidagi tengliklarga kelamiz:

$$\lim_{\Delta y\to 0}\frac{1}{\Delta y}\int_{\beta(y_0)}^{\beta(y_0+\Delta y)}f(x,y_0+\Delta y)dx=\lim_{\Delta y\to 0}f(x',y_0+\Delta y)\frac{[\beta(y_0+\Delta y)-\beta(y_0)]}{\Delta y}=$$
$$=f(\beta(y_0),y_0)\beta(y_0),$$
$$\lim_{\Delta y\to 0}\frac{1}{\Delta y}\int_{\alpha(y_0)}^{\alpha(y_0+\Delta y)}f(x,y_0+\Delta y)dx=\lim_{\Delta y\to 0}f(x'',y_0+\Delta y)\frac{[\alpha(y_0+\Delta y)-\alpha(y_0)]}{\Delta y}= \quad (8)$$
$$=f(\alpha(y_0),y_0)\alpha(y_0).$$

YUqoridagi (6) munosabatda $\Delta y\to 0$ da limitga o`tib, (7) va (8) tengliklarni e`tiborga olib, ushbu

$$\lim_{\Delta y\to 0}\frac{J_1(y_0+\Delta y)-J_1(y_0)}{\Delta y}=\int_{\alpha(y_0)}^{\beta(y_0)}f'_y(x,y_0)dx+$$
$$+f(\beta(y_0),y_0)\beta'(y_0)-f(\alpha(y_0),y_0)\alpha(y_0)$$

tenglikka kelamiz.

Demak,

$$J'_1(y_0)=\int_{\alpha(y_0)}^{\beta(y_0)}f'_y(x,y_0)dx+f(\beta(y_0),y_0)\beta'(y_0)-f(\alpha(y_0),y_0)\alpha'(y_0). \blacktriangleright$$

Misol. Ushbu

$$J_1(y)=\int_0^1 e^x|y-x|dx$$

funksiyaning hosilasi topilsin.

◄ Aytaylik, $y\in(-\infty,0]$ bo`lsin. Bu holda

$$J_1(y) = -y\int_0^1 e^x dx + \int_0^1 xe^x dx = y(1-e) + [e-(e-1)] = (1-e)y + 1$$

bo`lib,

$$J_1'(y) = 1 - e$$

bo`ladi.

Aytaylik, $y \in (0,1)$ bo`lsin. Bu holda

$$J_1(y) = \int_0^y (y-x)e^x dx - \int_y^1 (y-x)e^x dx = \int_0^y ye^x dx -$$

$$-\int_0^y xe^x dx - \int_y^1 ye^x dx + \int_y^1 xe^x dx = 2e^y - (e+1)y - 1$$

bo`lib,

$$J_1'(y) = 2e^y - e - 1$$

bo`ladi.

Aytaylik, $y \in [1, +\infty)$ bo`lsin. Bu holda

$$J_1(y) = y\int_0^1 e^x dx - \int_0^1 xe^x dx = y(e-1) - [e-(e-1)] = (e-1)y - 1$$

bo`lib, $J_1'(y) = e - 1$ bo`ladi.

Demak,

$$J_1'(y) = \begin{cases} 1-e, & \text{agar } -\infty < y \le 0, \\ 2e^y - e - 1, & \text{agar } 0 < y < 1 \\ e-1, & \text{agar } 1 \le y < +\infty \end{cases}$$

bo`ladi. ►

14-amaliy mashg'ulot

Misol. Ushbu

$$J_1(y) = \int_0^1 e^x |y-x| dx$$

funksiyaning hosilasi topilsin.

◄ Aytaylik, $y \in (-\infty, 0]$ bo'lsin. Bu holda

$$J_1(y) = -y\int_0^1 e^x dx + \int_0^1 xe^x dx = y(1-e) + [e-(e-1)] = (1-e)y + 1$$

bo'lib,

$$J_1'(y) = 1 - e$$

bo'ladi.

Aytaylik, $y \in (0,1)$ bo'lsin. Bu holda

$$J_1(y)=\int_0^y (y-x)e^x dx-\int_y^1 (y-x)e^x dx=\int_0^y ye^x dx-$$

$$-\int_0^y xe^x dx-\int_y^1 ye^x dx+\int_y^1 xe^x dx=2e^y-(e+1)y-1$$

bo‘lib,

$$J_1'(y)=2e^y-e-1$$

bo‘ladi.

Aytaylik, $y\in[1,+\infty)$ bo‘lsin. Bu holda

$$J_1(y)=y\int_0^1 e^x dx-\int_0^1 xe^x dx=y(e-1)-[e-(e-1)]=(e-1)y-1$$

bo‘lib,

$$J_1'(y)=e-1$$

bo‘ladi.

Demak,

$$J_1'(y)=\begin{cases}1-e, & \text{агар } -\infty<y\le 0,\\ 2e^y-e-1, & \text{агар } 0<y<1\\ e-1, & \text{агар } 1\le y<+\infty\end{cases}$$

bo‘ladi. ►

Mashqlar

1. Agar $J_1(y)=\int_0^y \frac{e^x dx}{\sqrt{y-z}}$ $(0\le x\le 1)$ bo`lsa, u holda $0<y<1$ uchun

$$J_1'(y)=\frac{1}{\sqrt{y}}+\int_0^y \frac{e^x dx}{\sqrt{y-z}}$$

bo`lishini isbotlansin.

2. Agar $J_1(y)=\int_y^{1+y} \frac{dx}{1+x^2+y^2}$ bo`lsa, $\lim\limits_{y\to 0} J_1(y)$ topilsin.

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Nazorat savollari

1. $J_1(y)$ funksiyaning uzluksizligi.
2. $J_1(y)$ funksiyani differensiallash.

GLOSSARY

$J_1(y)$ **funksiyaning uzluksizligi-** Faraz qilaylik, $f(x,y)$ funksiya M_0 to`plamda uzluksiz bo`lib, $\alpha(y)$ va $\beta(y)$ funksiyalar esa $[c,d]$ segmentda uzluksiz bo`lsin. U holda

$$J_1(y)=\int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya $[c,d]$ da uzluksiz bo`ladi.

$J_1(y)$ **funksiyani differentsiallash-** Aytaylik, $f(x,y)$, $\alpha(y)$ va $\beta(y)$ funksiyalar quyidagi shartlarni bajarsin:

1) $f(x,y)$ funksiya M_0 to`plamda uzluksiz;
2) $f(x,y)$ funksiya M_0 to`plamda uzluksiz $f'_y(x,y)$ xususiy hosilaga ega;
3) $\alpha(y)$ va $\beta(y)$ funksiyalar $[c,d]$ da $\alpha'(y)$ va $\beta'(y)$ hosilalarga ega.

U holda

$$J_1(y)=\int\limits_{\alpha(y)}^{\beta(y)} f(x,y)dx$$

funksiya $[c,d]$ segmentda $J_1'(y)$ hosilaga ega bo`lib,

$$J_1'(y)=\int\limits_{\alpha(y)}^{\beta(y)} f'_y(x,y)dx+\beta'(y)f(\beta(y),y)-\alpha'(y)f(\alpha(y),y)$$

bo`ladi.

KEYS BANKI

1-keys. Masala o`rtaga tashlanadi: Agar $J_1(y)=\int_0^y \frac{e^x dx}{\sqrt{y-z}}$ $(0\le x\le 1)$ bo`lsa, u holda $0<y<1$ uchun

$$J_1'(y)=\frac{1}{\sqrt{y}}+\int_0^y \frac{e^x dx}{\sqrt{y-z}}$$

bo`lishini isbotlansin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

15-Mavzu: Parametrga bog`liq xosmas integrallar
15-Ma'ruza
REJA:

1^0. Parametrga bog`liq xosmas integral tushunchasi.
2^0. Integralning tekis yaqinlashishi.
3^0. Parametrga bog`liq xosmas integrallarning parametr bo`yicha tekis yaqinlashish alomatlari.

Tayanch so'z va iboralar: *Parametrga bog'liq xosmas integral, integralning tekis yaqinlashishi, Veyershtrass, Abel, Dirixle alomatlari.*

1^0. Parametrga bog`liq xosmas integral tushunchasi. **[4, Definition, p.415]**
Faraz qilaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in E\subset R\}$$

to`plamda berilgan bo`lsin. Bu funksiya har bir tayin $y\in E$ da x o`zgaruvchining funksiyasi sifatida $[a,+\infty)$ da integrallanuvchi, ya`ni

$$\int_a^{+\infty} f(x,y)dx$$

xosmas integral yaqinlashuvchi. Ravshanki, integralning qiymati y o`zgaruvchiga bog`liq bo`ladi:

$$F(y)=\int_a^{+\infty} f(x,y)dx\,. \tag{1}$$

Masalan, $y>1$ bo`lganda

$$\int_1^{+\infty}\frac{dx}{x^y}=\lim_{t\to\infty}\int_1^t\frac{dx}{x^y}=\lim_{t\to\infty}\frac{1}{1-y}\left(t^{1-y}-1\right)=\frac{1}{y-1}$$

bo`ladi. Demak, bu holda

$$F(y)=\frac{1}{y-1}$$

bo`ladi.

(1) integral parametrga bog`liq chegarasi cheksiz xosmas integral, y esa parametr deyiladi.

Xuddi shunga o`xshash

$$F_1(y)=\int_{-\infty}^{a} f(x,y)dx, \quad F_2(y)=\int_{-\infty}^{+\infty} f(x,y)dx$$

parametrga bog`liq xosmas integrallar tushunchalari kiritiladi.

Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : x\in[a,b),\ y\in E\subset R\}$$

to`plamda berilgan bo`lsin. Bu funksiya har bir tayin $y\in E$ da x o`zgaruvchining funksiyasi sifatida qaralganda uning uchun $b$ maxsus nuqta bo`lib, u $[a,b)$ da integrallanuvchi, ya`ni

$$\int_a^b f(x,y)dx$$

xosmas integral yaqinlashuvchi bo`lsin. Ravshanki, bu holda ham integralning qiymati $y$ o`zgaruvchiga bog`liq bo`ladi:

$$\Phi(y)=\int_a^b f(x,y)dx \ . \qquad (2)$$

Masalan, $0<y<1$ bo`lganda

$$\int_1^2 \frac{dx}{(2-x)^y}=\lim_{t\to 2-0}\int_1^t (2-x)^{-y}dx=\lim_{t\to 2-0}\frac{1}{y-1}\left[(2-t)^{1-y}-1\right]=\frac{1}{1-y}$$

bo`ladi. Demak, bu holda

$$\Phi(y)=\frac{1}{1-y}$$

bo`ladi.

(2) integral parametrga bog`liq, chegaralanmagan funksiyaning xosmas integrali, y esa parametr deyiladi.

Umumiy holda, parametrga bog`liq, chegaralanmagan funksiyaning chegarasi cheksiz integrali tushunchasi ham yuqoridagidek kiritiladi.

Parametrga bog`liq xosmas integrallarning funktsional xossalari (limiti, uzluksizligi, differentsiallanishi integrallanishi)ni

$$F(y)=\int_a^{+\infty} f(x,y)dx$$

integral uchun keltirish bilan kifoyalanamiz.

2^0. Integralning tekis yaqinlashishi. Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in E\subset R\}$$

to`plamda berilgan bo`lib, har bir tayin $y \in E$ da

$$F(y) = \int_a^{+\infty} f(x,y)dx$$

xosmas integral yaqinlashuvchi bo`lsin. Ta`rifga binoan

$$F(y) = \int_a^{+\infty} f(x,y)dx = \lim_{t \to +\infty} \int_a^t f(x,y)dx \qquad (a < t < \infty)$$

bo`ladi.

Natijada berilgan $f(x,y)$ funksiya yordamida

$$G(y,t) = \int_a^t f(x,y)dx, \qquad (a < t < \infty)$$

$$F(y) = \int_a^{+\infty} f(x,y)dx$$

funksiyalar yuzaga keladi va

$$\lim_{t \to +\infty} G(y,t) = F(y) \qquad (y \in E)$$

munosabat bajariladi.

Demak, $G(y,t)$ funksiya $t \to +\infty$ da limit funksiya $F(y)$ga ega bo`ladi.

**1-ta`rif.** **[4, (17.13-17.14), p.416]** Agar $t \to +\infty$ da $G(y,t)$ funksiya limit funksiya $F(y)$ ga $E$ to`plamda tekis yaqinlashsa,

$$F(y) = \int_a^{+\infty} f(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi deyiladi.

Integralning E to`plamda tekis yaqinlashuvchiligini quyidagicha anglash lozim:

1) har bir tayin $y \in E$ da $\int_a^{+\infty} f(x,y)dx$ xosmas integral yaqinlashuvchi;

2) $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ topiladiki, $\forall t > \delta$ va $\forall y \in E$ uchun

$$\left| \int_t^{+\infty} f(x,y)dx \right| < \varepsilon$$

tengsizligi bajariladi.

