A.B. HASANOV

SHTURM-LIUVILL CHEGARAVIY MASALALARI NAZARIYASIGA KIRISH

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Monografiyada Shturm-Liuvill chegaraviy masalasi spektral nazariyasining to‘g‘ri va teskari masalalariga oid muammolar bayon etilgan. Monografiyaning asosiy maqsadi – Oliy o‘quv yurtlarida matematika, tatbiqiy matematika va informatika, mexanika va fizika bakalavr yo‘nalishlari bo‘yicha tahsil olayotgan talabalarda spektral analizning to‘g‘ri va teskari masalalariga bo‘lgan qiziqishini oshirishdan iborat. Monografiyadan matematik tahlil, differensial tenglamalar, matematik fizika va nazariy fizika mutaxassisliklari bo‘yicha ta’lim olayotgan magistrantlar, doktorantlar va ilmiy tadqiqotchilar ham foydalanishlari mumkin.

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Boris Moiseevich Levitan Ukrainaning Berdyansk shahrida, 1914 yilning 7 iyunida tavallud topgan. O'rta maktab yettinchi sinfini muvoffaqiyatli tugatganidan so'ng, Xarkov universitetining ikkinchi kursiga o'qishga qabul qilinib, uni 1936 yilda tamomlagan. U, 1940 yilda doktorlik dissertatsiyasini himoya qilgan va 1941 yilda professor unvonini olishga sazovor bo'lgan. Asosiy ilmiy ishlari deyarli davriy funksiyalar nazariyasi, umumlashgan siljitish operatorlari nazariyasi va differensial operatorlarning spektral nazariyasi hamda uning tatbiqlariga bag'ishlangan. Eng mashhur bo'lgan ishlaridan biri, 1951 yilda, I.M.Gelfand bilan hamkorlikda yozilgan, ikkinchi tartibli differensial tenglamani spektral funksiyasi bo'yicha tiklashga oid ilmiy ishidir. Ilmiy faoliyati, oldiniga Xarkov universiteti (1938-1941), so'ngra Moskvadagi F.E.Dzerjinskiy Artilleriya Akademiyasi (1944-1961), keyinchalik esa M.V.Lomonosov nomidagi Moskva davlat universiteti bilan bog'liq. Ilmiy faoliyati davomida 170 dan ortiq maqolasi va 8 ta monografiyasi chop qilingan. 1962 yilda Davlat mukofoti bilan taqdirlangan. 2004 yilning 4 aprelida AQSHning Minneapolis shahrida vafot etgan. Uning matematik g'oyalari mangulikka daxldor.

Ushbu kitobni ustozim B.M.Levitanning porloq xotirasiga bag'ishlayman.

SO'Z BOSHI

Mazkur kitobda chekli va cheksiz oraliqlarda berilgan Shturm-Liuvill operatori uchun to'g'ri va teskari spektral masalalarni yechish usullari keltirilgan. Matematik fizikaning bir qator masalalari Shturm-Liuvill operatorining xos qiymatlarini va ortonormallangan xos funksiyalarini topishga keltiriladi. Jumladan, klassik matematik fizikaning asosiy tenglamalari hisoblangan tor tebranishi va issiqlik o'tkazuvchanlik tenglamalarini Furye usuli bilan yechishda Shturm-Liuvill chegaraviy masalasining xos qiymatlarini, ortonormallangan xos funksiyalarini aniqlashga va ixtiyoriy funksiyani ular yordamida Furye qatoriga yoyishga to'g'ri keladi. Bu yo'nalishdagi ilk natijalar D.Bernulli, J.Dalamber, L.Eyler, J.Liuvill va C.Shturmlar tomonidan olingan. Shturm-Liuvill operatori spektral nazariyasining asosiy g'oyalari XX asrda G.D.Birkgof, G.Veyl, D.Gilbert, V.A.Steklov, E.Ch.Titchmarsh, N.Levinson, B.M.Levitan va boshqa olimlar tomonidan rivojlantirilgan.

Shturm-Liuvill chegaraviy masalasining $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlari va $\{\alpha_n\}_{n=0}^{\infty}$ normallovchi o'zgarmaslariga yoki spektral funksiyasiga, uning spektral xarakteristikalari deyiladi. Spektral xarakteristikalarni topish va ularning xossalarini o'rganish masalasiga spektral analizning to'g'ri masalasi deyiladi.

Mazkur qo'llanmaning birinchi bobidagi 5-7-paragraflarda Shturm-Liuvill operatori $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalari uchun asimptotikalar topilgan. Spektral xarakteristikalar yordamida Shturm-Liuvill tenglamasi koeffitsiyentlarini va chegaraviy shartlarini aniqlash masalasiga spektral analizning teskari masalasi deyiladi.

Bu sohaning kvant fizikasi, elektronika, chiziqli va nochiziqli xususiy hosilali tenglamalar nazariyasi, mexanika, kristallografiya, geologo-razvedka masalalariga muhim tatbiqlari topilganligi bois bu mavzu dolzarbligi yanada ortdi. Umuman olganda, teskari spektral masalani shunday izohlash mumkin:

1. Jismning xossalarini uning tebranish chastotasidan xulosa qilish.

2. Yerning tuzilishini zilzilalar natijasida hosil bo'lgan tebranishlar yordamida aniqlash.

Ushbu kitob birinchi bobining 11-12-paragraflarida Shturm-Liuvill chegaraviy masalasi xos funksiyalarining $L^2[0,\pi]$ fazoda to'laligi ko'rsatilgan. Bu teorema ilk bor XIX asrning oxirlarida V.A.Steklov tomonidan isbotlangan.

Qo'llanma birinchi bobining 13-paragrafida $f(x)$ funksiyaga mos keluvchi xos funksiyalardan tuzilgan qator va $\cos nx$ funksiyalar bo'yicha tuzilgan qator bir xil ma'noda yaqinlashishi, ya'ni teng yaqinlashish haqidagi teorema isbotlangan.

Shturm-Liuvill operatori uchun qo'yilgan teskari masalalar V.A.Ambarsumyan, G.Borg, A.N.Tixonov, N.Levinson, V.A.Marchenko, I.M.Gelfand, B.M.Levitan, M.G.Gasimov, M.G.Kreyn, L.D.Faddeev, F.S.Rofe-Beketov, V.A.Yurko va boshqa olimlar tomonidan yetarlicha o'rganilgan.

Teskari masalalar nazariyasining rivojiga muhim turtki bo'lgan ilk natija 1929 yilda V.A.Ambarsumyan tomonidan olingan.

Teorema (*1929 yil, V.A.Ambarsumyan*). *Agar ushbu*

$$-y'' + q(x)y = \lambda y, \quad q(x) \in C[0,\pi],$$

$$y'(0) = 0, \;\; y'(\pi) = 0,$$

haqiqiy koeffitsiyentli Shturm-Liuvill chegaraviy masalasining xos qiymatlari $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$ *bo'lsa, u holda* $q(x) \equiv 0$ *bo'ladi.*

Bu teorema chegaraviy masalaning bitta xos qiymatlar ketma-ketligini bilgan holda $q(x)$ koeffitsiyentni tiklash imkoni bor ekan degan g'oyaga sababchi bo'ldi. Bu taxmin noto'g'ri bo'lib chiqdi. Odatda, bitta spektr $q(x)$ koeffitsiyentni va chegaraviy shartlarni yagona aniqlash uchun yetarli emas.

Ambarsumyanning bu natijasi muhim ekanliligiga birinchi bo'lib shved matematigi G.Borg e'tibor bergan. 1946 yilda G.Borg Shturm-Liuvill chegaraviy masalasi uchun teskari spektral masalani o'zgacha qo'yish mumkinligini tavsiya qilgan. Jumladan, u Shturm-Liuvill differensial operatorini ikkita chegaraviy masalasining spektrlari, ya'ni umumiy differensial tenglama va faqat bitta chegaraviy shart bilan farq qiluvchi ikkita Shturm-Liuvill chegaraviy masalasining spektrlari yordamida qurishning yagonaligini ko'rsatib bergan. Demak, Ambarsumyanning natijasi umumiy qoidadan istisno ekan.

Teorema (*1946 yil, G.Borg*). *Agar* $\lambda_0 < \lambda_1 < ... < \lambda_n < ...$ *sonlar ushbu*

$$-y'' + q(x)y = \lambda y, \quad q(x) \in C[0, \pi],$$

$$\begin{cases} y'(0) - hy(0) = 0, & h \in R^1, \\ y'(\pi) + Hy(\pi) = 0, & H \in R^1 \end{cases}$$

chegaraviy masalaning xos qiymatlari va $\mu_0 < \mu_1 < ... < \mu_n < ...$ *sonlar ushbu*

$$-y'' + q(x)y = \lambda y, \quad q(x) \in C[0, \pi],$$

$$\begin{cases} y'(0) - hy(0) = 0, & h \in R^1, \\ y'(\pi) + H_1 y(\pi) = 0, & H_1 \in R^1, H_1 \neq H \end{cases}$$

chegaraviy masalaning xos qiymatlari bo'lsa, u holda λ_n *va* μ_n *xos qiymatlar ketma-ketligi* $q(x)$ *haqiqiy funksiyani va* h, H, H_1 *sonlarni yagona aniqlaydi.*

Keyinchalik, Borgning bu yagonalik teoremasi umumiy chegaraviy shart holida 1949 yilda L.A.Chudov tomonidan umumlashtirildi.

1949 yilda A.N.Tixonov yarim o'qda berilgan Shturm-Liuvill operatorini Veyl-Titchmarsh funksiyasi $m(z)$ yordamida yagona qurish mumkinligi haqidagi yagonalik teoremasini isbotlashga muvaffaq bo'ldi. Veyl-Titchmarsh funksiyasi bo'yicha chiziqli oddiy differensial operatorni qurish algoritmi V.A.Yurko tomonidan batafsil o'rganilgan.

Shturm-Liuvill operatorlari spektral nazariyasining teskari masalasini o'rganishda almashtirish operatorlari muhim rol o'ynaydi. Ular ikkita har xil Shturm-Liuvill tenglamalarining yechimlarini o'zaro bog'laydi. Almashtirish operatorlari ilk bor B.M.Levitan va J.Delsartlarning ilmiy ishlarida keltirilgan. Bu operator ixtiyoriy Shturm-Liuvill tenglamasi uchun A.Povzner tomonidan qurilgan. Spektral analizning teskari masalasini yechishda almashtirish operatorlari I.M.Gelfand, B.M.Levitan va V.A.Marchenkolar tomonidan qo'llanilgan. Almashtirish operatorining asosiy xossalari qo'llanma birinchi bobining 21-22 - paragraflarida keltirilgan.

1950 yilda V.A.Marchenko $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlar va $\{\alpha_n\}_{n=0}^{\infty}$ normallovchi o'zgarmaslar ketma-ketligi Shturm-Liuvill chegaraviy masalasini yagona ravishda aniqlashini ko'rsatib bergan. Qo'llanma birinchi bobining 23-paragrafida V.A.Ambarsumyan va V.A.Marchenko yagonalik teoremalari keltirilgan.

V.A.Marchenko yagonalik teoremasi e'lon qilingandan keyin spektral funksiya bo'yicha Shturm-Liuvill operatorini tiklash masalasi dolzarb bo'lib qolgan.

Bu masala 1951 yilda I.M.Gelfand va B.M.Levitan tomonidan yechilgan. So'ngra teskari masalani yechishning Gelfand-Levitan usuli B.M.Levitan, I.M.Gasimov va N.Levinson tomonidan mukammallashtirilgan.

Teskari masalani yechishning bir nechta usullari bor. Bu usullar ichida Gelfand-Levitan usuli muhim o'rinni egallaydi. Bu usulda almashtirish operatori asosiy rolni o'ynaydi. Usulning asosiy bosqichlaridan biri, almashtirish operatorlarining yadrosiga nisbatan olingan chiziqli integral tenglamadir. Bu integral tenglama teskari masalaning asosiy integral tenglamasi yoki Gelfand-Levitan integral tenglamasi deb yuritiladi.

Qo'llanma ikkinchi bobining 1-7-paragraflarida chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasining $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalari bo'yicha teskari masalani yechishning Gelfand-Levitan usuli bayon qilingan.

Ikki spektr yordamida Shturm-Liuvill chegaraviy masalasini qurish algoritmi ilk bor M.G.Kreyn tomonidan ishlab chiqildi. So'ngra bu algoritm berilgan spektrlar tilida 1964 yilda B.M.Levitan va M.G.Gasimovlar tomonidan takomillashtirildi. Mazkur kitobning ikkinchi bobidagi 8-9-paragraflarida G.Borgning yagonalik teoremasi va ikki spektr yordamida teskari masalani yechishning I.M.Gasimov va B.M.Levitan usuli keltirilgan.

1967 yilda C.Gardner, J.Grin, M.Kruskal, R.Miura zamonaviy matematik fizikaning asosiy tenglamalaridan biri bo'lgan ushbu

$$u_t - 6uu_x + u_{xxx} = 0$$

Korteveg-de Friz tenglamasiga qo'yilgan Koshi masalasining yechimini Shturm-Liuvill operatoriga qo'yilgan to'g'ri va teskari masalalardan foydalanib topishga muvaffaq bo'lishgan. Natijada Shturm-Liuvill operatori uchun qo'yilgan to'g'ri va teskari masalalarni o'rganishga bo'lgan qiziqish yana-da ortdi.

B.A.Dubrovin, S.P.Novikov, V.B.Matveev, A.R.Its, V.A.Marchenko, I.V.Ostrovskiy, B.M.Levitanlar Korteveg-de Friz tenglamasining yechimini davriy va deyarli davriy funksiyalar sinfida topishga muvaffaq bo'lganlar. Bunda B.A.Dubrovin, E.Trubovis va B.M.Levitanlar tomonidan Shturm-Liuvill operatorining regulyarlashtirilgan izlari formulalaridan foydalanilgan.

Shturm-Liuvill operatorining regulyarlashtirilgan izlari ilk bor I.M.Gelfand va B.M.Levitan tomonidan hisoblangan. Differensial operatorlarning regulyarlashtirilgan izlarini hisoblash usullari L.A.Dikiy, V.B.Lidskiy, V.A.Sadovnichiylar tomonidan takomillashtirilgan. Hozirgi kunda regulyarlashtirilgan izlarni hisoblashning bir qancha usullari mavjud. Biz bu kitobning birinchi bobida Shturm-Liuvill operatori izlarini hisoblashning B.M.Levitan va P.D.Laks usullari bilan tanishamiz, hamda Krum almashtirishini o'rganamiz.

Qo'llanmaning uchinchi va to'rtinchi boblarida yarim o'qda berilgan Shturm-Liuvill operatorini spektral funksiya bo'yicha tiklashning to'g'ri va teskari masalalari o'rganilgan va tatbiqiy ahamiyatga ega bo'lgan bir nechta misollar keltirilgan.

Butun o'qda berilgan Shturm-Liuvill operatori uchun to'g'ri va teskari masalalar qo'llanmaning beshinchi bobida o'rganilgan.

Mazkur qo'llanmani yozishda B.M.Levitanning "Обратные задачи Штурма-Лиувилля" va V.A.Yurkoning "Введение в теорию обратных спектральных задач"kitoblaridan hamda muallifning 1982-1993 yillari SamDU Mexanika-matematika va 1994-2007 yillari UrDU Fizika-matematika fakultetlarida maxsus kurs fanlaridan o'qigan ma'ruzalar matnlaridan foydalanildi.

Ushbu kitobda nazariy materiallar bilan bir qatorda mustaqil yechish uchun mashqlar hamda ularni yechish namunalari keltirilgan. Mazkur qo'llanmaning asosiy maqsadi, oliy o'quv yurtlarida matematika, tatbiqiy matematika va informatika, mexanika va fizika bakalavr yo'nalishlari bo'yicha tahsil olayotgan talabalarda spektral analizning to'g'ri va teskari masalalariga bo'lgan qiziqishni oshirishdan iborat:

Bu kitobdan matematik tahlil, differensial tenglamalar, matematik fizika va nazariy fizika mutaxasisligi bo'yicha ta'lim olayotgan talabalar va magistrantlar shuningdek doktorantlar va ilmiy tatqiqodchilar foydalanishlari mumkin.

Mazkur kitob yozlishida bergan qimmatli maslahatlari uchun O'zR FA akademiklari Sh.A.Alimov va M.S.Salohiddinovlarga hamda kitob matnini tahrir qilishda bergan yordamlari uchun shogirdlarim A.B.Yaxshimuratov, U.A.Xoitmetov va A.A.Reyimberganovlarga samimiy minnatdorchilik bildiraman. Kitobxonlarning kitob to'g'risidagi tanqidiy fikr va mulohazalarini mamnuniyat bilan qabul qilaman.

ahasanov2002@mail.ru *Muallif*

KIRISH

1. Tebranayotgan tor va Shturm-Liuvill tenglamasi. Dastlab biz ikki uchi mahkamlangan tor tebranishi jarayonining matematik modelini qanday qilib Shturm-Liuvill chegaraviy masalasiga kelishini tushuntirishga harakat qilamiz.

Faraz qilaylik bizga $x = 0$ va $x = l$ nuqtalarda Ox o'qi bo'yicha qattiq mahkamlangan, og'irligi torning tarangligiga nisbatan e'tiborga olinmaydigan, $t = 0$ vaqtda tinch bo'lgan Ox absissa o'qida joylashgan tor berilgan bo'lsin. Tor deganda egishda qarshilik ko'rsatmaydigan, ammo cho'zishda qarshilik ko'rsatuvchi elastik ip (ingichka sterjen) ni tushunamiz. Tor shunday qattiq jismki, uning uzunligi boshqa o'lchovlaridan anchagina ortiq. Torga ta'sir qilib turgan $T(x,t)$ taranglik kuchi yetarlicha katta deb faraz qilamiz. Shu sababli torning egilgandagi qarshiligini tarangligiga nisbatan hisobga olmasa ham bo'ladi. Absissasi x bo'lgan nuqtadagi taranglik kuchi, o'sha nuqtadagi torning grafigiga urinma yo'nalishida bo'ladi.

Biz tor siljishi (x,u) tekislikda yotuvchi va ta'sir kuchi Ox o'qiga perpendikulyar bo'lgan ko'ndalang tebranishlarnigina qaraymiz. Bu yerda u - torning muvozanat holatidan siljishini ifodalaydi. $u = u(x,t)$ siljish x va t vaqtning funksiyasidan iborat bo'ladi. Agar biz yetarlicha kichik tebranishni qarasak, u holda $u(x,t)$ va $u'_x(x,t)$ qiymatlar ham yetarlicha kichik bo'lib, $(u'_x)^2$ ni hisobga olmasak ham bo'ladi.

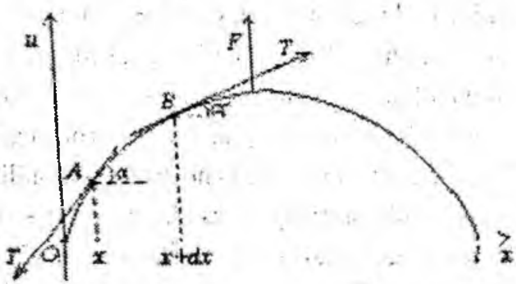
Tor tebranishining matematik modelini tuzish uchun $u(x,t)$ funksiyaga nisbatan differensial tenglama keltirib chiqaramiz.

Torning cheksiz kichik bo'lagi $(x, x+dx)$ ni qaraylik. Uning boshlang'ich holatdagi uzunligi dx bo'lib, t vaqtdagi uzunligi esa

$$AB = ds = \sqrt{dx^2 + du^2} = \sqrt{dx^2 + \left(\frac{\partial u}{\partial x}dx\right)^2} =$$

$$= dx\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} \approx dx,$$

bo'ladi. Demak, kichik tebranishda torning bu bo'lagida cho'zilish bo'lmaydi. U holda Guk qonuniga binoan taranglik kuchi t vaqtga bog'liq emas, ya'ni $T(x,t) \equiv T(x)$.



Torning $(x, x+dx)$ bo'lagiga ta'sir etuvchi kuchlarni Ox o'qiga proeksiyalab, Dalamber prinsipiga asosan

$$T(x+dx)\cos\alpha' - T(x)\cos\alpha = 0$$

tenglikni hosil qilamiz. Bu yerdan $\cos\alpha' \approx \cos\alpha \approx 1$ bo'lishini e'tiborga olsak, u holda $T(x+dx) \approx T(x)$, ya'ni T taranglik kuchining x ga bog'liq emasligi kelib chiqadi. Demak, $T(x,t) \equiv T$.

Faraz qilaylik torga T taranglik kuchidan tashqari yana yo'nalishi Ou o'qqa parallel $F(x,t)$ tashqi kuch ta'sir qilsin. Torning $(x, x+dx)$ bo'lagiga ta'sir etuvchi kuchlarni Ou o'qiga proeksiyasini topamiz:

$$T\sin\alpha' - T\sin\alpha + F(x,t)dx.$$

Bu yerda α va α' mos ravishda tayinlangan t uchun AB egri chiziqning $(x, u(x,t))$ va $(x+dx, u(x+dx,t))$ nuqtalariga o'tkazilgan urinmalarini Ox o'qi bilan tashkil etgan burchaklaridir. Nyutonning ikkinchi qonuniga asosan bu kuch o'z navbatida AB tor bo'lagining massasi bilan tezlanish ko'paytmasiga teng bo'lishi lozim. Agar $\rho(x)$ - torning chiziqli zichligi bo'lsa, u holda tor AB bo'lagining massasi ρdx ga teng bo'ladi. Tezlanish $\frac{\partial^2 u}{\partial t^2}$ ga teng bo'lishi ravshan. Bundan

$$T\sin\alpha' - T\sin\alpha + F(x,t)dx = \rho dx\frac{\partial^2 u}{\partial t^2},$$

tenglama kelib chiqadi.

Endi, ushbu

$$\sin\alpha = \frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2\alpha}} \approx \operatorname{tg}\alpha = \frac{\partial u}{\partial x},$$

$$\sin\alpha' = \frac{\operatorname{tg}\alpha'}{\sqrt{1+\operatorname{tg}^2\alpha'}} \approx \operatorname{tg}\alpha' = \left.\frac{\partial u}{\partial x}\right|_{x+dx},$$

tengliklardan foydalanib, yuqoridagi tenglamani quyidagicha yozish mumkin:

$$T\left[\left.\frac{\partial u}{\partial x}\right|_{x+dx} - \frac{\partial u}{\partial x}\right] + F(x,t)dx = \rho dx\frac{\partial^2 u}{\partial t^2},$$

Bu tenglikning chap tomonidagi

$$\left.\frac{\partial u}{\partial x}\right|_{x+dx} - \frac{\partial u}{\partial x},$$

ifodaga chekli orttirmalar haqidagi Lagranj teoremasini qo'llab,

$$\left.\frac{\partial u}{\partial x}\right|_{x+dx} - \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}u(x+dx,t) - \frac{\partial}{\partial x}u(x,t) = \frac{\partial^2 u}{\partial x^2}dx,$$

tenglikni topamiz. Bundan foydalanib, oxirgi tenglmani quyidagicha yozish mumkin:

$$T\frac{\partial^2 u}{\partial x^2} + F(x,t) = \rho(x)\frac{\partial^2 u}{\partial t^2}$$

Bu esa tor kichik bo'lagining ko'ndalang tebranishlarining tenglamasidir. Bu yerda $\rho(x) > 0$ va $T = \text{const}$. Yuqoridagi tenglamani ushbu

$$p^2(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{1}$$

ko'rinishda yozib olamiz. Bu yerda

$$p^2(x) = \frac{\rho(x)}{T}, \quad f(x,t) = \frac{F(x,t)}{T}.$$

Torga ta'sir qilayotgan tashqi kuch $F(x,t) = 0$ bo'lsa, torning erkin tebranish tenglamasi

$$p^2(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{2}$$

kelib chiqadi. Biz kuzatayotgan torning ikki uchi qattiq mahkamlangan bo'lgani uchun $x = 0$ va $x = l$ nuqtalarda chegaraviy shartlar

$$u(0,t) = 0, \quad u(l,t) = 0 \tag{3}$$

ko'rinishda bo'ladi. Bundan tashqari tor tebranish jarayonini bir qiymatli ifodalash uchun qo'shimcha u siljish va u_t tezlikning boshlang'ich vaqtdagi qiymatlarini berish zarur:

$$u(x,0) = a(x), \quad u_t(x,0) = b(x). \tag{4}$$

Endi (2) tenglama yechimini o'zgaruvchilarni ajratish yo'li bilan qidiramiz:

$$u(x,t) = \varphi(x)\psi(t) \tag{5}$$

Bu tenglikning o'ng tomonini (2) tenglamadagi $u(x,t)$ funksiyaning o'rniga qo'yib,

$$p^2(x)\varphi(x)\psi''(t) = \varphi''(x)\psi(t)$$

tenglikni hosil qilamiz. Oxirgi tenglikni

$$\frac{\varphi''(x)}{p^2(x)\varphi(x)} = \frac{\psi''(t)}{\psi(t)} \tag{6}$$

ko'rinishda yozish mumkin. Bundan $\dfrac{\varphi''(x)}{p^2(x)\varphi(x)}$ va $\dfrac{\psi''(t)}{\psi(t)}$ miqdorlarning birinchisi t ga, ikkinchisi esa x ga bog'liq emas, ya'ni ular o'zgarmas ekanligi kelib chiqadi. Bu o'zgarmasni $-\lambda$ orqali belgilab olamiz. U holda (6) tenglikdan

$$\varphi''(x) + \lambda p^2(x)\varphi(x) = 0, \tag{7}$$

$$\psi''(t) + \lambda\psi(t) = 0 \tag{8}$$

tenglamalarni topamiz. Shunday qilib, (2) tenglama ikkita tenglamaga ajraldi. Bulardan biri faqat x ga bog'liq, ikkinchisi esa faqat t ga bog'liq funksiyani o'z ichiga oladi. (8) tenglamani yechish oson:

$$\psi(t) = C_1 \cos\sqrt{\lambda}t + C_2 \sin\sqrt{\lambda}t. \tag{9}$$

(9) tenglikda qatnashayotgan $\cos\sqrt{\lambda}t$ va $\sin\sqrt{\lambda}t$ funksiyalar davriy funksiyalar bo'lgani uchun $\psi(t)$ funksiya ham davriy bo'lib, $\tau_0 = \frac{2\pi}{\sqrt{\lambda}}$ $(\lambda \neq 0)$ son uning davri bo'ladi. Shuning uchun $\sqrt{\lambda} = \frac{2\pi}{\tau_0}$ ga tor tebranish chastotasi deyiladi. (7) tenglama Shturm-Liuvill turidagi tenglama bo'lib, u kanonik ko'rinishda emas. Endi (7) tenglamada o'zgaruvchilarni

$$y = \int_0^x \rho(\xi)d\xi, \quad \nu(y) = \sqrt{p}\varphi \tag{10}$$

ko'rinishda almashtirib,

$$\frac{d^2\nu}{dy^2} + [\lambda - q(y)]\nu = 0 \tag{11}$$

kanonik ko'rinishdagi Shturm-Liuvill tenglamasiga ega bo'lamiz. Bu yerda $\rho(x)$ zichlik, T taranglik kuchi bilan Shturm-Liuvill operatorining $q(x)$ potentsial funksiyasi o'rtasidagi bog'lanish

$$q(x) = \frac{1}{\sqrt{p}}\frac{d^2\sqrt{p}}{dx^2}, \quad p(x) = \sqrt{\frac{\rho(x)}{T}} \tag{12}$$

formula orqali aniqlanadi. (3) chegaraviy shartlardan va (5) hamda (10) tengliklardan foydalanib,

$$\nu(0) = 0, \quad \nu(l) = 0 \tag{13}$$

tenglikni topamiz. (11), (13) chegaraviy masalaga kanonik ko'rinishdagi Shturm-Liuvill chegaraviy masalasi deyiladi. (11);

(13) Shturm-Liuvill operatori to'g'ri masalasi deganda biz, uning cheksiz ko'p $\{\lambda_n\}_{n=1}^{\infty}$ xos qiymatlarining mavjudligi va unga mos $\{\varphi_n\}_{n=1}^{\infty}$ xos funksiyalarining $L^2(0,l)$ fazoda to'laligini ko'rsatishdan iborat deb tushunamiz. Agar torning xossalari, ya'ni $\rho(x)$ zichligi va T taranglik kuchi ma'lum bo'lsa, u holda $\{\varphi_n, \lambda_n\}_{n=1}^{\infty}$ juftliklar ma'lum bo'ladi va t vaqt uchun torning holatini

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)(C_{1,n}\cos\sqrt{\lambda_n}t + C_{2,n}\sin\sqrt{\lambda_n}t)$$

formula orqali topish mumkin.

Teskari masalada har bir onda torning holati ma'lum bo'lmaydi, ammo tebranish chastotalari $\{\sqrt{\lambda_n}\}_{n=1}^{\infty}$ va turli tebranishlarda tor qanday tebranishi to'g'risidagi ma'lumotlar masalan, $\alpha_n = \|\varphi_n(x)\|^2_{L^2(0,l)}$, $n \geq 1$, ma'lum bo'ladi va torning $\rho(x)$ zichligi izlanadi.

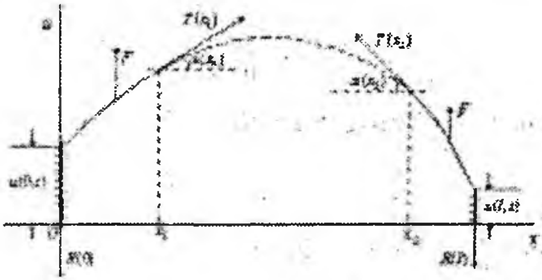
2. Chegaraviy shartlar haqida. Tor tebranish jarayoniga, uning chetki nuqtalarining mahkamlanish xarakterining ta'siri muhim rol o'ynaydi. Torning chetki nuqtalari qattiq, yumshoq yoki elastik mahkamlangan bo'lishi mumkin.

Agar torning chetki nuqtasi biror $\vec{v}$ vektor yo'nalishi bo'yicha siljimasa, u holda torning bu chetki nuqtasi $\vec{v}$ yo'nalishga nisbatan qattiq mahkamlangan deyiladi.

Agar torning chetki nuqtasi $\vec{v}$ vektor yo'nalishi bo'yicha erkin siljisa, u holda torning bu chetki nuqtasi yumshoq mahkamlangan deyiladi.

Agar $R = -k\mu$, $k > 0$ reaksiya kuchi ta'sirida torning chetki nuqtasi μ kattalikdagi masofaga siljisa, u holda torning bu uchi elastik mahkamlangan deyiladi.

$R = -k\mu$ reaksiya formulasi $k = \infty$ bo'lganda qattiq, $k = 0$ bo'lganda esa yumshoq mahkamlanishlarni ifodalaydi. Biz torning chetki $x = 0$ va $x = l$ nuqtalarini Ox o'qi bo'yicha qattiq mahkamlangan deb hisoblaymiz.