1-misol. Ushbu

$$\int_0^{+\infty} e^{-x} \cos xy dx$$

xosmas integralning $(-\infty,+\infty)$ da tekis yaqinlashuvchi ekani ko`rsatilsin.

◀ Har bir tayin $y\in(-\infty,+\infty)$ da qaralayotgan xosmas integralning yaqinlashuvchi ekanligi ravshan.

$\forall\varepsilon>0$ ga ko`ra $\delta=\ln\frac{2}{\varepsilon}$ deyilsa, unda $\forall t>\delta$ va $\forall y\in(-\infty,+\infty)$ uchun

$$\left|\int_t^{+\infty}e^{-x}\cos xy\,dx\right|\le\int_t^{+\infty}e^{-x}dx=e^{-t}\le e^{-\delta}=e^{-\ln\frac{2}{\varepsilon}}=\frac{\varepsilon}{2}<\varepsilon$$

bo`ladi. Demak, berilgan integral $(-\infty,+\infty)$ da tekis yaqinlashuvchi.▶

**2-ta`rif.** Agar $t\to+\infty$ da $G(y,t)$ funksiya limit funksiya $F(y)$ ga $E$ to`plamda tekis yaqinlashmasa,

$$F(y)=\int_a^{+\infty}f(x,y)dx$$

integral E to`plamda tekis yaqinlashmaydi deyiladi.

Integralning E to`plamda yaqinlashuvchi, ammo uning shu to`plamda tekis yaqinlashmaydi degani quyidagini anglatadi:

1) har bir tayin $y\in E$ da $\int_a^{+\infty}f(x,y)dx$ xosmas integral yaqinlashuvchi;

2) $\forall\delta>0$ olinganda ham, shunday $\varepsilon_0>0,\ y_0\in E$ va $t_1>\delta$ bo`lgan $t_1\in[a,+\infty)$ topiladiki,

$$\left|\int_{t_1}^{+\infty}f(x,y_1)dx\right|\ge\varepsilon_0$$

bo`ladi.

2-misol. Ushbu

$$\int_0^{+\infty}ye^{-xy}dx$$

xosmas integralning $(0,+\infty)$ da tekis yaqinlashmasligi ko`rsatilsin.

◀Ravshanki,

$$\int_0^{+\infty}ye^{-xy}dx=\lim_{t\to+\infty}\int_0^{t}ye^{-xy}dx=\lim_{t\to+\infty}\left(1-e^{-ty}\right)=1.$$

Demak, berilgan xosmas integral yaqinlashuvchi. Aytay-lik, $y\in E=(0,+\infty)$ bo`lsin. Ixtiyoriy musbat δ sonni olaylik. Agar $\varepsilon_0=\frac{1}{3}$, $t_0>\delta$ va $y_0=\frac{1}{t_0}$ deb olsak, u holda

$$\left| \int_{t_0}^{+\infty} y_0 e^{-xy_0} dx \right| = e^{-t_0 y_0} = e^{-1} > \frac{1}{3} = \varepsilon_0$$

bo`ladi. Bu esa $\int_0^{+\infty} ye^{-xy} dx$ integral $E = (0, +\infty)$ da tekis yaqinlashmasligini bildiradi.►

Yuqoridagi

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

parametrga bog`liq xosmas integralning parametr $y$ bo`yicha E to`plamda tekis yaqinlashishini quyidagicha ham ta`rif-lasa bo`ladi.

3-ta`rif. Agar

$$\lim_{t \to +\infty} \sup_{y \in E} \left| F(y) - \int_a^t f(x, y) dx \right| = \lim_{t \to +\infty} \sup_{y \in E} \left| \int_t^{+\infty} f(x, y) dx \right| = 0$$

$(a < t < +\infty)$ bo`lsa,

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

xosmas integral E to`plamda tekis yaqinlashuvchi deyiladi.

3-misol. Ushbu

$$F(y) = \int_1^{+\infty} \frac{dx}{x^y}$$

xosmas integralninig $E = [2, +\infty)$ to`plamda tekis yaqinlashuvchi ekani ko`rsatilsin.

◄ Ravshanki, $1 < t < +\infty$ uchun

$$0 \le \sup_{y \in [2, +\infty)} \left| \int_t^{+\infty} \frac{dx}{x^y} \right| = \sup_{y \in [2, +\infty)} \frac{1}{(y-1)t^{y-1}} \le \frac{1}{t}$$

bo`lib,

$$\lim_{t \to +\infty} \sup_{y \in [2, +\infty)} \left| \int_t^{+\infty} \frac{dx}{x^y} \right| = 0$$

bo`ladi. Demak, berilgan xosmas integral $E = [2, +\infty)$ to`plamda tekis yaqinlashuvchi.►

Endi integralning tekis yaqinlashishini ifodalovchi teoremani keltiramiz.

1-teorema. [4, Proposition 1, Cauchy criterion, p.418] Ushbu

$$F(y) = \int_a^{+\infty} f(x, y) dx$$

integralning E to`plamda tekis yaqinlashuvchi bo`lishi uchun $\forall \varepsilon > 0$ olinganda ham y ga bog`liq bo`lmagan shunday $\delta = \delta(\varepsilon) > 0$ topilib, $t' > \delta, t'' > \delta$ tengsizliklarni qanoatlantiruvchi $\forall t', t''$ va $\forall y \in E$ da

$$\left| \int_{t'}^{t''} f(x,y)dx \right| < \varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

Bu teoremaning isboti ravshan.

3^0. Parametrga bog`liq xosmas integrallarning parametr bo`yicha tekis yaqinlashish alomatlari.

2-teorema (Veyershtrass alomati). [4, Proposition 2, p.420] Aytaylik, $f(x,y)$ funksiya

$$M = \{(x,y) \in R^2 : x \in [a,+\infty), \ y \in E \subset R\}$$

to`plamda berilgan va har bir tayin $y \in E$ da $f(x,y)$ funksiya $[a,+\infty)$ da integral-lanuvchi bo`lsin.

Agar $[a,+\infty)$ da aniqlangan shunday $\varphi(x)$ funksiya topilsaki,

1) $\forall x \in [a,+\infty), \ \forall y \in E$ uchun $|f(x,y)| \le \varphi(x)$ bo`lsa,

2) ushbu $\int_a^{+\infty} \varphi(x)dx$ xosmas integral yaqinlashuvchi bo`lsa, u holda

$$F(y) = \int_a^{+\infty} f(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi bo`ladi.

◀Modomiki, $\int_a^{+\infty} \varphi(x)dx$ yaqinlashuvchi ekan, unda $\forall \varepsilon > 0$ olinganda ham, shunday $\delta = \delta(\varepsilon) > 0$ topiladiki, $t' > \delta, t'' > \delta$ bo`lganda

$$\left| \int_{t'}^{t''} \varphi(x)dx \right| < \varepsilon$$

tengsizlik bajariladi.

Ayni paytda,

$$\left| \int_{t'}^{t''} f(x,y)dx \right| \le \int_{t'}^{t''} |f(x,y)|dx \le \int_{t'}^{t''} \varphi(x)dx \qquad (t' < t'')$$

bo`lganligi sababli

$$\left|\int_{t'}^{t''} f(x,y)dx\right| < \varepsilon$$

bo`ladi. YUqorida keltirilgan 1-teoremaga muvofiq

$$F(y) = \int_a^{+\infty} f(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi bo`ladi.►

4-misol. Ushbu

$$\int_0^{+\infty} \frac{\cos xy}{1+x^2}dx \qquad (y \in E = (-\infty,+\infty))$$

integralning tekis yaqinlashuvchi ekani ko`rsatilsin.

◄Ravshanki, $\forall x \in [0,+\infty)$ va $\forall y \in (-\infty,+\infty)$ uchun

$$|f(x,y)| = \left|\frac{\cos xy}{1+x^2}\right| \le \frac{1}{1+x^2}$$

bo`ladi. Ayni paytda,

$$\int_0^{+\infty} \frac{1}{1+x^2}dx$$

xosmas integral yaqinlashuvchi bo`lganligi sababli Veyersh-trass alomatiga ko`ra berilgan integral $E = (-\infty,+\infty)$ da tekis yaqinlashuvchi bo`ladi.►

Integrallarning tekis yaqinlashishini aniqlashda ko`p foydalaniladigan Abel' hamda Dirixle alomatlarini isbot-siz keltiramiz.

3-teorema (Abel' alomati). [4, Proposition 3, Abel-Dirichlet test, p.421] $f(x,y)$ va $g(x,y)$ funksiyalar

$$M = \{(x,y) \in R^2 : x \in [a,+\infty),\ y \in E \subset R\}$$

to`plamda berilgan bo`lib, quyidagi shartlar bajarilsin:

1) har bir tayin $y \in E$ da $g(x,y)$ funksiya $[a,+\infty)$ da monoton bo`lsin;
2) $\forall (x,y) \in M$ uchun $|g(x,y)| \le c,\ (c = const)$ bo`lsin;
3) ushbu

$$\int_a^{+\infty} f(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi bo`lsin.

U holda

$$\int_a^{+\infty} f(x,y)g(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi bo`ladi.

4-teorema (Dirixle alomati). $f(x,y)$ va $g(x,y)$ funksiyalar M to`plamda berilgan bo`lib, quyidagi shartlar bajarilsin:

1) $\forall t \geq a$ hamda $\forall t \in E$ da

$$\left|\int_a^t f(x,y)dx\right| \leq c \qquad (c = const)$$

tengsizlik bajarilsin;

2) har bir tayin $y \in E$ da $g(x,y)$ funksiya limit funksiya $\varphi(x) = 0$ ga tekis yaqinlashsin.

U holda

$$\int_a^{+\infty} f(x,y)g(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi bo`ladi.

5-misol. Ushbu

$$\int_0^{+\infty} \frac{\sin xy}{x} dx \qquad (y \in E = [1,2])$$

integral tekis yaqinlashuvchilikka tekshirilsin.

◄ Berilgan integralda

$$f(x,y) = \sin xy, \quad g(x,y) = \frac{1}{x}$$

deyilsa, unda

1) $\forall t > 0,\ \forall y \in [1,2]$ uchun

$$\left|\int_0^t f(x,y)dx\right| = \left|\int_0^t \sin xy dx\right| = \left|\frac{1 - \cos ty}{y}\right| \leq 2,$$

2) $x \to +\infty$ da $g(x,y) = \dfrac{1}{x}$ funksiya $E = [1,2]$ da nolga tekis yaqinlashuvchi.

Dirixle alomatiga ko`ra berilgan integral $E = [1,2]$ da tekis yaqinlashuvchi bo`ladi.►

15-Amaliy mashg'ulot

1-misol. Ushbu

$$\int_{0}^{+\infty} e^{-x}\cos xy dx$$

xosmas integralning $(-\infty,+\infty)$ da tekis yaqinlashuvchi ekani ko`rsatilsin.

◀ Har bir tayin $y\in(-\infty,+\infty)$ da qaralayotgan xosmas integralning yaqinla-shuvchi ekanligi ravshan.

$\forall\varepsilon>0$ ga ko`ra $\delta=\ln\frac{2}{\varepsilon}$ deyilsa, unda $\forall t>\delta$ va $\forall y\in(-\infty,+\infty)$ uchun

$$\left|\int_{t}^{+\infty} e^{-x}\cos xy dx\right|\le\int_{t}^{+\infty} e^{-x}dx=e^{-t}\le e^{-\delta}=e^{-\ln\frac{2}{\varepsilon}}=\frac{\varepsilon}{2}<\varepsilon$$

bo`ladi. Demak, berilgan integral $(-\infty,+\infty)$ da tekis yaqinlashuvchi.▶

2-misol. Ushbu

$$\int_{0}^{+\infty} ye^{-xy}dx$$

xosmas integralning $(0,+\infty)$ da tekis yaqinlashmasligi ko`rsatilsin.

◀Ravshanki,

$$\int_{0}^{+\infty} ye^{-xy}dx=\lim_{t\to+\infty}\int_{0}^{t} ye^{-xy}dx=\lim_{t\to+\infty}\left(1-e^{-ty}\right)=1.$$

Demak, berilgan xosmas integral yaqinlashuvchi. Aytay-lik, $y\in E=(0,+\infty)$ bo`lsin. Ixtiyoriy musbat δ sonni olaylik. Agar $\varepsilon_0=\frac{1}{3}$, $t_0>\delta$ va $y_0=\frac{1}{t_0}$ deb olsak, u holda

$$\left|\int_{t_0}^{+\infty} y_0e^{-xy_0}dx\right|=e^{-t_0y_0}=e^{-1}>\frac{1}{3}=\varepsilon_0$$

bo`ladi. Bu esa $\int_{0}^{+\infty} ye^{-xy}dx$ integral $E=(0,+\infty)$ da tekis yaqinlashmasligini bildiradi.▶

3-misol. Ushbu

$$F(y)=\int_{1}^{+\infty}\frac{dx}{x^y}$$

xosmas integralninig $E=[2,+\infty)$ to`plamda tekis yaqinlashuvchi ekani ko`rsatilsin.

◀ Ravshanki, $1<t<+\infty$ uchun

$$0\le \sup_{y\in[2,+\infty)}\left|\int_t^{+\infty}\frac{dx}{x^y}\right|=\sup_{y\in[2,+\infty)}\frac{1}{(y-1)t^{y-1}}\le\frac{1}{t}$$

bo`lib,

$$\lim_{t\to+\infty}\sup_{y\in[2,+\infty)}\left|\int_t^{+\infty}\frac{dx}{x^y}\right|=0$$

bo`ladi. Demak, berilgan xosmas integral $E=[2,+\infty)$ to`plamda tekis yaqinlashuvchi.▶

4-misol. Ushbu

$$\int_0^{+\infty}\frac{\cos xy}{1+x^2}dx \qquad (y\in E=(-\infty,+\infty))$$

integralning tekis yaqinlashuvchi ekani ko`rsatilsin.

◀Ravshanki, $\forall x\in[0,+\infty)$ va $\forall y\in(-\infty,+\infty)$ uchun

$$|f(x,y)|=\left|\frac{\cos xy}{1+x^2}\right|\le\frac{1}{1+x^2}$$

bo`ladi. Ayni paytda,

$$\int_0^{+\infty}\frac{1}{1+x^2}dx$$

xosmas integral yaqinlashuvchi bo`lganligi sababli Veyersh-trass alomatiga ko`ra berilgan integral $E=(-\infty,+\infty)$ da tekis yaqinlashuvchi bo`ladi.▶

5-misol. Ushbu

$$\int_0^{+\infty}\frac{\sin xy}{x}dx \qquad (y\in E=[1,2])$$

integral tekis yaqinlashuvchilikka tekshirilsin.

◀ Berilgan integralda

$$f(x,y)=\sin xy,\quad g(x,y)=\frac{1}{x}$$

deyilsa, unda

1) $\forall t>0,\ \forall y\in[1,2]$ uchun

$$\left|\int_0^t f(x,y)dx\right|=\left|\int_0^t\sin xy dx\right|=\left|\frac{1-\cos ty}{y}\right|\le 2,$$

2) $x\to+\infty$ da $g(x,y)=\dfrac{1}{x}$ funksiya $E=[1,2]$ da nolga tekis yaqinlashuvchi.

Dirixle alomatiga ko`ra berilgan integral $E=[1,2]$ da tekis yaqinlashuvchi bo`ladi.▶

Mashqlar

1. Ushbu

$$\int_0^{+\infty}\frac{y\cos xy^2dx}{y+x^y}$$

integralning $E=[2,10]$ to`plamda tekis yaqinlashishi isbotlansin.

2. Ushbu

$$\int_0^{+\infty}\frac{dx}{1+x^y}\qquad (y>1)$$

integral tekis yaqinlashishga tekshirilsin.

3. Aytaylik, $f(x)$ funksiya R da uzluksiz bo`lib, $\forall x\in R$ da $f(x)\geq 0$ bo`lsin. Ushbu

$$\int_0^{+\infty}f(y-x)dx,\quad \int_{-\infty}^{0}f(y-x)dx$$

integrallarning y parametr bo`yicha ixtiyoriy chekli $[a,b]\subset R$ segmentda tekis yaqinlashuvchi bo`lishi isbotlansin.

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Nazorat savollari

1. Parametrga bog'liq xosmas integral tushunchasi.
2. Integralning tekis yaqinlashishi.
3. Veyershtrass alomati.
4. Abel alomati.
5. Dirixle alomati.

GLOSSARY

Parametrga bog`liq xosmas integral - Faraz qilaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in E\subset R\}$$

to`plamda berilgan bo`lsin. Bu funksiya har bir tayin $y\in E$ da x o`zgaruvchining funksiyasi sifatida $[a,+\infty)$ da integrallanuvchi, ya`ni

$$\int_a^{+\infty} f(x,y)dx$$

xosmas integral yaqinlashuvchi. Ravshanki, integralning qiymati y o`zgaruvchiga bog`liq bo`ladi:

$$F(y)=\int_a^{+\infty} f(x,y)dx\,. \tag{1}$$

Integralning tekis yaqinlashishi - Agar $t\to+\infty$ da $G(y,t)$ funksiya limit funksiya $F(y)$ ga E to`plamda tekis yaqinlashsa,

$$F(y)=\int_a^{+\infty} f(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi deyiladi.

Integralning to`plamda tekis yaqinlashishi - Agar

$$\lim_{t\to+\infty}\sup_{y\in E}\left|F(y)-\int_a^{t} f(x,y)dx\right|=\lim_{t\to+\infty}\sup_{y\in E}\left|\int_t^{+\infty} f(x,y)dx\right|=0$$

$(a<t<+\infty)$ bo`lsa,

$$F(y)=\int_a^{+\infty} f(x,y)dx$$

xosmas integral E to`plamda tekis yaqinlashuvchi deyiladi.

Tekis yaqinlashishning zarur va yetarli sharti - Ushbu

$$F(y)=\int_a^{+\infty} f(x,y)dx$$

integralning E to`plamda tekis yaqinlashuvchi bo`lishi uchun $\forall\varepsilon>0$ olinganda ham y ga bog`liq bo`lmagan shunday $\delta=\delta(\varepsilon)>0$ topilib, $t'>\delta, t''>\delta$ tengsizliklarni qanoatlantiruvchi $\forall t', t''$ va $\forall y\in E$ da

$$\left|\int_{t'}^{t''} f(x,y)dx\right|<\varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

Veyershtrass alomati. Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in E\subset R\}$$

to`plamda berilgan va har bir tayin $y\in E$ da $f(x,y)$ funksiya $[a,+\infty)$ da integrallanuvchi bo`lsin.

Agar $[a,+\infty)$ da aniqlangan shunday $\varphi(x)$ funksiya topilsaki,

1) $\forall x\in[a,+\infty),\ \forall y\in E$ uchun $|f(x,y)|\le\varphi(x)$ bo`lsa,

2) ushbu $\int_a^{+\infty}\varphi(x)dx$ xosmas integral yaqinlashuvchi bo`lsa, u holda

$$F(y)=\int_a^{+\infty}f(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi bo`ladi.

Abel' alomati. $f(x,y)$ va $g(x,y)$ funksiyalar

$$M=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in E\subset R\}$$

to`plamda berilgan bo`lib, quyidagi shartlar bajarilsin:

1) har bir tayin $y\in E$ da $g(x,y)$ funksiya $[a,+\infty)$ da monoton bo`lsin;

2) $\forall(x,y)\in M$ uchun $|g(x,y)|\le c,\ (c=const)$ bo`lsin;

3) ushbu

$$\int_a^{+\infty}f(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi bo`lsin.

U holda

$$\int_a^{+\infty}f(x,y)g(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi bo`ladi.

Dirixle alomati. $f(x,y)$ va $g(x,y)$ funksiyalar M to`plamda berilgan bo`lib, quyidagi shartlar bajarilsin:

1) $\forall t\ge a$ hamda $\forall t\in E$ da

$$\left|\int_a^t f(x,y)dx\right|\le c \qquad (c=const)$$

tengsizlik bajarilsin;

2) har bir tayin $y\in E$ da $g(x,y)$ funksiya limit funksiya $\varphi(x)=0$ ga tekis yaqinlashsin.

U holda

$$\int_{a}^{+\infty} f(x,y)g(x,y)dx$$

integral E to\`plamda tekis yaqinlashuvchi bo\`ladi.

KEYS BANKI

1-keys. Masala o\`rtaga tashlanadi: Aytaylik, $f(x)$ funksiya $R$ da uzluksiz bo\`lib, $\forall x \in R$ da $f(x) \geq 0$ bo\`lsin. Ushbu

$$\int_{0}^{+\infty} f(y-x)dx, \quad \int_{-\infty}^{0} f(y-x)dx$$

integrallarning y parametr bo\`yicha ixtiyoriy chekli $[a,b] \subset R$ segmentda tekis yaqinlashuvchi bo\`lishi isbotlansin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo\`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to\`plangan ma'lumotlardan foydalanib, qo\`yilgan masalani yeching (individual).

16-Mavzu: Parametrga bog`liq xosmas integrallarning funksional xossalari

16-Ma`ruza

REJA:

1^0. $F(y)$ funksiyaning limiti.
2^0. $F(y)$ funksiyaning uzluksizligi.
3^0. $F(y)$ funksiyani differentsiallash.
4^0. $F(y)$ funksiyani integrallash.

**Tayanch so`z va iboralar:** *limit nuqta, parametrga bog`liq xosmas integralning limiti, uzluksizligi, differensiallanishi hamda integrallanishi.*

1^0. $F(y)$ funksiyaning limiti. Aytaylik, $f(x,y)$ funksiya

$$M = \{(x,y) \in R^2 : x \in [a,+\infty),\ y \in E \subset R\}$$

to`plamda berilgan, $y_0 \in R$ esa $E$ to`plamning limit nuqtasi bo`lsin.

1-teorema. **[4, Proposition 4, p.423]** $f(x,y)$ funksiya qo`yidagi shartlarni bajarsin:

1) har bir tayin $y \in E$ da $f(x,y)$ funksiya x o`zgaruvchining funksiyasi sifatida $[a,+\infty)$ da uzluksiz;

2) $y \to y_0$ da $f(x,y)$ funksiya ixtiyoriy $[a,t]$ da $(a < t < \infty)$ limit funksiya $\varphi(x)$ ga tekis yaqinlashsin;

3) ushbu $F(y) = \int_a^{+\infty} f(x,y)dx$ integral E to`plamda tekis yaqinlashuvchi bo`lsin. U holda $y \to y_0$ da $F(y)$ funksiya limitga ega va

$$\lim_{y \to y_0} F(y) = \int_a^{+\infty} \varphi(x)dx$$

bo`ladi.

◄Teoremaning 1- va 2- shartlarining bajarilishidan $\varphi(x)$ funksiyaning $[a,\infty)$ da uzluksiz bo`lishini topamiz. Binobarin, $\varphi(x)$ ixtiyoriy $[a,t]$ da $(a < t < \infty)$ integrallanuvchi bo`ladi.

Modomiki,

$$F(y) = \int_a^{+\infty} f(x,y)dx$$

integral E to`plamda tekis yaqinlashuvchi ekan, unda 77-ma`ruzadagi 1-teoremaga ko`ra

$$\forall\varepsilon>0, \exists\delta=\delta(\varepsilon)>0,\ t'>\delta,\ t''>\delta,\ \forall t', t'',\ \forall y\in E:$$

$$\left|\int_{t'}^{t''} f(x,y)dx\right|<\varepsilon$$

bo`ladi.

Keyingi tengsizlikda, $y\to y_0$ da limitga o`tsak, u holda

$$\left|\int_{t'}^{t''}\varphi(x)dx\right|\le\varepsilon$$

tengsizlik hosil bo`ladi. Bundan $\varphi(x)$ funksiyaning $[a,\infty)$ da integrallanuvchiligi kelib chiqadi.

Ushbu

$$\left|\int_a^{+\infty} f(x,y)dx-\int_a^{+\infty}\varphi(x)dx\right|$$

ayirmani qaraymiz. Uning uchun quyidagi tengsizlik bajariladi:

$$\left|\int_a^{+\infty} f(x,y)dx-\int_a^{+\infty}\varphi(x)dx\right|\le\int_a^t\left|f(x,y)-\varphi(x)\right|dx+\left|\int_t^{+\infty} f(x,y)dx\right|+\left|\int_t^{+\infty}\varphi(x)dx\right|.\ (a<t<\infty)\quad(1)$$

Bu tengsizlikning o`ng tomonidagi qo`shiluvchilarni baholaymiz.

$$\int_a^{+\infty} f(x,y)dx$$

integral E to`plamda tekis yaqinlashuv-chi bo`lganligi sababli,

$$\forall\varepsilon>0, \exists\delta_1=\delta_1(\varepsilon)>0,\ \forall t>\delta_1,\ \forall y\in E$$

$$\left|\int_t^{+\infty} f(x,y)dx\right|<\frac{\varepsilon}{3}\qquad(2)$$

bo`ladi.

$$\int_a^{+\infty}\varphi(x)dx$$

integral yaqinlashuvchi bo`lganligi sababli $\forall\varepsilon>0, \exists\delta_2=\delta_2(\varepsilon)>0,\ \forall t>\delta_2:$

$$\left|\int_t^{+\infty}\varphi(x)dx\right|<\frac{\varepsilon}{3}\qquad(3)$$

bo`ladi.

Ravshanki, $\forall t > \delta_0$: $(\delta_0 = \max(\delta_1 \delta_2))$ da (2) va (3) tensizliklar bir yo`la bajariladi. Funksiya $y \to y_0$ da $f(x,y)$ $[a,t]$ da $(t > \delta_0)$ limit funksiya $\varphi(x)$ ga tekis yaqinlashuvchi bo`lganligi sababli

$$\forall \varepsilon > 0, \exists \delta' = \delta'(\varepsilon) > 0, \ |y - y_0| < \delta' \ \ \forall y \in E, \forall x \in [a,t] \ \ (a < t < \infty)$$

$$|f(x,y) - \varphi(x)| < \frac{\varepsilon}{3(t-a)} \tag{4}$$

bo`ladi.