Endi yuqoridagi holatlar uchun tor chetki nuqtalarida paydo bo'ladigan shartlarni keltirib chiqaramiz. Buning uchun torning $0 \le x \le x_1$ ($x_2 \le x \le l$) kichik bo'lagini olamiz. Tor bo'lagidagi mahkamlanish tugunlarida $R(0) = -k_1 u(0,t)$ ($R(l) = -k_2 u(l,t)$) reaksiya va ikkinchi chetida T taranglik kuchlarini qaraymiz. Bu holda torning $[0, x_1]$ va $[x_1, l]$ bo'laklariga ta'sir etuvchi kuchlarni Ou o'qiga proektsiyalab, mos ravishda

$$-k_1 u(0,t) + T(x_1)\sin\alpha(x_1) + F(x,t)x_1 - \rho(x)x_1 \frac{\partial^2 u}{\partial t^2} = 0$$

va

$$-k_2 u(l,t) + T(x_2)\sin\alpha(x_2) + F(x,t)(l - x_2) - \rho(x)(l - x_2)\frac{\partial^2 u}{\partial t^2} = 0$$

tenglamalarni topamiz. Bu yerda ushbu

$$\sin\alpha(x_1) \approx u'_x(x_1,t), \quad \sin\alpha(x_2) \approx u'_x(x_2,t)$$

tengliklarni e'tiborga olib, yuqoridagi tenglamalarda mos ravishda $x_1 \to 0$, $x_2 \to l$ da limitga o'tib

$$-k_1 u(0,t) + Tu'_x(0,t) = 0, \quad -k_2 u(l,t) + Tu'_x(l,t) = 0,$$

chegaraviy shartlarni topamiz. Ushbu

$$\frac{k_1}{T} = h, \quad \frac{k_2}{T} = H$$

belgilashlardan foydalansak, bu chegaraviy shartlar quyidagi ko'rinishni oladi:

$$u'_x(0,t) - hu(0,t) = 0,$$
$$u'_x(l,t) + Hu(l,t) = 0.$$

3.Kvant fizikasining Shredinger tenglamasi. Quyidagi tenglama

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) + E_p(x,y,z)\Psi = i\hbar\frac{\partial\Psi}{\partial t},$$

kvant fizikasining asosiy tenglamasi bo'lib, bu tenglamaga Shredinger tenglamasi deyiladi. Bu yerda $\Psi = \Psi(x,y,z,t)$ – zarracha harakatini tavsiflovchi funksiya bo'lib, unga to'lqin funksiyasi deyiladi, $h = 2\pi\hbar$ – Plank doimiysi, m – zarracha massasi, $E_p(x,y,z)$ – potensial energiya. Agar bu tenglamaning quyidagi ko'rinishdagi

$$\Psi(x,y,z,t) = \psi(x,y,z)e^{\frac{-iEt}{\hbar}},$$

yechimini izlasak, ushbu

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + E_p(x,y,z)\psi = E\psi,$$

tenglama hosil bo'ladi. Bu yerda E o'zgarmas bo'lib, u to'liq energiyani bildiradi. Bu tenglamaga Shredingerning statsionar (vaqtga bog'liq bo'lmagan) tenglamasi deyiladi. $E_p(x,y,z)$ va $\psi(x,y,z)$ funksiyalar faqat x o'zgaruvchiga bog'liq bo'lsa, u holda ushbu

$$-\frac{\hbar^2}{2m}\cdot\frac{d^2\psi(x)}{dx^2} + E_p(x)\psi(x) = E\psi(x),$$

tenglamaga ega bo'lamiz. Bu tenglama esa, Shturm-Liuvill tenglamasiga keltirilishi ravshan. Shuning uchun Shturm-Liuvill tenglamasi ayrim adabiyotlarda Shredinger tenglamasi deb yuritiladi. Shturm-Liuvill tenglamasining $q(x)$ koeffitsiyentiga potensial deyilishi ham ana shundan kelib chiqqan.

I BOB. CHEKLI ORALIQDA BERILGAN SHTURM-LIUVILL CHEGARAVIY MASALASI

1-§. Xos qiymatlarning va xos funksiyalarning sodda xossalari

Quyidagi masalaga

$$Ly \equiv -y'' + q(x)y = \lambda y\,, \quad x \in [0,\, \pi], \tag{1.1.1}$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.1.2}$$

Shturm-Liuvill chegaraviy masalasi deyiladi. Bu yerda $q(x) \in C[0,\, \pi]$ haqiqiy uzluksiz funksiya bo'lib, α va β berilgan haqiqiy sonlardir, λ esa kompleks parametr.

Agar (1.1.1) tenglamani $y(0) = 0,\ y(\pi) = 0$ chegaraviy shartlar bilan qarasak, hosil bo'ladigan chegaraviy masalaga Dirixle masalasi deyiladi, agar $y'(0) = 0,\ y'(\pi) = 0$ chegaraviy shartlar bilan qarasak, hosil bo'ladigan chegaraviy masalaga Neyman masalasi deyiladi.

(1.1.1) tenglamaning $q(x)$ koeffitsiyentiga (1.1.1)+(1.1.2) Shturm-Liuvill masalasining potensiali deyiladi.

Ta'rif 1.1.1. Agar λ parametrning biror $\lambda = \lambda_0$ qiymatida (1.1.1)+(1.1.2) chegaraviy masala noldan farqli $y(x, \lambda_0) \not\equiv 0$ yechimga ega bo'lsa, λ_0 songa (1.1.1)+(1.1.2) chegaraviy masalaning xos qiymati deyiladi, $y(x, \lambda_0)$ yechimga esa λ_0 xos qiymatga mos keluvchi xos funksiyasi deyiladi.

(1.1.1)+(1.1.2) Shturm-Liuvill masalasining barcha xos qiymatlaridan tuzilgan to'plamga uning spektri deyiladi.

1-xossa. $y_1(x,\, \lambda)$ va $y_2(x,\, \lambda)$ funksiyalar (1.1.1) tenglamaning ixtiyoriy yechimlari bo'lsin. U holda ulardan tuzilgan

$$W\{y_1(x,\, \lambda),\, y_2(x,\, \lambda)\} = \begin{vmatrix} y_1(x,\, \lambda) & y_2(x,\, \lambda) \\ y_1'(x,\, \lambda) & y_2'(x,\, \lambda) \end{vmatrix},$$

Vronskiy determinanti x o'zgaruvchiga bog'liq bo'lmaydi.

Isbot. Buning uchun ushbu

$$\frac{dW}{dx} \equiv 0,$$

tenglik bajarilishini ko'rsatish yetarli:

$$\frac{dW}{dx} \equiv (y_1 y_2' - y_1' y_2)' =$$

$$= y_1 y_2'' - y_1'' y_2 = y_1[q(x)y_2 - \lambda\, y_2] - y_2[q(x)y_1 - \lambda\, y_1] = 0\,. \blacksquare$$

2-xossa. (1.1.1) tenglamaning ikki yechimi chiziqli bog'liq bo'lishi uchun ulardan tuzilgan Vronskiy determinanti nolga teng bo'lishi zarur va etarli.

Isbot. Ushbu

$$\frac{d}{dx}\left\{\frac{y_1(x,\lambda)}{y_2(x,\lambda)}\right\} = \frac{y_1'(x,\lambda)y_2(x,\lambda) - y_1(x,\lambda)y_2'(x,\lambda)}{y_2^2(x,\lambda)} =$$

$$= -\frac{1}{y_2^2(x,\lambda)} W\left\{y_1(x,\lambda), y_2(x,\lambda)\right\}$$

ayniyatdan quyidagi

$$\frac{y_1(x,\lambda)}{y_2(x,\lambda)} = \text{const}$$

munosabatning bajarilishi uchun $W\left\{y_1(x,\lambda), y_2(x,\lambda)\right\} = 0$ bo'lishi zarur va yetarli ekani kelib chiqadi.

3-xossa. (*Grin ayniyati*). Ixtiyoriy $y(x)$, $z(x) \in C^2[0,\pi]$ funksiyalar uchun ushbu

$$\int_0^\pi Ly \cdot \bar{z}\, dx = W_\pi\{y, \bar{z}\} - W_0\{y, \bar{z}\} + \int_0^\pi y \cdot \overline{Lz}\, dx,$$

ayniyat bajariladi.

Isbot. Quyidagi ayirmani hisoblaymiz:

$$\int_0^\pi (\bar{z}Ly - y\overline{Lz})dx = \int_0^\pi \{\bar{z}[-y'' + q(x)y] - y[-\bar{z}'' + q(x)\bar{z}]\}dx =$$

$$= \int_0^{\pi} (\bar{z}''y - y''\bar{z})dx = \int_0^{\pi} (\bar{z}'y - y'\bar{z})'dx =$$

$$= \left| \begin{matrix} y(x) & \bar{z}(x) \\ y'(x) & \bar{z}'(x) \end{matrix} \right| \Bigg|_0^{\pi} = W_\pi\{y, \bar{z}\} - W_0\{y, \bar{z}\}. \blacksquare$$

4-xossa. Ixtiyoriy $y(x)$, $z(x) \in C^2[0, \pi]$ funksiyalar uchun ushbu

$$\int_0^{\pi} Ly \cdot \bar{z}\, dx = [y(0)\cos\alpha + y'(0)\sin\alpha] \cdot [\bar{z}(0)\sin\alpha - \bar{z}'(0)\cos\alpha] -$$

$$-[y(0)\sin\alpha - y'(0)\cos\alpha] \cdot [\bar{z}(0)\cos\alpha + \bar{z}'(0)\sin\alpha] +$$

$$+[y(\pi)\cos\beta + y'(\pi)\sin\beta] \cdot [-\bar{z}(\pi)\sin\beta + \bar{z}'(\pi)\cos\beta] +$$

$$+[y(\pi)\sin\beta - y'(\pi)\cos\beta] \cdot [\bar{z}(\pi)\cos\beta + \bar{z}'(\pi)\sin\beta] + \int_0^{\pi} y \cdot \overline{Lz}\, dx, \tag{1.1.3}$$

tenglik bajariladi.

Isbot. Grin ayniyatidagi $W_\pi\{y, \bar{z}\} - W_0\{y, \bar{z}\}$ ifodani kerakli ko'rinishda yozamiz. Buning uchun quyidagi sistemani tuzib olamiz:

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = U_1, \\ y(0)\sin\alpha - y'(0)\cos\alpha = U_2, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = U_3, \\ y(\pi)\sin\beta - y'(\pi)\cos\beta = U_4, \end{cases}$$

va undan ushbu

$$\begin{cases} y(0) = U_1\cos\alpha + U_2\sin\alpha, \\ y'(0) = U_1\sin\alpha - U_2\cos\alpha, \\ y(\pi) = U_3\cos\beta + U_4\sin\beta, \\ y'(\pi) = U_3\sin\beta - U_4\cos\beta, \end{cases}$$

tengliklarni hosil qilamiz. Bularni Grin ayniyatiga qo'ysak, (1.1.3) tenglik hosil bo'ladi.$\blacksquare$

Natija 1.1.1. Agar $y(x)$, $z(x) \in C^2[0,\pi]$ bo'lib, $y(x)$ funksiya (1.1.2) chegaraviy shartlarni qanoatlantirsa, u holda

$$\int_0^\pi Ly \cdot \bar{z}\,dx = \int_0^\pi y \cdot \overline{Lz}dx,$$

tenglik bajarilishi uchun $z(x)$ funksiya ham (1.1.2) chegaraviy shartlarni qanoatlantirishi zarur va yetarlidir.

Yuqoridagi natija, (1.1.1)+(1.1.2) chegaraviy masala yordamida aniqlangan L chiziqli operator $L^2(0,\pi)$ Gilbert fazosida o'z-o'ziga qo'shma operatorni ifodalashini ko'rsatadi.

5-xossa. (1.1.1)+(1.1.2) Shturm-Liuvill masalasining xos qiymatlari haqiqiydir.

Isbot. $\lambda = u + iv$, $i = \sqrt{-1}$, $(v \neq 0)$ son (1.1.1)+(1.1.2) Shturm-Liuvill chegaraviy masalasining xos qiymati bo'lsin deb faraz qilaylik va unga mos keluvchi xos funksiyani $y(x)$ bilan belgilaylik. U holda $\bar{\lambda} = u - iv$ son ham shu chegaraviy masalasining xos qiymati bo'ladi va unga $\bar{y}(x)$ xos funksiya mos keladi. Quyidagi

$$(\lambda - \bar{\lambda})\int_0^\pi |y(x)|^2 dx = \int_0^\pi (\lambda - \bar{\lambda})y(x)\bar{y}(x)dx =$$

$$= \int_0^\pi \left[(\lambda y)\bar{y} - y(\bar{\lambda}\bar{y})\right] dx =$$

$$= \int_0^\pi \{\bar{y}[-y'' + q(x)y] - y[-\bar{y}'' + q(x)\bar{y}]\}dx = \int_0^\pi (\bar{y}''y - y''\bar{y})dx =$$

$$= \int_0^\pi (\bar{y}'y - y'\bar{y})'dx = \begin{vmatrix} y(\pi) & \bar{y}(\pi) \\ -y(\pi)\operatorname{ctg}\beta & -\bar{y}(\pi)\operatorname{ctg}\beta \end{vmatrix} -$$

$$- \begin{vmatrix} y(0) & \bar{y}(0) \\ -y(0)\operatorname{ctg}\alpha & -\bar{y}(0)\operatorname{ctg}\alpha \end{vmatrix} = 0,$$

tenglikdan $\bar{\lambda} = \lambda$ ekanligi kelib chiqadi. Bu esa farazimizga zid.■

Natija 1.1.2. Xos funksiyani haqiqiy qilib tanlash mumkin. Chunki xos qiymat haqiqiy ekanligidan qaralayotgan tenglamaning haqiqiyligi kelib chiqadi. Chegaraviy shartlar esa hamisha haqiqiy.

6-xossa. (1.1.1)+(1.1.2) Shturm-Liuvill masalasining turli xos qiymatlariga mos keluvchi xos funksiyalari o'zaro ortogonaldir, ya'ni $\lambda_1 \neq \lambda_2$ xos qiymatlarga mos keluvchi $y_1(x)$, $y_2(x)$ xos funksiyalar uchun ushbu

$$\int_0^\pi y_1(x) \cdot y_2(x)\, dx = 0, \tag{1.1.4}$$

tenglik o'rinli bo'ladi.

Isbot. Ushbu

$$(\lambda_1 - \lambda_2)\int_0^\pi y_1(x)y_2(x)dx = \int_0^\pi [(\lambda_1 y_1)y_2 - y_1(\lambda_2 y_2)]\, dx =$$

$$= \int_0^\pi \{y_2[-y_1'' + q(x)y_1] - y_1[-y_2'' + q(x)y_2]\}dx =$$

$$= \int_0^\pi (y_2'' y_1 - y_1'' y_2)dx = \int_0^\pi (y_2' y_1 - y_1' y_2)' dx = \left.\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\right|_0^\pi = 0,$$

ayniyatda $\lambda_1 \neq \lambda_2$ bo'lgani uchun (1.1.4) tenglik o'rinli bo'lishligi kelib chiqadi.■

7-xossa. (1.1.1)+(1.1.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlari oddiy (karrasiz), ya'ni bitta xos qiymatga mos keluvchi xos funksiyalar bir-biriga proporsionaldir.

Isbot. λ xos qiymatga $y_1(x)$, $y_2(x)$ chiziqli erkli xos funksiyalar mos keladi deb faraz qilaylik. U holda

$$W\{y_1, y_2\} = \lim_{x\to 0} W\{y_1, y_2\} = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} =$$

$$= \begin{vmatrix} y_1(0) & y_2(0) \\ -y_1(0)\operatorname{ctg}\alpha & -y_2(0)\operatorname{ctg}\alpha \end{vmatrix} = 0,$$

bo'lgani uchun, $y_1(x)$, $y_2(x)$ chiziqli bog'liq bo'ladi. Bu esa farazimizga ziddir.■

8-xossa. $y(x, \lambda)$ funksiya Shturm-Liuvill tenglamasining λ bo'yicha uzluksiz differensiallanuvchi ixtiyoriy yechimi bo'lsin. U holda

$$\int_0^{\pi} y^2(x, \lambda)dx = W_{\pi}\{\dot{y}, y\} - W_0\{\dot{y}, y\},$$

tenglik bajariladi. Bu yerda $\dot{y} = \dfrac{\partial y(x,\lambda)}{\partial \lambda}$.

Isbot. Ushbu

$$-y'' + q(x)y = \lambda y, \tag{1.1.5}$$

ayniyatdan λ bo'yicha hosila olsak,

$$-\dot{y}'' + q(x)\dot{y} = y + \lambda\dot{y}, \tag{1.1.6}$$

tenglik kelib chiqadi. (1.1.5) va (1.1.6) tengliklarni mos ravishda $\dot{y}$ va y funksiyalarga ko'paytirib, bir-biridan ayirsak, ushbu

$$\dot{y}''y - y''\dot{y} = -y^2,$$

ayniyat hosil bo'ladi. Bu tenglikni $[0, \pi]$ kesmada integrallasak, ushbu

$$\int_0^{\pi} y^2(x, \lambda)dx = \int_0^{\pi} (y'\dot{y} - \dot{y}'y)'dx = W_{\pi}\{\dot{y}, y\} - W_0\{\dot{y}, y\},$$

formula kelib chiqadi.■

9-xossa. Agar quyidagi chegaraviy masalaning

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases}$$

xos qiymatlari λ_0, λ_1, $\lambda_2, \ldots$ va xos funksiyalari $y_0(x)$, $y_1(x)$, $y_2(x), \ldots$ bo'lsa, u holda ushbu

$$\begin{cases} -y'' + [q(x) + c]y = \lambda y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.1.7}$$

chegaraviy masalaning xos qiymatlari λ_0+c, λ_1+c, λ_2+c, ... va xos funksiyalari $y_0(x)$, $y_1(x)$, $y_2(x)$, ... bo'ladi. Bu yerda c o'zgarmas son.

Isbot. Ushbu

$$\begin{cases} -y'' + q(x)y = (\lambda - c)y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases}$$

chegaraviy masala noldan farqli yechimga ega bo'lishi uchun $\lambda - c = \lambda_n$ bo'lishi zarur va yetarli. Shartga ko'ra, bu holda oxirgi chegaraviy masala $y_n(x) \not\equiv 0$ yechimga ega. Demak, (1.1.7) masalaning xos qiymatlari $\lambda_0 + c$, $\lambda_1 + c$, $\lambda_2 + c$, ... va xos funksiyalari $y_0(x)$, $y_1(x)$, $y_2(x)$, ... bo'ladi.■

(1.1.1) differensial tenglamaning quyidagi

$$\varphi(0, \lambda) = -\sin\alpha, \ \ \varphi'(0, \lambda) = \cos\alpha$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini $\varphi(x, \lambda)$ orqali belgilaymiz. Xuddi shuningdek, (1.1.1) tenglamaning ushbu

$$\psi(\pi, \lambda) = -\sin\beta, \ \ \psi'(\pi, \lambda) = \cos\beta$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini $\psi(x, \lambda)$ orqali belgilab olamiz. Bu yerda $\varphi(x, \lambda)$ yechim (1.1.2) chegaraviy shartlardan birinchisini, $\psi(x,\lambda)$ yechim esa ikkinchisini qanoatlantiradi. Bu $\varphi(x,\lambda)$ va $\psi(x,\lambda)$ yechimlarni mos ravishda (1.1.2) chegaraviy shartlardan ikkinchisiga va birinchisiga qo'yib, ushbu

$$\Delta(\lambda) \equiv \varphi(\pi,\lambda)\cos\beta + \varphi'(\pi,\lambda)\sin\beta = 0,$$

$$\tilde{\Delta}(\lambda) \equiv \psi(0,\lambda)\cos\alpha + \psi'(0,\lambda)\sin\alpha = 0,$$

tenglamalarni hosil qilamiz. Bu tenglamalarga (1.1.1)+(1.1.2) Shturm-Liuvill chegaraviy masalasining xarakteristik tenglamalari deyiladi. Shturm-Liuvill tenglamasining $\varphi(x,\lambda)$ va $\psi(x,\lambda)$ yechimlaridan tuzilgan ushbu

$$\omega(\lambda) = W\{\varphi(x,\lambda),\ \psi(x,\lambda)\} \equiv \begin{vmatrix} \varphi(x,\lambda) & \psi(x,\lambda) \\ \varphi'(x,\lambda) & \psi'(x,\lambda) \end{vmatrix},$$

Vronskiy determinantini qaraymiz. Biz yuqorida bu determinant x o'zgaruvchiga bog'liq emasligini ko'rsatgan edik. Shuning uchun ushbu

$$\omega(\lambda) = W\{\varphi(x,\lambda),\ \psi(x,\lambda)\} = \lim_{x\to 0} W\{\varphi(x,\lambda),\ \psi(x,\lambda)\} =$$

$$= \lim_{x\to\pi} W\{\varphi(x,\lambda),\ \psi(x,\lambda)\},$$

tengliklarni yozishimiz mumkin. Bu tengliklardan

$$\omega(\lambda) = \Delta(\lambda) = -\tilde{\Delta}(\lambda),$$

kelib chiqadi. Bu yerdagi $\omega(\lambda)$, $\Delta(\lambda)$, $\tilde{\Delta}(\lambda)$ funksiyalar λ o'zgaruvchining butun funksiyalari bo'lib, sanoqlita $\{\lambda_n\}_{n=0}^{\infty}$ nollarga ega ekanligini keyinchalik ko'rsatamiz.

$\Delta(\lambda) = 0$ xarakteristik tenglamaning λ_n, $n = 0,1,2,\ldots$ ildizlari Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan iborat bo'lib, $\varphi(x,\lambda_n)$ va $\psi(x,\lambda_n)$ funksiyalar uning xos funksiyalari bo'ladi va ushbu

$$\psi(x,\lambda_n) = c_n\varphi(x,\lambda_n), \quad c_n \neq 0 \qquad (1.1.8)$$

tenglik bajariladi. Haqiqatan ham, $\lambda = \lambda_n$ soni $\Delta(\lambda) = 0$ tenglamaning ildizi bo'lsa, u holda $\omega(\lambda_n) = 0$ bo'lgani uchun (1.1.8) tenglik o'rinli bo'ladi. $\varphi(x, \lambda_n)$ va $\psi(x, \lambda_n)$ funksiyalar (1.1.2) chegaraviy shartlarni qanoatlantiradi, bundan esa $\lambda = \lambda_n$ son xos qiymat hamda $\varphi(x, \lambda_n)$ va $\psi(x, \lambda_n)$ funksiyalar Shturm-Liuvill chegaraviy masalasining xos funksiyalari ekanligi kelib chiqadi.

Izoh 1.1.2. Odatda, agar (1.1.2) chegaraviy shartlardan birinchisi ushbu $y'(0) - hy(0) = 0$ ko'rinishda bo'lsa, u holda $\varphi(x, \lambda)$ yechim $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = h$ boshlang'ich shartlarni qanoatlantiradigan qilib olinadi, agar (1.1.2) chegaraviy shartlardan birinchisi $y(0) = 0$ ko'rinishda bo'lsa, u holda $\varphi(x, \lambda)$ yechim $\varphi(0, \lambda) = 0$, $\varphi'(0, \lambda) = 1$ boshlang'ich shartlarni qanoatlantiradigan qilib olinadi.

Agar $\lambda = \lambda_0$ soni Shturm-Liuvill chegaraviy masalasining xos qiymati bo'lib, $y(x, \lambda_0)$ unga mos keluvchi xos funksiya bo'lsa, u holda $y(0, \lambda_0)$ va $y'(0, \lambda_0)$ qiymatlardan kamida bittasi noldan farqli bo'ladi, aks xolda yechimning yagonaligi haqidagi Koshi teoremasidan $y(x, \lambda_0) \equiv 0$ ekanligi kelib chiqadi. Bu esa xos funksiya ta'rifiga ziddir. Xuddi shuningdek, $y(\pi, \lambda_0)$ va $y'(\pi, \lambda_0)$ qiymatlardan kamida bittasi noldan farqli bo'lishi ko'rsatiladi.

Quyidagi

$$\alpha_n = \sqrt{\int_0^\pi \varphi^2(x, \lambda_n)dx}, \quad n = 0,\ 1,\ 2,\ ...,$$

sonlarga (1.1.1)+(1.1.2) chegaraviy masalaning normallovchi o'zgarmaslari deyiladi. (1.1.1)+(1.1.2) masalaning ortonormallangan xos funksiyalari quyidagi tengliklardan topiladi:

$$u_n(x) = \frac{1}{\alpha_n}\varphi(x, \lambda_n), \quad n = 0, 1, 2, ...$$

Ta'rif 1.1.2. Ushbu $\{\lambda_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ sonli ketma-ketliklar juftligiga Shturm-Liuvill chegaraviy masalasining spektral berilganlari (spektral xarakteristikalari) deyiladi.

Ta'rif 1.1.3. Monoton o'suvchi, chapdan uzluksiz ushbu

$$\rho(\lambda) = \begin{cases} 0, & \lambda = 0, \\ -\sum\limits_{\lambda \le \lambda_n < 0} \dfrac{1}{\alpha_n^2}, & \lambda < 0, \\ \sum\limits_{0 < \lambda_n < \lambda} \dfrac{1}{\alpha_n^2}, & \lambda > 0, \end{cases} \tag{1.1.9}$$

funksiyaga Shturm-Liuvill chegaraviy masalasining spektral funksiyasi deyiladi.

10-xossa. Shturm-Liuvill chegaraviy masalasining normallovchi o'zgarmaslari uchun ushbu

$$\alpha_n^2 = \begin{cases} \varphi'(\pi, \lambda_n)\dot{\Delta}(\lambda_n), & \sin\beta = 0, \\ -\dfrac{\varphi(\pi, \lambda_n)}{\sin\beta}\dot{\Delta}(\lambda_n), & \sin\beta \neq 0, \end{cases} \tag{1.1.10}$$

tenglik o'rinli bo'ladi. Bu yerda

$$\Delta(\lambda) = \varphi(\pi, \lambda)\cos\beta + \varphi'(\pi, \lambda)\sin\beta,$$

funksiya (1.1.1)+(1.1.2) Shturm-Liuvill masalasining xarakteristik funksiyasi, $\varphi(x, \lambda)$ funksiya esa (1.1.1) tenglamaning

$$\varphi(0, \lambda) = -\sin\alpha, \quad \varphi'(0, \lambda) = \cos\alpha$$

boshlang'ich shartlarni qanoatlantiruvchi yechimidir.

Isbot. 8-xossada $y(x, \lambda)$ yechim o'rnida $\varphi(x, \lambda)$ yechimni olib, $\lambda = \lambda_n$ desak, ushbu

$$\alpha_n^2 = \int_0^{\pi} \varphi^2(x, \lambda_n)dx = \begin{vmatrix} \dot{\varphi}(\pi, \lambda_n) & \varphi(\pi, \lambda_n) \\ \dot{\varphi}'(\pi, \lambda_n) & \varphi'(\pi, \lambda_n) \end{vmatrix} -$$

$$- \begin{vmatrix} \dot{\varphi}(0, \lambda_n) & \varphi(0, \lambda_n) \\ \dot{\varphi}'(0, \lambda_n) & \varphi'(0, \lambda_n) \end{vmatrix} = \begin{vmatrix} \dot{\varphi}(\pi, \lambda_n) & \varphi(\pi, \lambda_n) \\ \dot{\varphi}'(\pi, \lambda_n) & \varphi'(\pi, \lambda_n) \end{vmatrix} - \begin{vmatrix} 0 & -\sin\alpha \\ 0 & \cos\alpha \end{vmatrix} =$$

$$= \begin{vmatrix} \dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n) \\ \dot\varphi'(\pi,\lambda_n) & \varphi'(\pi,\lambda_n) \end{vmatrix},$$

tenglik hosil bo'ladi.

Quyidagi ikkita holni ko'rib chiqamiz.

1) $\sin\beta = 0$ bo'lsin. Bu holda $\Delta(\lambda) = \varphi(\pi,\lambda)$, $\varphi(\pi,\lambda_n) = 0$ bo'lgani uchun

$$\alpha_n^2 = \varphi'(\pi,\lambda_n)\dot\Delta(\lambda_n)$$

tenglik o'rinli.

2) $\sin\beta \neq 0$ bo'lsin. Bu holda $\varphi'(\pi,\lambda_n) = -\varphi(\pi,\lambda_n)\operatorname{ctg}\beta$ bo'lgani uchun

$$\alpha_n^2 = \begin{vmatrix} \dot\varphi(\pi,\lambda_n) & \varphi(\pi,\lambda_n) \\ \dot\varphi'(\pi,\lambda_n) & -\varphi(\pi,\lambda_n)\operatorname{ctg}\beta \end{vmatrix} =$$

$$= \varphi(\pi,\lambda_n)[-\dot\varphi(\pi,\lambda_n)\operatorname{ctg}\beta - \dot\varphi'(\pi,\lambda_n)] =$$

$$= -\frac{\varphi(\pi,\lambda_n)}{\sin\beta}[\dot\varphi(\pi,\lambda_n)\cos\beta + \dot\varphi'(\pi,\lambda_n)\sin\beta] = -\frac{\varphi(\pi,\lambda_n)}{\sin\beta}\dot\Delta(\lambda_n)$$

tenglik bajariladi.■

Natija 1.1.4. (1.1.8) tenglikdan foydalanib, (1.1.10) tenglikni quyidagi ko'rinishda yozish mumkin:

$$\alpha_n^2 \cdot c_n = \begin{cases} \cos\beta \cdot \dot\Delta(\lambda_n), & \sin\beta = 0, \\ \dot\Delta(\lambda_n), & \sin\beta \neq 0. \end{cases}$$

Natija 1.1.5. (1.1.10) formuladan (1.1.1)+(1.1.2) Shturm-Liuvill chegaraviy masalasining

$$\Delta(\lambda) = \varphi(\pi,\lambda)\cos\beta + \varphi'(\pi,\lambda)\sin\beta,$$

xarakteristik funksiyasi karrali ildizga ega emasligi kelib chiqadi.

Misol. Ushbu

$$\begin{cases} -y'' = \lambda y, \quad 0 \le x \le \pi, \\ y'(0) - hy(0) = 0, \\ y'(\pi) - hy(\pi) = 0, \end{cases} \tag{1.1.11}$$

chegaraviy masalaning xos qiymatlarini, ortonormallangan xos funksiyalarini va spektral funksiyasini topamiz. Avvalo ushbu

$$y'' + \lambda y = 0,$$

tenglamaning umumiy yechimini topamiz:

$$y(x, \lambda) = C_1 \cos\sqrt{\lambda}x + C_2\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

So'ngra, quyidagi

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h,$$

boshlang'ich shartlarni qanoatlantiruvchi $\varphi(x, \lambda)$ yechimni aniqlaymiz:

$$\varphi(x, \lambda) = \cos\sqrt{\lambda}x + h\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

Topilgan bu yechim chegaraviy shartlardan birinchisini qanoatlantirishi ravshan. Bu yechimni chegaraviy shartlardan ikkinchisiga qo'yib, xarakteristik tenglamani keltirib chiqaramiz:

$$\varphi'(x, \lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + h\cos\sqrt{\lambda}x,$$

$$-\sqrt{\lambda}\sin\sqrt{\lambda}\pi + h\cos\sqrt{\lambda}\pi - h \cdot \left\{\cos\sqrt{\lambda}\pi + h\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}\right\} = 0,$$

$$-\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}(\lambda + h^2) = 0.$$

Oxirgi xarakteristik tenglamadan xos qiymatlarni topamiz:

$$\lambda_0 = -h^2, \quad \lambda_n = n^2, \quad n = 1,\ 2,\ \ldots\ .$$

Bu xos qiymatlarga quyidagi xos funksiyalar mos keladi:

$$\varphi(x, \lambda_0) = e^{hx},$$

$$\varphi(x, \lambda_n) = \cos nx + h \cdot \frac{\sin nx}{n}, \quad n = 1,\ 2,\ \ldots\ .$$

Endi esa, normallovchi o'zgarmaslarni hisoblaymiz:

$$\alpha_0^2 = \int_0^\pi \varphi^2(x,\lambda_0)dx = \int_0^\pi e^{2hx}dx = \frac{1}{2h}e^{2hx}\Big|_0^\pi = \frac{1}{2h}(e^{2h\pi}-1),$$

$$\alpha_n^2 = \int_0^\pi \varphi^2(x,\lambda_n)dx = \int_0^\pi \left\{\cos nx + h\frac{\sin nx}{n}\right\}^2 dx =$$

$$= \int_0^\pi \left\{\cos^2 nx + \frac{h}{n}\sin 2nx + \frac{h^2}{n^2}\sin^2 nx\right\}dx =$$

$$= \int_0^\pi \left\{\frac{1}{2} + \frac{h^2}{2n^2} + \frac{h}{n}\sin 2nx + \frac{1}{2}\cos 2nx - \frac{h^2}{2n^2}\cos 2nx)\right\}dx =$$

$$= \frac{(n^2+h^2)\pi}{2n^2}, \quad n = 1,\ 2,\ \ldots\ .$$

Demak, (1.1.11) Shturm-Liuvill chegaraviy masalasining ortonormallangan xos funksiyalari quyidagi funksiyalardan iborat:

$$u_0(x) = \sqrt{\frac{2h}{e^{2h\pi}-1}}e^{hx},$$

$$u_n(x) = \sqrt{\frac{2}{(n^2+h^2)\pi}}\,\{n\cos nx + h\sin nx\}, \qquad n = 1,\ 2,\ \ldots,$$

spektral funksiyasi esa ($h \neq 0$ bo'lganida) ushbu

$$\rho(\lambda) = \begin{cases} -\dfrac{2h}{e^{2h\pi}-1}, & \lambda \le -h^2, \\ 0, & -h^2 < \lambda \le 1, \\ \dfrac{2}{\pi}\sum\limits_{0<n<\sqrt{\lambda}} \dfrac{n^2}{n^2+h^2}, & \lambda > 1, \end{cases}$$

formula bilan aniqlanadi.

Xususan, $h = 0$ bo'lganda, (1.1.11) Shturm-Liuvill chegaraviy masalasi quyidagi ko'rinishni oladi:

$$\begin{cases} -y'' = \lambda y, & 0 \le x \le \pi, \\ y'(0) = 0, \\ y'(\pi) = 0. \end{cases} \tag{1.1.12}$$

Bu chegaraviy masalaning xos qiymatlari $\lambda_n = n^2, n = 0, 1, 2, ...$ bo'lib, ularga mos keluvchi ortonormallangan xos funksiyalar

$$u_0(x) = \sqrt{\frac{1}{\pi}}, \quad u_n(x) = \sqrt{\frac{2}{\pi}} \cos nx, \quad n = 1, 2, ...,$$

bo'ladi. Normallovchi o'zgarmaslar ketma-ketligi esa

$$\alpha_0 = \sqrt{\pi}, \quad \alpha_n = \sqrt{\frac{\pi}{2}}, \quad n = 1, 2, ...,$$

bo'ladi. (1.1.9) formuladan foydalanib, (1.1.12) Neyman chegaraviy masalasining spektral funksiyasini topamiz:

$$\rho(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \frac{1}{\pi}, & 0 < \lambda \le 1, \\ \frac{1}{\pi} + \frac{2}{\pi} \sum\limits_{0<n<\sqrt{\lambda}} 1, & \lambda > 1. \end{cases}$$

Buni quyidagicha yozish mumkin

$$\rho(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \frac{1}{\pi}, & 0 < \lambda \le 1, \\ \frac{1}{\pi} + \frac{2}{\pi}\left[\sqrt{\lambda}\right], & \lambda > 1. \end{cases}$$

Bu yerda $[a]$ belgilash a sonining butun qismini bildiradi.

Mustaqil yechish uchun mashqlar

1. Quyidagi o'zgarmas potensialli Shturm-Liuvill chegaraviy masalalarining xos qiymatlarini, xos funksiyalarini, normallovchi o'zgarmaslarini, ortonormallangan xos funksiyalarini va spektral funksiyasini toping ($0 \le x \le \pi$):

$$a)\begin{cases} -y'' = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases} \quad b)\begin{cases} -y'' = \lambda y, \\ y'(0) = 0, \\ y'(\pi) = 0, \end{cases} \quad c)\begin{cases} -y'' = \lambda y, \\ y'(0) = 0, \\ y(\pi) = 0, \end{cases}$$

$$d)\begin{cases} -y'' = \lambda y, \\ y(0) = 0, \\ y'(\pi) = 0, \end{cases} \quad e)\begin{cases} -y'' = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) - hy(\pi) = 0. \end{cases}$$

2. Davriy va antidavriy chegaraviy shartli quyidagi Shturm-Liuvill masalalarining xos qiymatlarini, xos funksiyalarini, normallovchi o'zgarmaslarini, ortonormallangan xos funksiyalarini toping ($0 \le x \le \pi$):

$$a)\begin{cases} -y'' = \lambda y, \\ y(0) = y(\pi), \\ y'(0) = y'(\pi), \end{cases} \quad b)\begin{cases} -y'' = \lambda y, \\ y(0) = -y(\pi), \\ y'(0) = -y'(\pi). \end{cases}$$

3. Potensialga 5 sonini qo'shganda quyidagi

$$a)\begin{cases} -y'' = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases} \quad b)\begin{cases} -y'' = \lambda y, \\ y'(0) = 0, \\ y'(\pi) = 0, \end{cases} \quad c)\begin{cases} -y'' = \lambda y, \\ y'(0) = 0, \\ y(\pi) = 0, \end{cases}$$

Shturm-Liuvill chegaraviy masalalarining xos qiymatlari va xos funksiyalari qanday o'zgaradi?

4. Bitta yechimi quyidagi ko'rinishda bo'lgan barcha Shturm-Liuvill tenglamalarini toping:

$$y = \cos\sqrt{\lambda}x + a(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

5. Bitta yechimi quyidagi ko'rinishda bo'lgan barcha Shturm-Liuvill tenglamalarini toping:

$$y = \sqrt{\lambda}\sin\sqrt{\lambda}x + b(x)\cos\sqrt{\lambda}x.$$

6. Ushbu

$$\varphi(x,\lambda) = \frac{\sin kx}{k} - \frac{3x}{x^3 - a}\left(\frac{1}{k^3}\sin kx - \frac{1}{k^2}x\cos kx\right), \quad (k = \sqrt{\lambda})$$

funksiya quyidagi

$$-y'' + \frac{6x^4 + 12ax}{(x^3 - a)^2}y = \lambda y,$$

differensial tenglamani qanoatlantirishini tekshiring.

7. Ushbu

$$-y'' + \frac{2}{(x-a)^2}y = \lambda y,$$

Shturm-Liuvill tenglamasining umumiy yechimi quyidagi formulalar bilan berilishini tekshiring:

$$y = C_1 \cdot \left(\cos\sqrt{\lambda}x - \frac{1}{x-a}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right) +$$

$$+C_2 \cdot \left(\sqrt{\lambda}\sin\sqrt{\lambda}x + \frac{1}{x-a}\cos\sqrt{\lambda}x\right), \quad \lambda \neq 0;$$

$$y = C_1 \cdot (x-a)^2 + C_2 \cdot \frac{1}{x-a}, \quad \lambda = 0.$$

8. Ushbu

$$-y'' - \frac{2m^2}{ch^2(mx+\alpha)}y = \lambda y,$$

Shturm-Liuvill tenglamasining umumiy yechimi quyidagi formulalar bilan berilishini tekshiring:

$$y = C_1 \cdot \left(\cos\sqrt{\lambda}x - m \cdot th(mx+\alpha) \cdot \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right) +$$

$$+C_2 \cdot \left(\sqrt{\lambda}\sin\sqrt{\lambda}x + m \cdot th(mx+\alpha) \cdot \cos\sqrt{\lambda}x\right), \quad \lambda \neq -m^2;$$

$$= C_1 \cdot \frac{1}{ch(mx+\alpha)} + C_2 \cdot \left(sh(mx+\alpha) + \frac{mx}{ch(mx+\alpha)} \right), \ \lambda = -m^2.$$

9. Ushbu

$$-y'' + \frac{2m^2}{sh^2(mx-\alpha)} y = \lambda y,$$

Shturm-Liuvill tenglamasining umumiy yechimi quyidagi formulalar bilan berilishini tekshiring:

$$y = C_1 \cdot \left(\cos\sqrt{\lambda}x - m \cdot cth(mx-\alpha) \cdot \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right) +$$

$$+ C_2 \cdot \left(\sqrt{\lambda}\sin\sqrt{\lambda}x + m \cdot cth(mx-\alpha) \cdot \cos\sqrt{\lambda}x \right), \ \lambda \neq -m^2;$$

$$y = C_1 \cdot \frac{1}{sh(mx-\alpha)} + C_2 \cdot \left(ch(mx-\alpha) - \frac{mx}{sh(mx-\alpha)} \right), \ \lambda = -m^2.$$

10. Ushbu

$$-y'' + \frac{2m^2}{\cos^2(mx+\alpha)} y = \lambda y,$$

Shturm-Liuvill tenglamasining umumiy yechimi quyidagi formulalar bilan berilishini tekshiring:

$$y = C_1 \cdot \left(\cos\sqrt{\lambda}x + m \cdot \mathrm{tg}(mx+\alpha) \cdot \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right) +$$

$$+ C_2 \cdot \left(\sqrt{\lambda}\sin\sqrt{\lambda}x - m \cdot \mathrm{tg}(mx+\alpha) \cdot \cos\sqrt{\lambda}x \right), \ \lambda \neq m^2;$$

$$y = C_1 \cdot \frac{1}{\cos(mx+\alpha)} + C_2 \cdot \left(\sin(mx+\alpha) + \frac{mx}{\cos(mx+\alpha)} \right), \ \lambda = m^2.$$

11. Ushbu

$$-y'' + \frac{2m^2}{\sin^2(mx+\alpha)} y = \lambda y,$$

Shturm-Liuvill tenglamasining umumiy yechimi quyidagi formulalar bilan berilishini tekshiring:

$$y = C_1 \cdot \left(\cos\sqrt{\lambda}x - m \cdot \mathrm{ctg}(mx+\alpha) \cdot \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right) +$$

$$+C_2\cdot\left(\sqrt{\lambda}\sin\sqrt{\lambda}x+m\cdot\mathrm{ctg}(mx+\alpha)\cdot\cos\sqrt{\lambda}x\right),\quad \lambda\neq m^2;$$

$$y=C_1\cdot\frac{1}{\sin(mx+\alpha)}+C_2\cdot\left(\cos(mx+\alpha)-\frac{mx}{\sin(mx+\alpha)}\right),\quad \lambda=m^2.$$

12. Quyidagi o'zgaruvchan potensialli Shturm-Liuvill chegaraviy masalalarining xos qiymatlarini, xos funksiyalarini, normallovchi o'zgarmaslarini, ortonormallangan xos funksiyalarini va spektral funksiyasini toping ($0\le x\le\pi$):

$$a)\begin{cases}-y''+\dfrac{2}{(x+1)^2}y=\lambda y,\\ y'(0)+y(0)=0,\\ y'(\pi)+\dfrac{1}{\pi+1}y(\pi)=0,\end{cases}$$

$$b)\begin{cases}-y''+\dfrac{2h^2}{(hx-1)^2}y=\lambda y,\\ y'(0)-hy(0)=0,\\ y'(\pi)+\dfrac{h}{\pi h-1}y(\pi)=0,\ \dfrac{1}{h}\notin[0,\ \pi],\end{cases}$$

$$c)\begin{cases}-y''-\dfrac{2}{\mathrm{ch}^2\,x}y=\lambda y,\\ y(0)=0,\\ y'(\pi)+\mathrm{th}\,\pi y(\pi)=0,\end{cases}\qquad d)\begin{cases}-y''-\dfrac{2}{\mathrm{ch}^2\,x}y=\lambda y,\\ y'(0)=0,\\ y'(\pi)+\mathrm{th}\,\pi y(\pi)=0,\end{cases}$$

$$e)\begin{cases}-y''-\dfrac{2}{\mathrm{ch}^2(x+\alpha)}y=\lambda y,\\ y'(0)+\mathrm{th}\,\alpha\cdot y(0)=0,\\ y'(\pi)+\mathrm{th}(\pi+\alpha)\cdot y(\pi)=0,\end{cases}$$

$$f)\begin{cases}-y''-\dfrac{2m^2}{\mathrm{ch}^2\,mx}y=\lambda y,\\ y(0)=0,\\ y'(\pi)+m\,\mathrm{th}(m\pi)y(\pi)=0,\end{cases}$$

$$g)\begin{cases} -y'' - \dfrac{2m^2}{\mathrm{ch}^2\, mx} y = \lambda y, \\ y'(0) = 0, \\ y'(\pi) + m\,\mathrm{th}(m\pi) y(\pi) = 0, \end{cases}$$

$$h)\begin{cases} -y'' - \dfrac{2m^2}{\mathrm{ch}^2(mx+\alpha)} y = \lambda y, \\ y'(0) + m\,\mathrm{th}\,\alpha \cdot y(0) = 0, \\ y'(\pi) + m\,\mathrm{th}(m\pi+\alpha) y(\pi) = 0, \end{cases}$$

$$i)\begin{cases} -y'' + \dfrac{2m^2}{\mathrm{sh}^2(mx-\alpha)} y = \lambda y, \\ y'(0) + m \cdot \mathrm{cth}\,\alpha \cdot y(0) = 0, \qquad \dfrac{\alpha}{m} \notin [0,\ \pi] \\ y'(\pi) + m \cdot \mathrm{cth}(m\pi - \alpha) \cdot y(\pi) = 0. \end{cases}$$

13. Quyidagi o‘zgaruvchan potensialli Shturm-Liuvill chegaraviy masalalarining xos qiymatlarini, xos funksiyalarini, normallovchi o‘zgarmaslarini, ortonormallangan xos funksiyalarini va spektral funksiyasini toping ($0 \le x \le \pi$):

$$a)\begin{cases} -y'' + \dfrac{2}{x^2} y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + \dfrac{1}{\pi} y(\pi) = 0, \end{cases} \qquad b)\begin{cases} -y'' + \dfrac{2}{\mathrm{sh}^2\, x} y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + \mathrm{cth}\,\pi \cdot y(\pi) = 0, \end{cases}$$

$$c)\begin{cases} -y'' + \dfrac{2m^2}{sh^2 mx} y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + m \cdot \mathrm{cth}(m\pi) y(\pi) = 0, \end{cases}$$

$$d)\begin{cases} -y'' + \dfrac{2}{\sin^2 x} y = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases} \qquad e)\begin{cases} -y'' + \dfrac{1}{2\sin^2\left(\frac{x}{2}\right)} y = \lambda y, \\ y(0) = 0, \\ y'(\pi) = 0, \end{cases}$$

$$f)\begin{cases} -y'' + \dfrac{2m^2}{\sin^2(mx+\alpha)}y = \lambda y, \\ y'(0) - m\cdot\operatorname{ctg}\alpha\cdot y(0) = 0, \\ y'(\pi) + m\cdot\operatorname{ctg}(m\pi+\alpha)\cdot y(\pi) = 0. \end{cases}$$

$$g)\begin{cases} -y'' + \dfrac{2m^2}{\cos^2 mx}y = \lambda y, \\ y'(0) = 0, \\ y'(\pi) - m\cdot\operatorname{tg} m\pi\cdot y(\pi) = 0, \end{cases}$$

$$h)\begin{cases} -y'' + \dfrac{2m^2}{\sin^2 mx}y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + m\cdot\operatorname{ctg} m\pi\cdot y(\pi) = 0, \end{cases}$$

$$i)\begin{cases} -y'' + \dfrac{2m^2}{\cos^2(mx+\alpha)}y = \lambda y, \\ y'(0) - m\cdot\operatorname{tg}\alpha\cdot y(0) = 0, \\ y'(\pi) - m\cdot\operatorname{tg}(m\pi+\alpha)\cdot y(\pi) = 0, \end{cases}$$

$$j)\begin{cases} -y'' + \dfrac{2}{\cos^2 x}y = \lambda y, \\ y'(0) = 0, \\ y'(\pi) = 0, \end{cases}$$

2-§. Shturm-Liuvill tenglamasi uchun qo'yilgan Koshi masalasi

Quyidagi Koshi masalasini ko'rib chiqamiz:

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.2.1}$$

$$\begin{cases} y(0) = y_0, \\ y'(0) = y_1. \end{cases} \tag{1.2.2}$$

Bu yerda $q(x) \in C[0, \pi]$ haqiqiy funksiya bo'lib, y_0, y_1 ixtiyoriy haqiqiy sonlar.

(1.2.1)+(1.2.2) Koshi masalasiga ekvivalent bo'ladigan integral tenglama tuzamiz. $y(x)$ funksiya (1.2.1)+(1.2.2) masalaning biror yechimi bo'lsin. (1.2.1) tenglamani avvalo ushbu

$$y'' = f(x), \tag{1.2.3}$$

ko'rinishda yozib olamiz. Bu yerda

$$f(x) = [q(x) - \lambda]\, y(x). \tag{1.2.4}$$

So'ngra (1.2.3) tenglama uchun Koshi funksiyasini tuzamiz, ya'ni tarkibida t parametr qatnashgan ushbu

$$\begin{cases} y'' = 0, \\ y|_{x=t} = 0, \\ y'|_{x=t} = 1, \end{cases}$$

Koshi masalasining yechimini topamiz: $y = c_1x + c_2$,

$$\begin{cases} c_1t + c_2 = 0, \\ c_1 = 1, \end{cases} \qquad \begin{cases} c_1 = 1, \\ c_2 = -t, \end{cases}$$

$$y = K(x, t) = x - t.$$

Shuning uchun (1.2.3) tenglamaning umumiy yechimi

$$y(x) = A_0 + A_1x + \int_0^x (x - t)f(t)dt, \tag{1.2.5}$$

ko'rinishda bo'ladi. (1.2.2) boshlang'ich shartlardan

$$A_0 = y_0, \quad A_1 = y_1, \tag{1.2.6}$$

bo'lishi kelib chiqadi. (1.2.4) va (1.2.6) tengliklardan foydalanib, (1.2.5) ayniyatni ushbu

$$y(x) = y_0 + y_1 x + \int_0^x (x-t)[q(t) - \lambda]\, y(t) dt \tag{1.2.7}$$

ko'rinishda yozamiz. (1.2.7) tenglik izlangan integral tenglamadir. Bu tenglama Volterraning ikkinchi turdagi integral tenglamasidir.

Shunday qilib, (1.2.1)+(1.2.2) Koshi masalasining yechimi mavjud bo'lsa, u (1.2.7) integral tenglamani qanoatlantirar ekan. Aksincha, $y(x)$ funksiya (1.2.7) integral tenglamaning uzluksiz yechimi bo'lsa, u (1.2.1)+(1.2.2) Koshi masalasining ham yechimi bo'ladi. Haqiqatan ham, $y(x)$ uzluksiz ekanligidan (1.2.7) ayniyatning o'ng tomoni differensiallanuvchi bo'lishi kelib chiqadi, bundan esa chap tomon ham hosilaga ega bo'lishi ko'rinadi. Undan hosila olsak, ushbu

$$y'(x) = y_1 + \int_0^x [q(t) - \lambda]\, y(t)\, dt \tag{1.2.8}$$

tenglik hosil bo'ladi. (1.2.8) ayniyatdan yana hosila olsak,

$$y'' = [q(x) - \lambda]\, y(x),$$

ya'ni

$$-y'' + q(x)\, y = \lambda y,$$

tenglik kelib chiqadi.(1.2.7) va (1.2.8) tengliklarda $x = 0$ desak, (1.2.2) boshlang'ich shartlarni olamiz.

Teorema 1.2.1. *Agar $q(x) \in C[0,\pi]$, y_0, $y_1 \in R^1$ bo'lsa, u holda (1.2.1)+ (1.2.2) Koshi masalasining $[0,\pi]$ kesmada aniqlangan $\varphi(x,\lambda)$ yechimi mavjud va yagona bo'lib, u x o'zgaruvchi-*

ning har bir tayinlangan qiymatida λ bo'yicha $\frac{1}{2}$ tartibdagi butun funksiyadir, ya'ni tayinlangan x da $\varphi(x,\lambda)$ funksiya kompleks tekislikning ixtiyoriy chegaralangan soxasida kompleks ma'noda differensiallanuvchidir.

Isbot.(*Mavjudligi*). Koshi masalasi yechimining mavjudligini isbotlash uchun (1.2.7) integral tenglamaning uzluksiz yechimi mavjud ekanligini ko'rsatish yetarli. Buning uchun quyidagi funksiyalar ketma-ketligini tuzib olamiz:

$$\begin{aligned}\varphi_0(x,\lambda) = y_0 + y_1 x, \quad \varphi_n(x,\lambda) = y_0 + y_1 x + \\ + \int_0^x (x-t)[q(t)-\lambda]\varphi_{n-1}(t,\lambda)dt, \quad n \in N.\end{aligned} \tag{1.2.9}$$

Bu funksiyalar ketma-ketmaligi $x \in [0,\pi]$, $\lambda \in \mathbb{C}$ qiymatlarda aniqlangan. $R > 0$ ixtiyoriy son bo'lsin. $\varphi_n(x,\lambda)$ ketma-ketlik $x \in [0,\pi]$, $|\lambda| \le R$ bo'lganda tekis yaqinlashishini ko'rsatamiz. Shu maqsadda ushbu

$$\varphi_0 + \sum_{n=1}^{\infty} [\varphi_n - \varphi_{n-1}], \tag{1.2.10}$$

funksional qatorni tuzib olamiz. Bu qatorning xususiy yig'indisi $\varphi_n(x,\lambda)$ funksiyaga teng bo'lishi ravshan. Quyidagi

$$M = \max_{[0,\,\pi]}|q(x)|, \quad K = \max_{[0,\,\pi]}|\varphi_0(x,\lambda)|,$$

belgilashlarni kiritib olamiz. U holda ushbu

$$|\varphi_1(x,\lambda) - \varphi_0(x,\lambda)| = \left|\int_0^x (x-t)[q(t)-\lambda]\varphi_0(t,\lambda)dt\right| \le$$

$$\le \int_0^x \pi\,(M+R)\,K\,dt = \pi\,(M+R)\,Kx,$$

$$|\varphi_2(x,\lambda)-\varphi_1(x,\lambda)|=$$

$$=\left|\int_0^x (x-t)[q(t)-\lambda]\cdot[\varphi_1(t,\lambda)-\varphi_0(t,\lambda)]\,dt\right|\le$$

$$\le\int_0^x (x-t)\,(M+R)\cdot\pi\,(M+R)\,K\cdot t\,dt\le\pi^2\,(M+R)^2K\,\frac{x^2}{2},$$

tengsizliklar o'rinli bo'ladi. Umuman $|\lambda|\le R$, $x\in[0,\pi]$ bo'lganida quyidagi

$$|\varphi_n(x,\lambda)-\varphi_{n-1}(x,\lambda)|\le\frac{K[x\pi\,(M+R)]^n}{n!},\qquad(1.2.11)$$

baholash o'rinli bo'ladi. Bu tengsizlik induksiya usulida osongina isbot qilinadi. (1.2.11) baholashga asosan ushbu

$$K+\sum_{n=1}^{\infty}K\frac{[\pi^2(M+R)]^n}{n!}<\infty,$$

sonli qator (1.2.10) funksional qator uchun majoranta qator bo'ladi. Demak, $|\lambda|\le R$, $x\in[0,\pi]$ to'plamda (1.2.10) qator Veyershtrass alomatiga asosan tekis yaqinlashuvchi bo'ladi. Uning yig'indisini $\varphi(x,\lambda)$ orqali belgilaymiz. $\varphi_n(x,\lambda)$ funksiyalarning uzluksizligidan $\varphi(x,\lambda)$ funksiyaning uzluksizligi kelib chiqadi.

Agar (1.2.9) tenglikda $n\to\infty$ da limitga o'tsak,

$$\varphi(x,\lambda)=y_0+y_1x+\int_0^x (x-t)[q(t)-\lambda]\,\varphi(t,\lambda)\,dt,$$

ayniyat hosil bo'ladi. Demak, $\varphi(x,\lambda)$ funksiya (1.2.7) integral tenglamaning uzluksiz yechimi bo'lar ekan.

(*Yagonaligi*). Endi (1.2.7) integral tenglamaning yechimi yagona bo'lishini isbotlaymiz. Buning uchun ikkita $\varphi(x,\lambda)\not\equiv\psi(x,\lambda)$ yechim mavjud deb faraz qilamiz. Bu yechimlarni integral tenglamaga qo'yib, hosil bo'lgan ayniyatlarni bir-biridan ayiramiz:

$$\varphi(x,\lambda)-\psi(x,\lambda)=\int_0^x (x-t)\,[q(t)-\lambda]\,[\varphi(t,\lambda)-\psi(t,\lambda)]\,dt.$$

Bu tenglikka asosan

$$|\varphi(x,\lambda)-\psi(x,\lambda)|\leq \pi\,(M+R)\int_0^x |\varphi(t,\lambda)-\psi(t,\lambda)|\,dt, \tag{1.2.12}$$

bo'ladi. Agar

$$z(x)=\int_0^x |\varphi(t,\lambda)-\psi(t,\lambda)|\,dt,$$

belgilash kiritsak, (1.2.12) tengsizlik ushbu

$$z'(x)-\pi\,(M+R)\,z\,(x)\leq 0, \tag{1.2.13}$$

ko'rinishni oladi. (1.2.13) tengsizlikni $e^{-\pi\,(M+R)\,x}$ funksiyaga ko'paytiramiz va chap tomonni ko'paytmaning hosilasi ko'rinishida yozamiz:

$$(z(x)\,e^{-\pi\,(M+R)\,x})'\leq 0. \tag{1.2.14}$$

(1.2.14) tengsizlikda $x=t$ desak, va hosil bo'ladigan tengsizlikni $[0,\ x]$ oraliqda integrallasak,

$$z(x)\leq 0$$

kelib chiqadi. (1.2.12) baholashdan

$$|\varphi(x,\lambda)-\psi(x,\lambda)|\leq 0,$$

ya'ni $\varphi(x,\lambda)\equiv\psi(x,\lambda)$ kelib chiqadi. Bu esa farazimizga ziddir.

(*Butunligi*). $\varphi(x,\lambda)$ yechimning λ ga nisbatan butun funksiya bo'lishini isbotlaymiz. $\varphi_n(x,\lambda)$ funksiyalarning har biri $|\lambda|<$

R sohada golomorf bo'lishi ravshan. Veyershtrassning kompleks analizdagi teoremasiga ko'ra $\varphi(x,\lambda)$ funksiya $|\lambda| < R$ sohada golomorf bo'ladi. $R > 0$ son ixtiyoriy bo'lganligi uchun $\varphi(x,\lambda)$ butun funksiya bo'ladi. $\varphi(x,\lambda)$ ning $\frac{1}{2}$ tartibdagi butun funksiya bo'lishini keyinchalik yechimning asimptotikasidan foydalanib ko'rsatamiz.■

Teorema 1.2.2. *Ixtiyoriy kompleks $\lambda \in C$ da (1.2.1) tenglama yechimlaridan tuzilgan chiziqli fazoning o'lchami ikkiga teng.*

Isbot. Bu tasdiqning isboti teorema 1.2.1 va birinchi paragrafdagi 2-xossadan kelib chiqadi. ■

Mustaqil yechish uchun mashqlar

1. Koshi funksiyasi yordamida quyidagi Koshi masalalariga ekvivalent bo'lgan integral tenglamalar tuzing ($0 \le x \le \pi$):

$$a)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(0) = 1, \end{cases} \qquad b)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 1, \\ y'(0) = 0, \end{cases}$$

$$c)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 1, \\ y'(0) = h. \end{cases}$$

2. O'zgarmasni variatsiyalash usuli yordamida quyidagi Koshi masalalariga ekvivalent bo'lgan integral tenglamalar tuzing ($0 \le x \le \pi$):

$$a)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(0) = 1, \end{cases} \qquad b)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 1, \\ y'(0) = 0, \end{cases}$$

$$c)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 1, \\ y'(0) = h. \end{cases}$$

3-§. Shturm-Liuvill tenglamasi yechimining asimptotikasi

Quyidagi Koshi masalasini ko'rib chiqamiz

$$-y'' + q(x)\,y = \lambda y, \quad 0 \le x \le \pi, \tag{1.3.1}$$

$$\begin{cases} y(0) = y_0, \\ y'(0) = y_1. \end{cases} \tag{1.3.2}$$

Bu yerda $q(x) \in C\,[0,\pi]$ haqiqiy funksiya bo'lib, y_0, y_1 haqiqiy sonlar.

(1.3.1)÷(1.3.2) Koshi masalasining $y(x,\lambda)$ yechimi mavjud, yagona va λ ga nisbatan butun funksiya bo'lishini isbot qilgan edik.

Endi, $y(x,\lambda)$ yechimning $|\lambda| \to \infty$ bo'lganda asimptotikasini o'rganish maqsadida (1.3.1)÷(1.3.2) Koshi masalasiga ekvivalent bo'lgan integral tenglama tuzamiz. Buning uchun avvalo (1.3.1) tenglamani ushbu

$$y'' + \lambda\,y = f(x), \tag{1.3.3}$$

ko'rinishda yozib olamiz. Bu yerda

$$f(x) = q(x)\,y(x,\lambda). \tag{1.3.4}$$

So'ngra (1.3.3) tenglama uchun Koshi funksiyasini tuzamiz, ya'ni tarkibida t parametr qatnashgan ushbu

$$\begin{cases} y'' + \lambda\,y = 0, \\ y\,|_{x=t} = 0, \\ y'\,|_{x=t} = 1, \end{cases}$$

Koshi masalasining yechimini topamiz:

$$y(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$y'(x) = -c_1\sqrt{\lambda}\,\sin\sqrt{\lambda}\,x + c_2\cos\sqrt{\lambda}x,$$

$$\begin{cases} c_1 \cos\sqrt{\lambda}\,t + c_2\dfrac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}} = 0, \\ -c_1\sqrt{\lambda}\, \sin\sqrt{\lambda}\,t + c_2\cos\sqrt{\lambda}\,t = 1, \end{cases}$$

$$\Delta = \begin{vmatrix} \cos\sqrt{\lambda}\,t & \dfrac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}} \\ -\sqrt{\lambda}\,\sin\sqrt{\lambda}\,t & \cos\sqrt{\lambda}\,t \end{vmatrix} = 1,$$

$$\Delta_1 = \begin{vmatrix} 0 & \dfrac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}} \\ 1 & \cos\sqrt{\lambda}\,t \end{vmatrix} = -\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}},$$

$$\Delta_2 = \begin{vmatrix} \cos\sqrt{\lambda}\,t & 0 \\ -\sqrt{\lambda}\,\sin\sqrt{\lambda}\,t & 1 \end{vmatrix} = \cos\sqrt{\lambda}\,t,$$

$$c_1 = -\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}, \quad c_2 = \cos\sqrt{\lambda}\,t,$$

$$K(x,t) = -\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}\cos\sqrt{\lambda}\,x + \cos\sqrt{\lambda}\,t\,\frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} =$$

$$= \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}\,(x-t).$$

Koshi funksiyasi yordamida (1.3.3) tenglamaning umumiy yechimi quyidagicha ifodalanadi

$$y(x,\lambda) = A_0\cos\sqrt{\lambda}\,x + A_1\frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} +$$

$$+\frac{1}{\sqrt{\lambda}}\int_0^x f(t)\,\sin\sqrt{\lambda}\,(x-t)\,dt. \tag{1.3.5}$$

(1.3.4) belgilashni va boshlang‘ich shartlarni inobatga olsak, (1.3.5) tenglik ushbu

$$y(x,\lambda) = y_0\cos\sqrt{\lambda}\,x + y_1\frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} +$$

$$+\frac{1}{\sqrt{\lambda}}\int_0^x q(t)\,y(t,\lambda)\,\sin\sqrt{\lambda}\,(x-t)\,dt, \qquad (1.3.6)$$

ko'rinishni oladi. Bu tenglik biz izlagan integral tenglamadir. Bu tenglama Volterraning ikkinchi turdagi integral tenglamasidir, unga Liuvill integral tenglamasi deb ham aytiladi.

$c(x,\lambda)$ va $s(x,\lambda)$ orqali (1.3.1) tenglamaning quyidagi

$$\begin{cases} c(0,\lambda)=1, \\ c'(0,\lambda)=0 \end{cases} \quad \text{va} \quad \begin{cases} s(0,\lambda)=0, \\ s'(0,\lambda)=1 \end{cases}$$

boshlang'ich shartlarini qanoatlantiruvchi yechimlarini belgilaymiz. Bu yechimlar uchun (1.3.6) integral tenglama ushbu

$$c(x,\lambda)=\cos\sqrt{\lambda}\,x+\frac{1}{\sqrt{\lambda}}\int_0^x q(t)\,c(t,\lambda)\,\sin\sqrt{\lambda}\,(x-t)\,dt, \quad (1.3.7)$$

$$s(x,\lambda)=\frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\int_0^x q(t)\,s(t,\lambda)\,\sin\sqrt{\lambda}(x-t)\,dt, \quad (1.3.8)$$

ko'rinishlarda bo'ladi.

Lemma 1.3.1. *Agar* $x\in[0,\pi]$, $\sqrt{\lambda}=k=\sigma+i\,\tau$, $|k|>2\int_0^\pi|q(t)|\,dt$ *bo'lsa, u holda*

$$|c\,(x,\lambda)|<2e^{|\tau|x}, \qquad (1.3.9)$$

$$|s\,(x,\lambda)|<2\frac{e^{|\tau|x}}{|k|}, \qquad (1.3.10)$$

baholashlar o'rinli bo'ladi.