(1), (2), (3) va (4) munosabatlardan

$$\left| \int_a^{+\infty} f(x,y)dx - \int_a^{+\infty} \varphi(x)dx \right| < \varepsilon$$

bo`lishi kelib chiqadi. Demak

$$\lim_{y \to y_0} F(y) = \lim_{y \to y_0} \int_a^{+\infty} f(x,\ y)dx = \int_a^{+\infty} \varphi(x)dx\ . \ ►$$

Keyingi tenglikni quyidagicha ham yozish mumkin

$$\lim_{y \to y_0} \int_a^{+\infty} f(x,\ y)dx = \int_a^{+\infty} \left[\lim_{y \to y_0} f(x,\ y) \right] dx\ .$$

1- misol. Ushbu

$$\lim_{y \to +0} \int_a^{+\infty} e^{-xy} \frac{\sin x}{x} dx = \int_a^{+\infty} \frac{\sin x}{x} dx$$

tenglik isbotlansin.

◄ Agar $\varphi(x) = \dfrac{\sin x}{x}$ funksiyaning $x = 0$ nuqtadagi qiymati-ni $\varphi(0) = 1$ deb olinsa, unda

$$f(x,y) = e^{-xy} \frac{\sin x}{x}$$

funksiya $\{(x,\ y) \in R^2 : x \in [0, +\infty),\ y \in [0, +\infty)\}$ to`plamda uzluksiz bo`ladi.

Ravshanki, har bir tayin $y \in [0, +\infty)$ da $f(x,\ y)$ funksiya x o`zgaruvchining funksiyasi sifatida $[0, +\infty)$ da uzluksiz bo`lib, $y \to +\infty$ da bu funksiya ixtiyoriy $[0,\ t]$ da $(0 < t < +\infty)$ $\varphi(x) = \dfrac{\sin x}{x}$ funksiyaga tekis yaqinlashadi.

Endi,

$$\int_0^{+\infty} e^{-xy} \frac{\sin x}{x} dx$$

xosmas integralni parametr y bo`yicha $[0,+\infty)$ da tekis yaqinlashuvchi bo`lishini ko`rsatamiz.

Agar avvalgi ma`ruzada keltirilgan Abel' alomatida $f(x,y)$ funksiya sifatida $\frac{\sin x}{x}$, $g(x,y)$ funksiya sifatida $e^{-xy}$ funksiyalar olinsa, ular uchun Abel' alomatining barcha shartlarining o`rinli bo`lishini ko`rsatish qiyin emas. Demak, alomatga ko`ra

$$\int_0^{+\infty} e^{-xy}\frac{\sin x}{x}dx$$

integral tekis yaqinlashuvchi.

Yuqorida keltirilgan 1-teoremaga binoan

$$\lim_{y\to+0}\int_0^{+\infty} e^{-xy}\frac{\sin x}{x}dx=\int_0^{+\infty}\left(\lim_{y\to+0} e^{-xy}\frac{\sin x}{x}\right)dx$$

bo`lib, undan

$$\lim_{y\to+0}\int_0^{+\infty} e^{-xy}\frac{\sin x}{x}dx=\int_0^{+\infty}\frac{\sin x}{x}dx$$

bo`lishi kelib chiqadi. ►

2⁰. $F(y)$ funksiyaning uzluksizligi. Aytaylik, $f(x,y)$ funksiya

$$M_0=\{(x,y)\in R^2 : x\in[a,+\infty),\ y\in[c,d]\}$$

to`plamda berilgan bo`lsin.

2-teorema. **[4, Proposition 5, p.425]** Agar $f(x,y)$ funksiya M_0 to`plamda uzluksiz bo`lib,

$$F(y)=\int_a^{+\infty} f(x,y)dx$$

integral $[c,d]$ da tekis yaqinlashuvchi bo`lsa, u holda $F(y)$ funksiya $[c,d]$ da uzluksiz bo`ladi.

◄Ixtiyoriy $y_0\in[c,d]$, $y_0+\Delta y\in[c,d]$ nuqtalarni olib, $F(y)$ funksiyaning orttirmasini topamiz:

$$\Delta F(y_0)=F(y_0+\Delta y)-F(y_0)=\int_a^{+\infty}[f(x,y_0+\Delta y)-f(x,y_0)]dx.$$

SHartga ko`ra $\int_a^{+\infty} f(x,y)dx$ integral $[c,d]$ da tekis yaqinla-shuvchi. Unda

$$\forall\varepsilon>0,\exists\delta=\delta(\varepsilon)>0,\ \forall t_0>\delta,\forall\ y\in[c,d]:$$

$$\left|\int_{t_0}^{+\infty} f(x,y)dx\right| < \frac{\varepsilon}{3} \qquad (5)$$

bo`ladi. Ravshanki, $f(x,y)$ funksiya

$$M_{t_0} = \{(x,y)\in R^2 : x\in[a,t_0],\ y\in[c,d]\}\ (a<t_0<+\infty)$$

to`plamda tekis uzluksiz bo`ladi. Unda

$$\forall\varepsilon>0, \exists\delta_1=\delta_1(\varepsilon)>0,\ \Delta y<\delta_1(\varepsilon)$$

$$|f(x,y_0+\Delta y)-f(x,y_0)| < \frac{\varepsilon}{3(t_0-a)} \qquad (6)$$

bo`ladi. Agar $\delta_0 = \max\{\delta, \delta_1\}$ deyilsa uning uchun (5) va (6) tengsizliklar bir yo`la bajariladi. (5) va (6) munosabatlarni e`tiborga olib topamiz:

$$|\Delta F(y_0)=| = \left|\int_a^{+\infty}[f(x,y_0+\Delta y)-f(x,y_0)]dx\right| \le$$

$$\le \int_a^{t_0}|f(x,y_0+\Delta y)-f(x,y_0)|dx + \left|\int_{t_0}^{+\infty} f(x,y_0+\Delta y)dx\right| + \left|\int_{t_0}^{+\infty} f(x,y_0)\right| < \varepsilon.$$

Demak,

$$\lim_{\Delta y\to 0}\Delta F(y_0)=0.$$

Bu esa $F(y)$ funksiyaning $[c,d]$ oraliqda uzluksizligini bildiradi. ►

2-misol. Ushbu

$$F(y)=\int_0^{+\infty} e^{-(x-y)^2}dx$$

integral parametr y ning uzluksiz funksiyasi bo`lishi ko`rsatilsin.

◄ Berilgan integralda

$$x-y=t$$

almashtirish bajaramiz. Unda

$$F(y)=\int_{-y}^{+\infty} e^{-t^2}dt = \int_{-y}^{0} e^{-t^2}dt + \int_0^{+\infty} e^{-t^2}dt = \int_0^{y} e^{-t^2}dt + \int_0^{+\infty} e^{-t^2}dt$$

bo`lib, bu yig`indining har bir qo`shiluvchisi $y$ ning uzluksiz funksiyasi bo`lgani uchun berilgan integral parametr y ning uzluksiz funksiyasi bo`ladi. ►

3^0. $F(y)$ funksiyani differentsiallash. Faraz qilaylik $f(x,y)$ funksiya M_0 to`plamda berilgan bo`lsin.

3-teorema. **[4, Proposition 6, p.426]** $f(x,y)$ funksiya quyidagi shartlarni qanoatlantirsin:

1) $f(x,y)$ funksiya M_0 to`plamda uzluksiz;

2) $f'_y(x,y)$ xususiy hosila mavjud va u M_0 to`plamda uzluksiz;

3) Har bir tayin $y\in[c,d]$ da

$$F(y)=\int_a^{+\infty} f(x,y)dx$$

integral yaqinlashuvchi;

4) Ushbu

$$\int_a^{+\infty} f'_y(x,y)dx$$

integral $[c,d]$ da tekis yaqinlashuvchi.

U holda $F(y)$ funksiya $[c,d]$ da $F'(y)$ hosilaga ega va

$$F'(y)=\int_a^{+\infty} f'_y(x,y)dx$$

bo`ladi.

◀ $y_0\in[c,d]$, $y_0+\Delta y\in[c,d]$ nuqtalarni olib, topamiz:

$$\frac{F(y_0+\Delta y)-F(y_0)}{\Delta y}=\int_a^{+\infty}\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}dx.$$

Lagranj teoremasiga ko`ra

$$\frac{f(x,y_0+\Delta y)-f(x,y_0)}{\Delta y}=f'_y(x,y_0+\theta\Delta y),\ (0<\theta<1),$$

$$\frac{F(y_0+\Delta y)-F(y_0)}{\Delta y}=\int_a^{+\infty} f'_y(x,y_0+\theta\Delta y)dx$$

bo`ladi. Demak,

$$\lim_{\Delta y\to 0}\frac{F(y_0+\Delta y)-F(y_0)}{\Delta y}=\lim_{\Delta y\to 0}\int_a^{+\infty} f'_y(x,y_0+\theta\Delta y)dx. \qquad (7)$$

SHartga ko`ra $f'_y(x,y)$ funksiya $M_0$ to`plamda uzluksiz. Kantor teoremasiga binoan u

$$M_t=\{(x,y)\in R^2: x\in[a,t]\ \ y\in[c,d]\}\,(a<t<\infty)$$

to`plamda tekis uzluksiz bo`ladi. Unda

$$\forall\varepsilon>0,\exists\delta=\delta(\varepsilon)>0,\ |\Delta y|<\delta(\varepsilon)\ \forall x\in[a,t]$$

$$\left|f'_y(x,y_0+\theta\Delta y)-f'_y(x,y_0)\right|<\varepsilon$$

bo`ladi. Demak $\Delta y \to 0$ da $f'_y(x, y_0 + \theta\Delta y)$ funksiya $f'_y(x, y_0)$ ga tekis yaqinlashadi. SHartga ko`ra

$$\int_a^{+\infty} f'_y(x, y_0)dx$$

integral tekis yaqinlashuvchi. Unda

$$\forall\varepsilon > 0, \exists\delta = \delta(\varepsilon) > 0,\ t' > \delta; t'' > \delta;\ \forall t', t'',\ \forall\ y \in [c, d]:$$

$$\left|\int_{t'}^{t''} f'_y(x, y)dx\right| < \varepsilon$$

jumladan,

$$\left|\int_{t'}^{t''} f'_y(x, y_0 + \theta\,\Delta y)dx\right| < \varepsilon$$

bo`ladi. Keyingi tengsizlikning bajarilishidan esa

$$\int_a^{+\infty} f'_y(x, y_0 + \theta\,\Delta y)dx$$

integralning tekis yaqinlashuviligi kelib chiqadi. Ushbu ma`ruzada keltirilgan 1-teoremani (7) tenglikning o`ng tomoniga qo`llab, topamiz.

$$\lim_{\Delta y \to 0}\int_a^{+\infty} f'_y(x, y_0 + \theta\Delta y)dx = \int_a^{+\infty}[\lim_{\Delta y \to 0} f'_y(x, y_0 + \theta\Delta y)]dx = \int_a^{+\infty} f'_y(x, y_0)dx. \quad (8)$$

(7) va (8) munosabatlardan

$$F'(y_0) = \int_a^{+\infty} f'_y(x, y_0)dx \quad (9)$$

bo`lishi kelib chiqadi.

(9) munosabatni quyidagicha ham yozish mumkin:

$$\frac{d}{dy}\int_a^{+\infty} f(x, y)dx = \int_a^{+\infty}\frac{\partial f(x, y)}{\partial y}dx.$$

Bu differentsiallash amalini integral ostiga o`tkazish qoidasini ifodalaydi. ►

4^0. $F(y)$ funksiyani integrallash. Aytaylik, $f(x, y)$ funksiya

$$M_0 = \{(x, y) \in R^2 : x \in [a, +\infty),\ y \in [c, d]\}$$

to`plamda berilgan bo`lsin.

4-teorema. **[4, Proposition 7, p.429]** Agar $f(x, y)$ funksiya M_0 to`plamda uzluksiz va $F(y) = \int_a^{+\infty} f(x, y)dx$ integral $[c, d]$ da tekis yaqinlshuvchi bo`lsa, u holda $F(y)$ funksiya $[c, d]$ da integrallanuvchi va

$$\int_c^d F(y)dy = \int_c^d [\int_a^{+\infty} f(x,y)dx]dy = \int_a^{+\infty} [\int_c^d f(x,y)dy]dx$$

bo`ladi.

◀Ravshanki, $F(y)$ funksiya $[c,d]$ da uzluksiz bo`ladi. Binobarin, u $[c,d]$ da integrallanuvchi.