Isbot. Quyidagi

$$F(x,\lambda)=\frac{c(x,\lambda)}{e^{|\tau|x}},$$

belgilashni kiritib olamiz. U holda

$$c(x,\,\lambda)=e^{|\tau|x}F(x,\,\lambda)$$

bo'ladi. Buni (1.3.7) tenglamaga qo'yamiz:

$$e^{|\tau|x}F(x,\lambda)=\cos kx+\frac{1}{k}\int\limits_0^x q(t)e^{|\tau|t}F(t,\lambda)\sin k(x-t)dt,$$

$$F(x,\lambda)=\frac{\cos kx}{e^{|\tau|x}}+\frac{1}{k}\int\limits_0^x q(t)F(t,\lambda)\frac{\sin k(x-t)}{e^{|\tau|(x-t)}}dt. \qquad (1.3.11)$$

Quyidagi baholashlarni bajaramiz:

$$|\cos kx|=\left|\frac{e^{ikx}+e^{-ikx}}{2}\right|=\left|\frac{e^{i\sigma x-\tau x}+e^{-i\sigma x+\tau x}}{2}\right|\le$$

$$\le\frac{1}{2}(e^{-\tau x}+e^{\tau x})\le e^{|\tau|x},$$

$$|\sin kx|=\left|\frac{e^{ikx}-e^{-ikx}}{2i}\right|=\left|\frac{e^{i\sigma x-\tau x}-e^{-i\sigma x+\tau x}}{2}\right|\le$$

$$\le\frac{1}{2}(e^{-\tau x}+e^{\tau x})\le e^{|\tau|x}.$$

$M(\lambda)=\max\limits_{0\le x\le\pi}|F(x,\lambda)|$ bo'lsin. U holda yuqorida olingan baholashlarga ko'ra (1.3.11) tenglikdan ushbu

$$|F(x,\lambda)|\le 1+\frac{1}{|k|}\int\limits_0^\pi |q(t)|M(\lambda)dt,$$

tengsizlik kelib chiqadi. Bu tengsizlikdan esa

$$M(\lambda)\le 1+M(\lambda)\cdot\frac{1}{|k|}\int\limits_0^\pi |q(t)|dt,$$

ya'ni

$$M(\lambda)\left(1-\frac{1}{|k|}\int\limits_0^\pi |q(t)|dt\right)\le 1, \qquad (1.3.12)$$

hosil bo'ladi. Lemma shartiga ko'ra

$$\frac{1}{|k|}\int\limits_0^\pi |q(t)|dt<\frac{1}{2},$$

$$1-\frac{1}{|k|}\int\limits_0^{\pi}|q(t)|dt>\frac{1}{2}, \qquad (1.3.13)$$

bo'ladi. (1.3.12) va (1.3.13) tengsizliklardan ushbu $M(\lambda)<2$ baholash kelib chiqadi, ya'ni $|F(x,\lambda)|<2$ bo'ladi. Shunday qilib (1.3.9) baholash isbot qilindi. (1.3.10) baholash ham shu tarzda isbot qilinadi.■

Lemma 1.3.2. *Agar $x\in[0,\pi]$, $\sqrt{\lambda}=k=\sigma+i\tau$, $|k|>2\int\limits_0^{\pi}|q(t)|dt$ bo'lsa, u holda*

$$\left|c(x,\lambda)-\cos\sqrt{\lambda}x\right|\le\frac{2}{|k|}\int\limits_0^{x}|q(t)|dt\cdot e^{|\tau|x}, \qquad (1.3.14)$$

$$\left|s(x,\lambda)-\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right|\le\frac{2}{|k|^2}\int\limits_0^{x}|q(t)|dt\cdot e^{|\tau|x}, \qquad (1.3.15)$$

$$\left|c'(x,\lambda)+\sqrt{\lambda}\sin\sqrt{\lambda}x\right|\le 2\int\limits_0^{x}|q(t)|dt\cdot e^{|\tau|x}, \qquad (1.3.16)$$

$$\left|s'(x,\lambda)-\cos\sqrt{\lambda}x\right|\le\frac{2}{|k|}\int\limits_0^{x}|q(t)|dt\cdot e^{|\tau|x}, \qquad (1.3.17)$$

baholashlar o'rinli bo'ladi.

Isbot. (1.3.7) integral tenglama va (1.3.9) tengsizlikka ko'ra

$$\left|c(x,\lambda)-\cos\sqrt{\lambda}x\right|\le\frac{1}{|k|}\int\limits_0^{x}|q(t)|\cdot|c(t,\lambda)|\cdot|\sin k(x-t)|dt\le$$

$$\le\frac{1}{|k|}\int\limits_0^{x}|q(t)|\cdot 2e^{|\tau|t}\cdot e^{|\tau|(x-t)}dt\le\frac{2}{|k|}\int\limits_0^{x}|q(t)|dt\cdot e^{|\tau|x},$$

bo'ladi. (1.3.8) integral tenglama va (1.3.10) baholashga ko'ra

$$\left|s(x,\lambda)-\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right|\le\frac{1}{|k|}\int_0^x|q(t)|\cdot|s(t,\lambda)|\cdot|\sin k(x-t)|dt\le$$

$$\le\frac{1}{|k|}\int_0^x|q(t)|\cdot 2\frac{e^{|\tau|t}}{|k|}\cdot e^{|\tau|(x-t)}dt\le\frac{2}{|k|^2}\int_0^x|q(t)|dt\cdot e^{|\tau|x},$$

bo'ladi. (1.3.14) va (1.3.15) baholashlar isbotlandi. (1.3.16) va (1.3.17) tengsizliklarni isbot qilish maqsadida, avvalo (1.3.7) va (1.3.8) tenglamalardan hosila olamiz:

$$c'(x,\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}\,x+\int_0^x q(t)\,c(t,\lambda)\,\cos\sqrt{\lambda}\,(x-t)\,dt,$$

$$s'(x,\lambda)=\cos\sqrt{\lambda}\,x+\int_0^x q(t)\,s(t,\lambda)\,\cos\sqrt{\lambda}(x-t)\,dt.$$

Bu tengliklardan hamda (1.3.9) va (1.3.10) tengsizliklardan foydalanib, ushbu

$$\left|c'(x,\lambda)+\sqrt{\lambda}\sin\sqrt{\lambda}x\right|\le\int_0^x|q(t)|\cdot|c(t,\lambda)|\cdot|\cos k(x-t)|dt\le$$

$$\le\int_0^x|q(t)|\cdot 2e^{|\tau|t}\cdot e^{|\tau|(x-t)}dt\le 2\int_0^x|q(t)|dt\cdot e^{|\tau|x},$$

$$\left|s'(x,\lambda)-\cos\sqrt{\lambda}x\right|\le\int_0^x|q(t)|\cdot|s(t,\lambda)|\cdot|\cos k(x-t)|dt\le$$

$$\le\int_0^x|q(t)|\cdot 2\frac{e^{|\tau|t}}{|k|}\cdot e^{|\tau|(x-t)}dt\le\frac{2}{|k|}\int_0^x|q(t)|dt\cdot e^{|\tau|x},$$

baholashlarni olamiz.■

Natija 1.3.1. Agar $x \in [0, \pi]$, $\sqrt{\lambda} = k = \sigma + i\tau$, $|k| > 2\int\limits_0^\pi |q(t)|dt$ bo'lsa, u holda quyidagi

$$c(x, \lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k}\right),$$

$$s(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right),$$

$$c'(x, \lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \underline{\underline{O}}\left(e^{|\tau|x}\right),$$

$$s'(x, \lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k}\right),$$

asimptotik formulalar o'rinli bo'ladi.

Natija 1.3.2. Agar $\varphi(x, \lambda)$ orqali (1.3.1) tenglamaning ushbu

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilasak, u holda

$$\varphi(x, \lambda) = c(x, \lambda) + h\, s(x, \lambda)$$

bo'ladi. Bundan $x \in [0, \pi]$, $\sqrt{\lambda} = k = \sigma + i\tau$, $|k| > 2\int\limits_0^\pi |q(t)|dt$ bo'lganda ushbu

$$\varphi(x, \lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\operatorname{Im}\sqrt{\lambda}|x}}{\sqrt{\lambda}}\right),$$

$$\varphi'(x, \lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \underline{\underline{O}}\left(e^{|\operatorname{Im}\sqrt{\lambda}|x}\right),$$

asimptotik formulalar o'rinli bo'lishi kelib chiqadi.

Mustaqil yechish uchun mashqlar

1. Koshi funksiyasi yordamida quyidagi Koshi masalalariga ekvivalent bo'lgan integral tenglamalar tuzing va u yordamida yechimning asimptotikasini o'rganing ($0 \le x \le \pi$):

$$a)\begin{cases} -y'' + xy = \lambda y, \\ y(0) = 0, \\ y'(0) = 1, \end{cases} \qquad b)\begin{cases} -y'' + xy = \lambda y, \\ y(0) = 1, \\ y'(0) = 0, \end{cases}$$

$$c)\begin{cases} -y'' + xy = \lambda y, \\ y(0) = 1, \\ y'(0) = h, \end{cases} \qquad d)\begin{cases} -y'' + x^2 y = \lambda y, \\ y(0) = 0, \\ y'(0) = 1, \end{cases}$$

$$e)\begin{cases} -y'' + x^2 y = \lambda y, \\ y(0) = 1, \\ y'(0) = 0, \end{cases} \qquad f)\begin{cases} -y'' + x^2 y = \lambda y, \\ y(0) = 1, \\ y'(0) = h. \end{cases}$$

4-§. Shturm-Liuvill tenglamasi yechimining asimptotikasini aniqlashtirish

Quyidagi

$$-y'' + q(x)\, y = \lambda y, \quad 0 \le x \le \pi, \tag{1.4.1}$$

Shturm-Liuvill tenglamasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya, λ kompleks parametr.

$c(x, \lambda)$ va $s(x, \lambda)$ orqali (1.4.1) tenglamaning quyidagi

$$\begin{cases} c(0, \lambda) = 1 \\ c'(0, \lambda) = 0 \end{cases} \quad \text{va} \quad \begin{cases} s(0, \lambda) = 0 \\ s'(0, \lambda) = 1 \end{cases}$$

boshlang'ich shartlarini qanoatlantiruvchi yechimlarini belgilaymiz.

Birinchi bob uchinchi paragrafining ikkinchi lemmasida, agar $x \in [0, \pi]$, $\sqrt{\lambda} \equiv k = \sigma + i\tau$, $|k| > 2\int_0^\pi |q(t)|dt$ bo'lsa, u holda

$c(x, \lambda)$ va $s(x, \lambda)$ yechimlar uchun ushbu

$$c(x,\ \lambda) = \cos kx + \frac{1}{k} \cdot A(x, \lambda), \quad |A(x, \lambda)| \le 2\int_0^x |q(t)|dt \cdot e^{|\tau|x}, \tag{1.4.2}$$

$$s(x,\ \lambda) = \frac{1}{k} \sin kx + \frac{1}{k^2} \cdot B(x, \lambda), \quad |B(x, \lambda)| \le 2\int_0^x |q(t)|dt \cdot e^{|\tau|x}, \tag{1.4.3}$$

baholashlar o'rinli ekanligi ko'rsatilgan edi. (1.4.2) va (1.4.3) ifodalarni quyidagi

$$c(x, \lambda) = \cos k\, x + \frac{1}{k}\int_0^x q(t)\, c(t, \lambda)\, \sin k(x-t)\, dt, \tag{1.4.4}$$

$$s(x, \lambda) = \frac{\sin kx}{k} + \frac{1}{k}\int_0^x q(t)\, s(t, \lambda)\, \sin k(x-t)\, dt, \tag{1.4.5}$$

$$c'(x, \lambda) = -k \sin kx + \int_0^x q(t)\, c(t, \lambda)\, \cos k(x-t)\, dt, \tag{1.4.6}$$

$$s'(x, \lambda) = \cos kx + \int_0^x q(t)\, s(t, \lambda)\, \cos k(x-t)\, dt, \tag{1.4.7}$$

tengliklarning o'ng tomonlariga qo'yib,

$$c(x, \lambda) = \cos kx + \frac{1}{2k}\left(\int_0^x q(t)dt\right) \sin kx -$$

$$-\frac{1}{2k}\int_0^x q(t)\, \sin k(2t-x)\, dt +$$

$$+\frac{1}{k^2}\int_0^x q(t) A(t, \lambda)\, \sin k(x-t)\, dt, \tag{1.4.8}$$

$$s(x,\lambda)=\frac{\sin kx}{k}-\frac{1}{2k^2}\left(\int\limits_0^x q(t)dt\right)\cos kx+$$

$$+\frac{1}{2k^2}\int\limits_0^x q(t)\cos k(2t-x)dt+$$

$$+\frac{1}{k^3}\int\limits_0^x q(t)\,B(t,\lambda)\,\sin k(x-t)\,dt, \qquad (1.4.9)$$

$$c'(x,\lambda)=-k\sin kx+\frac{1}{2}\left(\int\limits_0^x q(t)\;dt\right)\cos kx+$$

$$+\frac{1}{2}\int\limits_0^x q(t)\cos k(2t-x)\,dt+$$

$$+\frac{1}{k}\int\limits_0^x q(t)A(t,\lambda)\,\cos k(x-t)\,dt, \qquad (1.4.10)$$

$$s'(x,\lambda)=\cos kx+\frac{1}{2k}\left(\int\limits_0^x q(t)dt\right)\sin kx+$$

$$+\frac{1}{2k}\int\limits_0^x q(t)\,\sin k(2t-x)\,dt+$$

$$+\frac{1}{k^2}\int\limits_0^x q(t)B(t,\lambda)\,\cos k(x-t)\,dt, \qquad (1.4.11)$$

tasvirlarni hosil qilamiz. (1.4.8) tenglikning oxirida turgan integralni baholaymiz:

$$\left|\int\limits_0^x q(t)\,A(t,\lambda)\,\sin k\,(x-t)\,dt\right|\leq$$

$$\leq \int_0^x |q(t)|\,|A(t,\lambda)|\,|\sin k\,(x-t)\,|dt \leq$$

$$\leq \int_0^x |q(t)| \cdot \left\{ 2\int_0^t |q(s)|ds \cdot e^{|\tau|t} \right\} \cdot e^{|\tau|(x-t)}dt =$$

$$\leq 2e^{|\tau|x} \cdot \int_0^x |q(t)| \cdot \left\{ \int_0^t |q(s)|ds \right\} dt = \left\{ \int_0^x |q(t)|dt \right\}^2 \cdot e^{|\tau|x}.$$

Xuddi shuningdek, ushbu

$$\left| \int_0^x q(t)\,B(t,\lambda)\,\sin k\,(x-t)\,dt \right| \leq \left\{ \int_0^x |q(t)|dt \right\}^2 \cdot e^{|\tau|x},$$

$$\left| \int_0^x q(t)\,A(t,\lambda)\cos k\,(x-t)\,dt \right| \leq \left\{ \int_0^x |q(t)|dt \right\}^2 \cdot e^{|\tau|x};$$

$$\left| \int_0^x q(t)\,B(t,\lambda)\cos k\,(x-t)\,dt \right| \leq \left\{ \int_0^x |q(t)|dt \right\}^2 \cdot e^{|\tau|x},$$

tengsizliklarning o'rinli ekanligini ko'rsatish mumkin. (1.4.8)-(1.4.11) tasvirlarda yuqoridagi baholashlardan foydalansak, quyidagi asimptotikalarga ega bo'lamiz:

$$c(x,\lambda) = \cos kx + \frac{1}{2k}\left(\int_0^x q(t)dt \right) \sin kx -$$

$$-\frac{1}{2k}\int_0^x q(t)\sin k(2t-x)dt + \underline{O}\left(\frac{e^{|\tau|x}}{k^2} \right), \qquad (1.4.12)$$

$$s(x,\lambda) = \frac{\sin kx}{k} - \frac{1}{2k^2}\left(\int_0^x q(t)dt \right) \cos kx +$$

$$+\frac{1}{2k^2}\int_0^x q(t)\cos k(2t-x)dt+\underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^3}\right), \qquad (1.4.13)$$

$$c'(x,\lambda)=-k\sin k\,x+\frac{1}{2}\left(\int_0^x q(t)\,dt\right)\cos kx+$$

$$+\frac{1}{2}\int_0^x q(t)\,\cos k(2t-x)\,dt+\underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k}\right), \qquad (1.4.14)$$

$$s'(x,\lambda)=\cos k\,x+\frac{1}{2k}\left(\int_0^x q(t)\,dt\right)\sin kx+$$

$$+\frac{1}{2k}\int_0^x q(t)\,\sin k(2t-x)\,dt+\underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right). \qquad (1.4.15)$$

Teorema 1.4.1. *Agar $\varphi(x,\lambda)$ orqali (1.4.1) tenglamaning*

$$\varphi(0,\lambda)=1, \quad \varphi'(0,\lambda)=h$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilasak, u holda

$$\varphi(x,\lambda)=c(x,\lambda)+h\,s(x,\lambda)$$

bo'ladi. Agar $q(x)\in C[0,\pi]$, $x\in[0,\pi]$, $\sqrt{\lambda}=k=\sigma+i\tau$ bo'lsa, u holda $\varphi(x,\lambda)$ va $\varphi'(x,\lambda)$ funksiyalar uchun ushbu

$$\varphi(x,\lambda)=\cos kx+\left(h+\frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin kx}{k}-$$

$$-\frac{1}{2k}\int_0^x q(t)\sin k(2t-x)dt+\underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right),$$

$$\varphi'(x,\lambda)=-k\sin k\,x+\left(h+\frac{1}{2}\int_0^x q(t)\;dt\right)\cos kx+$$

$$+\frac{1}{2}\int_0^x q(t)\,\cos k(2t-x)\,dt+\underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k}\right),$$

asimptotikalar o'rinli bo'ladi.■

Lemma 1.4.1. *Agar $f(t) \in L^2(0,\ x)$, $x > 0$, $k = \sigma + i\tau$ bo'lsa, u holda ushbu*

$$\int_0^x f(t)\,\cos k(2t-x)\,dt = \bar{\bar{o}}\left(e^{|\tau|x}\right),\quad |k| \to \infty, \qquad (1.4.16)$$

$$\int_0^x f(t)\,\sin k(2t-x)\,dt = \bar{\bar{o}}\left(e^{|\tau|x}\right),\quad |k| \to \infty, \qquad (1.4.17)$$

asimptotikalar o'rinli.

Isbot. Ma'lumki, berilgan $f(t) \in L^2(0,\ x)$ funksiyaga $L^2(0,\ x)$ fazoning normasi bo'yicha yaqinlashuvchi $f_n(t) \in C^1[0,\ x]$ finit funksiyalar ketma-ketligi mavjud. Demak, ixtiyoriy $\varepsilon > 0$ son uchun shunday $n_0 = n_0(\varepsilon)$ nomer topilib, $n \geq n_0(\varepsilon)$ bo'lganida ushbu

$$\sqrt{\int_0^x |f(t)-f_n(t)|^2 dt} < \varepsilon,$$

tengsizlik o'rinli bo'ladi. Bundan foydalanib quyidagi integralni baholaymiz:

$$\left|\int_0^x f(t)\cos k(2t-x)dt\right| \leq \int_0^x |f(t)-f_{n_0}(t)||\cos k(2t-x)|dt+$$

$$+\left|\int_0^x f_{n_0}(t)\cos k(2t-x)dt\right|,$$

$$\int_0^x |f(t)-f_{n_0}(t)||\cos k(2t-x)|dt \leq$$

$$\leq \sqrt{\int_0^x |f(t)-f_{n_0}(t)|^2 dt} \cdot e^{|\tau|x}\sqrt{x} \leq \varepsilon \cdot e^{|\tau|x}\sqrt{x}.$$

$$\left|\int_0^x f_{n_0}(t)\cos k(2t-x)dt\right| = \left|\int_0^x f_{n_0}(t)d\left(\frac{\sin k(2t-x)}{2k}\right)\right|,$$

$$\left|-\frac{1}{2k}\int_0^x f'_{n_0}(t)\sin k(2t-x)dt\right| \leq$$

$$\leq \frac{1}{2|k|}\int_0^x |f'_{n_0}(t)|e^{|\tau||2t-x|}dt \leq \frac{M}{|k|}e^{|\tau|x} < \varepsilon \cdot e^{|\tau|x}, \quad (|k| > R(\varepsilon)).$$

Natijada ushbu

$$\left|\int_0^x f(t)\cos k(2t-x)dt\right| < \varepsilon \cdot e^{|\tau|x}(\sqrt{x}+1), \quad (|k| > R(\varepsilon)),$$

tengsizlikka ega bo'lamiz. Bu esa (1.4.16) asimptotik tenglik o'rinli bo'lishini bildiradi. (1.4.17) tenglik ham shu tarzda isbot qilinadi.■

Lemma 1.4.2. *Agar $q(x) \in C^1[0,\pi]$, $x \in [0,\pi]$, $\sqrt{\lambda} = k = \sigma + i\tau$ bo'lsa, u holda quyidagi asimptotikalar o'rinli:*

$$c(x,\lambda) = \cos kx + \frac{1}{2k}\left(\int_0^x q(t)dt\right)\sin kx + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right), (|k| \to \infty),$$

$$s(x,\lambda) = \frac{\sin kx}{k} - \frac{1}{2k^2}\left(\int_0^x q(t)dt\right)\cos kx + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^3}\right), (|k| \to \infty),$$

$$c'(x,\lambda) = -k\sin kx + \frac{1}{2}\left(\int_0^x q(t)\,dt\right)\cos kx + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k}\right), (|k| \to \infty),$$

(1.4.18)

$$s'(x,\lambda)=\cos kx+\frac{1}{2k}\left(\int\limits_0^x q(t)\,dt\right)\sin kx+\underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right),\ (|k|\to\infty).$$

Isbot. Bo'laklab integrallash qoidasidan foydalanib, quyidagi integrallarni hisoblaymiz:

$$\int\limits_0^x q(t)\sin k(2t-x)dt=\int\limits_0^x q(t)d\left(-\frac{\cos k(2t-x)}{2k}\right)=$$

$$=-\frac{1}{2k}[q(x)-q(0)]\cos kx+\frac{1}{2k}\int\limits_0^x q'(t)\cos k(2t-x)dt,\quad (1.4.19)$$

$$\int\limits_0^x q(t)\cos k(2t-x)dt=\int\limits_0^x q(t)d\left(\frac{\sin k(2t-x)}{2k}\right)=$$

$$=\frac{1}{2k}[q(x)+q(0)]\sin kx-\frac{1}{2k}\int\limits_0^x q'(t)\sin k(2t-x)dt.\quad (1.4.20)$$

Lemma 1.4.1 dan foydalanib, ushbu

$$\int\limits_0^x q(t)\sin k(2t-x)dt=$$

$$=-\frac{1}{2k}[q(x)-q(0)]\cos kx+\bar{\bar{o}}\left(\frac{e^{|\tau|x}}{k}\right),(|k|\to\infty),$$

$$\int\limits_0^x q(t)\cos k(2t-x)dt=$$

$$=\frac{1}{2k}[q(x)+q(0)]\sin kx+\bar{\bar{o}}\left(\frac{e^{|\tau|x}}{k}\right),(|k|\to\infty),$$

asimptotikalarni topamiz. Bu ifodalarni (1.4.12)-(1.4.15) formulalarga qo'ysak, (1.4.18) asimptotik tengliklar kelib chiqadi.■

Teorema 1.4.2. *Agar* $q(x) \in C^1[0,\pi]$, $x \in [0,\pi]$, $\sqrt{\lambda} = k = \sigma + i\tau$ *bo'lsa, u holda quyidagi*

$$c(x,\lambda) = \cos kx + \frac{1}{2k}a(x)\sin kx +$$

$$+\frac{1}{4k^2}\left[q(x) - q(0) - \frac{1}{2}a^2(x)\right]\cos kx -$$

$$-\frac{1}{4k^2}\int\limits_0^x q'(t)\cos k(2t-x)dt + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^3}\right), \quad (|k| \to \infty),$$

$$s(x,\lambda) = \frac{1}{k}\sin kx - \frac{1}{2k^2}a(x)\cos kx +$$

$$+\frac{1}{4k^3}\left[q(x) + q(0) - \frac{1}{2}a^2(x)\right]\sin kx -$$

$$-\frac{1}{4k^3}\int\limits_0^x q'(t)\sin k(2t-x)dt + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^4}\right), \quad (|k| \to \infty),$$

$$c'(x,\lambda) = -k\sin kx + \frac{1}{2}a(x)\cos kx +$$

$$+\frac{1}{4k}\left[q(x) + q(0) + \frac{1}{2}a^2(x)\right]\sin kx -$$

$$-\frac{1}{4k}\int\limits_0^x q'(t)\sin k(2t-x)dt + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right), \quad (|k| \to \infty),$$

$$s'(x,\lambda) = \cos kx + \frac{1}{2k}a(x)\sin kx -$$

$$-\frac{1}{4k^2}\left[q(x) - q(0) + \frac{1}{2}a^2(x)\right]\cos kx +$$

$$+\frac{1}{4k^2}\int\limits_0^x q'(t)\cos k(2t-x)dt + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^3}\right), \quad (|k| \to \infty),$$

asimptotikalar o'rinli bo'ladi. Bu yerda

$$a(x) = \int_0^x q(t)dt.$$

Isbot. Quyidagi

$$c(x,\lambda) = \cos kx + \frac{1}{2k}a(x)\sin kx + \frac{1}{k^2}A_1(x,\lambda),$$
$$|A_1(x,\lambda)| \le C \cdot e^{|\tau|x},$$
$$s(x,\lambda) = \frac{1}{k}\sin kx - \frac{1}{2k^2}a(x)\cos kx + \frac{1}{k^3}B_1(x,\lambda),$$
$$|B_1(x,\lambda)| \le C \cdot e^{|\tau|x},$$

ifodalarni (1.4.4)-(1.4.7) tengliklarning o'ng tomoniga qo'yib, yuqoridagi kabi mulohaza yuritsak, natijada teoremada keltirilgan tasdiq kelib chiqadi.■

Natija 1.4.2. Agar $q(x) \in C^1[0,\pi]$, $x \in [0,\pi]$, $\sqrt{\lambda} = k = \sigma + i\tau$ bo'lsa, u holda quyidagi

$$\varphi(x,\lambda) = \cos kx + \left(h + \frac{1}{2}a(x)\right)\frac{\sin kx}{k} +$$
$$+ \left\{\frac{1}{4}[q(x) - q(0)] - \frac{h}{2}a(x) - \frac{1}{8}a^2(x)\right\} \cdot \frac{\cos kx}{k^2} -$$
$$-\frac{1}{4k^2}\int_0^x q'(t)\cos k(2t-x)dt + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^3}\right), \quad (|k| \to \infty),$$

$$\varphi'(x,\lambda) = -k\sin kx + \left(h + \frac{1}{2}a(x)\right)\cos kx +$$
$$+\left\{\frac{1}{4}[q(x) + q(0)] + \frac{h}{2}a(x) + \frac{1}{8}a^2(x)\right\} \cdot \frac{\sin kx}{k} -$$
$$-\frac{1}{4k}\int_0^x q'(t)\sin k(2t-x)dt + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right), \quad (|k| \to \infty),$$

asimptotik formulalar o'rinli.

Mustaqil yechish uchun mashqlar

1. Koshi funksiyasi yordamida quyidagi Koshi masalalariga ekvivalent bo'lgan integral tenglamalar tuzing va u yordamida yechim asimptotikasining dastlabki ikkita hadini toping ($0 \le x \le \pi$):

$$a)\begin{cases} -y'' - xy = \lambda y, \\ y(0) = 0, \\ y'(0) = 1, \end{cases} \qquad b)\begin{cases} -y'' - xy = \lambda y, \\ y(0) = 1, \\ y'(0) = 0, \end{cases}$$

$$c)\begin{cases} -y'' - xy = \lambda y, \\ y(0) = 1, \\ y'(0) = h, \end{cases} \qquad d)\begin{cases} -y'' - x^2y = \lambda y, \\ y(0) = 0, \\ y'(0) = 1, \end{cases}$$

$$e)\begin{cases} -y'' - x^2y = \lambda y, \\ y(0) = 1, \\ y'(0) = 0, \end{cases} \qquad f)\begin{cases} -y'' - x^2y = \lambda y, \\ y(0) = 1, \\ y'(0) = h. \end{cases}$$

5-§. Shturm-Liuvill masalasining xos qiymatlari uchun asimptotik formulalar

Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.5.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.5.2}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy funksiya va h, H chekli haqiqiy sonlar.

$\varphi(x, \lambda)$ orqali (1.5.1) tenglamaning

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

Aniqlanishiga ko'ra, $\varphi(x,\lambda)$ funksiya chegaraviy shartlardan birinchisini qanoatlantiradi. (1.5.1)+(1.5.2) chegaraviy masalaning xos qiymatlarini topish uchun bu funksiyani chegaraviy shartlardan ikkinchisiga qo'yamiz:

$$\Delta(\lambda) \equiv \varphi'(\pi,\lambda) + H\varphi(\pi,\lambda) = 0. \tag{1.5.3}$$

Hosil bo'lgan tenglamaga (1.5.1)+(1.5.2) Shturm-Liuvill masalasining xarakteristik tenglamasi deyiladi. Uning ildizlari xos qiymatlardan iborat bo'ladi.

Teorema 1.5.1. *(1.5.1)+(1.5.2) Shturm-Liuvill masalasining xos qiymatlaridan tuzilgan to'plam quyidan chegaralangan, ya'ni shunday son mavjudki, undan kichik xos qiymat yo'q.*

Isbot. Teskarisini faraz qilaylik, ya'ni Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan tuzilgan to'plam quyidan chegaralanmagan deb hisoblaylik. U holda, bu to'plamdan $-\infty$ ga intiladigan xos qiymatlardan tuzilgan qismiy ketma-ketlik ajratib olish mumkin bo'ladi. Aytaylik, $\{\lambda_n\}_{n=0}^{\infty}$ shu shartni qanoatlantiruvchi ketma-ketlik bo'lsin. $\mu_n = -\lambda_n > 0$ belgilash kiritamiz.

Agar ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{\sqrt{\lambda}}\right), \quad (|\lambda| \to \infty), \tag{1.5.4}$$

$$\varphi'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \underline{\underline{O}}(e^{|\mathrm{Im}\sqrt{\lambda}|x}), \quad (|\lambda| \to \infty), \tag{1.5.5}$$

asimptotik formulalarda, λ parametrning haqiqiy manfiy qiymatlarini qarasak va $\mu = -\lambda > 0$ belgilashni kiritsak, quyidagi formulalarni hosil qilamiz:

$$\varphi(x,\lambda) = ch\sqrt{\mu}x + \underline{\underline{O}}\left(\frac{e^{\sqrt{\mu}x}}{\sqrt{\mu}}\right), \quad \mu \to +\infty, \tag{1.5.6}$$

$$\varphi'(x,\lambda) = \sqrt{\mu}sh\sqrt{\mu}x + \underline{\underline{O}}\left(e^{\sqrt{\mu}x}\right), \quad \mu \to +\infty. \tag{1.5.7}$$

(1.5.6) va (1.5.7) asimptotik formulalarni (1.5.3) xarakteristik tenglamaga qo'yib,

$$\sqrt{\mu_n} sh\sqrt{\mu_n}\pi + Hch\sqrt{\mu_n}\pi + \underline{\underline{O}}(e^{\sqrt{\mu_n}\pi}) = 0, \quad n \to \infty,$$

$$\sqrt{\mu_n}\frac{sh\sqrt{\mu_n}\pi}{e^{\sqrt{\mu_n}\pi}} + H\frac{ch\sqrt{\mu_n}\pi}{e^{\sqrt{\mu_n}\pi}} + \underline{\underline{O}}(1) = 0, \quad n \to \infty,$$

bo'lishini ko'ramiz. $\mu_n \to \infty$ bo'lgani uchun oxirgi tenglikning chap tomoni ∞ ga intiladi, o'ng tomoni esa nolga teng, ziddiyat.■

Natija 1.5.1. Butun funksiyaning nollar to'plami chekli limitik nuqtaga ega bo'lmaganligi uchun Shturm-Liuvill chegaraviy masalasi xos qiymatlari to'plami chekli limitik nuqtaga ega bo'lmaydi. Chunki xarakteristik tenglamaning chap tomoni $\Delta(\lambda)$ butun funksiya. Xarakteristik tenglama faqat haqiqiy ildizlarga ega bo'lgani uchun xos qiymatlar ketma-ketligi faqat $+\infty$ ga intilishi mumkin.

Teorema 1.5.2. *(1.5.1)+(1.5.2) Shturm-Liuvill chegaraviy masalasining cheksizta xos qiymati mavjud. Bu xos qiymatlarni o'sib borish tartibida λ_n, $n = 0, 1, 2, \dots$ orqali belgilasak, ushbu*

$$\sqrt{\lambda_n} = n + \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty, \tag{1.5.8}$$

asimptotik formula o'rinli bo'ladi.