SHartga ko`ra

$$F(y) = \int_a^{+\infty} f(x,y)dx$$

integral $[c,d]$ da tekis yaqinlashuvchi. Unda

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0,\ \forall t > \delta,\ \forall y \in [c,d]:$$

$$\left| \int_t^{+\infty} f(x,y)dx \right| < \varepsilon$$

bo`ladi SHu munosabatdagi t ni olib topamiz:

$$\int_a^{+\infty} f(x,y)dx = \int_a^t f(x,y)dx + \int_t^{+\infty} f(x,y)dx .$$

Natijada

$$\int_c^d [\int_a^{+\infty} f(x,y)dx]dy = \int_c^d [\int_a^t f(x,y)dx]dy + \int_c^d [\int_t^{+\infty} f(x,y)dx]dy =$$

$$= \int_a^t [\int_c^d f(x,y)dy]dx + \int_c^d [\int_t^{+\infty} f(x,y)dx]dy$$

bo`ladi.

Agar

$$\left| \int_c^d F(y) - \int_a^t [\int_c^d f(x,y)dy]dx \right| \le \int_c^d \left| \int_t^{+\infty} f(x,y)dx \right| dy < \varepsilon(d-c)$$

bo`lishini e`tiborga olsak, unda

$$\int_c^d F(y) = \lim_{t\to\infty} \int_a^t [\int_c^d f(x,y)dy]dx = \int_a^{\infty} [\int_c^d f(x,y)dy]dx$$

bo`lib,

$$\int_c^d [\int_a^{+\infty} f(x,y)dx]dy = \int_a^{+\infty} [\int_c^d f(x,y)dy]dx$$

ekanligi kelib chiqadi. ▶

3-misol. **[4, Example 13, p.427]** Ushbu

$$J=\int_0^{+\infty}\frac{\sin x}{x}dx$$

integral hisoblansin.

◀Bu xosmas integralning yaqinlashuchi bo'lishi 15-bobning 2-§ ida ko'rsatilgan edi. Endi berilgan integralni hisoblaymiz. Buning uchun quyidagi

$$J(a)=J=\int_0^{+\infty}e^{-ax}\frac{\sin x}{x}dx$$

parametrga bog'liq xosmas integralni qaraymiz.

Ravshanki,

$$f(x,a)=e^{-ax}\frac{\sin x}{x}\quad (f(0,a)=1)$$

funksiya

$$\{(x,a)\in R^2:\ \ x\in[0,+\infty),\ a\in[0,\ c],\ c>0\}$$

to'plamda uzluksiz,

$$f_a'(x,a)=-e^{-ax}\sin x$$

xususiy hosilaga ega va u ham uzluksiz funksiya. Quyidagi

$$\int_0^{+\infty}f_a'(x,a)dx=-\int_0^{+\infty}e^{-ax}\sin xdx$$

integral esa $a\geq a_0$. $(a_0>0)$ da tekis yaqinlashuvchi. 16-teoremaga ko'ra

$$J'(a)=\int_0^{+\infty}\left(e^{-ax}\frac{\sin x}{x}\right)_a' dx=-\int_0^{+\infty}e^{-ax}\sin xdx=-\frac{1}{1+a^2}$$

bo'ladi. Demak,

$$J(a)=-arctga+c.$$

$a=+\infty$ bo'lganda, $J(+\infty)=0$ bo'lib, $-\frac{\pi}{2}+c=0$ ya'ni $c=\frac{\pi}{2}$ bo'ladi.

Demak,

$$J(a)=\frac{\pi}{2}-arctga.$$

Bu tenglikda $a\to 0$ da limitga o'tib quyidagini topamiz:

$$\lim_{a\to 0}J(a)=\frac{\pi}{2}.$$

Shunday qilib, $J(0)=\frac{\pi}{2}$ ya'ni

$$J=\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$$

bo’ladi. ►

16- Amaliy mashg’ulot

1- misol. Ushbu

$$\lim_{y\to+0}\int_a^{+\infty}e^{-xy}\frac{\sin x}{x}dx=\int_a^{+\infty}\frac{\sin x}{x}dx$$

tenglik isbotlansin.

◄ Agar $\varphi(x)=\dfrac{\sin x}{x}$ funksiyaning $x=0$ nuqtadagi qiymati-ni $\varphi(0)=1$ deb olinsa, unda

$$f(x,y)=e^{-xy}\frac{\sin x}{x}$$

funksiya $\{(x,\ y)\in R^2 : x\in[0,+\infty),\ y\in[0,+\infty)\}$ to‘plamda uzluksiz bo‘ladi.

Ravshanki, har bir tayin $y\in[0,+\infty)$ da $f(x,\ y)$ funksiya x o‘zgaruvchining funksiyasi sifatida $[0,+\infty)$ da uzluksiz bo‘lib, $y\to+\infty$ da bu funksiya ixtiyoriy $[0,\ t]$ da $(0<t<+\infty)$ $\varphi(x)=\dfrac{\sin x}{x}$ funksiyaga tekis yaqinlashadi.

Endi,

$$\int_0^{+\infty}e^{-xy}\frac{\sin x}{x}dx$$

xosmas integralni parametr y bo‘yicha $[0,+\infty)$ da tekis yaqinlashuvchi bo‘lishini ko‘rsatamiz.

Agar 77-ma'ruzada keltirilgan Abel alomatida $f(x,\ y)$ funksiya sifatida $\dfrac{\sin x}{x}$, $g(x,\ y)$ funksiya sifatida e^{-xy} funksiyalar olinsa, ular uchun Abel alomati-ning barcha shartlarining o‘rinli bo‘lishini ko‘rsatish qiyin emas. Demak, alomatga ko‘ra

$$\int_0^{+\infty}e^{-xy}\frac{\sin x}{x}dx$$

integral tekis yaqinlashuvchi.

Yuqorida keltirilgan 1-teoremaga binoan

$$\lim_{y\to+0}\int_0^{+\infty} e^{-xy}\frac{\sin x}{x}dx=\int_0^{+\infty}\left(\lim_{y\to+0} e^{-xy}\frac{\sin x}{x}\right)dx$$

bo‘lib, undan

$$\lim_{y\to+0}\int_0^{+\infty} e^{-xy}\frac{\sin x}{x}dx=\int_0^{+\infty}\frac{\sin x}{x}dx$$

bo‘lishi kelib chiqadi. ►

2-misol. Ushbu

$$F(y)=\int_0^{+\infty} e^{-(x-y)^2}dx$$

integral parametr y ning uzluksiz funksiyasi bo‘lishi ko‘rsatilsin.

◄ Berilgan integralda

$$x-y=t$$

almashtirish bajaramiz. Unda

$$F(y)=\int_{-y}^{+\infty} e^{-t^2}dt=\int_{-y}^{0}e^{-t^2}dt+\int_0^{+\infty} e^{-t^2}dt=\int_0^{y}e^{-t^2}dt+\int_0^{+\infty} e^{-t^2}dt$$

bo‘lib, bu yig‘indining har bir qo‘shiluvchisi y ning uzluksiz funksiyasi bo‘lgani uchun berilgan integral parametr y ning uzluksiz funksiyasi bo‘ladi. ►

3-misol. Ushbu

$$J=\int_0^{+\infty}\frac{\sin x}{x}dx$$

integral hisoblansin.

◄Bu xosmas integralning yaqinlashuchi bo’lishi 15-bobning 2-§ ida ko’rsatilgan edi. Endi berilgan integralni hisoblaymiz. Buning uchun quyidagi

$$J(a)=J=\int_0^{+\infty} e^{-ax}\frac{\sin x}{x}dx$$

parametrga bog’liq xosmas integralni qaraymiz.

Ravshanki,

$$f(x,a)=e^{-ax}\frac{\sin x}{x}\qquad (f(0,a)=1)$$

funksiya

$$\{(x,a)\in R^2:\ \ x\in[0,+\infty),\ a\in[0,\ c],\ c>0\}$$

to’plamda uzluksiz,

$$f_a'(x,a)=-e^{-ax}\sin x$$

xususiy hosilaga ega va u ham uzluksiz funksiya. Quyidagi

$$\int_0^{+\infty} f'_a(x,a)dx = -\int_0^{+\infty} e^{-ax}\sin x dx$$

integral esa $a \geq a_0$. $(a_0 > 0)$ da tekis yaqinlashuvchi. 16-teoremaga ko'ra

$$J'(a) = \int_0^{+\infty}\left(e^{-ax}\frac{\sin x}{x}\right)'_a dx = -\int_0^{+\infty} e^{-ax}\sin x dx = -\frac{1}{1+a^2}$$

bo'ladi. Demak,

$$J(a) = -arctga + c.$$

$a = +\infty$ bo'lganda, $J(+\infty) = 0$ bo'lib, $-\frac{\pi}{2} + c = 0$ ya'ni $c = \frac{\pi}{2}$ bo'ladi.

Demak,

$$J(a) = \frac{\pi}{2} - arctga.$$

Bu tenglikda $a \to 0$ da limitga o'tib quyidagini topamiz:

$$\lim_{a\to 0} J(a) = \frac{\pi}{2}.$$

Shunday qilib, $J(0) = \frac{\pi}{2}$ ya'ni

$$J = \int_0^{+\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$$

bo'ladi.▶

Mashqlar

1. Agar $f(x)$ funksiya $(0,\infty)$ da integrallanuvchi bo`lib,

$$F(y) = \int_0^{\infty} e^{-yx} f(x)dx$$

bo`lsa,

$$\lim_{y\to+0} F(y) = \lim_{y\to+0}\int_0^{\infty} e^{-yx} f(x)dx = \int_0^{\infty} f(x)dx$$

bo`lishi isbotlansin.

2. Ushbu

$$F(y) = \int_0^{\infty} e^{-y^2x} y dx \quad (-\infty < y < +\infty)$$

funksiyani uzluksizlikka tekshirilsin.

Foydalanish uchun adabiyotlar

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Nazorat savollari

1. $F(y)$ funksiyaning limiti haqida ayting.
2. $F(y)$ funksiyaning uzluksizligi haqida gapiring.
3. $F(y)$ funksiyani differentsiallash nima?
4. $F(y)$ funksiyani integrallash haqida ayting.

GLOSSARY

Ko'p o'zgaruvchili funksiya uzluksizligi - Agar

$$\lim_{x\to x_0} f(x)=f\left(x^0\right)$$

bo'lsa, $f(x)$ funksiya x^0 nuqtada uzluksiz deyiladi.

Tekis uzluksiz - Agar $\forall\varepsilon>0$ son olinganda ham shunday $\delta=\delta(\varepsilon)>0$ son topilsaki,

$$\rho(x',x'')<\delta$$

tengsizlikni qanoatlantiruvchi ixtiyoriy $x'\in E, x''\in E$ uchun

$$|f(x'')-f(x')|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya E to'plamda tekis uzluksiz deyiladi.

KEYS BANKI

1-keys. Masala o`rtaga tashlanadi: Ushbu

$$\lim_{y\to+0}\int_a^{+\infty} e^{-xy}\frac{\sin x}{x}dx=\int_a^{+\infty}\frac{\sin x}{x}dx$$

tenglik isbotlansin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

17-Mavzu: Eyler integrallari
17-ma`ruza
REJA:

1^0. Beta funksiya va uning tekis yaqinlashuvchiligi.
2^0. $B(a,b)$ funksiyaning xossalari.
3^0. Gamma funksiya va uning yaqinlashuvchiligi.
4^0. $\Gamma(a)$ funksiyaning xossalari.
5^0. Beta va gamma funksiyalar orasidagi bog`lanish.

**Tayanch so`z va iboralar:** *Beta funksiya, tekis yaqinlashuvchilik, xossalari, Gamma funksiya, xossalari, Beta va gamma funksiyalar orasidagi bog`lanish.*

1^0. Beta funksiya va uning tekis yaqinlashuvchiligi. **[4, (17.27), p.437]**
Ushbu

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

parametrga bog`liq xosmas integral beta funksiya (I-tur eyler integrali) deyiladi va $B(a,b)$ kabi belgilanadi:

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx \quad (a>0, b>0).$$

Demak, beta funksiya

$$\{(a,b)\in R^2 : a\in(0,+\infty), b\in(0,+\infty)\}$$

to`plamda aniqlangan funksiya.

1-teorema. Ushbu

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

integral $M_0=\{(a,b)\in R^2 : a\in[a_0,+\infty), b\in[b_0,+\infty), a_0>0, b_0>0\}$ to`plamda tekis yaqinlashuvchi bo`ladi.