Isbot. (1.5.1)+(1.5.2) masalaning xos qiymatlari orasida cheklitasi manfiy bo'lishi mumkin. Demak, n ning biror qiymatidan boshlab barchasi musbat bo'ladi. Biz λ_n ketma-ketlik uchun n ning yetarlicha katta qiymatlaridagi asimptotikasini keltirib chiqaramiz. Buning uchun ushbu

$$\varphi(\pi, \lambda) = \cos k\pi + \underline{\underline{O}}\left(\frac{1}{k}\right), \quad k \to +\infty, \tag{1.5.9}$$

$$\varphi'(\pi, \lambda) = -k\sin k\pi + \underline{\underline{O}}(1), \quad k \to +\infty, \tag{1.5.10}$$

asimptotik formulalardan foydalanamiz. Bu yerda $k = \sqrt{\lambda} > 0$.

(1.5.9) va (1.5.10) ifodalarni xarakteristik tenglamaga qo'yamiz:

$$-k\sin k\pi + H\cos k\pi + \underline{\underline{O}}(1) = 0, \quad k \to +\infty.$$

Bu tenglamadan

$$\sin k\pi = H\cdot\frac{\cos k\pi}{k} + \underline{\underline{O}}\left(\frac{1}{k}\right), \quad k \to +\infty,$$

$$\sin k\pi = \underline{\underline{O}}\left(\frac{1}{k}\right), \quad k \to +\infty, \qquad (1.5.11)$$

kelib chiqadi. (1.5.11) tenglamaning cheksizta ildizi bo'lib, bu ildizlar butun sonlar atroflarida joylashgan bo'ladi, aks holda (1.5.11) tenglama o'ng tomoni nolga intiladi va chap tomoni nolga intilmaydi. Shuning uchun

$$k_n = m_n + \delta_n,$$

bo'ladi. Bu yerda m_n butun son va

$$\delta_n \to 0, \quad (n \to \infty).$$

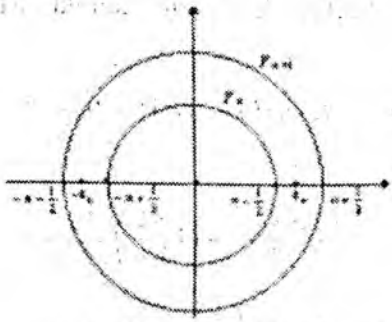
Rushe teoremasiga asoslanib, $m_n = n$ bo'lishini ko'rsatamiz. Buning uchun

$f(k) = -k\sin k\pi$ va $g(k) = \varphi'(\pi, k^2) + H\varphi(\pi, k^2) + k\sin k\pi$ funksiyalarni ko'rib chiqamiz. Bu funksiyalarning yig'indisi (1.5.1)+(1.5.2) chegaraviy masalaning xarakteristik funksiyasini beradi. $g(k)$ funksiya uchun (1.5.4) va (1.5.5) asimptotik formulalarga ko'ra

$$g(k) = \underline{\underline{O}}\left(e^{|\operatorname{Im} k|\pi}\right), \quad (|k| \to \infty),$$

topamiz.

γ_n orqali ushbu $k = \left(\left(n - \frac{1}{2}\right)\right)^{i\alpha}$, $0 \le \alpha \le 2\pi$ aylanani belgilaymiz. Bu yerda n natural son.



1-rasm.

n natural sonning biror n_0 qiymatidan boshlab, γ_n aylana ustida $|f(k)| > |g(k)|$ bo'ladi. Haqiqatan ham,

$$|f(k)| = |-k \sin k\pi| = \left| \left(n - \frac{1}{2} \right) e^{i\alpha} \sin \left\{ \pi \left(n - \frac{1}{2} \right) e^{i\alpha} \right\} \right| =$$

$$= \underline{\underline{O}} \left(\left(n - \frac{1}{2} \right) e^{\pi \left(n - \frac{1}{2} \right) |\sin \alpha|} \right),$$

$$|g(k)| = \underline{\underline{O}} \left(e^{\pi \left(n - \frac{1}{2} \right) |\sin \alpha|} \right),$$

bo'lgani uchun

$$\frac{|f(k)|}{|g(k)|} = \underline{\underline{O}}(n) \to \infty,$$

bo'ladi; xususan n natural sonning biror n_0 qiymatidan boshlab, ushbu $\frac{|f(k)|}{|g(k)|} > 1$ tengsizlik bajariladi.

$f(k)$ funksiyaning γ_n aylana ichidagi karrasiz ildizlari ± 1, ± 2, ..., $\pm(n-1)$ bo'lib, 0 soni ikki karrali ildiz bo'ladi, barcha ildizlarining soni $2n$ ta. Rushe teoremasiga ko'ra qaralayotgan aylana ichida $f(k)$ va $f(k)+g(k)$ funksiyalar bir xil sondagi ildizlarga ega bo'ladi. Bunga asosan, $f(k)+g(k)$ funksiya γ_n aylana ichidagi ildizlarining soni $2n$ ta. γ_{n+1} aylananing ichida esa $2(n+1) = 2n+2$ ta ildizga ega, ya'ni, γ_n va γ_{n+1} aylanalar bilan chegaralangan halqada 2 ta ildiz joylashgan. $f(k)+g(k) = \varphi'(\pi, k^2) + H\varphi(\pi, k^2)$

juft funksiya bo'lgani uchun uning γ_{n+1} aylana ichidagi barcha ildizlari: $\pm k_0$, $\pm k_1$, ..., $\pm k_{n-1}$ bo'ladi. $\pm k_n$ ildizlarning qaralayotgan halqada yotishini va haqiqiyligini e'tiborga olsak, $k_n \in \left(n-\frac{1}{2}, n+\frac{1}{2}\right)$, $n > n_0$, ya'ni $|k_n - n| < \frac{1}{2}$, $n > n_0$ bo'lishi ko'rinadi. Demak,

$$k_n = n + \delta_n, \quad |\delta_n| < \frac{1}{2}, \quad n > n_0. \tag{1.5.12}$$

δ_n ning asimptotikasini o'rganish maqsadida (1.5.12) ifodani (1.5.11) tenglikka qo'yamiz:

$$\sin(\pi n + \pi\delta_n) = \underline{\underline{O}}\left(\frac{1}{n+\delta_n}\right), \quad n \to \infty,$$

$$\sin(\pi\delta_n) = \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty,$$

$$\delta_n = \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty. \tag{1.5.13}$$

(1.5.13) ifodani (1.5.12) tenglikka qo'ysak, (1.5.8) formulaga ega bo'lamiz.■

(1.5.1)+(1.5.2) Shturm-Liuvill chegaraviy masalasining λ_n, $n = 0, 1, 2, \ldots$ xos qiymatlari uchun topilgan (1.5.8) asimptotik formulani yanada aniqlashtirish mumkin.

Teorema 1.5.3. *Agar $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya bo'lib, (1.5.1)+(1.5.2) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\sqrt{\lambda_n} = n + \frac{c_1}{n} + \frac{\alpha_n}{n}, \tag{1.5.14}$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_1 = \frac{h+H}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\alpha_n\} \in l_2. \tag{1.5.15}$$

Isbot. Quyidagi

$$\varphi(x,\lambda) = \cos kx + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin kx}{k} -$$

$$-\frac{1}{2k}\int_0^x q(t)\sin k(2t-x)dt + \underline{\underline{O}}\left(\frac{1}{k^2}\right),$$

$$\varphi'(x,\lambda) = -k\sin k\,x + \left(h + \frac{1}{2}\int_0^x q(t)\ dt\right)\cos kx +$$

$$+\frac{1}{2}\int_0^x q(t)\ \cos k(2t-x)\ dt + \underline{\underline{O}}\left(\frac{1}{k}\right),$$

asimptotik formulalar yordamida xarakteristik funksiyaning asimptotikasini keltirib chiqaramiz:

$$\Delta(\lambda) \equiv \varphi'(\pi,\lambda) + H\varphi(\pi,\lambda) =$$

$$= -k\sin k\pi + \left(h + H + \frac{1}{2}\int_0^\pi q(t)dt\right)\cos k\pi +$$

$$+\frac{1}{2}\int_0^\pi q(t)\ \cos k(2t-\pi)\ dt + \underline{\underline{O}}\left(\frac{1}{k}\right).$$

Bunga ko'ra, quyidagi

$$F(k) = \Delta(k^2) + k\sin k\pi - \left(h + H + \frac{1}{2}\int_0^\pi q(t)dt\right)\cos k\pi, \tag{1.5.16}$$

funksiya uchun ushbu

$$F(k) = \frac{1}{2}\int_0^\pi q(t)\ \cos k(2t-\pi)\ dt + \underline{\underline{O}}\left(\frac{1}{k}\right), \tag{1.5.17}$$

asimptotik formula o'rinli bo'ladi.

Agar $\tilde{\alpha}_n = F(k_n)$ belgilash kiritsak, $k_n = n+\delta_n$ bo'lgani uchun (1.5.17) tenglikka asosan $\{\tilde{\alpha}_n\} \in l_2$ ekanligi kelib chiqadi. (1.5.16) ifodaga asosan

$$(n+\delta_n)\sin(n+\delta_n)\pi - \left(h + H + \frac{1}{2}\int\limits_0^\pi q(t)dt\right)\cos(n+\delta_n)\pi = \tilde{\alpha}_n,$$

ya'ni

$$(n+\delta_n)\sin\delta_n\pi - \left(h + H + \frac{1}{2}\int\limits_0^\pi q(t)dt\right)\cos\delta_n\pi = (-1)^n\cdot\tilde{\alpha}_n, \tag{1.5.18}$$

bo'ladi. Ushbu

$$\delta_n = \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n\to\infty,$$

asimptotikadan

$$\sin\delta_n\pi = \delta_n\pi + \underline{\underline{O}}\left(\frac{1}{n^3}\right), \quad \cos\delta_n\pi = 1 + \underline{\underline{O}}\left(\frac{1}{n^2}\right), \tag{1.5.19}$$

kelib chiqadi. (1.5.19) ifodalarni (1.5.18) tenglikka qo'ysak,

$$n\delta_n\pi + \underline{\underline{O}}\left(\frac{1}{n^2}\right) - \left(h + H + \frac{1}{2}\int\limits_0^\pi q(t)dt\right) + \underline{\underline{O}}\left(\frac{1}{n^2}\right) = (-1)^n\cdot\tilde{\alpha}_n,$$

ya'ni

$$\delta_n = \frac{1}{n}\left(\frac{h+H}{\pi} + \frac{1}{2\pi}\int\limits_0^\pi q(t)dt\right) + \frac{1}{n\pi}\left\{(-1)^n\cdot\tilde{\alpha}_n + \underline{\underline{O}}\left(\frac{1}{n^2}\right)\right\},$$

tenglik hosil bo'ladi. Buni $k_n = n+\delta_n$ tenglikka qo'ysak, (1.5.14) formulaga ega bo'lamiz.■

Izoh 1.5.1. (1.5.1) + (1.5.2) chegaraviy masalaning $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlari uchun olingan (1.5.14) + (1.5.15) asimptotik formula $q(x)$ haqiqiy funksiya $L^2(0,\pi)$ fazoga tegishli bo'lgan holda ham o'rinli bo'ladi.

Teorema 1.5.4. *Agar $q(x) \in C^1[0, \pi]$ haqiqiy funksiya bo'lib, (1.5.1)+ (1.5.2) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\sqrt{\lambda_n} = n + \frac{c_1}{n} + \frac{\beta_n}{n^2}, \tag{1.5.20}$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_1 = \frac{h+H}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\beta_n\} \in l_2.$$

Isbot. Bu holda $\varphi(x, \lambda)$ yechim va uning hosilasi uchun quyidagi asimptotik formulalar o'rinli

$$\varphi(x,\lambda) = \cos kx + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin kx}{k} +$$

$$+\left\{\frac{1}{4}[q(x) - q(0)] - \frac{h}{2}\int_0^x q(t)dt - \frac{1}{8}\left(\int_0^x q(t)dt\right)^2\right\}\cdot\frac{\cos kx}{k^2} -$$

$$-\frac{1}{4k^2}\int_0^x q'(t)\cos k(2t-x)dt + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^3}\right),$$

$$\varphi'(x,\lambda) = -k\sin kx + \left(h + \frac{1}{2}\int_0^x q(t)\ dt\right)\cos kx +$$

$$+\left\{\frac{1}{4}[q(x) + q(0)] + \frac{h}{2}\int_0^x q(t)dt + \frac{1}{8}\left(\int_0^x q(t)dt\right)^2\right\}\cdot\frac{\sin kx}{k} -$$

$$-\frac{1}{4k}\int_0^x q'(t)\sin k(2t-x)dt + \underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^2}\right).$$

Bu formulararga asosan, xarakteristik funksiya ushbu

$$\Delta(k^2) = -k\sin k\pi + \left(h + H + \frac{1}{2}\int_0^\pi q(t)dt\right)\cos k\pi +$$

$$+c\cdot\frac{\sin k\pi}{k} - \frac{1}{4k}\int_0^\pi q'(t)\sin k(2t-\pi)dt + \underline{\underline{O}}\left(\frac{1}{k^2}\right),$$

asimptotikaga ega bo'lishi ko'rinadi. Bu yerda

$$c = Hh + \frac{1}{4}[q(\pi) + q(0] + \frac{h+H}{2}\int_0^\pi q(t)dt + \frac{1}{8}\left(\int_0^\pi q(t)dt\right)^2.$$

Demak, ushbu

$$F(k) = \Delta(k^2) + k\sin k\pi - \pi\, c_1\cos k\pi - c\cdot\frac{\sin k\pi}{k}, \qquad (1.5.21)$$

funksiya uchun quyidagi asimptotika o'rinli:

$$F(k) = -\frac{1}{4k}\int_0^\pi q'(t)\sin k(2t-\pi)dt + \underline{\underline{O}}\left(\frac{1}{k^2}\right).$$

Endi $k = k_n$, $k_n = n+\delta_n$ deb, ushbu $\tilde{\beta}_n = k_n\cdot F(k_n)$ belgilashni kiritsak, u holda $\{\tilde{\beta}_n\} \in l_2$ bo'ladi, chunki teoremaning shartiga ko'ra $q'(x) \in L^2[0,\pi]$. (1.5.21) tenglikda $k = k_n$ desak, u ushbu

$$(n+\delta_n)\sin(n+\delta_n)\pi - \pi\, c_1\cos(n+\delta_n)\pi - c\cdot\frac{\sin(n+\delta_n)\pi}{n+\delta_n} = \frac{\tilde{\beta}_n}{n+\delta_n},$$

ko'rinishni oladi. Oxirgi tenglikdan

$$(n+\delta_n)^2\sin\delta_n\pi - \pi\, c_1(n+\delta_n)\cos\delta_n\pi - c\cdot\sin\delta_n\pi = (-1)^n\cdot\tilde{\beta}_n, \qquad (1.5.22)$$

kelib chiqadi. Oldingi teoremaga ko'ra ushbu

$$\delta_n = \frac{c_1}{n} + \frac{\alpha_n}{n}, \qquad (1.5.23)$$

asimptotik formula o'rinli. Bundan esa quyidagi tengliklar kelib chiqadi:

$$\sin\delta_n\pi = \frac{\pi c_1}{n} + \frac{\pi\alpha_n}{n} + \underline{\underline{O}}\left(\frac{1}{n^3}\right), \quad \cos\delta_n\pi = 1 + \underline{\underline{O}}\left(\frac{1}{n^2}\right). \tag{1.5.24}$$

(1.5.23) va (1.5.24) ifodalarni (1.5.22) ga qo'yamiz:

$$\left(n + \frac{c_1}{n} + \frac{\alpha_n}{n}\right)^2 \left(\frac{\pi c_1}{n} + \frac{\pi\alpha_n}{n} + \underline{\underline{O}}\left(\frac{1}{n^3}\right)\right) -$$

$$-\pi c_1\left(n + \frac{c_1}{n} + \frac{\alpha_n}{n}\right)\left(1 + \underline{\underline{O}}\left(\frac{1}{n^2}\right)\right) - c\cdot\underline{\underline{O}}\left(\frac{1}{n}\right) = (-1)^n\cdot\tilde{\beta}_n,$$

$$\left(n^2 + \underline{\underline{O}}(1)\right)\left(\frac{\pi c_1}{n} + \frac{\pi\alpha_n}{n} + \underline{\underline{O}}\left(\frac{1}{n^3}\right)\right) - \pi c_1 n + \underline{\underline{O}}\left(\frac{1}{n}\right) = (-1)^n\cdot\tilde{\beta}_n.$$

Oxirgi tenglikdan

$$\alpha_n = \frac{1}{n\pi}\left\{(-1)^n\cdot\tilde{\beta}_n + \underline{\underline{O}}\left(\frac{1}{n}\right)\right\},$$

kelib chiqadi. Demak, quyidagilar o'rinli:

$$\alpha_n = \frac{\beta_n}{n}, \quad \delta_n = \frac{c_1}{n} + \frac{\beta_n}{n^2}, \quad \{\beta_n\} \in l_2.$$

Buni $k_n = n + \delta_n$ tenglikka qo'ysak, (1.5.20) formulaga ega bo'lamiz.■

Natija 1.5.2. Agar $q(x) \in C^1[0, \pi]$ bo'lsa, u holda (1.5.1)+(1.5.2) Shturm-Liuvill masalasining λ_n, $n = 0, 1, 2, \ldots$ xos qiymatlari uchun quyidagi asimptotik formula o'rinli:

$$\lambda_n = n^2 + c_0 + \frac{\gamma_n}{n}, \quad \{\gamma_n\} \in l_2.$$

Bu yerda

$$c_0 = \frac{2h + 2H}{\pi} + \frac{1}{\pi}\int_0^\pi q(t)dt.$$

Izoh 1.5.2. Shturm-Liuvill tenglamasidagi $q(x)$ koeffitsiyentning silliqlik darajasini oshirish hisobiga (1.5.1)+(1.5.2) chegaraviy masalaning xos qiymatlari uchun olingan (1.5.20) asimptotik formulani ancha aniqlashtirish mumkin. Quyidagi teoremani isbotsiz keltiramiz, bu teoremani isbotlash uchun avvalo Shturm-Liuvill tenglamasining yechimi uchun yanada aniqroq asimptotik formula olish lozim ([97]).

Ta'rif 1.5.1. n marta differensiallanuvchi, hamda n-tartibli hosilasi $L^2(0, \pi)$ fazoga tegishli bo'lgan funksiyalar to'plamiga $W_2^n(0, \pi)$ Sobolev fazosi deyiladi.

Teorema 1.5.5. *Agar $q(x) \in W_2^m[0, \pi]$, $(m \geq 1)$, ya'ni $q(x)$ potensial Sobolev fazosiga qarashli bo'lsa, u holda (1.5.1)+(1.5.2) chegaraviy masalaning $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlari uchun ushbu*

$$\sqrt{\lambda_n} = n + \frac{c_1}{n} + \frac{c_2}{n^2} + ... + \frac{c_{m+1}}{n^{m+1}} + \frac{\beta_n}{n^{m+1}},$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$\{\beta_n\} \in l_2, \quad c_{2p} = 0, \quad p \geq 0.$$

Teorema 1.5.6. *Agar $q(x) \in C^{\infty}[0, \pi]$ bo'lsa, u holda (1.5.1)+(1.5.2) chegaraviy masalaning $\{\lambda_n\}_{n=0}^{\infty}$ xos qiymatlari uchun*

$$\sqrt{\lambda_n} = n + \sum_{k=1}^{\infty} \frac{c_k}{n^k}, \quad c_{2p} = 0, \;\; p \geq 0,$$

asimptotik formula o'rinli bo'ladi.

Isbot. Hisoblashlarni soddalashtirish maqsadida $h = H = 0$ hol bilan cheklanamiz. Bu holda (1.5.1)+(1.5.2) chegaraviy masala quyidagi ko'rinishni oladi:

$$-y'' + q(x)y = \lambda y, \quad x \in [0, \pi], \tag{1.5.25}$$

$$y'(0) = 0, \;\; y'(\pi) = 0\,, \tag{1.5.26}$$

Bu yerda $q(x) \in C^{\infty}[0, \pi]$ cheksiz marta differensiallanuvchi haqiqiy funksiya. (1.5.25) tenglamaning yechimini ushbu

$$y(x, \lambda) = \exp\left\{ i\mu x + \int_0^x \sigma(t, \mu) dt \right\}, \quad \mu = \sqrt{\lambda}, \tag{1.5.27}$$

ko'rinishda izlaymiz. (1.5.27) ifodani (1.5.26) tenglamaga qo'yib, $\sigma(\mu, x)$ funksiyaga nisbatan

$$\sigma' + 2i\mu\sigma + \sigma^2 - q(x) = 0, \qquad (1.5.28)$$

birinchi tartibli differensial tenglamani hosil qilamiz.

Faraz qilaylik, $\sigma(\mu, x)$ funksiya uchun ushbu

$$\sigma(\mu, x) = \sum_{k=1}^{\infty} \frac{\sigma_k(x)}{(2i\mu)^k}, \qquad (1.5.29)$$

yoyilma o'rinli bo'lsin. Bu yerdagi $\sigma_k(x)$ noma'lum funksiyalarni topish uchun (1.5.29) ifodani (1.5.28) tenglamaga qo'yamiz. Natijada hosil bo'lgan tenglikda $(2i\mu)$ parametrning mos darajalari oldidagi koeffitsiyentlarni solishtirib $\sigma_k(x)$ funksiyaga nisbatan quyidagi rekkurent tenglamalarni hosil qilamiz:

$$\sigma_1(x) = q(x), \quad \sigma_1'(x) = -\sigma_2(x),$$

$$\sigma_m(x) = -\sigma_{m-1}'(x) - \sum_{j=1}^{m-1} \sigma_{m-j-1}(x)\sigma_j(x), m = 3, 4, 5, \ldots. \qquad (1.5.30)$$

Bu formulalardan foydalanib, $\sigma_k(x)$ funksiyalarning bir nechtasini topishimiz mumkin:

$$\begin{aligned} &\sigma_2(x) = -q'(x), \\ &\sigma_3(x) = q''(x) - q^2(x), \\ &\sigma_4(x) = -q'''(x) + 4q(x)q'(x), \\ &\sigma_5(x) = q^{(IV)}(x) - 5q'^2(x) - 6q(x)q''(x) + 2q^3(x), \end{aligned} \qquad (1.5.31)$$

Bu yerda $q(x)$ funksiyaning haqiqiyligini hisobga olsak, (1.5.30)+(1.5.31) tengliklardan $\sigma_k(x)$ funksiyalarning ham haqiqiyligi kelib chiqadi.

Agar $\lambda \neq 0$ bo'lsa, u holda (1.5.27) formula (1.5.25) tenglamaning ikkita chiziqli erkli yechimlarini beradi:

$$y_1(x,\lambda) = \exp\left\{ i\mu x + \int_0^x \sigma(t,\mu)dt \right\},$$

$$y_2(x,\lambda) = \exp\left\{ -i\mu x + \int_0^x \sigma(t,-\mu)dt \right\}, \quad \mu = \sqrt{\lambda}. \qquad (1.5.32)$$

Bu yerda $\sqrt{\lambda}$ ildiz qiymati quyidagicha tanlanadi: agar $\lambda < 0$ bo'lsa, u holda $\sqrt{\lambda} = \mu = i\tau$, $\tau > 0$. Boshqa hollarda ildizning qiymati analitik davom qildirish yordamida aniqlanadi. $y_1(x,\lambda)$ va $y_2(x,\lambda)$ yechimlar yordamida ushbu

$$z(x,\mu) = y_1(x,\mu)y_2'(0,\mu) - y_2(x,\mu)y_1'(0,\mu), \qquad (1.5.33)$$

funksiyani tuzib olamiz. Bu funksiya (1.5.26) chegaraviy shartlarning birinchisini qanoatlantiradi, ya'ni $z'(0,\mu) = 0$. (1.5.33) ifodani (1.5.26) chegaraviy shartlarning ikkinchisiga qo'yib, (1.5.25) + (1.5.26) chegaraviy masalaning xarakteristik tenglamasini hosil qilamiz:

$$y_1'(\pi,\mu)y_2'(0,\mu) - y_2'(\pi,\mu)y_1'(0,\mu) = 0. \qquad (1.5.34)$$

(1.5.32) tenglikdan foydalanib (1.5.34) tenglamani quyidagi ko'rinishda yozib olamiz:

$$\exp\left\{ 2i\mu\pi + \int_0^\pi [\sigma(t,\mu) - \sigma(t,-\mu)]dt \right\} = \tau(\mu). \qquad (1.5.35)$$

Bu yerda

$$\tau(\mu) = \frac{[-i\mu + \sigma(t,-\mu)][i\mu + \sigma(0,-\mu)]}{[i\mu + \sigma(t,\mu)][-i\mu + \sigma(0,-\mu)]}. \qquad (1.5.36)$$

(1.5.36) tenglik yordamida aniqlangan $\tau(\mu)$ funksiya uchun ushbu

$$\tau(\mu) = 1 + \sum_{k=1}^{\infty} \frac{\tau_k}{(2i\mu)^k}, \quad \tau_k = \text{const}, \qquad (1.5.37)$$

asimptotik yoyilma o'rinli. (1.5.35) va (1.5.37) tengliklardan

$$2\pi i\mu_k - 2i\sum_{j=0}^{\infty}\frac{(-1)^j a_{2j+1}}{(2\mu_k)^{2j+1}} = \ln\tau(\mu_k) = 2\pi ik + \sum_{j=0}^{\infty}\frac{b_j}{(2i\mu_k)^j} \tag{1.5.38}$$

hosil bo'ladi. Bu yerda

$$a_{2j+1} = \int_0^{\pi}\sigma_{2j+1}(x)dx, \quad b_j = \text{const}, \quad \mu_k = \sqrt{\lambda_k}.$$

$\tau(\mu)$ funksiyaning aniqlanishiga ko'ra

$$\tau(-\mu) = \frac{1}{\tau(\mu)},$$

bo'ladi. Agar μ_k (1.5.36) tenglamaning ildizi bo'lsa, u holda $-\mu_k$ ham shu tenglamani qanoatlantiradi. Bundan (1.5.38) tenglikning o'ng tarafida μ_k ning faqat toq darajalari qatnashishi kelib chiqadi. Haqiqatan ham, (1.5.38) tenglamada μ_k ni $-\mu_k$ bilan, k ni esa $-k$ bilan almashtirsak

$$-2\pi i\mu_k + 2i\sum_{j=0}^{\infty}\frac{a_{2j+1}}{(2\mu_k)^{2j+1}} = -2\pi ik + \sum_{j=0}^{\infty}\frac{(-1)^j b_j}{(2i\mu_k)^j}, \tag{1.5.39}$$

tenglik hosil bo'ladi. (1.5.38) va (1.5.39) tengliklarni qo'shish natijasida

$$\sum_{j=0}^{\infty}\frac{b_{2j}}{(2i\mu_k)^{2j}} = 0,$$

hosil bo'ladi. Bundan $\mu_k \to \infty$ bo'lganda $b_{2j} = 0$, $j = 0, 1, 2, 3, \dots$ kelib chiqadi. Shunday qilib, (1.5.38) tenglikni ushbu

$$\mu_k = -\frac{1}{\pi}\sum_{j=0}^{\infty}\frac{a_{2j+1}}{(2\mu_k)^{2j+1}} = k - \frac{1}{2\pi}\sum_{j=0}^{\infty}\frac{(-1)^j b_{2j+1}}{(2i\mu_k)^{2j+1}}, \tag{1.5.40}$$

ko'rinishda yozish mumkin ekan. Bu tenglikdan

$$\mu_k = k + \underline{O}\left(\frac{1}{k}\right), \tag{1.5.41}$$

kelib chiqadi, chunki $\mu_k \to \infty$. (1.5.41) ifodani (1.5.40) tenglikka qo'yamiz va yana shu jarayonni davom ettirib, μ_k uchun

$$\mu_k = k + \sum_{j=0}^{\infty} \frac{c_{2j+1}}{k^{2j+1}}, \tag{1.5.42}$$

asimptotik yoyilmani topamiz. Bu yerda c_{2j+1} - k ga bog'liq bo'lmaydi, lekin u a_{2j+1}, $\sigma_k(0)$ va $\sigma_k(\pi)$ sonlarga bog'liq bo'ladi.■

6-§. Chegaraviy shartlarida $y(0) = 0$ yoki $y(\pi) = 0$ bo'lgan Shturm-Liuvill masalasining xos qiymatlari uchun asimptotik formulalar

1. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.6.1}$$

$$\begin{cases} y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.6.2}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya va H chekli haqiqiy son.

$s(x, \lambda)$ orqali (1.6.1) tenglamaning ushbu

$$s(0, \lambda) = 0, \quad s'(0, \lambda) = 1,$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. $s(x, \lambda)$ funksiya aniqlanishiga ko'ra chegaraviy shartlardan birinchisini qanoatlantiradi. (1.6.1)+ (1.6.2) masalaning xos qiymatlarini topish uchun bu funksiyani chegaraviy shartlarning ikkinchisiga qo'yamiz:

$$s'(\pi, \lambda) + Hs(\pi, \lambda) = 0. \tag{1.6.3}$$

Hosil bo'lgan tenglamaga (1.6.1)+(1.6.2) Shturm-Liuvill masalasining xarakteristik tenglamasi deyiladi. Uning ildizlari xos qiymatlardan iborat.

Teorema 1.6.1. *(1.6.1)+(1.6.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan tuzilgan to'plam quyidan chegaralangan, ya'ni shunday son mavjudki, undan kichik xos qiymat yo'q.*

Isbot. Teskarisini faraz qilaylik, ya'ni Shturm-Liuvill masalasining xos qiymatlaridan tuzilgan to'plam quyidan chegaralanmagan deb hisoblaylik. U holda, bu to'plamdan $-\infty$ ga intiladigan xos qiymatlardan tuzilgan qismiy ketma-ketlik ajratib olish mumkin bo'ladi. Aytaylik, $\{\lambda_n\}_{n=0}^{\infty}$ shu shartni qanoatlantiruvchi ketma-ketlik bo'lsin. $\mu_n = -\lambda_n > 0$ belgilash kiritamiz.

Agar ushbu

$$s(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \underline{\underline{O}}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{\lambda}\right), \quad (|\lambda| \to \infty), \qquad (1.6.4)$$

$$s'(x,\lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{\sqrt{\lambda}}\right), \quad (|\lambda| \to \infty), \qquad (1.6.5)$$

asimptotik formulalarda, λ parametrning haqiqiy manfiy qiymatlarini qarasak va $\mu = -\lambda > 0$ belgilashni kiritsak, quyidagi formulalarni hosil qilamiz:

$$s(x,\lambda) = \frac{sh\sqrt{\mu}x}{\sqrt{\mu}} + \underline{\underline{O}}\left(\frac{e^{\sqrt{\mu}x}}{\mu}\right), \quad \mu \to +\infty, \qquad (1.6.6)$$

$$s'(x,\lambda) = ch\sqrt{\mu}x + \underline{\underline{O}}\left(\frac{e^{\sqrt{\mu}x}}{\sqrt{\mu}}\right), \quad \mu \to +\infty. \qquad (1.6.7)$$

(1.6.6) va (1.6.7) asimptotik formulalarni xarakteristik tenglamaga qo'yib,

$$ch\sqrt{\mu_n}\pi + \underline{\underline{O}}\left(\frac{e^{\sqrt{\mu_n}\pi}}{\sqrt{\mu_n}}\right) = 0, \qquad n \to \infty,$$

$$\frac{ch\sqrt{\mu_n}\pi}{e^{\sqrt{\mu_n}\pi}} + O\left(\frac{1}{\sqrt{\mu_n}}\right) = 0, \qquad n \to \infty.$$

bo'lishini ko'ramiz. $\mu_n \to \infty$ bo'lgani uchun oxirgi tenglikning chap tomoni $\frac{1}{2}$ ga intiladi, o'ng tomoni esa nolga teng, ziddiyat.■

Natija 1.6.1. Butun funksiyaning nollar to'plami chekli limit nuqtaga ega bo'lmaganligi uchun (1.6.1)+(1.6.2) Shturm-Liuvill chegaraviy masalasi xos qiymatlarining to'plami chekli limit nuqtaga ega bo'lmaydi. Chunki xarakteristik tenglamaning chap tomoni butun funksiya. Xarakteristik tenglama faqat haqiqiy ildizlarga ega bo'lgani uchun xos qiymatlar ketma-ketligi faqat $+\infty$ ga intilishi mumkin.

Teorema 1.6.2. *(1.6.1)+(1.6.2) chegaraviy masalaning cheksizta xos qiymati mavjud. Bu xos qiymatlarni o'sib borish tartibida λ_n, $n = 0, 1, 2, \ldots$ orqali belgilasak, ushbu*

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty, \tag{1.6.8}$$

asimptotik formula o'rinli bo'ladi.

Isbot. (1.6.1)+(1.6.2) chegaraviy masalaning xos qiymatlari orasida cheklitasi manfiy bo'lishi mumkin. Demak, n ning biror qiymatidan boshlab barchasi musbat bo'ladi. Biz λ_n ketma-ketlik uchun n ning yetarlicha katta qiymatlaridagi asimptotikasini keltirib chiqaramiz. Buning uchun ushbu

$$s(\pi, \lambda) = \frac{\sin k\pi}{k} + \underline{\underline{O}}\left(\frac{1}{k^2}\right), \quad k \to +\infty, \tag{1.6.9}$$

$$s'(\pi, \lambda) = \cos k\pi + \underline{\underline{O}}\left(\frac{1}{k}\right), \quad k \to +\infty, \tag{1.6.10}$$

asimptotik formulalardan foydalanamiz. Bu yerda $k = \sqrt{\lambda} > 0$.

(1.6.9) va (1.6.10) ifodalarni xarakteristik tenglamaga qo'yamiz:

$$\cos k\pi + \underline{\underline{O}}\left(\frac{1}{k}\right) = 0, \quad k \to +\infty. \tag{1.6.11}$$

Bu tenglamadan

$$\cos k\pi = \underline{\underline{O}}\left(\frac{1}{k}\right), \quad k \to +\infty, \tag{1.6.12}$$

kelib chiqadi. (1.6.12) tenglamaning cheksizta ildizi bo‘lib, bu ildizlar $\cos k\pi = 0$ tenglamaning ildizlari atrofida joylashgan bo‘ladi, aks holda (1.6.12) tenglama o‘ng tomoni nolga intiladi va chap tomoni nolga intilmaydi. Shuning uchun

$$k_n = m_n + \frac{1}{2} + \delta_n, \tag{1.6.13}$$

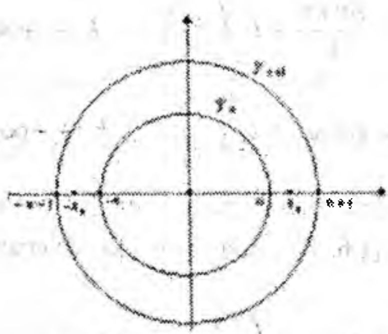
bo‘ladi. Bu yerda m_n butun son va

$$\delta_n \to 0, \quad (n \to \infty). \tag{1.6.14}$$

Rushe teoremasiga asoslanib, $m_n = n$ bo‘lishini ko‘rsatamiz. Buning uchun $f(k) = \cos k\pi$ va $g(k) = s'(\pi, k^2) + Hs(\pi, k^2) - \cos k\pi$ funksiyalarni ko‘rib chiqamiz. Bu funksiyalarning yig‘indisi (1.6.1)+(1.6.2) chegaraviy masalaning xarakteristik funksiyasini beradi. $g(k)$ funksiya uchun (1.6.4) va (1.6.5) asimptotik formulalarga ko‘ra

$$g(k) = \underline{\underline{O}}\left(\frac{e^{|\operatorname{Im} k|\pi}}{k}\right), \quad (|k| \to \infty), \tag{1.6.15}$$

topamiz. γ_n orqali ushbu $k = ne^{i\alpha}$, $0 \leq \alpha \leq 2\pi$ aylanani belgilaymiz. Bu yerda n natural son.



2- rasm.

n natural sonning biror n_0 qiymatidan boshlab, γ_n aylana ustida $|f(k)| > |g(k)|$ bo'ladi. Haqiqatan, ham

$$|f(k)| = |\cos k\pi| = \left|\cos\left\{\pi n e^{i\alpha}\right\}\right| = \underline{\underline{O}}\left(e^{\pi n|\sin\alpha|}\right),$$

$$|g(k)| = \underline{\underline{O}}\left(\frac{e^{\pi n|\sin\alpha|}}{n}\right),$$

bo'lgani uchun

$$\frac{|f(k)|}{|g(k)|} = \underline{\underline{O}}(n) \to \infty$$

bo'ladi, xususan n natural sonning biror n_0 qiymatidan boshlab, ushbu $\frac{|f(k)|}{|g(k)|} > 1$ tengsizlik bajariladi.

$f(k)$ funksiyaning ildizlari karrasiz bo'lib, γ_n aylana ichidagi ildizlar $\pm\frac{1}{2}, \pm\frac{3}{2}, ..., \pm\left(n - \frac{1}{2}\right)$ sonlardan iborat. Ularning soni $2n$ ta. Rushe teoremasiga ko'ra qaralayotgan aylana ichida $f(k)$ va $f(k) + g(k)$ funksiyalar bir xil sondagi ildizlarga ega bo'ladi. Bunga asosan, $f(k) + g(k)$ funksiya γ_n aylana ichidagi ildizlarining soni $2n$ ta. γ_{n+1} aylananing ichida esa $2(n + 1) = 2n + 2$ ta ildizga ega, ya'ni, γ_n va γ_{n+1} aylanalar bilan chegaralangan halqada 2 ta ildiz joylashgan. $f(k) + g(k) = s'(\pi, k^2) + Hs(\pi, k^2)$ juft funksiya bo'lgani uchun uning γ_{n+1} aylana ichidagi barcha ildizlari: $\pm k_0$, $\pm k_1$, ..., $\pm k_{n-1}$, $\pm k_n$ bo'ladi. $\pm k_n$ ildizlarning qaralayotgan halqada yotishini va haqiqiyligini e'tiborga olsak, $k_n \in (n, n+1), n > n_0$, ya'ni $\left|k_n - \left(n + \frac{1}{2}\right)\right| < \frac{1}{2}, n > n_0$ bo'lishi ko'rinadi. Demak,

$$k_n = n + \frac{1}{2} + \delta_n, \quad |\delta_n| < \frac{1}{2}, \quad n > n_0. \qquad (1.6.16)$$

δ_n ning asimptotikasini o'rganish maqsadida (1.6.16) ifodani (1.6.12) tenglikka qo'yamiz:

$$\cos\left(\frac{\pi}{2} + \pi n + \pi\delta_n\right) = \underline{\underline{O}}\left(\frac{1}{n + \frac{1}{2} + \delta_n}\right), \quad n \to \infty,$$

$$\sin(\pi\delta_n) = \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty,$$

$$\delta_n = \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty. \tag{1.6.17}$$

(1.6.17) ifodani (1.6.16) tenglikka qo'ysak, ushbu

$$k_n = n + \frac{1}{2} + \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty, \tag{1.6.18}$$

formulaga ega bo'lamiz. Demak,

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty. \tag{1.6.19}$$

asimptotik formula o'rinli ekan. ■

Quyidagi teoremalarni isbotlashni o'quvchiga qoldiramiz.

Teorema 1.6.3. *Agar $q(x) \in C[0,\pi]$ haqiqiy uzluksiz funksiya bo'lib, (1.6.1)+ (1.6.2) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \frac{c_1}{n + \frac{1}{2}} + \frac{\beta_n}{n}, \tag{1.6.20}$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_1 = \frac{H}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\beta_n\} \in l_2. \tag{1.6.21}$$

Natija 1.6.2. *Agar $q(x) \in C[0, \pi]$ haqiqiy funksiya bo'lib, (1.6.1)+(1.6.2) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 + c_0 + \chi_n,$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_0 = \frac{2H}{\pi} + \frac{1}{\pi}\int_0^{\pi} q(t)dt, \quad \{\chi_n\} \in l_2.$$

Teorema 1.6.3′. *Agar $q(x) \in C^1[0, \pi]$ haqiqiy funksiya bo'lib, (1.6.1)+ (1.6.2) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \frac{c_1}{n + \frac{1}{2}} + \frac{\gamma_n}{n^2},$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_1 = \frac{H}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\gamma_n\} \in l_2.$$

Natija 1.6.2′. *Agar $q(x) \in C^1[0, \pi]$ haqiqiy funksiya bo'lib, (1.6.1)+(1.6.2) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 + c_0 + \frac{\omega_n}{n},$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_0 = \frac{2H}{\pi} + \frac{1}{\pi}\int_0^{\pi} q(t)dt, \quad \{\omega_n\} \in l_2.$$

2. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.6.22}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{1.6.23}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy funksiya va h chekli haqiqiy son.

Teorema 1.6.4. *(1.6.22)+(1.6.23) Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan tuzilgan to'plam quyidan chegaralangan, ya'ni shunday son mavjudki, undan kichik xos qiymat yo'q.*

Natija 1.6.3. Butun funksiyaning nollar to'plami chekli limit nuqtaga ega bo'lmaganligi uchun Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan tuzilgan to'plami chekli limit nuqtaga ega bo'lmaydi. Chunki xarakteristik tenglamaning chap tomoni butun funksiya. Xarakteristik tenglama faqat haqiqiy ildizlarga ega bo'lgani uchun xos qiymatlar ketma-ketligi faqat $+\infty$ ga intilishi mumkin.

Teorema 1.6.5. *(1.6.22)+(1.6.23) chegaraviy masalaning cheksizta xos qiymati mavjud. Bu xos qiymatlarni o'sib borish tartibida* λ_n, $n = 0, 1, 2, \ldots$ *orqali belgilasak, ushbu*

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \underline{O}\left(\frac{1}{n}\right), \quad n \to \infty, \tag{1.6.24}$$

asimptotik formula o'rinli bo'ladi.

Teorema 1.6.6. *Agar* $q(x) \in C[0,\pi]$ *haqiqiy uzluksiz funksiya bo'lib, (1.6.22)+ (1.6.23) chegaraviy masalaning xos qiymatlari* $\{\lambda_n\}_{n=0}^{\infty}$ *bo'lsa, u holda ushbu*

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \frac{c_2}{n+\frac{1}{2}} + \frac{\beta_n}{n}, \tag{1.6.25}$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_2 = \frac{h}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\beta_n\} \in l_2. \tag{1.6.26}$$

Isbot. (1.6.22)+(1.6.23) masalada $t = \pi - x$ almashtirish bajarsak va

$$q_1(t) = q(\pi - t), \quad z(t) = y(\pi - t), \quad H_1 = h$$

belgilashlarni kiritsak, ushbu

$$-z'' + q_1(t)z = \lambda z, \quad 0 \le x \le \pi, \tag{1.6.27}$$

$$\begin{cases} z(0) = 0, \\ z'(\pi) + H_1 z(\pi) = 0, \end{cases} \tag{1.6.28}$$

chegaraviy masala hosil bo'ladi. (1.6.27)+(1.6.28) chegaraviy masala xos qiymatlarining asimptotikasi o'rganilgan edi. Bundan (1.6.25) va (1.6.26) kelib chiqadi.■

Natija 1.6.3′. *Agar $q(x) \in C[0, \pi]$ haqiqiy funksiya bo'lib, (1.6.22)+(1.6.23) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 + c_0 + \chi_n,$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_0 = \frac{2h}{\pi} + \frac{1}{\pi}\int_0^{\pi} q(t)dt, \quad \{\chi_n\} \in l_2.$$

Teorema 1.6.6′. *Agar $q(x) \in C^1[0, \pi]$ haqiqiy funksiya bo'lib, (1.6.22)+ (1.6.23) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\sqrt{\lambda_n} = n + \frac{1}{2} + \frac{c_2}{n + \frac{1}{2}} + \frac{\gamma_n}{n^2},$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_2 = \frac{h}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\gamma_n\} \in l_2.$$

Natija 1.6.3″. *Agar $q(x) \in C^1[0, \pi]$ haqiqiy funksiya bo'lib, (1.6.22)+ (1.6.23) chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsa, u holda ushbu*

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 + c_0 + \frac{\omega_n}{n},$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_0 = \frac{2h}{\pi} + \frac{1}{\pi}\int_0^{\pi} q(t)dt, \quad \{\omega_n\} \in l_2.$$

3. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.6.29}$$

$$\begin{cases} y(0) = 0, \\ y(\pi) = 0, \end{cases} \qquad (1.6.30)$$

Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya.

Quyida keltirilgan teoremalar xuddi yuqoridagi usullar yordamida isbotlanadi.

Teorema 1.6.7. *(1.6.29)÷(1.6.30) Shturm-Liuvill chegaraviy masalasining xos qiymatlaridan tuzilgan to‘plam quyidan chegaralangan, ya’ni shunday son mavjudki, undan kichik xos qiymat yo‘q.*

Teorema 1.6.8. *(1.6.29)÷(1.6.30) Shturm-Liuvill chegaraviy masalasining cheksizta xos qiymati mavjud. Bu xos qiymatlarni o‘sib borish tartibida* λ_n, $n = 1, 2, \ldots$ *orqali belgilasak, ushbu*

$$\sqrt{\lambda_n} = n + \underline{\underline{O}}\left(\frac{1}{n}\right), \quad n \to \infty, \qquad (1.6.31)$$

asimptotik formula o‘rinli bo‘ladi.

Teorema 1.6.9. *Agar* $q(x) \in C[0, \pi]$ *haqiqiy uzluksiz funksiya bo‘lib, (1.6.29)÷(1.6.30) masalaning xos qiymatlari* $\{\lambda_n\}_{n=1}^{\infty}$ *bo‘lsa, u holda ushbu*

$$\sqrt{\lambda_n} = n + \frac{c_3}{n} + \frac{\beta_n}{n}, \qquad (1.6.32)$$

asimptotik formula o‘rinli bo‘ladi. Bu yerda

$$c_3 = \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\beta_n\} \in l_2. \qquad (1.6.33)$$

Natija 1.6.4. *Agar* $q(x) \in C[0, \pi]$ *haqiqiy uzluksiz funksiya bo‘lib, (1.6.29)÷(1.6.30) masalaning xos qiymatlari* $\{\lambda_n\}_{n=1}^{\infty}$ *bo‘lsa, u holda ushbu*

$$\lambda_n = n^2 + c_0 + \omega_n$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_0 = \frac{1}{\pi}\int_0^{\pi} q(t)dt, \quad \{\omega_n\} \in l_2.$$

Teorema 1.6.9'. *Agar* $q(x) \in C^1[0, \pi]$ *haqiqiy funksiya bo'lib, (1.6.29)+ (1.6.30) masalaning xos qiymatlari* $\{\lambda_n\}_{n=1}^{\infty}$ *bo'lsa, u holda ushbu*

$$\sqrt{\lambda_n} = n + \frac{c_3}{n} + \frac{\gamma_n}{n^2},$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_3 = \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \quad \{\gamma_n\} \in l_2.$$

Natija 1.6.4'. *Agar* $q(x) \in C^1[0, \pi]$ *haqiqiy funksiya bo'lib, (1.6.29)+ (1.6.30) masalaning xos qiymatlari* $\{\lambda_n\}_{n=1}^{\infty}$ *bo'lsa, u holda ushbu*

$$\lambda_n = n^2 + c_0 + \frac{\chi_n}{n}$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_0 = \frac{1}{\pi}\int_0^{\pi} q(t)dt, \quad \{\chi_n\} \in l_2.$$

7-§. Normallovchi o'zgarmaslarning va normallangan xos funksiyalarning asimptotikasi

1. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.7.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.7.2}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya va H, h chekli haqiqiy sonlar.

$\varphi(x, \lambda)$ orqali (1.7.1) tenglamaning ushbu

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. Bu yechim uchun quyidagi

$$\varphi(x, \lambda) = \cos\sqrt{\lambda}x + \frac{h}{\sqrt{\lambda}}\sin\sqrt{\lambda}x + \frac{\sin\sqrt{\lambda}x}{2\sqrt{\lambda}}\int_0^x q(t)dt -$$

$$-\frac{1}{2\sqrt{\lambda}}\int_0^x q(t)\sin\sqrt{\lambda}(2t - x)dt + \underline{\underline{O}}\left(\frac{1}{\lambda}\right), \quad \lambda \to +\infty, \tag{1.7.3}$$

asimptotik formula olingan edi. (1.7.1)+(1.7.2) chegaraviy masalaning xos qiymatlarini $\{\lambda_n\}_{n=0}^{\infty}$ orqali belgilaymiz. Oldingi paragraflarda xos qiymatlar uchun quyidagi asimptotik formula keltirib chiqarilgan edi:

$$k_n = \sqrt{\lambda_n} = n + \frac{c_0}{n} + \frac{\gamma_n}{n}. \tag{1.7.4}$$

Bu yerda

$$c_0 = \frac{h + H}{\pi} + \frac{1}{2\pi}\int_0^\pi q(t)dt, \quad \{\gamma_n\} \in l_2. \tag{1.7.5}$$

Endi ortonormallangan xos funksiyalar uchun asimptotik formulalarni topamiz. Buning uchun avvalo $\varphi(x,\lambda_n)$ funksiyaning asimptotikasini o'rganamiz. (1.7.4) ni (1.7.3) ga qo'ysak,

$$\varphi(x,\lambda_n)=\cos(n+\delta_n)x+\left(h+\frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin(n+\delta_n)x}{n+\delta_n}-$$

$$-\frac{1}{2(n+\delta_n)}\int_0^x q(t)\sin[(n+\delta_n)(2t-x)]dt+\underline{\underline{O}}\left(\frac{1}{(n+\delta_n)^2}\right), \tag{1.7.6}$$

bo'ladi. Bu yerda $\delta_n=\frac{c_0}{n}+\frac{\gamma_n}{n}$.

(1.7.6) formulani soddalashtiramiz. Buning uchun avvalo quyidagi tengliklarni olamiz:

$$\frac{1}{n+\delta_n}=\frac{1}{n}\left\{1-\frac{\delta_n}{n}+\left(-\frac{\delta_n}{n}\right)^2+\ldots\right\}=\frac{1}{n}+\underline{\underline{O}}\left(\frac{1}{n^3}\right),$$

$$\cos(n+\delta_n)x=\cos nx\cos\delta_n x-\sin nx\sin\delta_n x=$$

$$=\cos nx\left[1-\frac{\delta_n^2x^2}{2!}+\ldots\right]-\sin nx\left[\delta_n x-\frac{\delta_n^3x^3}{3!}+\ldots\right]=$$

$$=\cos nx-\delta_n x\sin nx+\underline{\underline{O}}\left(\frac{1}{n^2}\right),\quad n\to\infty, \tag{1.7.7}$$

$$\sin(n+\delta_n)x=\sin nx\left[1-\frac{\delta_n^2x^2}{2!}+\ldots\right]+\cos nx\left[\delta_n x-\frac{\delta_n^3x^3}{3!}+\ldots\right]=$$

$$=\sin nx+\delta_n x\cos nx+\underline{\underline{O}}\left(\frac{1}{n^2}\right),\quad n\to\infty. \tag{1.7.8}$$

(1.7.7) va (1.7.8) ifodalarni (1.7.6) tenglikka qo'ysak, ushbu

$$\varphi(x,\lambda_n)=\cos nx-c_0x\frac{\sin nx}{n}+\left(h+\frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin nx}{n}+\frac{\gamma_n(x)}{n},$$

formula kelib chiqadi, ya'ni

$$\varphi(x,\lambda_n)=\cos nx+a(x)\frac{\sin nx}{n}+\frac{\gamma_n(x)}{n}, \tag{1.7.9}$$

bu yerda

$$a(x) = -c_0 x + h + \frac{1}{2}\int\limits_0^x q(t)dt, \ \{\gamma_n(x)\} \in l_2. \qquad (1.7.10)$$

Endi normallovchi o'zgarmaslar uchun asimptotik formulalarni topamiz:

$$\alpha_n^2 = \int\limits_0^\pi \varphi^2(x, \lambda_n)dx = \int\limits_0^\pi \cos^2 nxdx + \frac{1}{n}\int\limits_0^\pi a(x)\sin 2nxdx +$$

$$+\frac{\tilde{\beta}_n}{n} + \underline{\underline{O}}\left(\frac{1}{n^2}\right), \ \{\tilde{\beta}_n\} \in l_2. \qquad (1.7.11)$$

(1.7.11) formulada darajani pasaytirish formulasini va bo'laklab integrallash qoidasini qo'llab, ushbu

$$\alpha_n^2 = \frac{\pi}{2} + \frac{\beta_n}{n}, \ \{\beta_n\} \in l_2.$$

tenglik o'rinli bo'lishini ko'ramiz. Bunga ko'ra

$$\alpha_n = \left\{\frac{\pi}{2}\left(1 + \frac{2\beta_n}{\pi n}\right)\right\}^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}\left(1 + \frac{\omega_n}{n}\right), \ \{\omega_n\} \in l_2 \quad (1.7.12)$$

bo'ladi. (1.7.12) formulaga asosan

$$\frac{1}{\alpha_n} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1 + \frac{\omega_n}{n}} =$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(1 - \frac{\omega_n}{n} + \underline{\underline{O}}\left(\frac{1}{n^2}\right)\right) = \sqrt{\frac{2}{\pi}} + \frac{\tilde{\omega}_n}{n}, \ \{\tilde{\omega}_n\} \in l_2 \quad (1.7.13)$$

bo'lishi ravshan.

Nihoyat, (1.7.13) va (1.7.9) ifodalarni ushbu

$$u_n(x) = \frac{1}{\alpha_n}\varphi(x, \lambda_n),$$

tenglikka qo'yib, $u_n(x)$ normallangan xos funksiyalar uchun quyidagi asimptotik formulani keltirib chiqaramiz:

$$u_n(x) = \left[\sqrt{\frac{2}{\pi}} + \frac{\tilde{\omega}_n}{n}\right] \cdot \left\{\cos nx + a(x)\frac{\sin nx}{n} + \frac{\gamma_n(x)}{n}\right\},$$

$$u_n(x) = \sqrt{\frac{2}{\pi}}\left\{\cos nx + a(x)\frac{\sin nx}{n}\right\} + \frac{\tilde{\gamma}_n(x)}{n}, \ \{\tilde{\gamma}_n(x)\} \in l_2.$$

Oldingi paragraflarda $q(x) \in C^1[0,\pi]$ bo'lgan holda $\varphi(x,\lambda)$ yechim va $\{\lambda_n\}$ xos qiymatlar ketma-ketligi uchun quyiyidagi asimptotik formulalar olingan edi:

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + A(x)\cdot\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + B(x)\cdot\frac{\cos\sqrt{\lambda}x}{\lambda} -$$

$$-\frac{1}{4\lambda}\int\limits_0^x q'(t)\cos\sqrt{\lambda}(2t-x)dt + \underline{\underline{O}}\left(\frac{1}{\lambda\sqrt{\lambda}}\right), \qquad (1.7.13')$$

$$k_n = \sqrt{\lambda_n} = n + \frac{c_0}{n} + \frac{\beta_n}{n^2}, \ \{\beta_n\} \in l_2. \qquad (1.7.13'')$$

Bu yerda

$$A(x) = h + \frac{1}{2}\int\limits_0^x q(t)dt,$$

$$B(x) = \frac{1}{4}[q(x) - q(0)] - \frac{h}{2}\int\limits_0^x q(t)dt - \frac{1}{8}\left(\int\limits_0^x q(t)dt\right)^2.$$

Xuddi oldingi banddagidek, (1.7.13") ifodani (1.7.13') tenglikka qo'yib, ushbu

$$\varphi(x,\lambda_n) = \cos nx + \{c_0 x + A(x)\}\cdot\frac{\sin nx}{n} +$$

$$+\left\{\frac{c_0^2 x^2}{2} + A(x)c_0 x + B(x)\right\}\cdot\frac{\cos nx}{n^2} + \frac{\omega_n(x)}{n^2}, \quad \{\omega_n(x)\} \in l_2$$

asimptotik formulani topamiz. Bunga ko'ra

$$\alpha_n^2 = \frac{\pi}{2}\left(1 + \frac{c}{n^2} + \frac{\tilde{\omega}_n}{n^2}\right), \ \{\tilde{\omega}_n\} \in l_2, \ c = \text{const}.$$

Demak, normallangan xos funksiyalar uchun quyidagi

$$u_n(x) = \sqrt{\frac{2}{\pi}}\left\{\cos nx + (-c_0 x + A(x))\cdot\frac{\sin nx}{n} +\right.$$

$$+C(x)\cdot\frac{\cos nx}{n^2}\Big\}+\frac{\varepsilon_n(x)}{n^2},\quad \{\varepsilon_n(x)\}\in l_2$$

asimptotika o'rinli. Bunda $C(x)$ funksiya $A(x)$ va $B(x)$ funksiyalar orqali ifodalanadi.

Shturm-Liuvill tenglamasidagi $q(x)$ koeffitsiyentning silliqlik darajasini oshirish natijasida α_n normallovchi o'zgarmaslar uchun asimptotikani yanada aniqlashtirish mumkin.

Quyidagi teoremani isbotsiz keltiramiz.

Teorema 1.7.1. *1) Agar $q(x)\in W_2^N[0,\pi]$, $N\geq 1$ Sobolev fazosiga tegishli bo'lsa, u holda*

$$\alpha_n^2=\frac{\pi}{2}+\sum_{j=1}^{N+1}\frac{c_j}{n^j}+\frac{\chi_n}{n^{N+1}},\ \{\chi_n\}\in l_2 \tag{1.7.14}$$

asimptotik formula o'rinli bo'ladi. Bu yerda

$$c_{2k+1}=0,\quad k\geq 0.$$

2) Agar $q(x)\in C^{\infty}[0,\pi]$ cheksiz differensiallanuvchi funksiya bo'lsa, u holda (1.7.1)+(1.7.2) chegaraviy masalaning normallovchi o'zgarmaslari uchun

$$\alpha_n^2=\frac{\pi}{2}+\frac{a_0}{n^2}+\frac{a_1}{n^4}+\frac{a_2}{n^6}+\ldots$$

yoyilma o'rinli bo'ladi.

2. Quyidagi

$$-y''+q(x)y=\lambda y,\quad 0\leq x\leq\pi, \tag{1.7.15}$$

$$\begin{cases} y(0)=0,\\ y'(\pi)+Hy(\pi)=0,\end{cases} \tag{1.7.16}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x)\in C[0,\ \pi]$ haqiqiy funksiya va H chekli haqiqiy son.

$s(x,\lambda)$ orqali (1.7.15) tenglamaning ushbu

$$\begin{cases} s(0,\lambda)=0,\\ s'(0,\lambda)=1,\end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. Bu yechim uchun quyidagi

$$s(x,\lambda)=\frac{\sin kx}{k}-\frac{1}{2k^2}\left(\int_0^x q(t)dt\right)\cos kx+$$

$$+\frac{1}{2k^2}\int_0^x q(t)\cos k(2t-x)dt+\underline{\underline{O}}\left(\frac{e^{|\tau|x}}{k^3}\right), \tag{1.7.17}$$

asimptotik formula olingan edi. (1.7.15)+(1.7.16) masalaning xos qiymatlarini $\{\lambda_n\}_{n=0}^{\infty}$ orqali belgilaymiz. Oldingi paragraflarda xos qiymatlar uchun quyidagi asimptotik formula keltirib chiqarilgan edi:

$$k_n=\sqrt{\lambda_n}=n+\frac{1}{2}+\frac{c_1}{n+\frac{1}{2}}+\frac{\gamma_n}{n},\quad \{\gamma_n\}\in l_2. \tag{1.7.18}$$

Bu yerda

$$c_1=\frac{H}{\pi}+\frac{1}{2\pi}\int_0^{\pi} q(t)dt. \tag{1.7.19}$$

Endi ortonormallangan xos funksiyalar uchun asimptotik formulalarni topamiz. Shu maqsadda $s(x,\lambda_n)$ funksiyaning asimptotikasini o'rganamiz. (1.7.18) ifodani (1.7.17) tenglikka qo'ysak,

$$s(x,\lambda_n)=\frac{\sin\left(n+\frac{1}{2}+\delta_n\right)x}{n+\frac{1}{2}+\delta_n}-$$

$$-\frac{1}{2\left(n+\frac{1}{2}+\delta_n\right)^2}\cdot\int_0^x q(t)dt\cos\left(n+\frac{1}{2}+\delta_n\right)x+$$

$$+\frac{1}{2\left(n+\frac{1}{2}+\delta_n\right)^2}\int_0^x q(t)\cos\left(n+\frac{1}{2}+\delta_n\right)(2t-x)dt+$$

$$+\underline{\underline{O}}\left(\frac{1}{\left(n+\frac{1}{2}+\delta_n\right)^3}\right), \tag{1.7.20}$$

bo'ladi. (1.7.20) formulani soddalashtiramiz. Buning uchun avvalo quyidagi tengliklarni olamiz:

$$\frac{1}{n+\frac{1}{2}+\delta_n}=\frac{1}{n+\frac{1}{2}}\left\{1-\frac{\delta_n}{n+\frac{1}{2}}+\left(-\frac{\delta_n}{n+\frac{1}{2}}\right)^2+\ldots\right\}=$$

$$=\frac{1}{n+\frac{1}{2}}+\underline{\underline{O}}\left(\frac{1}{n^3}\right),$$

$$\sin\left(n+\frac{1}{2}+\delta_n\right)x=\sin\left(n+\frac{1}{2}\right)x\left[1-\frac{\delta_n^2x^2}{2!}+\ldots\right]+$$

$$+\cos\left(n+\frac{1}{2}\right)x\cdot\left[\delta_nx-\frac{\delta_n^3x^3}{3!}+\ldots\right]=$$

$$=\sin\left(n+\frac{1}{2}\right)x+\frac{c_1(x)}{n+\frac{1}{2}}\cos\left(n+\frac{1}{2}\right)x+\frac{\beta_n(x)}{n},\quad\{\beta_n(x)\}\in l_2,$$

$$\cos\left(n+\frac{1}{2}+\delta_n\right)x=\cos\left(n+\frac{1}{2}\right)x\left[1-\frac{\delta_n^2x^2}{2!}+\ldots\right]-$$

$$-\sin\left(n+\frac{1}{2}\right)x\left[\delta_nx-\frac{\delta_n^3x^3}{3!}+\ldots\right]=$$

$$=\cos\left(n+\frac{1}{2}\right)x-\frac{c_1(x)}{n+\frac{1}{2}}\sin\left(n+\frac{1}{2}\right)x+\frac{\tilde{\beta}_n(x)}{n},\quad\{\tilde{\beta}_n(x)\}\in l_2 \tag{1.7.21}$$

(1.7.21) ifodalarni (1.7.20) tenglikka qo'ysak, ushbu

$$s(x,\lambda_n)=\frac{\sin\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)}+$$

$$+\left(c_1x-\frac{1}{2}\int\limits_0^x q(t)dt\right)\frac{\cos\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)^2}+\frac{\omega_n(x)}{n^2},\quad\{\omega_n(x)\}\in l_2 \tag{1.7.22}$$

formula kelib chiqadi.

Endi normallovchi o'zgarmaslar uchun asimptotik formulalarni topamiz:

$$\alpha_n^2 = \int\limits_0^\pi s^2(x,\lambda_n)dx = \frac{1}{\left(n+\frac{1}{2}\right)^2}\cdot\frac{\pi}{2} + \frac{\tilde{\gamma}_n}{n^3}, \quad \{\tilde{\gamma}_n\} \in l_2. \qquad (1.7.23)$$

Bunga ko'ra

$$\alpha_n = \frac{1}{n+\frac{1}{2}}\cdot\sqrt{\frac{\pi}{2}}\cdot\left(1+\frac{\chi_n}{n}\right), \quad \{\chi_n\} \in l_2. \qquad (1.7.24)$$

bo'ladi. (1.7.24) formulaga asosan

$$\frac{1}{\alpha_n} = \left(n+\frac{1}{2}\right)\sqrt{\frac{2}{\pi}}\cdot\left(1+\frac{\tilde{\chi}_n}{n}\right), \quad \{\tilde{\chi}_n\} \in l_2, \qquad (1.7.25)$$

bo'lishi ravshan. Nihoyat, (1.7.25) va (1.7.22) ifodalarni ushbu

$$u_n(x) = \frac{1}{\alpha_n}s(x,\lambda_n),$$

tenglikka qo'yib, $u_n(x)$ normallangan xos funksiyalar uchun asimptotik formula keltirib chiqaramiz:

$$u_n(x) = \sqrt{\frac{2}{\pi}}\left\{\sin\left(n+\frac{1}{2}\right)x + \right.$$

$$\left. + \left[c_1 x - \frac{1}{2}\int\limits_0^x q(t)dt\right]\cdot\frac{\cos\left(n+\frac{1}{2}\right)}{n+\frac{1}{2}}\right\} + \frac{\omega_n(x)}{n}.$$

3. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \qquad (1.7.26)$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases} \qquad (1.7.27)$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C^1[0,\ \pi]$ haqiqiy funksiya va h chekli haqiqiy son.