◄ $B(a,b)$ funksiyani ifodalovchi integralni ikki qismga

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx=\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx+\int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}dx$$

ajratib, har bir integralning tekis yaqinlashishga tekshiramiz. Parametr $a \geq a_0$ $(a_0 > 0)$, $\forall b > 0$, $\forall x \in \left(0, \frac{1}{2}\right]$ da

$$x^{a-1}(1-x)^{b-1} \leq x^{a_0-1}(1-x)^{b-1} \leq 2x^{a_0-1}$$

va $a > 0$ bo\`lganda

$$\int_0^{\frac{1}{2}} x^{a-1} dx$$

integralning yaqinlashuvchi bo\`lishidan Veyershtrass alomatiga ko\`ra

$$\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1} dx$$

integralning $a \geq a_0$, $(a_0 > 0)$ da tekis yaqinlashuvchiligini topamiz. SHuningdek, parametr $b \geq b_0$ $(b_0 > 0)$, $\forall a > 0$, $\forall x \in \left[\frac{1}{2}, 1\right)$ da

$$x^{a-1}(1-x)^{b-1} \leq x^{a-1}(1-x)^{b_0-1} \leq 2(1-x)^{b_0-1}$$

va $b > 0$ bo\`lganda

$$\int_{\frac{1}{2}}^{1} (1-x)^{b-1} dx$$

integralning yaqinlashuvchi bo\`lishidan Veyershtrass alomati-ga ko\`ra

$$\int_{\frac{1}{2}}^{1} x^{a-1}(1-x)^{b-1} dx$$

integralning $b \geq b_0$ $(b_0 > 0)$ da tekis yaqinlashuvchi bo\`lishini topamiz. Demak,

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

integral

$$M_0 = \{(a,b) \in R^2 : a \in [a_0, +\infty), b \in [b_0, +\infty), a_0 > 0, b_0 > 0\}$$

to\`plamda tekis yaqinlashuvchi bo\`ladi. ►

Natija. $B(a,b)$ funksiya

$$M = \{(a,b) \in R^2 : a \in (0, +\infty), b \in (0, +\infty)\}$$

to\`plamda uzluksiz bo\`ladi.

◄Bu tasdiq

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

integralning tekis yaqinlashuvchiligi hamda integral ostidagi funksiyaning M to`plamda uzluksiz bo`lishidan kelib chiqadi. ►

2⁰. $B(a,b)$ ***funksiyaning xossalari.*** **[4, (17.29), p.438]** endi

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

funksiyaning xossalarini keltiramiz.

1) **[4, (17.29), p.438]** $B(a,b)$ funksiya a va b argumentlariga nisbatan simmetrik funksiya, ya`ni,

$$B(a,b)=B(b,a) \ \ (a>0, b>0)$$

bo`ladi.

◄ $B(a,b)$ ni ifodalovchi integralda $x=1-t$ almashtirish bajarib topamiz:

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx=-\int_1^0 (1-t)^{a-1}t^{b-1}dt=\int_0^1 t^{b-1}(1-t)^{a-1}dt=B(b,a). \ ►$$

2) $B(a,b)$ funksiya quyidagicha ham ifoda qilinadi:

$$B(a,b)=\int_0^{+\infty}\frac{t^{a-1}}{(1+t)^{a+b}}dt. \qquad (1)$$

◄ $B(a,b)$ ni ifodalovchi integralda $x=\frac{t}{1+t}$ almashtirish bajarib topamiz:

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx=\int_0^{+\infty}\left(\frac{t}{1+t}\right)^{a-1}\left(1-\frac{t}{1+t}\right)^{b-1}\frac{dt}{(1+t)^2}=\int_0^{+\infty}\frac{t^{a-1}}{(1+t)^{a+b}}dt. \ ►$$

Agar (1) da $b=1-a$ $(0<a<1)$ deyilsa, unda

$$B(a,1-a)=\int_0^{+\infty}\frac{t^{a-1}}{1+t}dt=\frac{\pi}{\sin\pi a}$$

bo`ladi. Xususan,

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\pi$$

bo`ladi.

3) **[4, (17.30`), p.438]** $B(a,b)$ funksiya uchun ushbu

$$B(a+1,b)=\frac{a}{a+b}B(a,b) \ \ (a>0, b>0)$$

formula o`rinli bo`ladi.

◄Ravshanki,

$$B(a+1,b)=\int_0^1 x^a(1-x)^{b-1}dx.$$

Bu integralni bo`laklab integrallaymiz:

$$B(a+1,b)=\int_0^1 x^a(1-x)^{b-1}dx=-\frac{1}{b}\int_0^1 x^a d((1-x)^b)=-\frac{1}{b}x^a(1-x)^b\Big|_0^1+\frac{a}{b}\int_0^1 x^a(1-x)^b dx=$$

$$=\frac{a}{b}\int_0^1 x^{a-1}(1-x)^b=\frac{a}{b}\left[\int_0^1 x^{a-1}(1-x)^{b-1}dx-\int_0^1 x^a(1-x)^{b-1}dx\right]=\frac{a}{b}B(a,b)-\frac{a}{b}B(a+1,b).$$

Natijada

$$B(a+1,b)=\frac{a}{b}B(a,b)-\frac{a}{b}B(a+1,b) \qquad (2)$$

bo`lib, undan

$$B(a+1,b)=\frac{a}{a+b}B(a,b)$$

bo`lishi kelib chiqadi. ►

$B(a,b)$ funksiya simmetrik bo`lganligidan ushbu

$$B(a,b+1)=\frac{b}{a+b}B(a,b) \qquad (3)$$

bo`ladi.

Natija. [4, (17.32), p.438] $B(m,n)$ funksiyaga $(m\in N, n\in N)$ (2) va (3) formulalarni takror qo`llash natijasida

$$B(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!}$$

bo`lishi kelib chiqadi.

3⁰. Gamma funksiya va uning yaqinlashuvchiligi. [4, (17.28), p.437] Ushbu

$$\int_0^{+\infty} x^{a-1}e^{-x}dx$$

parametrga bog`liq xosmas integral gamma funksiya (II-tur eyler integrali) deyiladi va $\Gamma(a)$ kabi belgilanadi:

$$\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx.$$

Demak, gamma funksiya $(0,+\infty)$ da aniqlangan funksiya.

2-teorema. Ushbu

$$\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx$$

integral $[a_0, b_0]$ da $(0 < a_0 < b_0 < +\infty)$ tekis yaqinlashuvchi bo`ladi.

◄ $\Gamma(a)$ funksiyani ifodalovchi integralni ikki integral yig`indisi sifatida yozib olamiz:

$$\Gamma(a) = \int_0^1 x^{a-1}e^{-x}dx + \int_1^{+\infty} x^{a-1}e^{-x}dx .$$

So`ng ikkala integralning ixtiyoriy $[a_0, b_0]$ segmentda $(0 < a_0 < b_0 < +\infty)$ tekis yaqinlashuvchi bo`lishini ko`rsatamiz. Parametr $a \geq a_0$ $(a_0 > 0)$, $\forall x \in (0,1]$ da

$$x^{a-1}e^{-x} \leq x^{a_0-1}$$

va $a_0 > 0$ da

$$\int_0^1 x^{a_0-1}dx$$

integralning yaqinlashuvchi bo`lishidan Veyershtrass alomati-ga ko`ra

$$\int_0^1 x^{a-1}e^{-x}dx$$

integralning $a \geq a_0$ da tekis yaqinlashuvchi bo`lishi kelib chiqadi. SHuningdek, parametr $a \leq b_0$, $\forall x \in [1, +\infty)$ da

$$x^{a-1}e^{-x} \leq x^{b_0-1}e^{-x}$$

va

$$\int_1^{+\infty} x^{b_0-1}e^{-x}dx$$

integralning yaqinlashuvchi bo`lishidan yana Veyershtrass alomatiga ko`ra

$$\int_1^{+\infty} x^{a-1}e^{-x}dx$$

integralning $a \leq b_0$ da tekis yaqinlashuvchi bo`lishini topamiz. Demak,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}e^{-x}dx$$

xosmas integral $[a_0, b_0]$ da tekis yaqinlashuvchi bo`ladi. ►

Natija. $\Gamma(a)$ funksiya $(0, +\infty)$ da uzluksiz bo`ladi.

◄Bu tasdiq $\int_0^{+\infty} x^{a-1}e^{-x}dx$ integralning tekis yaqinlashuvchiligi hamda integral ostida-gi funksiyaning $M = \{(x, a) \in R^2 : x \in (0, +\infty), a \in (0, +\infty)\}$ da uzluksiz bo`li-shidan kelib chiqadi. ►

4⁰. $\Gamma(a)$ ***funksiyaning xossalari.*** 1) **[4, (17.34), p.439]** Gamma funksiya $(0,+\infty)$ da barcha tartibdagi uzluksiz hosilalarga ega va

$$\Gamma^{(n)}(a)=\int_0^{+\infty}x^{a-1}e^{-x}(\ln x)^n\,dx\quad(n=1,2,3....)$$

bo`ladi.

◀Ravshanki,

$$\Gamma(a)=\int_0^{+\infty}x^{a-1}e^{-x}dx$$

integral ostidagi $f(x,a)=x^{a-1}e^{-x}$ funksiya

$$M=\{(x,a)\in R^2:x\in(0,+\infty),a\in(0,+\infty)\}$$

to`plamda uzluksiz bo`lib, uzluksiz $f'_a(x,a)=x^{a-1}e^{-x}\ln x$ hosilaga ega bo`ladi. YUqorida aytganimizdek

$$\Gamma(a)=\int_0^1 x^{a-1}e^{-x}dx+\int_1^{+\infty}x^{a-1}e^{-x}dx$$

tenglikning o`ng tomonidagi integrallar ixtiyoriy $[a_0,b_0]$ da $(0<a_0<b_0<+\infty)$ tekis yaqinlashuvchi.

Ushbu $\int_0^1 x^{a-1}\ln x\cdot e^{-x}dx$, $\int_1^{+\infty}x^{a-1}\ln x\cdot e^{-x}dx$ integrallarni qaray-lik. Bu integrallardan birinchisi, $a\geq a_0>0$ da

$$\left|x^{a-1}\ln x\cdot e^{-x}\right|\leq x^{a_0-1}|\ln x|$$

va

$$\int_0^1 x^{a_0-1}|\ln x|dx$$

integral yaqinlashuvchi bo`lganligidan Veyershtrass alomatiga ko`ra tekis yaqinla-shuvchi bo`ladi. SHuningdek ikkinchi integral ham, $a\leq b_0<+\infty$ da

$$\left|x^{a-1}\ln x\cdot e^{-x}\right|\leq x^{b_0-1}\ln x\cdot e^{-x}=x^{b_0}\cdot e^{-x}$$

va

$$\int_1^{+\infty}x^{b_0-1}\ln x\cdot e^{-x}dx$$

integral yaqinlashuvchi bo`lganligidan yana Veyershtrass alomatiga ko`ra tekis yaqinlashuvchi bo`ladi. Parametrga bog`liq xosmas integralning parametr bo`yicha differentsiallash haqidagi teoremadan foydalanib topamiz:

$$\frac{d}{da}\Gamma(a)=\frac{d}{da}\left[\int_0^1 x^{a-1}e^{-x}dx+\int_1^{+\infty}x^{a-1}e^{-x}dx\right]=\int_0^1\frac{d}{da}\left(x^{a-1}e^{-x}\right)dx+$$

$$+\int_1^{+\infty}\frac{d}{da}\left(x^{a-1}e^{-x}\right)dx=\int_1^{+\infty}x^{a-1}\cdot\ln x\cdot e^{-x}dx.$$

Demak,

$$\Gamma'(a)=\int_0^{+\infty}x^{a-1}\cdot\ln x\cdot e^{-x}dx.$$

$\Gamma'(a)$ funksiyaning $[a_0,b_0]$ da uzluksiz bo`lishi ravshan.

Xuddi shu yo`l bilan $\Gamma(a)$ funksiyaning ikkinchi, uchinchi va hokazo tartib-dagi hosilalarining mavjudligi, uzluksiz-ligi hamda

$$\Gamma^{(n)}(a)=\int_0^{+\infty}x^{a-1}e^{-x}(\ln x)^n dx$$

bo`lishi ko`rsatiladi. ►

2) **[4, (17.35), p.440]** $\Gamma(a)$ funksiya uchun ushbu

$$\Gamma(a+1)=a\Gamma(a) \tag{4}$$

formula o`rinli bo`ladi.

◄Ravshanki,

$$\Gamma(a)=\int_0^{+\infty}x^{a-1}e^{-x}dx.$$

Bu integralni bo`laklab integrallaymiz. Natijada

$$\Gamma(a+1)=\int_0^{+\infty}x^{a}e^{-x}dx=-\int_0^{+\infty}x^{a}d(e^{-x})=-x^{a}e^{-x}\Big|_0^{+\infty}+a\int_0^{+\infty}x^{a-1}e^{-x}dx=a\Gamma(a)$$

bo`ladi. ►

Ma`lumki, $a\in(0,1]$ bo`lsa, $a+1\in(1,2]$ bo`ladi. $\Gamma(a)$ funksiyaning bu xossasini ifodalovchi (4) munosabat $\Gamma(a)$ funksiyaning $(0,1]$ dagi qiymatlariga ko`ra uning $(1,2]$ oraliqdagi qiymatlarini, umuman ixtiyoriy $(n,n+1]$ dagi qiymatla-rini topish imkonini beradi.

Natija. [4, (17.36), p.440] $\Gamma(n)$ funksiyaga $(n\in N)$ (4) formulani takror qo`llash natijasida $(\Gamma(1)=1)$

$$\Gamma(n)=(n-1)!$$

bo`lishi kelib chiqadi.

3) $\Gamma(a)$ funksiyaning o`zgarish xarakteri. Ravshanki,

$$\Gamma(1)=\int_0^{+\infty} e^{-x}dx=1.$$

YUqoridagi (4) formulaga ko`ra

$$\Gamma(2)=1\cdot\Gamma(1)=1$$

bo`ladi. Roll' teoremasiga muvofiq, shunday a_0 $(1<a_0<2)$ nuqta topiladiki,

$$\Gamma'(a_0)=0$$

bo`ladi. Ayni paytda, $\forall a\in(0,+\infty)$ da

$$\Gamma''(a)=\int_0^{+\infty} x^{a-1}e^{-x}\ln^2 x dx>0$$

bo`lganligi uchun $\Gamma'(a)$ funksiya $(0,+\infty)$da qat`iy o`suvchi bo`ladi. Binobarin, $\Gamma'(a)$ funksiya a_0 nuqtadan boshqa nuqtalarda nolga aylanmaydi. Demak,

$$\Gamma'(a)=\int_0^{+\infty} x^{a-1}e^{-x}\ln x dx=0$$

tenglama $(0,+\infty)$ oraliqda yagona echimga ega. Unda,

$$0<a<a_0 \text{ da } \Gamma'(a)<0,\quad a_0<a<+\infty \text{ da } \Gamma'(a)>0$$

bo`lib, $\Gamma(a)$funksiya $a_0$ nuqtada minimumga ega bo`ladi. ($a_0=1{,}4616...$, $\Gamma(a_0)=\min\Gamma(a)=0{,}8856...$bo`lishi topilgan).

$\Gamma(a)$ funksiya $a>a_0$ da o`suvchi bo`lganligi sababli $a>n+1$ bo`lganda $\Gamma(a)>\Gamma(n+1)=n!$ bo`lib, undan

$$\lim_{a\to+\infty}\Gamma(a)=+\infty$$

bo`lishi kelib chiqadi.

Agar $a\to+0$ da $\Gamma(a+1)\to\Gamma(1)=1$ hamda

$$\Gamma(a)=\frac{\Gamma(a+1)}{a}$$

bo`lishini e`tiborga olsak, unda

$$\lim_{a\to+0}\Gamma(a)=+\infty$$

ekanligini topamiz.

***5⁰. Beta va gamma funksiyalar orasidagi bog`lanish.*** Beta va gamma funksiyalar orasidagi bog`lanishni quyidagi teorema ifodalaydi.

3-teorema. $\forall(a,b)\in\{(a,b)\in R^2 : a\in(0,+\infty), b\in(0,+\infty)\}$ uchun

$$B(a,b)=\frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)} \quad (5)$$

formula o`rinli bo`ladi.

◀Ushbu

$$\Gamma(s)=\int_0^{+\infty} x^{s-1}e^{-x}dx$$

integralda $x=(1+u)t$, $(t>0)$ almashtirish bajarib, s ni $a+b$ ga almashtiramiz. Natijada

$$\Gamma(a+b)=\int_0^{+\infty}(1+u)^{a+b-1}t^{a+b-1}e^{-(1+u)t}(1+u)dt$$

bo`lib,

$$\frac{\Gamma(a+b)}{(1+u)^{a+b}}=\int_0^{+\infty}t^{a+b-1}e^{-(1+u)t}dt$$

bo`ladi.

Endi bu tenglikning har ikki tomonini u^{a-1} ga ko`paytirib, so`ng $(0,+\infty)$ oraliq bo`yicha integrallab topamiz:

$$\Gamma(a+b)\int_0^{+\infty}\frac{u^{a-1}}{(1+u)^{a+b}}du=\int_0^{+\infty}\left[\int_0^{+\infty}t^{a+b-1}e^{-(1+u)t}dt\right]u^{a-1}du$$

ya`ni,

$$\Gamma(a+b)\cdot B(a,b)=\int_0^{+\infty}\left[\int_0^{+\infty}t^{a+b-1}e^{-(1+u)t}dt\right]u^{a-1}du\,.$$

[1], 17-bob, 8-§da keltirilgan teoremadan foydalanib, keyingi tenglikning o`ng tomonidagi integrallarning o`rinlarini almashtiramiz. Natijada

$$\Gamma(a+b)\cdot B(a,b)=\int_0^{+\infty}\left[\int_0^{+\infty}u^{a-1}e^{-(1+u)t}du\right]t^{a+b-1}dt$$

bo`ladi. Integralda $ut=y$ almashtirish bajarib topamiz:

$$\Gamma(a+b)\cdot B(a,b)=\int_0^{+\infty}\left[\int_0^{+\infty}y^{a-1}t^{b-1}e^{-t}e^{-y}dy\right]dt=\int_0^{+\infty}t^{b-1}e^{-t}dt\cdot\int_0^{+\infty}y^{a-1}e^{-y}dy=\Gamma(b)\cdot\Gamma(a).$$

Demak,

$$B(a,b)=\frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}\,.\ ▶$$

Natija. **[4, The Complement Formula (17.40), p.441]** $\forall a\in(0,1)$ uchun

$$\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin a\pi} \qquad (6)$$

bo`ladi.

◀ (5) tenglikda $b=1-a$ $(0<a<1)$ deb olinsa, unda

$$B(a,1-a)=\frac{\Gamma(a)\cdot\Gamma(1-a)}{\Gamma(1)}$$

bo`ladi. Ma`lumki,

$$B(a,1-a)=\frac{\pi}{\sin a\pi},\ \Gamma(1)=1.$$

Demak,

$$\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin a\pi},\ (0<a<1).\ ►$$

Agar (6) formulada $a=\frac{1}{2}$ deyilsa, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ bo`lishi kelib chiqadi.

1-misol. 1. Ushbu $\int_0^{+\infty} e^{-x^2}dx$ integral hisoblansin.

◄Bu integralda $x^2=t$ almashtirish bajaramiz. Unda

$$dx=\frac{1}{2\sqrt{t}}dt=\frac{1}{2}t^{-\frac{1}{2}}dt$$

bo`lib,

$$\int_0^{+\infty} e^{-x^2}dx=\frac{1}{2}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt=\frac{1}{2}\int_0^{+\infty} t^{\frac{1}{2}-1}e^{-t}dt=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$$

bo`ladi. ►

2-misol. Ushbu $\int_0^{+\infty}\frac{dx}{1+x^3}$ integral hisoblansin.

◄Bu integralda $1+x^3=\frac{1}{y}$ almashtirish bajaramiz. Unda

$$x=\left(\frac{1-y}{y}\right)^{\frac{1}{3}},\ dx=\frac{1}{3}\left(\frac{1-y}{y}\right)^{-\frac{2}{3}}\cdot\left(-\frac{dy}{y^2}\right),$$

$$\int_0^{+\infty}\frac{dx}{1+x^3}=-\frac{1}{3}\int_1^0 y\frac{1}{3}\left(\frac{1-y}{y}\right)^{-\frac{2}{3}}\cdot\frac{dy}{y^2}=\frac{1}{3}\int_0^1 y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}dy=$$

$$=\frac{1}{3}B(\frac{2}{3},\frac{1}{3})=\frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}=\frac{1}{3}\cdot\frac{\pi}{\sin\frac{1}{3}\pi}=\frac{\pi}{3\cdot\frac{\sqrt{3}}{2}}=\frac{2\pi}{3\sqrt{3}}$$

bo`ladi. ►

17-Amaliy mashg'ulotlar

1-misol. Ushbu

$$\int_{0}^{+\infty} e^{-x^2}dx$$

integral hisoblansin.

◀Bu integralda $x^2 = t$ almashtirish bajaramiz. Unda

$$dx = \frac{1}{2\sqrt{t}}dt = \frac{1}{2}t^{-\frac{1}{2}}dt$$

bo'lib,

$$\int_{0}^{+\infty} e^{-x^2}dx = \frac{1}{2}\int_{0}^{+\infty} t^{-\frac{1}{2}}e^{-t}dt = \frac{1}{2}\int_{0}^{+\infty} t^{\frac{1}{2}-1}e^{-t}dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

bo'ladi. ▶

2-misol. Ushbu

$$\int_{0}^{+\infty} \frac{dx}{1+x^3}$$

integral hisoblansin.

◀Bu integralda

$$1+x^3 = \frac{1}{y}$$

almashtirish bajaramiz. Unda

$$x = \left(\frac{1-y}{y}\right)^{\frac{1}{3}},\ dx = \frac{1}{3}\left(\frac{1-y}{y}\right)^{-\frac{2}{3}} \cdot \left(-\frac{dy}{y^2}\right),$$

$$\int_{0}^{+\infty} \frac{dx}{1+x^3} = -\frac{1}{3}\int_{1}^{0} y\frac{1}{3}\left(\frac{1-y}{y}\right)^{-\frac{2}{3}} \cdot \frac{dy}{y^2} = \frac{1}{3}\int_{0}^{1} y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}dy =$$

$$= \frac{1}{3}B(\frac{2}{3},\frac{1}{3}) = \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)} = \frac{1}{3}\cdot\frac{\pi}{\sin\frac{1}{3}\pi} = \frac{\pi}{3\cdot\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}}$$

bo'ladi. ▶

Mashqlar

1. $\forall a \in R \setminus \{0,-1,...,-n,...\}$ da

$$\Gamma(a) = \lim_{n\to\infty} \frac{n^a \cdot n!}{a(a+1)...(a+n)}$$

bo`lishi isbotlansin.

2. Ushbu

$$\int_0^{+\infty} \frac{x^m dx}{(a+bx^n)^p} \quad (a>0, b>0, n>0)$$

integral beta funksiya orqali ifodalansin.

3. Quyidagi

$$\left(\Gamma'(a)\right)^2 < \Gamma(a)\cdot\Gamma''(a)$$

tengsizlik isbotlansin.

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Nazorat savollar

1.Beta funksiya va uning tekis yaqinlashuvchiligi.

2. $B(a,b)$ funksiyaning xossalari.

3. Gamma funksiya va uning yaqinlashuvchiligi.

4. $\Gamma(a)$ funksiyaning xossalari.

5. Beta va gamma funksiyalar orasidagi bog`lanish.

G L O S S A R Y

I-tur Eyler integrali- Ushbu

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

parametrga bog‘liq xosmas integral beta funksiya (I-tur Eyler integrali) deyiladi va $B(a,b)$ kabi belgilanadi:

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx \quad (a>0, b>0).$$

II-tur Eyler integrali - Ushbu

$$\int_0^{+\infty} x^{a-1}e^{-x}dx$$

parametrga bog‘liq xosmas integral gamma funksiya (II -tur Eyler integrali) deyila-di va $\Gamma(a)$ kabi belgilanadi:

$$\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx.$$

K E Y S   B A N K I

1-keys. Masala o‘rtaga tashlanadi: Ushbu

$$\int_0^{+\infty}\frac{dx}{x^2+a}=\frac{\pi}{2\sqrt{a}}$$

$(a>0)$ tenglikdan foydalanib,

$$\int_0^{+\infty}\frac{dx}{\left(x^2+a\right)^{n+1}}, \quad (n\in N)$$

integral hisoblansin.

Keysni bajarish bosqichlari va topshiriqlar:

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

Ba`zi xosmas integrallarni hisoblash

REJA:

1^0. $\int\limits_0^{+\infty}\frac{\sin x}{x}dx$ integralni hisoblash.

2^0. $\int\limits_0^{+\infty}\frac{\sin yx}{x}dx$ integralni hisoblash.

3^0. $\int\limits_0^{+\infty}\frac{x^{a-1}}{1+x}dx$ integralni hisoblash.

4^0. $\int\limits_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx$ integralni hisoblash.

5^0. Ba`zi xosmas integrallarning qiymatlari.

**Tayanch so`z va iboralar:** *Parametrga bog`liq integrallar, funksional xossalai, ba`zi xosmas integrallar.*

1^0. $\int\limits_0^{+\infty}\frac{\sin x}{x}dx$ integralni hisoblash. **[4, Example 13, p.427]** Bu integralning yaqinlashuvchiligi avvalgi ma`ruzada keltirilgan.