(1.7.26)+(1.7.27) chegaraviy masalaning xos qiymatlari uchun quyidagi

$$k_n = \sqrt{\lambda_n} = n + \frac{1}{2} + \frac{c_2}{n+\frac{1}{2}} + \frac{\gamma_n}{n^2}, \quad \{\gamma_n\} \in l_2. \qquad (1.7.28)$$

asimptotik formula keltirib chiqarilgan edi. Bu yerda

$$c_2 = \frac{h}{\pi} + \frac{1}{2\pi}\int_0^\pi q(t)dt. \qquad (1.7.29)$$

Endi ortonormallangan xos funksiyalar uchun asimptotik formulalarni topamiz. Buning uchun avvalo $\varphi(x, \lambda_n)$ funksiyaning asimptotikasini o'rganamiz. (1.7.28) ifodani (1.7.3) tenglikka qo'ysak,

$$\varphi(x, \lambda_n) = \cos\left(n + \frac{1}{2} + \delta_n\right)x +$$

$$+\left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin\left(n + \frac{1}{2} + \delta_n\right)x}{n + \frac{1}{2} + \delta_n} + \underline{\underline{O}}\left(\frac{1}{\left(n + \frac{1}{2} + \delta_n\right)^2}\right)$$

$$(1.7.30)$$

bo'ladi. (1.7.30) formulani soddalashtiramiz. Buning uchun avvalo quyidagi tenglikni olamiz:

$$\cos\left(n + \frac{1}{2} + \delta_n\right)x = \cos\left(n + \frac{1}{2}\right)x\left[1 - \frac{\delta_n^2 x^2}{2!} + \ldots\right] -$$

$$-\sin\left(n + \frac{1}{2}\right)x\left[\delta_n x - \frac{\delta_n^3 x^3}{3!} + \ldots\right] =$$

$$= \cos\left(n + \frac{1}{2}\right)x - \delta_n x \sin\left(n + \frac{1}{2}\right)x +$$

$$+\underline{\underline{O}}\left(\frac{1}{n^2}\right) = \cos\left(n + \frac{1}{2}\right)x - c_2 x\frac{\sin\left(n + \frac{1}{2}\right)x}{n + \frac{1}{2}} + \underline{\underline{O}}\left(\frac{1}{n^2}\right), \quad n \to \infty.$$

$$(1.7.31)$$

(1.7.21) va (1.7.31) ifodalarni (1.7.30) tenglikka qo'ysak, ushbu

$$\varphi(x, \lambda_n) = \cos\left(n + \frac{1}{2}\right)x - c_2 x\frac{\sin\left(n + \frac{1}{2}\right)x}{n + \frac{1}{2}} +$$

$$+\left(h + \frac{1}{2}\int_0^x q(t)dt\right)\cdot\frac{\sin\left(n + \frac{1}{2}\right)x}{n + \frac{1}{2}} + \underline{\underline{O}}\left(\frac{1}{n^2}\right),$$

formula kelib chiqadi, ya'ni

$$\varphi(x,\lambda_n) = \cos\left(n+\frac{1}{2}\right)x + b(x)\frac{\sin\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} + \underline{\underline{O}}\left(\frac{1}{n^2}\right). \tag{1.7.32}$$

Bu yerda

$$b(x) = -c_2 x + h + \frac{1}{2}\int_0^x q(t)dt. \tag{1.7.33}$$

Endi normallovchi o'zgarmaslar uchun asimptotik formula topamiz

$$\alpha_n^2 = \int_0^\pi \varphi^2(x,\lambda_n)dx = \int_0^\pi \cos^2\left(n+\frac{1}{2}\right)xdx +$$

$$+\frac{1}{n+\frac{1}{2}}\int_0^\pi b(x)\sin(2n+1)xdx + \underline{\underline{O}}\left(\frac{1}{n^2}\right). \tag{1.7.34}$$

(1.7.34) tenglikda darajani pasaytirish formulasini va bo'laklab integrallash qoidasini qo'llab, ushbu

$$\alpha_n^2 = \frac{\pi}{2} + \underline{\underline{O}}\left(\frac{1}{n^2}\right),$$

tenglik o'rinli bo'lishini ko'ramiz. Bunga ko'ra

$$\alpha_n = \left\{\frac{\pi}{2}\left(1+\underline{\underline{O}}\left(\frac{1}{n^2}\right)\right)\right\}^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}\left(1+\underline{\underline{O}}\left(\frac{1}{n^2}\right)\right),$$

va

$$\frac{1}{\alpha_n} = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+\underline{\underline{O}}\left(\frac{1}{n^2}\right)} = \sqrt{\frac{2}{\pi}}\cdot\left(1+\underline{\underline{O}}\left(\frac{1}{n^2}\right)\right) = \sqrt{\frac{2}{\pi}}+\underline{\underline{O}}\left(\frac{1}{n^2}\right), \tag{1.7.35}$$

bo'lishi ravshan. Nihoyat (1.7.32) va (1.7.35) ifodalarni ushbu

$$u_n(x) = \frac{1}{\alpha_n}\varphi(x,\lambda_n),$$

tenglikka qo'yib, $u_n(x)$ normallangan xos funksiyalar uchun asimptotik formula keltirib chiqaramiz:

$$u_n(x)=\sqrt{\frac{2}{\pi}}\cdot\left\{\cos\left(n+\frac{1}{2}\right)x+b(x)\frac{\sin\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}}\right\}+$$

$$+\underline{\underline{O}}\left(\frac{1}{n^2}\right),\quad n\to\infty.$$

4. Quyidagi

$$-y''+q(x)y=\lambda y,\quad 0\le x\le\pi, \tag{1.7.36}$$

$$\begin{cases} y(0)=0,\\ y(\pi)=0,\end{cases} \tag{1.7.37}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x)\in C^1[0,\ \pi]$ haqiqiy funksiya.

Ushbu

$$s(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\underline{\underline{O}}\left(\frac{1}{\lambda}\right),\quad \lambda\to+\infty,$$

$$k_n=\sqrt{\lambda_n}=n+\frac{c_3}{n}+\frac{\gamma_n}{n^2},\qquad \{\gamma_n\}\in l_2,$$

$$c_3=\frac{1}{2\pi}\int\limits_0^{\pi}q(t)dt,$$

asimptotik formulalardan

$$s(x,\lambda_n)=\frac{\sin nx}{n}+\underline{\underline{O}}\left(\frac{1}{n^2}\right),\quad n\to+\infty,$$

$$\frac{1}{\alpha_n}=n\sqrt{\frac{2}{\pi}}\cdot\left(1+\underline{\underline{O}}\left(\frac{1}{n}\right)\right),\quad n\to+\infty,$$

va

$$u_n(x)=\frac{1}{\alpha_n}s(x,\lambda_n)=\sqrt{\frac{2}{\pi}}\cdot\sin nx+\underline{\underline{O}}\left(\frac{1}{n}\right),\quad n\to+\infty,$$

formulalar xuddi yuqoridagi bandlardagidek keltirilib chiqariladi.

Izoh 1.7.1. Yuqorida ko'rilgan chegaraviy masalalarning xos qiymatlari va normallovchi o'zgarmaslari uchun olingan asimptotik formulalar $q(x) \in L^2[0, \ \pi]$ haqiqiy funksiya bo'lgan holda ham o'rinli.

Mustaqil yechish uchun mashqlar

1. $q(x) \in C[0, \pi]$ bo'lgan holda, quyidagi Shturm-Liuvill masalalarining ortonormallangan xos funksiyalari uchun asimptotik formulalarni keltirib chiqaring $(0 \le x \le \pi)$:

1. $\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases}$ 2. $\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases}$

3. $\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0. \end{cases}$

8-§. Shturm-Liuvill masalasi bilan bog'liq bo'lgan cheksiz ko'paytmalar

1. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.8.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.8.2}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \ \pi]$ haqiqiy uzluksiz funksiya va $H, \ h$ chekli haqiqiy sonlar.

$\varphi(x, \lambda)$ orqali (1.8.1) tenglamaning ushbu

$$\begin{cases} \varphi(0, \lambda) = 1, \\ \varphi'(0, \lambda) = h, \end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

Aniqlanishiga ko'ra $\varphi(x, \lambda)$ funksiya chegaraviy shartlardan birinchisini qanoatlantiradi. Bu funksiyani chegaraviy shartlardan ikkinchisiga qo'ysak,

$$\Delta(\lambda) \equiv \varphi'(\pi, \lambda) + H\varphi(\pi, \lambda) = 0, \tag{1.8.3}$$

(1.8.1)+(1.8.2) Shturm-Liuvill chegaraviy masalasining xarakteristik tenglamasi hosil bo'ladi.

Teorema 1.8.1. *(1.8.1)+(1.8.2) Shturm-Liuvill chegaraviy masalasining $\Delta(\lambda)$ xarakteristik funksiyasi uchun quyidagi*

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda)\prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2},$$

formula o'rinli. Bu yerda $\{\lambda_n\}_{n=0}^{\infty}$ orqali (1.8.1)+(1.8.2) Shturm-Liuvill chegaraviy masalasining xos qiymatlari belgilangan.

Isbot. Ushbu $\Delta(\lambda) = \varphi'(\pi, \lambda) + H\varphi(\pi, \lambda)$ funksiyaning butun funksiya bo'lishi yuqoridagi paragraflarda ko'rsatilgan edi. $\Delta(z^2)$ funksiya juft va uning nollari $\pm\sqrt{\lambda_n}$, $n = 0,\ 1,\ 2,\ \ldots$ sonlardan iborat. Butun funksiyalar nazariyasidagi Veyershtrass teoremasiga ko'ra

$$\Delta(z^2) = C\prod_{n=0}^{\infty}\left(1 - \frac{z^2}{\lambda_n}\right), \tag{1.8.4}$$

bo'ladi. Bu yerda $C = \text{const}$. $\Delta(z^2)$ juft funksiya bo'lgani uchun Veyershtrass yoyilmasidagi eksponensial had qatnashmaydi. O'zgarmas C sonining aniq qiymatini topish uchun quyidagi

$$-y'' = \lambda y,$$

$$y'(0) = 0, \quad y'(\pi) = 0,$$

Shturm-Liuvill chegaraviy masalasining xarkateristik funksiyasini

$$\Delta_0(\lambda) \equiv \varphi'(\pi, \lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}\pi,$$

tuzib olamiz. Endi birinchi bobning beshinchi paragrafidagi (1.5.4)+(1.5.5) asimptotikalardan

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\operatorname{Im}\sqrt{\lambda}|x}}{\sqrt{\lambda}}\right), \quad |\lambda|\to\infty,$$
$$\varphi'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \underline{\underline{O}}\left(e^{|\operatorname{Im}\sqrt{\lambda}|x}\right), \quad |\lambda|\to\infty,$$

foydalanib, $\Delta(\lambda)$ funksiya uchun ushbu

$$\Delta(\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}\pi + \underline{\underline{O}}\left(e^{|\operatorname{Im}\sqrt{\lambda}|\pi}\right), \quad |\lambda|\to\infty,$$

asimptotikani topamiz. Quyidagi

$$\lambda = z^2,\ z = it,\ (t>0)$$

belgilashni kiritamiz. Natijada, ushbu

$$\Delta(z^2) = -it\sin it\pi + \underline{\underline{O}}\left(e^{t\pi}\right),\ t\to\infty,$$
$$\Delta_0(z^2) = -it\sin it\pi,$$

ifodalarni hosil qilamiz. $\Delta(z^2)$ va $\Delta_0(z^2)$ xarakteristik funksiyalarning nisbati uchun

$$\frac{\Delta(z^2)}{\Delta_0(z^2)} = 1 - \frac{\underline{\underline{O}}\left(e^{t\pi}\right)}{it\sin it\pi}, \tag{1.8.5}$$

formulani hosil qilamiz. Bu yerda quyidagi

$$\sin it\pi = \frac{i}{2}e^{t\pi}\left(1-e^{-2t\pi}\right),$$

formuladan foydalanib, (1.8.5) tenglikda $t\to\infty$ bo'lganda limitga o'tsak

$$\lim_{t\to\infty}\frac{\Delta\left(z^2\right)}{\Delta_0\left(z^2\right)} = 1,$$

hosil bo'ladi. Bundan esa ushbu

$$\Delta(z^2)\sim -it\sin it\pi,\quad (z=it,\ \ t\to\infty), \tag{1.8.6}$$

asimptotikaning o'rinli ekani kelib chiqadi.

Endi (1.8.4) tenglikdan C o'zgarmasning qiymatini topamiz:

$$C = \lim_{t\to\infty} \frac{\Delta(z^2)}{\prod\limits_{n=0}^{\infty}\left(1 - \frac{z^2}{\lambda_n}\right)} = \lim_{t\to\infty} \frac{-it\sin i\pi t}{\prod\limits_{n=0}^{\infty}\left(1 + \frac{t^2}{\lambda_n}\right)} =$$

$$= \lim_{t\to\infty} \frac{-it\cdot i\pi t\prod\limits_{n=1}^{\infty}\left(1 + \frac{t^2}{n^2}\right)}{\prod\limits_{n=0}^{\infty}\left(1 + \frac{t^2}{\lambda_n}\right)} =$$

$$= \lambda_0\pi\lim_{t\to\infty}\left\{\prod_{n=1}^{\infty}\left(1 + \frac{\lambda_n - n^2}{n^2}\right)\cdot\prod_{n=1}^{\infty}\left(1 + \frac{n^2 - \lambda_n}{\lambda_n + t^2}\right)\right\}.$$

Quyidagi

$$\lambda_n = n^2 + \frac{2}{\pi}\left(h + H + \frac{1}{2}\int_0^{\pi} q(s)ds\right) + \gamma_n, \ \{\gamma_n\}\in l_2$$

asimptotikadan yuqorida qatnashayotgan cheksiz ko'paytmalarning yaqinlashuvchiligi kelib chiqadi. Ushbu

$$\lim_{t\to\infty}\prod_{n=1}^{\infty}\left(1 + \frac{n^2 - \lambda_n}{\lambda_n + t^2}\right) = 1,$$

tenglikka asosan

$$C = \lambda_0\pi\prod_{n=1}^{\infty}\frac{\lambda_n}{n^2},$$

bo'ladi. Bundan

$$\Delta(\lambda) = \pi\lambda_0\left(\prod_{n=1}^{\infty}\frac{\lambda_n}{n^2}\right)\cdot\left(1 - \frac{z^2}{\lambda_0}\right)\cdot\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n}\right) =$$

$$= \pi(\lambda_0 - z^2)\prod_{n=1}^{\infty}\frac{\lambda_n - z^2}{n^2} = \pi(\lambda_0 - \lambda)\cdot\prod_{n=1}^{\infty}\frac{\lambda_n - \lambda}{n^2},$$

kelib chiqadi.■

Xulosa. Shturm-Liuvill chegaraviy masalasining λ_n, $n = 0, 1, 2, 3, \ldots$ xos qiymatlari yordamida $\Delta(\lambda)$ xarakteristik funksiya bir qiymatli aniqlanadi.

2. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq \pi, \tag{1.8.7}$$

$$\begin{cases} y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.8.8}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \ \pi]$ haqiqiy uzluksiz funksiya va H chekli haqiqiy son.

$s(x, \lambda)$ orqali (1.8.7) tenglamaning ushbu

$$s(0, \lambda) = 0, \quad s'(0, \lambda) = 1,$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

(1.8.7)+(1.8.8) chegaraviy masalaning xarakteristik tenglamasi

$$\Delta(\lambda) \equiv s'(\pi, \lambda) + Hs(\pi, \lambda) = 0 \tag{1.8.9}$$

bo'ladi.

Teorema 1.8.2. *(1.8.7)+(1.8.8) chegaraviy masalaning* $\Delta(\lambda)$ *xarakteristik funksiyasi uchun ushbu*

$$\Delta(\lambda) = \prod_{n=0}^{\infty} \frac{\mu_n - \lambda}{\left(n + \frac{1}{2}\right)^2}, \tag{1.8.10}$$

formula o'rinli bo'ladi. Bu yerda μ_n, $n = 0, 1, 2, 3, \ldots$ *orqali (1.8.7)+(1.8.8) chegaraviy masalaning xos qiymatlari belgilangan.*

Isbot. Ushbu $\Delta(\lambda) = s'(\pi, \lambda) + Hs(\pi, \lambda)$ funksiya butun funksiya bo'lishi yuqoridagi paragraflarda ko'rsatilgan edi. $\Delta(z^2)$ funksiya juft va uning nollari $\pm\sqrt{\mu_n}$, $n = 0, \ 1, \ 2, \ \ldots$ sonlardan iborat bo'ladi. Butun funksiyalar nazariyasidagi Veyershtrass teoremasiga ko'ra

$$\Delta(z^2) = C\prod_{n=0}^{\infty} \left(1 - \frac{z^2}{\mu_n}\right), \tag{1.8.11.}$$

bo'ladi. Bu yerda $C = \text{const}$. Bu yoyilmada $\Delta(z^2)$ juft funksiya bo'lgani uchun eksponentsial had qatnashmaydi. O'zgarmas C

sonining aniq qiymatini topish uchun quyidagi

$$-y'' = \lambda y,$$

$$y(0) = 0, \quad y'(\pi) = 0,$$

Shturm-Liuvill chegaraviy masalasining xarkateristik funksiyasini

$$\Delta_0(\lambda) \equiv s'(\pi, \lambda) = \cos\sqrt{\lambda}\pi,$$

tuzib olamiz. Agar ushbu

$$s(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \underline{\underline{O}}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{\lambda}\right), \quad |\lambda| \to \infty,$$

$$s'(x, \lambda) = \cos\sqrt{\lambda}x + \underline{\underline{O}}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{\sqrt{\lambda}}\right), \quad |\lambda| \to \infty,$$

asimptotikalardan foydalansak, $\Delta(\lambda)$ funksiya uchun ushbu

$$\Delta(\lambda) = \cos\sqrt{\lambda}\pi + \underline{\underline{O}}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|\pi}}{\sqrt{\lambda}}\right), \quad |\lambda| \to \infty,$$

asimptotikani topamiz. Quyidagi

$$\lambda = z^2, \ z = it, \ (t > 0)$$

belgilashni kiritamiz. Natijada, ushbu

$$\Delta(z^2) = \cos it\pi + \underline{\underline{O}}\left(\frac{e^{t\pi}}{t}\right), \ t \to \infty,$$

$$\Delta_0(z^2) = \cos it\pi,$$

ifodalarni hosil qilamiz. $\Delta(z^2)$ va $\Delta_0(z^2)$ xarakteristik funksiyalarning nisbati uchun

$$\frac{\Delta(z^2)}{\Delta_0(z^2)} = 1 + \frac{\underline{\underline{O}}\left(\frac{e^{t\pi}}{t}\right)}{\cos it\pi}, \tag{1.8.12}$$

formulani hosil qilamiz. Bu yerda quyidagi

$$\cos it\pi = \frac{1}{2}e^{\pi t}\left(1 + e^{-2t\pi}\right),$$

formuladan foydalanib, (1.8.12) tenglikda $t \to \infty$ bo'lganda limitga o'tsak,

$$\lim_{t\to\infty} \frac{\Delta(z^2)}{\Delta_0(z^2)} = 1,$$

tenglik hosil bo'ladi. Bundan esa ushbu

$$\Delta(z^2) \sim \cos it\pi, \quad (z = it, \ t \to \infty),$$

asimptotikaning o'rinli ekani kelib chiqadi.

Endi (1.8.11) tenglikdan C o'zgarmasning qiymatini topamiz:

$$C = \lim_{t\to\infty} \frac{\Delta(z^2)}{\prod\limits_{n=0}^{\infty}\left(1 - \frac{z^2}{\mu_n}\right)} = \lim_{t\to\infty} \frac{\cos i\pi t}{\prod\limits_{n=0}^{\infty}\left(1 + \frac{t^2}{\mu_n}\right)} =$$

$$= \lim_{t\to\infty} \frac{\prod\limits_{n=0}^{\infty}\left(1 + \frac{t^2}{(n+1/2)^2}\right)}{\prod\limits_{n=0}^{\infty}\left(1 + \frac{t^2}{\mu_n}\right)} = \lim_{t\to\infty} \prod_{n=0}^{\infty} \frac{\mu_n \cdot ((n+1/2)^2 + t^2)}{(n+1/2)^2 \cdot (\mu_n + t^2)} =$$

$$= \lim_{t\to\infty} \left\{ \prod_{n=0}^{\infty} \left(1 + \frac{\mu_n - (n+1/2)^2}{(n+1/2)^2}\right) \cdot \prod_{n=0}^{\infty} \left(1 + \frac{(n+1/2)^2 - \mu_n}{\mu_n + t^2}\right) \right\}.$$

Quyidagi

$$\mu_n = \left(n + \frac{1}{2}\right)^2 + \frac{2H}{\pi} + \frac{1}{\pi}\int_0^{\pi} q(t)dt + \gamma_n, \ \{\gamma_n\} \in l_2,$$

asimptotikadan yuqorida qatnashayotgan cheksiz ko'paytmalarning yaqinlashuvchiligi kelib chiqadi. Ushbu

$$\lim_{t\to\infty} \prod_{n=0}^{\infty} \left(1 + \frac{(n+1/2)^2 - \mu_n}{\mu_n + t^2}\right) = 1,$$

tenglik bajarilishi ravshan. Demak,

$$C = \prod_{n=0}^{\infty} \frac{\mu_n}{(n+1/2)^2},$$

ekan. Buni (1.8.11) tenglikka qo'yib,

$$\Delta(\lambda)=\prod_{n=0}^{\infty}\frac{\mu_n}{(n+1/2)^2}\cdot\prod_{n=0}^{\infty}\frac{\mu_n-\lambda}{\mu_n}=\prod_{n=0}^{\infty}\frac{\mu_n-\lambda}{(n+1/2)^2},$$

yoyilmani olamiz. ■

3. Quyidagi

$$-y''+q(x)y=\lambda y,\quad 0\le x\le\pi, \tag{1.8.13}$$

$$\begin{cases} y'(0)-hy(0)=0,\\ y(\pi)=0, \end{cases} \tag{1.8.14}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x)\in C[0,\ \pi]$ haqiqiy funksiya va h chekli haqiqiy son.

(1.8.13)+(1.8.14) masalaning xarakteristik tenglamasi quyidagi

$$\Delta(\lambda)\equiv\varphi(\pi,\lambda)=0, \tag{1.8.15}$$

ko'rinishda bo'ladi.

Teorema 1.8.3. *(1.8.13)+(1.8.14) chegaraviy masalaning xarakteristik funksiyasi uchun*

$$\Delta(\lambda)=\prod_{n=0}^{\infty}\frac{\xi_n-\lambda}{\left(n+\frac{1}{2}\right)^2}, \tag{1.8.16}$$

formula o'rinli bo'ladi. Bu yerda ξ_n, $n=0,1,2,...$ orqali (1.8.10)+(1.8.11) Shturm-Liuvill chegaraviy masalasining xos qiymatlari belgilangan.

Isbot. Ushbu $\Delta(\lambda)=\varphi(\pi,\lambda)$ funksiya $\frac{1}{2}$-tartibli butun funksiya bo'lishi yuqoridagi paragraflarda qayd etilgan edi. $\Delta(z^2)$ funksiya juft va uning nollari $\pm\sqrt{\xi_n}$, $n=0,\ 1,\ 2,\ ...$ sonlardan iborat. Butun funksiyalar nazariyasidagi Veyershtrass teoremasiga ko'ra

$$\Delta(z^2)=C\prod_{n=0}^{\infty}\left(1-\frac{z^2}{\xi_n}\right), \tag{1.8.17}$$

bo'ladi. Bu yerda C o'zgarmas son. $\Delta(z^2)$ juft funksiya bo'lgani uchun (1.8.17) da eksponensial had qatnashmaydi.

Ushbu

$$\Delta(z^2)\sim \cos i\pi t, \quad (z = it, \quad t \to \infty),$$

asimptotikadan

$$C = \lim_{t\to\infty} \frac{\Delta(z^2)}{\prod\limits_{n=0}^{\infty}\left(1 - \frac{z^2}{\xi_n}\right)} = \lim_{t\to\infty} \frac{\cos i\pi t}{\prod\limits_{n=0}^{\infty}\left(1 + \frac{t^2}{\xi_n}\right)} = \lim_{t\to\infty} \frac{\prod\limits_{n=0}^{\infty}\left(1 + \frac{t^2}{(n+1/2)^2}\right)}{\prod\limits_{n=0}^{\infty}\left(1 + \frac{t^2}{\xi_n}\right)} =$$

$$= \lim_{t\to\infty}\prod_{n=0}^{\infty} \frac{\xi_n \cdot ((n+1/2)^2 + t^2)}{(n+1/2)^2 \cdot (\xi_n + t^2)} =$$

$$= \lim_{t\to\infty}\left\{\prod_{n=0}^{\infty}\left(1 + \frac{\xi_n - (n+1/2)^2}{(n+1/2)^2}\right) \cdot \prod_{n=0}^{\infty}\left(1 + \frac{(n+1/2)^2 - \xi_n}{\xi_n + t^2}\right)\right\},$$

kelib chiqadi. Quyidagi

$$\xi_n = \left(n + \frac{1}{2}\right)^2 + \frac{2h}{\pi} + \frac{1}{\pi}\int_0^{\pi} q(t)dt + \gamma_n, \ \{\gamma_n\} \in l_2,$$

asimptotikadan yuqorida qatnashayotgan cheksiz ko'paytmalarning yaqinlashuvchiligi kelib chiqadi. Ushbu

$$\lim_{t\to\infty}\prod_{n=0}^{\infty}\left(1 + \frac{(n+1/2)^2 - \xi_n}{\xi_n + t^2}\right) = 1,$$

tenglik bajarilishi ravshan. Demak,

$$C = \prod_{n=0}^{\infty} \frac{\xi_n}{(n+1/2)^2},$$

ekan. Bundan

$$\Delta(\lambda) = \prod_{n=0}^{\infty} \frac{\xi_n}{(n+1/2)^2} \cdot \prod_{n=0}^{\infty} \frac{\xi_n - \lambda}{\xi_n} = \prod_{n=0}^{\infty} \frac{\xi_n - \lambda}{(n+1/2)^2},$$

kelib chiqadi.■

4. Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq \pi, \tag{1.8.18}$$

$$\begin{cases} y(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{1.8.19}$$

Shturm-Liuvill chegaraviy masalasini ko‘rib chiqamiz. Bu yerda $q(x) \in C[0, \ \pi]$ haqiqiy funksiya.

Bu holda (1.8.18)+(1.8.19) masalaning xarakteristik tenglamasi quyidagicha bo‘ladi:

$$\Delta(\lambda) \equiv s(\pi, \lambda) = 0. \tag{1.8.20}$$

Teorema 1.8.4. *(1.8.15)+(1.8.16) chegaraviy masalaning xarakteristik funksiyasi uchun quyidagi tenglik o‘rinli:*

$$\Delta(\lambda) = \pi \prod_{n=1}^{\infty} \frac{\eta_n - \lambda}{n^2}. \tag{1.8.21}$$

Bu yerda η_n, $n = 1, 2, ...$ orqali (1.8.15)+(1.8.16) chegaraviy masalaning xos qiymatlari belgilangan.

Isbot. Ushbu $\Delta(\lambda) = s(\pi, \lambda)$ funksiya $\frac{1}{2}$ - tartibli butun funksiya bo‘lishi ko‘rsatilgan edi. $\Delta(z^2)$ funksiya juft va uning nollari $\pm\sqrt{\eta_n}$, $n = 0, 1, 2, ...$ sonlardan iborat. Butun funksiyalar nazariyasidagi Veyershtrass teoremasiga ko‘ra

$$\Delta(z^2) = C \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\eta_n}\right), \tag{1.8.22}$$

bo‘ladi. Bu yerda C o‘zgarmas. $\Delta(z^2)$ juft funksiya bo‘lgani uchun (1.8.22) yoyilmada eksponensial had qatnashmaydi.

Ushbu

$$\Delta(z^2) \sim \frac{\sin i\pi t}{it}, \quad (z = it, \quad t \to \infty),$$

asimptotikadan

$$C=\lim_{t\to\infty}\frac{\Delta(z^2)}{\prod\limits_{n=1}^{\infty}\left(1-\frac{z^2}{\eta_n}\right)}=\lim_{t\to\infty}\frac{\frac{\sin i\pi t}{it}}{\prod\limits_{n=1}^{\infty}\left(1+\frac{t^2}{\eta_n}\right)}=\lim_{t\to\infty}\frac{\pi\prod\limits_{n=1}^{\infty}\left(1+\frac{t^2}{n^2}\right)}{\prod\limits_{n=1}^{\infty}\left(1+\frac{t^2}{\eta_n}\right)}=$$

$$=\pi\lim_{t\to\infty}\left\{\prod_{n=1}^{\infty}\left(1+\frac{\eta_n-n^2}{n^2}\right)\cdot\prod_{n=1}^{\infty}\left(1+\frac{n^2-\eta_n}{\eta_n+t^2}\right)\right\}=\pi\prod_{n=1}^{\infty}\frac{\eta_n}{n^2}$$

kelib chiqadi. Quyidagi

$$\eta_n=n^2+\frac{1}{\pi}\int\limits_0^{\pi}q(s)ds+\gamma_n,\quad\{\gamma_n\}\in l_2$$

asimptotikadan yuqorida qatnashayotgan cheksiz ko'paytmalarning yaqinlashuvchiligi kelib chiqadi.

Demak,

$$\Delta(\lambda)=\pi\left(\prod_{n=1}^{\infty}\frac{\eta_n}{n^2}\right)\cdot\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\eta_n}\right)=\pi\prod_{n=1}^{\infty}\frac{\eta_n-\lambda}{n^2},$$

bo'lar ekan.■

9-§. Shturm-Liuvill chegaraviy masalasi uchun Grin funksiyasi va uning xossalari

Ushbu

$$-y''+q(x)y=\lambda y+f(x),\tag{1.9.1}$$

$$\begin{cases}y(0)\cos\alpha+y'(0)\sin\alpha=0,\\ y(\pi)\cos\beta+y'(\pi)\sin\beta=0,\end{cases}\tag{1.9.2}$$

Shturm-Liuvill chegaraviy masalasi berilgan bo'lsin.

Ta'rif 1.9.1. (1.9.1)+(1.9.2) chegaraviy masalaning Grin funksiyasi deb, quyidagi shartlarni qanoatlantiruvchi $G(x,t,\lambda)$ funksiyaga aytiladi:

1) $G(x,t,\lambda)$ funksiya $[0,\pi]\times[0,\pi]$ to'plamda uzluksiz;

2) $t\in[0,\pi]$ parametrning ixtiyoriy tayinlangan qiymatida $G(x,t,\lambda)$ funksiya $[0,t)$ va $(t,\pi]$ oraliqlarda ushbu

$$-y''+q(x)y=\lambda y \tag{1.9.3}$$

bir jinsli tenglamani qanoatlantiradi;

3) $G'_x(x,t,\lambda)$ funksiyaning $x=t$ nuqtadagi sakrashi (-1) ga teng, ya'ni

$$G'_x(x,t,\lambda)|_{x=t+0}-G'_x(x,t,\lambda)|_{x=t-0}=-1;$$

4) $G(x,t,\lambda)$ funksiya (1.9.2) chegaraviy shartlarni qanoatlantiradi.

Quyidagi Koshi masalalarining yechimlarini mos ravishda $\varphi(x,\lambda)$ va $\psi(x,\lambda)$ orqali belgilaymiz:

$$\begin{cases} -y''+q(x)y=\lambda y,\\ y(0)=-\sin\alpha,\\ y'(0)=\cos\alpha, \end{cases} \qquad \begin{cases} -y''+q(x)y=\lambda y,\\ y(\pi)=-\sin\beta,\\ y'(\pi)=\cos\beta. \end{cases}$$

$\omega(\lambda)=W\{\varphi(x,\lambda),\psi(x,\lambda)\}$ bo'lsin. U holda

$$\omega(\lambda)=\begin{vmatrix} \varphi(0,\lambda) & \psi(0,\lambda)\\ \varphi'(0,\lambda) & \psi'(0,\lambda)\end{vmatrix}=\begin{vmatrix} -\sin\alpha & \psi(0,\lambda)\\ \cos\alpha & \psi'(0,\lambda)\end{vmatrix}=$$

$$=-[\psi(0,\lambda)\cos\alpha+\psi'(0,\lambda)\sin\alpha], \tag{1.9.4}$$

va

$$\omega(\lambda)=\begin{vmatrix} \varphi(\pi,\lambda) & \psi(\pi,\lambda)\\ \varphi'(\pi,\lambda) & \psi'(\pi,\lambda)\end{vmatrix}=\begin{vmatrix} \varphi(\pi,\lambda) & -\sin\beta\\ \varphi'(\pi,\lambda) & \cos\beta\end{vmatrix}=$$

$$=\varphi(\pi,\lambda)\cos\beta+\varphi'(\pi,\lambda)\sin\beta, \tag{1.9.5}$$

bo'ladi.