Ma`lumki,

$$\int\limits_0^{+\infty} e^{-xy}\sin x dx=\frac{1}{1+y^2}. \qquad (1)$$

Bu tenglikdagi

$$F(y)=\int\limits_0^{+\infty} e^{-xy}\sin x dx$$

parametrga bog`liq integral parametr $y$ bo`yicha ixtiyoriy $[t, A]$ da $(t>0)$ tekis yaqinlashuvchi bo`ladi. Bu tasdiq

$$\left|e^{-xy}\sin x\right|<e^{-ty},\quad \int\limits_0^{+\infty} e^{-ty}dx=\frac{1}{t}$$

bo`lishi hamda Veyershtrass alomatini qo`llashdan kelib chiqadi. (1) tenglikni integrallab topamiz:

$$\int\limits_t^A\left[\int\limits_0^{+\infty} e^{-xy}\sin x dx\right]dy=\int\limits_t^A\frac{1}{1+y^2}dy=arctgA-arctgt.$$

Bu tenglikni chap tomonidagi integral uchun

$$\int\limits_t^A\left[\int\limits_0^{+\infty} e^{-xy}\sin x dx\right]dy=\int\limits_0^{+\infty}\left[\int\limits_t^A e^{-xy}\sin x dy\right]dx=\int\limits_0^{+\infty}\frac{e^{-tx}-e^{-Ax}}{x}\sin x dx$$

va $\forall x \geq 0$ da $|\sin x| \leq x$ bo`lib,

$$\left|\int_0^{+\infty} \frac{e^{-Ax}}{x}\sin x dx\right| \leq \int_0^{+\infty} e^{-Ax}dx = \frac{1}{A}$$

bo`ladi. Natijada $A \to +\infty$ da

$$\frac{\pi}{2} - arctgt = \int_0^{+\infty} e^{-tx}\frac{\sin x}{x}dx \qquad (2)$$

bo`lishi kelib chiqadi. **[4, p.428]**

Endi

$$\lim_{t\to+0}\int_0^{+\infty} e^{-ty}\frac{\sin x}{x}dx = \int_0^{+\infty}\frac{\sin x}{x}dx$$

tenglikni o`rinli ekanini (qaralsin, avvalgi ma`ruza) e`tiborga olib (2) da $t \to +0$ da limitga o`tib topamiz:

$$\int_0^{+\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}.$$

2^0. $\int_0^{+\infty}\frac{\sin yx}{x}dx$ integralni hisoblash. Bu integralning $(-\infty,+\infty)$ da yaqinlashuvchi bo`lishi ravshan. Aytaylik, $y > 0$ bo`lsin. Bu holda integralda $yx = t$ almashtirish bajarib topamiz:

$$\int_0^{+\infty}\frac{\sin yx}{x}dx = \int_0^{+\infty}\frac{\sin t}{t}dt = \frac{\pi}{2}.$$

Aytaylik, $y < 0$ bo`lsin. Bu holda qaralayotgan integralda $yx = -t$ almashtirish bajarib topamiz:

$$\int_0^{+\infty}\frac{\sin yx}{x}dx = -\int_0^{+\infty}\frac{\sin t}{t}dt = -\frac{\pi}{2}.$$

Aytaylik, $y = 0$ bo`lsin. Bu holda

$$\int_0^{+\infty}\frac{\sin 0x}{x}dx = 0$$

bo`ladi. Demak,

$$\int_0^{+\infty}\frac{\sin yx}{x}dx = \begin{cases} \frac{\pi}{2},\ agar\ y > 0 \\ 0,\ agar\ y = 0 \\ -\frac{\pi}{2}, agar\, y < 0 \end{cases}$$

ya`ni,

$$\int_0^{+\infty}\frac{\sin yx}{x}dx = \frac{\pi}{2}signy$$

bo`ladi.

3⁰. $\int_0^{+\infty} \frac{x^{a-1}}{1+x}dx$ ***integralni hisoblash.*** Avvalo bu parametrga bog`liq xosmas integralni yaqinlashuvchilikka tekshiramiz. Uning uchun berilgan integralni quyidagicha yozib olamiz:

$$F(a) = \int_0^{+\infty} \frac{x^{a-1}}{1+x}dx = \int_0^{1} \frac{x^{a-1}}{1+x}dx + \int_1^{+\infty} \frac{x^{a-1}}{1+x}dx. \qquad (3)$$

Aytaylik, $0 < x < 1$ bo`lsin. Bu holda

$$\frac{x^{a-1}}{1+x} < x^{a-1}$$

bo`lib, $a > 0$ da ushbu

$$\int_0^1 x^{a-1}dx$$

integralning yaqinlashuvchi bo`lganligidan, $a > 0$ da

$$\int_0^1 \frac{x^{a-1}}{1+x}dx$$

integralning ham yaqinlashuvchi bo`lishi kelib chiqadi.

Aytaylik, $x \geq 1$ bo`lsin. Bu holda

$$\frac{x^{a-1}}{1+x} < x^{a-2}$$

bo`lib, $a < 1$ da ushbu

$$\int_1^{+\infty} x^{a-2}dx$$

integralning yaqinlashuvchi bo`lganligidan, $a < 1$da

$$\int_1^{+\infty} \frac{x^{a-1}}{1+x}dx$$

integralning ham yaqinlashuvchi bo`lishi kelib chiqadi.

Demak, qaralayotgan

$$F(a) = \int_0^{+\infty} \frac{x^{a-1}}{1+x}dx$$

integral $0 < a < 1$ da yaqinlashuvchi bo`ladi.

Endi $F(a)$ integralni hisoblaymiz. Ma`lumki, $0 < x < 1$ da

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k.$$

Bu tenglikdan

$$\frac{x^{a-1}}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^{a+k-1}$$

bo`lishini topamiz. Tenglikning o`ng tomonidagi qator $[a_0, b_0]$ da $(0 < a_0 \le x \le b_0 < 1)$ tekis yaqinlashuvchi bo`lib, uning qismiy yig`indisi

$$S_n(x) = \sum_{k=0}^{n-1} (-1)^k x^{a+k-1} = \frac{x^{a-1}\left(1-(-x)^n\right)}{1+x}$$

bo`ladi.

Agar $\forall n \in N,\ \forall x \in (0,\ 1)$ uchun

$$\frac{x^{a-1}\left(1-(-x)^n\right)}{1+x} < x^{a-1}$$

tengsizlikni hamda $\int_0^1 x^{a-1}dx \ \ (0 < a < 1)$ integralning yaqinlashuvchanligini e`tiborga olsak, unda Veyershtrass alomatiga ko`ra

$$\int_0^1 S_n(x)dx$$

tekis yaqinlashuvchi bo`ladi.

Demak,

$$\lim_{n\to\infty}\int_0^1 S_n(x)dx = \int_0^1 \left(\lim_{n\to\infty} S_n(x)\right)dx$$

ya`ni,

$$\lim_{n\to\infty}\int_0^1 \left(\sum_{k=0}^{n-1}(-1)^k x^{a+k-1}\right)dx = \int_0^1 \left(\lim_{n\to\infty}\sum_{k=0}^{n-1}(-1)^k x^{a+k-1}\right)dx = \int_0^1 \frac{x^{a-1}}{1+x}dx$$

bo`ladi.

Demak,

$$\int_0^1 \frac{x^{a-1}}{1+x}dx = \lim_{n\to\infty}\int_0^1 \left(\sum_{k=0}^{n-1}(-1)^k x^{a+k-1}\right)dx = \sum_{k=0}^{n-1}\int_0^1 \left((-1)^k x^{a+k-1}\right)dx = \sum_{k=0}^{n-1}\frac{(-1)^k}{a+k}. \qquad (4)$$

Endi

$$\int_1^{+\infty}\frac{x^{a-1}}{1+x}dx$$

integralda $x = \frac{1}{t}$ almashtirish bajarsak, unda

$$\int_1^{+\infty}\frac{x^{a-1}}{1+x}dx = \int_0^1 \frac{t^{-a}}{1+t}dt = \int_0^1 \frac{t^{(1-a)-1}}{1+t}dt$$

bo`lib, yuqoridagi (4) munosabatga ko`ra

$$\int_1^{+\infty}\frac{x^{a-1}}{1+x}dx = \sum_{k=1}^{\infty}\frac{(-1)^k}{a-k} \qquad (5)$$

bo`ladi. (3), (4) va (5) munosabatlardan

$$\int_{1}^{+\infty}\frac{x^{a-1}}{1+x}dx=\frac{1}{a}+\sum_{k=1}^{\infty}(-1)^k\left[\frac{1}{a+k}+\frac{1}{a-k}\right]$$

bo`lishi kelib chiqadi.

Ma`lumki,

$$\frac{1}{a}+\sum_{k=1}^{\infty}(-1)^k\left[\frac{1}{a+k}+\frac{1}{a-k}\right]=\frac{\pi}{\sin a\pi}\quad(0<a<1)$$

(qaralsin, 76-ma`ruza).

Demak,

$$\int_{0}^{+\infty}\frac{x^{a-1}}{1+x}dx=\frac{\pi}{\sin a\pi}\quad(0<a<1)$$

bo`ladi.

4^0. $\int_{0}^{+\infty}\frac{f(ax)-f(bx)}{x}dx$ ***integralni hisoblash.*** Bunda $f(x)$ funksiya $[0,+\infty)$ da uzluksiz, istalgan $A>0$ da $\int_{A}^{+\infty}\frac{f(x)}{x}dx$ integral yaqinlashuvchi va $a>0$, $b>0$.

Berilgan integralni quyidagi ikkita integralning limiti deb qaraymiz.

$$\int_{0}^{+\infty}\frac{f(ax)-f(bx)}{x}dx=\lim_{\delta\to0}\left[\int_{\delta}^{+\infty}\frac{f(ax)}{x}dx-\int_{\delta}^{+\infty}\frac{f(bx)}{x}dx\right].$$

Bu tenglikning o`ng tomonidagi birinchi integralda $ax=t$, ikkinchi integralda $bx=t$ almashtirishlarni bajarib topamiz:

$$\int_{0}^{+\infty}\frac{f(ax)-f(bx)}{x}dx=\lim_{\delta\to0}\int_{a\delta}^{+\infty}\frac{f(t)}{t}dt-\lim_{\delta\to0}\int_{b\delta}^{+\infty}\frac{f(t)}{t}dt=\lim_{\delta\to0}\int_{a\delta}^{b\delta}\frac{f(t)}{t}dt.$$

Ravshanki, $f(t)$ uzluksiz funksiya, $\frac{1}{t}$ funksiya esa ishora saqlaydi (chunki $a>0$, $b>0$, $x\in(0,+\infty)$). Demak,

$$\int_{a\delta}^{b\delta}\frac{f(t)}{t}dt$$

integralda o`rta qiymat haqidagi teoremani qo`llash mumkin:

$$\int_{a\delta}^{b\delta}\frac{f(t)}{t}dt=f(\xi)\int_{a\delta}^{b\delta}\frac{dt}{t}.$$

Natijada,

$$\lim_{\delta\to0}\int_{a\delta}^{b\delta}\frac{f(t)}{t}dt=\lim_{\delta\to0}f(\xi)\int_{a\delta}^{b\delta}\frac{1}{t}dt=\lim_{\delta\to0}f(\xi)\ln\frac{b}{a}\qquad(7)$$

bo`ladi. Modomiki, ξ nuqta $a\delta$ bilan $b\xi$ orasida ekan, $\delta\to0$ da $\xi\to0$ va

$$\lim_{\xi\to0}f(\xi)=f(0)\qquad(8)$$

bo`ladi.

(6), (7) va (8) munosabatlardan

$$\int_0^{+\infty} \frac{f(ax)-f(bx)}{x}dx = f(0)\ln\frac{b}{a}$$

bo`lishi kelib chiqadi.

***5⁰. Ba`zi xosmas integrallarning qiymatlari.*** **[4, Example 17, p.433]** Quyida ba`zi xosmas integrallarning qiymatlarini keltiramiz:

1. $\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, 2. $\int_0^{+\infty} \sin x^2 dx = \frac{\sqrt{\frac{\pi}{2}}}{2}$, 3. $\int_0^{+\infty} \cos x^2 dx = \frac{\sqrt{\frac{\pi}{2}}}{2}$,

4. $\int_0^{+\infty} \frac{\cos xy}{1+x^2}dx = \frac{\pi}{2}e^{-|y|}$, 5. $\int_0^{+\infty} \frac{x\sin xy}{1+x^2}dx = \frac{\pi}{2} sign\, y e^{-|y|}$,

6. $\int_0^{+\infty} \frac{\sin x^2}{x}dx = \frac{\pi}{4}$, 7. $\int_0^{+\infty} \frac{\sin^4 ax - \sin^4 bx}{x}dx = \frac{3}{8}\ln\frac{a}{b}$,

8. $\int_0^{+\infty} \frac{e^{-ax}-e^{-bx}}{x}\sin nx dx = arctg\frac{b}{n} - arctg\frac{a}{n} \quad (a>0,\ b>0,\ n\neq 0)$,

9. $\int_0^{+\infty} \frac{e^{-ax}-e^{-bx}}{x}\cos nx dx = \frac{1}{2}\ln\frac{b^2+n^2}{a^2+n^2}$.

Mashqlar

1. Ushbu

$$\int_0^{+\infty} \frac{1-\cos xy}{x}e^{-kx} = \frac{1}{2}\ln\left(1+\frac{y^2}{k^2}\right) (y>0, k>0)$$

tenglik isbotlansin.

2. Ushbu

$$\int_0^{+\infty} \frac{\ln(1+a^2x^2)}{b^2+x^2}dx, \quad (a>0, b>0)$$

integral hisoblansin.

3. Ushbu

$$\int_0^{+\infty} \frac{dx}{x^2+a} = \frac{\pi}{2\sqrt{a}}$$

$(a>0)$ tenglikdan foydalanib,

$$\int_0^{+\infty} \frac{dx}{(x^2+a)^{n+1}}, \quad (n\in N)$$

integral hisoblansin.