1-xossa. $\omega(\lambda)$ funksiyaning nollari xos qiymatlardan iborat bo'ladi.

Bu fikr (1.9.5) tenglikdan kelib chiqadi. Chunki (1.9.5) tenglikning o'ng tomonida (1.9.3)+(1.9.2) Shturm-Liuvill chegaraviy masalasining xarakteristik funksiyasi turibdi.■

Teorema 1.9.1. *1) Agar λ son (1.9.1)+(1.9.2) chegaraviy masalaga mos keluvchi bir jinsli masalaning xos qiymati bo'lmasa, u holda (1.9.1)+(1.9.2) masalaning Grin funksiyasi mavjud va yagona bo'lib, u ushbu*

$$G(x,t,\lambda) = \begin{cases} \dfrac{\varphi(x,\lambda)\psi(t,\lambda)}{\omega(\lambda)}, & x \le t, \\ \dfrac{\psi(x,\lambda)\varphi(t,\lambda)}{\omega(\lambda)}, & x \ge t, \end{cases}$$

formula bilan beriladi.

2) Agar λ son (1.9.1)+(1.9.2) masalaga mos keluvchi bir jinsli masalaning xos qiymati bo'lsa, u holda (1.9.1)+(1.9.2) masalasining Grin funksiyasi mavjud bo'lmaydi.

Isbot. Grin funksiyasi ta'rifidan kelib chiqib, uni ushbu

$$G(x,t,\lambda) = \begin{cases} A(t)\varphi(x,\lambda) + B(t)\psi(x,\lambda), & x \le t, \\ C(t)\varphi(x,\lambda) + D(t)\psi(x,\lambda), & x \ge t, \end{cases} \quad (1.9.6)$$

ko'rinishda izlaymiz.

Grin funksiyasi ta'rifining birinchi shartiga ko'ra u uzluksiz bo'ladi, demak, quyidagi tenglik bajariladi:

$$A(t)\varphi(t,\lambda)+B(t)\psi(t,\lambda)-C(t)\varphi(t,\lambda)-D(t)\psi(t,\lambda) = 0. \quad (1.9.7)$$

Uchinchi shartga ko'ra

$$A(t)\varphi'(t,\lambda) + B(t)\psi'(t,\lambda) - C(t)\varphi'(t,\lambda) - D(t)\psi'(t,\lambda) = 1 \quad (1.9.8)$$

bo'ladi. To'rtinchi shartdan

$$[A(t)\varphi(0,\lambda) + B(t)\psi(0,\lambda)]\cos\alpha +$$
$$+[A(t)\varphi'(0,\lambda) + B(t)\psi'(0,\lambda)]\sin\alpha = 0, \quad (1.9.9)$$

$$[C(t)\varphi(\pi,\lambda)+D(t)\psi(\pi,\lambda)]\cos\beta+$$
$$+[C(t)\varphi'(\pi,\lambda)+D(t)\psi'(\pi,\lambda)]\sin\beta=0 \qquad (1.9.10)$$

kelib chiqadi. (1.9.9) va (1.9.10) tengliklarni quyidagi ko‘rinishda yozib olamiz:

$$A(t)[\varphi(0,\lambda)\cos\alpha+\varphi'(0,\lambda)\sin\alpha]+$$
$$+B(t)[\psi(0,\lambda)\cos\alpha+\psi'(0,\lambda)\sin\alpha]=0, \qquad (1.9.11)$$
$$C(t)[\varphi(\pi,\lambda)\cos\beta+\varphi'(\pi,\lambda)\sin\beta]+$$
$$+D(t)[\psi(\pi,\lambda)\cos\beta+\psi'(\pi,\lambda)\sin\beta]=0. \qquad (1.9.12)$$

Agar ushbu

$$\omega(\lambda)=-[\psi(0,\lambda)\cos\alpha+\psi'(0,\lambda)\sin\alpha],$$
$$\omega(\lambda)=\varphi(\pi,\lambda)\cos\beta+\varphi'(\pi,\lambda)\sin\beta,$$

formulalarni va boshlang‘ich shartlarni e’tiborga olsak, (1.9.11) va (1.9.12) tengliklar quyidagi ko‘rinishni oladi:

$$B(t)\omega(\lambda)=0, \qquad (1.9.13)$$

$$C(t)\omega(\lambda)=0. \qquad (1.9.14)$$

Quyidagi hollarni ko‘rib chiqamiz.

1) $\omega(\lambda)\neq 0$ bo‘lsin. U holda

$$B(t)=0, \quad C(t)=0 \qquad (1.9.15)$$

bo‘ladi. Bularni (1.9.7) va (1.9.8) tengliklarga qo‘ysak, ushbu

$$\begin{cases} A\varphi(t,\lambda)-D\psi(t,\lambda)=0,\\ A\varphi'(t,\lambda)-D\psi'(t,\lambda)=1, \end{cases} \qquad (1.9.16)$$

sistema hosil bo‘ladi. (1.9.16) sistemani Kramer qoidasi yordamida yechamiz:

$$\Delta=-\omega(\lambda), \quad \Delta_1=\begin{vmatrix} 0 & -\psi(t,\lambda)\\ 1 & -\psi'(t,\lambda)\end{vmatrix}=\psi(t,\lambda),$$

$$\Delta_2 = \begin{vmatrix} \varphi(t,\lambda) & 0 \\ \varphi'(t,\lambda) & 1 \end{vmatrix} = \varphi(t,\lambda),$$

bo'lgani uchun

$$A(t) = -\frac{\psi(t,\lambda)}{\omega(\lambda)}, \quad D(t) = -\frac{\varphi(t,\lambda)}{\omega(\lambda)}, \tag{1.9.17}$$

bo'ladi. Topilgan (1.9.15) va (1.9.17) ifodalarni (1.9.6) tenglikka qo'yib,

$$G(x,t,\lambda) = \begin{cases} -\dfrac{\varphi(x,\lambda)\psi(t,\lambda)}{\omega(\lambda)}, & x \le t, \\ -\dfrac{\psi(x,\lambda)\varphi(t,\lambda)}{\omega(\lambda)}, & x \ge t, \end{cases} \tag{1.9.18}$$

bo'lishini ko'ramiz. Demak, bu holda Grin funksiyasi mavjud va yagona bo'lib, u (1.9.18) formula bilan beriladi.

2) $\omega(\lambda) = 0$ bo'lsin. Bu holda $\psi(t,\lambda) = \gamma\varphi(t,\lambda)$ bo'ladi. Buni (1.9.7) va (1.9.8) tengliklarga qo'yib,

$$(A + B\gamma - C - D\gamma)\varphi(t,\lambda) = 0, \tag{1.9.19}$$

$$(A + B\gamma - C - D\gamma)\varphi'(t,\lambda) = 1, \tag{1.9.20}$$

bo'lishini ko'ramiz. $\varphi(t,\lambda) \not\equiv 0$ bo'lgani uchun

$$A + B\gamma - C - D\gamma = 0$$

bo'ladi. Buni (1.9.20) tenglikka qo'ysak, $0 \cdot \varphi'(t,\lambda) = 1$ ziddiyat kelib chiqadi. Demak, bu holda Grin funksiyasi mavjud emas ekan.■

Natija 1.9.1. Grin funksiyasi uchun yozilgan (1.9.18) formuladan uning x va t ga nisbatan simmetrikligi, ya'ni $G(x,t,\lambda) = G(t,x,\lambda)$ kelib chiqadi.

Teorema 1.9.2. *(D.Gilbert). Agar λ son (1.9.1)+(1.9.2) chegaraviy masalaga mos keluvchi bir jinsli masalaning xos qiymati bo'lmasa, u holda ixtiyoriy $f(x) \in C[0,\pi]$ funksiya uchun*

(1.9.1)+(1.9.2) masalaning yechimi mavjud va yagona bo'ladi hamda u ushbu

$$y(x) = \int_0^{\pi} G(x,t,\lambda) f(t)dt, \tag{1.9.21}$$

formula bilan beriladi.

Isbot. (1.9.21) formula bilan beriladigan funksiya (1.9.1)+(1.9.2) chegaraviy masalaning yechimi ekanligini tekshirib ko'ramiz. Buning uchun uni ushbu

$$y(x) = -\frac{\psi(x,\lambda)}{\omega(\lambda)}\int_0^x \varphi(t,\lambda)f(t)dt - \frac{\varphi(x,\lambda)}{\omega(\lambda)}\int_x^{\pi} \psi(t,\lambda)f(t)dt \tag{1.9.22}$$

ko'rinishda yozib olamiz va uning hosilalarini hisoblaymiz:

$$y'(x) = -\frac{\psi'(x,\lambda)}{\omega(\lambda)}\int_0^x \varphi(t,\lambda)f(t)dt - \frac{\psi(x,\lambda)\varphi(x,\lambda)}{\omega(\lambda)}f(x) -$$

$$-\frac{\varphi'(x,\lambda)}{\omega(\lambda)}\int_x^{\pi} \psi(t,\lambda)f(t)dt + \frac{\varphi(x,\lambda)\psi(x,\lambda)}{\omega(\lambda)}f(x),$$

$$y''(x) = -\frac{\psi''(x,\lambda)}{\omega(\lambda)}\int_0^x \varphi(t,\lambda)f(t)dt - \frac{\psi'(x,\lambda)\varphi(x,\lambda)}{\omega(\lambda)}f(x) -$$

$$-\frac{\varphi''(x,\lambda)}{\omega(\lambda)}\int_x^{\pi} \psi(t,\lambda)f(t)dt + \frac{\varphi'(x,\lambda)\psi(x,\lambda)}{\omega(\lambda)}f(x). \tag{1.9.23}$$

Ushbu $\varphi'' = [q(x)-\lambda]\varphi$, $\psi'' = [q(x)-\lambda]\psi$ va $\varphi\psi' - \psi\varphi' = \omega(\lambda)$ ayniyatlardan foydalanib, (1.9.23) formuladan quyidagi tenglikni keltirib chiqaramiz:

$$y'' = [q(x) - \lambda]y - f(x).$$

Chegaraviy shartlar bajarilishi ravshan.

Yagonaligini isbot qilish uchun (1.9.1)+(1.9.2) chegaraviy masalaning ikkita $y_1(x) \not\equiv y_2(x)$ yechimi mavjud deb faraz qilamiz. $y_1(x)$ va $y_2(x)$ yechimlarni (1.9.1)+(1.9.2) masalaga qo'yib, hosil bo'lgan ayniyatlarni mos ravishda bir-biridan ayirsak, hamda $u(x) = y_1(x) - y_2(x)$ belgilash kiritsak, $u(x)$ funksiya quyidagi bir jinsli masalani qanoatlantirishini ko'ramiz:

$$-u'' + q(x)u = \lambda u,$$

$$\begin{cases} u(0)\cos\alpha + u'(0)\sin\alpha = 0, \\ u(\pi)\cos\beta + u'(\pi)\sin\beta = 0. \end{cases}$$

Bu masala faqat nol yechimga ega, chunki λ son xos qiymat emas. Demak, $u(x) \equiv 0$ ekan, ya'ni $y_1(x) \equiv y_2(x)$ ekan. Bu esa farazimizga zid.■

Ta'rif 1.9.1. (1.9.21) tenglik bilan beriladigan chiziqli integral operatorga Shturm-Liuvill chegaraviy masalasining rezolventasi deyiladi.

Izoh 1.9.1. Ushbu

$$-(p(x)y')' + q(x)y = f(x), \quad x \in [a,\ b], \quad p(x) > 0,$$

$$\begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0, \end{cases}$$

chegaraviy masalaning Grin funksiyasi deb, quyidagi shartlarni qanoatlantiruvchi $G(x,t)$ funksiyaga aytiladi:

1) $G(x,t)$ funksiya $[a,\ b] \times [a,\ b]$ to'plamda uzluksiz;

2) $t \in [a,\ b]$ parametrning ixtiyoriy tayinlangan qiymatida $G(x,t)$ funksiya $[a,\ t)$ va $(t,\ b]$ oraliqlarda bir jinsli

$$-(p(x)y')' + q(x)y = 0$$

tenglamani qanoatlantiradi;

3) $G'_x(x,t)$ funksiyaning $x = t$ nuqtadagi sakrashi $\left(-\dfrac{1}{p(t)}\right)$

ga teng, ya'ni

$$G'_x(x,t)|_{x=t+0} - G'_x(x,t)|_{x=t-0} = -\frac{1}{p(t)};$$

4) $G(x,t)$ funksiya chegaraviy shartlarni qanoatlantiradi.

Bir jinsli chegaraviy masalaning noldan farqli yechimi bo'lmasa, qaralayotgan masalaning Grin funksiyasi ushbu

$$G(x,t) = \begin{cases} -\dfrac{\varphi(x)\psi(t)}{p(a)\omega(a)}, & x \le t \\ -\dfrac{\psi(x)\varphi(t)}{p(a)\omega(a)}, & x \ge t, \end{cases}$$

formula orqali topiladi. Bu yerda $\varphi(x)$ va $\psi(x)$ funksiyalar quyidagi

$$-(p(x)y')' + q(x)y = 0$$

bir jinsli tenglamaning mos ravishda birinchi va ikkinchi chegaraviy shartlarni qanoatlantiruvchi biror yechimlari, $\omega(x) = W\{\varphi(x),\ \psi(x)\}$.

Mustaqil yechish uchun mashqlar

1. Quyidagi bir jinsli bo'lmagan chegaraviy masalalarning yechimlarini Grin funksiyasi yordamida toping:

$$a)\begin{cases} -y'' = \lambda y + f(x), \\ y(0) = 0, \\ y(\pi) = 0, \end{cases} \qquad b)\begin{cases} -y'' = \lambda y + f(x), \\ y'(0) = 0, \\ y'(\pi) = 0, \end{cases}$$

$$c)\begin{cases} -y'' = \lambda y + f(x), \\ y'(0) = 0, \\ y(\pi) = 0, \end{cases} \qquad d)\begin{cases} -y'' = \lambda y + f(x), \\ y(0) = 0, \\ y'(\pi) = 0, \end{cases}$$

$$e)\begin{cases} -y'' = \lambda y + f(x), \\ y'(0) = y(0), \\ y'(\pi) = y(\pi), \end{cases} \qquad f)\begin{cases} -y'' = \lambda y + f(x), \\ y'(0) = hy(0), \\ y'(\pi) = hy(\pi). \end{cases}$$

2. Quyidagi chegaraviy masalalarning yechimlarini Grin funksiyasi yordamida toping:

$$a)\ \begin{cases} -y'' = f(x), \\ y(0) = 0, \\ y(1) = 0, \end{cases} \quad b)\ \begin{cases} -y'' = f(x), \\ y(0) = 0, \\ y'(1) = 0, \end{cases}$$

$$c)\ \begin{cases} -y'' = f(x), \\ y(0) = 0, \\ y(1) + hy'(1) = 0, \end{cases} \quad d)\ \begin{cases} -y'' = f(x), \\ y'(0) = y(0), \\ y(1) + y'(1) = 0, \end{cases}$$

$$e)\ \begin{cases} -y'' = f(x), \\ y(0) = hy'(0), \quad h \geq 0, \\ y(1) = 0, \end{cases} \quad f)\ \begin{cases} -y'' + 4y = f(x), \\ y(0) = 0, \\ y(1) = 0, \end{cases}$$

$$g)\ \begin{cases} -y'' + y = f(x), \\ y(0) = 0, \\ y(1) = 0, \end{cases} \quad h)\ \begin{cases} -y'' + y = f(x), \\ y'(0) = 0, \\ y'(1) = 0, \end{cases}$$

$$i)\ \begin{cases} -y'' + y = f(x), \\ y(0) = 0, \\ y'(1) = 0, \end{cases} \quad j)\ \begin{cases} -y'' - y = f(x), \\ y(0) = 0, \\ y'(1) = 0, \end{cases}$$

$$k)\ \begin{cases} -y'' - y = f(x), \\ y(0) = y'(0), \\ y(1) = y'(1), \end{cases} \quad l)\ \begin{cases} -y'' + \dfrac{2}{x^2}y = f(x), \\ 2y(1) = y'(1), \\ y(2) + 2y'(2) = 0, \end{cases}$$

$$m)\ \begin{cases} -y'' + (1 + x^2)y = f(x), \\ y(0) = 0, \\ y'(1) = 0. \end{cases}$$

3. Quyidagi davriy va antidavriy chegaraviy masalalarning yechimlarini Grin funksiyasi yordamida toping:

$$a)\begin{cases} -y'' = \lambda y + f(x), \\ y(0) = y(\pi), \\ y'(0) = y'(\pi), \end{cases} \qquad b)\begin{cases} -y'' = \lambda y + f(x), \\ y(0) = -y(\pi), \\ y'(0) = -y'(\pi), \end{cases}$$

$$c)\begin{cases} -y'' + y = f(x), \\ y(0) = y(\pi), \\ y'(0) = y'(\pi), \end{cases} \qquad d)\begin{cases} -y'' = f(x), \\ y(0) = -y(1), \\ y'(0) = -y'(1), \end{cases}$$

$$e)\begin{cases} -y'' + k^2 y = f(x), \quad k \neq 0, \\ y(-1) = y(1), \\ y'(-1) = y'(1). \end{cases}$$

4. Quyidagi chegaraviy masalalarning yechimlarini Grin funksiyasi yordamida toping:

$$a)\begin{cases} -y'' + y' = f(x), \\ y'(0) = 0, \\ y(1) + y'(1) = 0, \end{cases}$$

$$b)\begin{cases} -x^2 y'' - \frac{1}{4} y = \lambda y + f(x) \\ y(1) = 0, \\ y(e) = 0, \end{cases}$$

$$c)\begin{cases} -\frac{xy''}{1+x} - \frac{y'}{(1+x)^2} = f(x), \\ y(1) = 0, \\ y(e) - ey'(e) = 0, \end{cases}$$

$$d)\begin{cases} -x^3 y'' - 3x^2 y' - xy = f(x), \\ y(1) = 0, \\ 2y'(2) + y(2) = 0, \end{cases}$$

$$e)\begin{cases} -x^4 y'' - 4x^3 y' - 2x^2 y = f(x), \\ y(1) + y'(1) = 0, \\ y(2) + 3y'(2) = 0, \end{cases}$$

$$f)\begin{cases} -\left(e^{-\frac{x^2}{2}}y'\right)' + e^{-\frac{x^2}{2}}y = f(x), \\ y(0) = 0, \\ y(1) = 0, \end{cases}$$

$$g)\begin{cases} -e^{x^2}y'' - 2xe^{x^2}y' = f(x), \\ y(0) = 2y'(0), \\ y(1) = 0, \end{cases}$$

$$h)\begin{cases} -(\cos^2 x \cdot y')' = f(x), \\ y(0) = 0, \\ y\left(\frac{\pi}{4}\right) = 0 \end{cases}$$

$$i)\begin{cases} -\left(\frac{1}{\cos x}y'\right)' = f(x), \\ y(0) = 0, \\ y\left(\frac{\pi}{4}\right) = 0, \end{cases}$$

$$j)\begin{cases} -\cos^2 x \cdot y'' + \sin 2x \cdot y' = f(x), \\ y(0) = y'(0), \\ y\left(\frac{\pi}{4}\right) + y'\left(\frac{\pi}{4}\right) = 0, \end{cases}$$

$$k)\begin{cases} -(1+x^2)y'' - 2xy' = f(x), \\ y(0) = y'(0), \\ y(1) = 0, \end{cases}$$

$$l)\begin{cases} -(1+x^2)y'' - 2xy' = f(x), \\ y(0) = 0, \\ y(1) + y'(1) = 0, \end{cases}$$

$$m)\begin{cases} -(3+x^2)y'' - 2xy' = f(x), \\ y(0) = y'(0), \\ y(1) = 0, \end{cases}$$

$$n)\begin{cases} -\dfrac{1}{x^2}y'' + \dfrac{2}{x^3}y' - \dfrac{2}{x^4}y = f(x), \\ y'(0) = 0, \\ y(1) = 0, \end{cases}$$

$$o)\begin{cases} -(1+x)^2y'' - 2(x+1)y' + 2y = f(x), \\ y(0) = 0, \\ y(1) = 0, \end{cases}$$

$$p)\begin{cases} -(xy')' + \dfrac{4}{x}y = f(x), \\ y(0) = 0, \\ y(1) = 0, \end{cases}$$

$$r)\begin{cases} -\left(\dfrac{1}{x-2}y'\right)' + \dfrac{3}{(x-2)^3}y = f(x), \\ y(0) = 0, \\ y(1) = 0, \end{cases}$$

$$s)\begin{cases} -(1+\cos x)y'' + \sin x \cdot y' = f(x), \\ y(0) - 2y'(0) = 0, \\ y\left(\dfrac{\pi}{2}\right) = 0. \end{cases}$$

5. Quyidagi chegaraviy masalalarning yechimlarini Grin funksiyasi yordamida toping:

$$a)\begin{cases} -(xy')' = f(x), \\ |y(0)| < \infty, \\ y(1) = 0, \end{cases} \qquad b)\begin{cases} -(\mathrm{tg}^2\, x \cdot y')' = f(x), \\ |y(0)| < \infty, \\ y\left(\dfrac{\pi}{4}\right) = 0, \end{cases}$$

$$c)\begin{cases} -(\mathrm{tg}\, x \cdot y')' = f(x), \\ |y(0)| < \infty, \\ y\left(\dfrac{\pi}{4}\right) = 0, \end{cases}$$

$$d)\begin{cases} -\cos^2 x\cdot y''+\sin 2x\cdot y'=f(x),\\ y(0)=0\,,\\ \left|y\left(\dfrac{\pi}{2}\right)\right|<\infty. \end{cases}$$

$$e)\begin{cases} -\sin^2 x\cdot y''-\sin 2x\cdot y'=f(x),\\ |y(0)|<\infty\,,\\ y\left(\dfrac{\pi}{2}\right)=0, \end{cases}$$

$$f)\begin{cases} -\sin^2 x\cdot y''-\sin 2x\cdot y'=f(x),\\ |y(0)|<\infty\,,\\ y\left(\dfrac{\pi}{2}\right)+y'\left(\dfrac{\pi}{2}\right)=0. \end{cases}$$

$$g)\begin{cases} -y''+\dfrac{2}{x^2}y=f(x).\\ |y(0)|<\infty,\\ y(1)=0, \end{cases}$$

$$h)\begin{cases} -x^2y''-2xy+6y=f(x),\\ |y(0)|<\infty\,,\\ y'(1)+3y(1)=0, \end{cases}$$

$$i)\begin{cases} -x^2y''-2xy+2y=f(x),\\ |y(0)|<\infty\,,\\ y'(1)=0, \end{cases}$$

$$j)\begin{cases} -xy''-y'=f(x),\\ |y(0)|<\infty,\\ y'(1)+y(1)=0, \end{cases}$$

$$k)\begin{cases} -(xy')'+(1+x)y=f(x),\\ |y(0)|<\infty,\\ y(1)=0, \end{cases}$$

$$l)\begin{cases} -x^2y'' - 2xy' + 2y = f(x), \\ |y(0)| < \infty, \\ y(1) + y'(1) = 0, \end{cases}$$

$$m)\begin{cases} -x^2y'' - 2xy' + 2y = f(x), \\ |y(0)| < \infty\,, \\ 2y(1) + y'(1) = 0, \end{cases}$$

$$n)\begin{cases} -(x+1)y'' - y' = f(x), \\ |y(-1)| < \infty, \\ y(0) = 0, \end{cases}$$

$$o)\begin{cases} xy'' + y' = f(x), \\ |y(0)| < \infty, \\ y(1) + y'(1) = 0, \end{cases}$$

$$p)\begin{cases} -(\sqrt{x}y')' + 3x^{-\frac{3}{2}}y = f(x), \\ |y(0)| < \infty, \\ y(2) = 0, \end{cases}$$

$$q)\begin{cases} -(xy')' = f(x), \\ |y(0)| < \infty, \\ y(1) = 0, \end{cases}$$

$$r)\begin{cases} -x^2y'' - xy' + n^2y = f(x), \\ |y(0)| < \infty, \\ y(1) = 0, \end{cases}$$

$$s)\begin{cases} -((x^2-1)y')' + 2y = f(x), \\ |y(1)| < \infty, \\ y(2) = 0, \end{cases}$$

$$t)\begin{cases} -\dfrac{x}{1-x}y'' - \dfrac{1}{(1-x)^2}y' = f(x), \\ 2y(-1) + y'(-1) = 0, \\ |y(0)| < \infty, \end{cases}$$

$$u)\begin{cases} -y'' + \dfrac{a(a-1)}{x^2}y = f(x), \ (a > 1), \\ |y(0)| < \infty, \\ y(1) = 0. \end{cases}$$

10-§. Umumlashgan Grin funksiyasi

Ushbu

$$Ly \equiv -y'' + q(x)y = f(x), \quad 0 \leq x \leq \pi, \tag{1.10.1}$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.10.2}$$

chegaraviy masalani ko‘rib chiqamiz.

Agar $\lambda = 0$ soni quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq \pi, \tag{1.10.3}$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.10.4}$$

bir jinsli masalaning xos qiymati bo‘lsa, Grin funksiyasi mavjud emasligini oldingi paragrafda ko‘rgan edik, ya'ni ushbu

$$-y'' + q(x)y = 0, \quad 0 \leq x \leq \pi, \tag{1.10.5}$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.10.6}$$

chegaraviy masala noldan farqli yechimga ega bo‘lsa, Grin funksiyasi mavjud bo‘lmasligini bilamiz.

Ammo bu fikr (1.10.1)÷(1.10.2) chegaraviy masalaning yechimi yo‘q degani emas. Bu holda $f(x)$ funksiya qanday shartlarni qanoatlantirganda (1.10.1)+(1.10.2) chegaraviy masala yechimga ega bo‘ladi va uni qanday qilib topish mumkin degan savollar tug‘iladi. Mazkur paragrafda bu savolga atroflicha javob beriladi.

Faraz qilaylik, (1.10.5)+(1.10.6) bir jinsli chegaraviy masala noldan farqli normallangan $\varphi_0(x)$ yechimga ega bo‘lsin. Agar yana boshqa bir noldan farqli yechim mavjud bo‘lsa, bu yechim $\varphi_0(x)$ xos funksiyaga proporsional bo‘ladi, chunki (1.10.3)+(1.10.4) chegaraviy masalaning xos qiymatlari oddiydir.

Lemma 1.10.1. *Agar (1.10.1)+(1.10.2) chegaraviy masala yechimga ega bo‘lsa, u holda ushbu*

$$\int_0^{\pi} f(x)\varphi_0(x)dx = 0,$$

tenglik bajariladi, ya’ni (1.10.1) tenglamaning o‘ng tomoni unga mos keluvchi bir jinsli chegaraviy masalaning xos funksiyasiga ortogonaldir.

Isbot. Grin ayniyatiga ko‘ra

$$\int_0^{\pi} f(x)\varphi_0(x)dx = \int_0^{\pi} [-y''(x) + q(x)y(x)]\varphi_0(x)dx =$$

$$= \int_0^{\pi} y(x)L\varphi_0(x)dx = \int_0^{\pi} y(x)\cdot 0dx = 0,$$

bo‘ladi.■

Lemma 1.10.2. *Agar $\tilde{y}(x)$ funksiya (1.10.1)+(1.10.2) chegaraviy masalaning biror yechimi bo‘lsa, u holda (1.10.1)+(1.10.2) chegaraviy masalaning ixtiyoriy yechimi ushbu*

$$y(x) = \tilde{y}(x) + C\cdot\varphi_0(x), \tag{1.10.7}$$

ko‘rinishda bo‘ladi. Bu yerda C ixtiyoriy o‘zgarmas son.

Isbot. $y(x)$ funksiya (1.10.1)+(1.10.2) chegaraviy masalaning ixtiyoriy yechimi bo‘lsin. U holda $\psi(x) = y(x) - \tilde{y}(x)$ funksiya

(1.10.5)+(1.10.6) bir jinsli chegaraviy masalaning yechimi bo'lishi ravshan. (1.10.5)+(1.10.6) chegaraviy masalaning ixtiyoriy yechimi $\varphi_0(x)$ funksiyaga proporsional, ya'ni $\psi(x) = C\cdot\varphi_0(x)$ bo'ladi. Demak, (1.10.7) tasvir o'rinli ekan.■

Lemma 1.10.3. *Agar (1.10.1)+(1.10.2) chegaraviy masalaning ushbu*

$$\int_0^\pi y(x)\varphi_0(x)dx = 0, \tag{1.10.8}$$

shartni qanoatlantiruvchi yechimi mavjud bo'lsa, u yagonadir.

Isbot. (1.10.1)+(1.10.2) chegaraviy masalaning (1.10.8) shartni qanoatlantiruvchi ikkita yechimi $y_1(x) \not\equiv y_2(x)$ mavjud deb faraz qilaylik. U holda $\psi(x) = y_1(x) - y_2(x)$ funksiya (1.10.5)+(1.10.6) bir jinsli chegaraviy masalaning (1.10.8) shartni qanoatlantiruvchi yechimi bo'ladi. Bu yechim uchun ushbu $\psi(x) = C\cdot\varphi_0(x)$ tenglik o'rinli bo'ladi. Buni (1.10.8) shartga qo'ysak,

$$C\int_0^\pi \varphi_0^2(x)dx = 0, \quad C = 0, \quad \psi(x)\equiv 0, \quad y_1(x)\equiv y_2(x),$$

hosil bo'ladi. Bu esa farazimizga ziddir, ya'ni yechim yagona ekan.■

Ta'rif 1.10.1. Quyidagi shartlarni qanoatlantiruvchi $H(x,t)$ funksiyaga (1.10.1)+(1.10.2) chegaraviy masalaning umumlashgan Grin funksiyasi deyiladi:

1) $H(x,t)$ funksiya $[0,\pi]\times[0,\pi]$ to'plamda uzluksiz;

2) $H(x,t)$ funksiya $[0,\ t)$ va $(t,\pi]$ oraliqlarda ushbu

$$-H_{xx} + q(x)H = \varphi_0(x)\varphi_0(t),$$

tenglamani qanoatlantiradi;

3) $H_x|_{x=t+0} - H_x|_{x=t-0} = -1;$

4) $H(x,t)$ funksiya (1.10.2) chegaraviy shartlarni qanoatlantiradi;

5) $\int\limits_0^\pi H(x,t)\varphi_0(x)dx = 0, \quad t \in [0,\pi]$.

Teorema 1.10.1. *(1.10.1)+(1.10.2) chegaraviy masalaning umumlashgan Grin funksiyasi mavjud va yagona.*

Isbot. Umumlashgan Grin funksiyasining ta'rifidan kelib chiqib, uni quyidagi

$$H(x,t) = \begin{cases} \theta(x,t) + A(t)\varphi_0(x) + B(t)\psi(x), & x \le t, \\ \theta(x,t) + C(t)\varphi_0(x) + D(t)\psi(x), & x \ge t, \end{cases} \qquad (1.10.9)$$

ko'rinishda izlaymiz. Bu yerda $\theta(x,t)$ ushbu

$$\begin{cases} -y'' + q(x)y = \varphi_0(x)\varphi_0(t), \\ y(0) = 0, \\ y'(0) = 0, \end{cases}$$

Koshi masalaning yechimi, $\psi(x)$ esa

$$-y'' + q(x)y = 0,$$

bir jinsli tenglamaning $\varphi_0(x)$ yechimga proporsional bo'lmagan yechimidir.

Umumlashgan Grin funksiyasi ta'rifidagi to'rtinchi shartga asosan

$$[\theta(0,t) + A(t)\varphi_0(0) + B(t)\psi(0)]\cos\alpha +$$
$$+[\theta'(0,t) + A(t)\varphi_0'(0) + B(t)\psi'(0)]\sin\alpha = 0, \qquad (1.10.10)$$

$$[\theta(\pi,t) + C(t)\varphi_0(\pi) + D(t)\psi(\pi)]\cos\beta +$$
$$+[\theta(\pi,t) + C(t)\varphi_0'(\pi) + D(t)\psi'(\pi)]\sin\beta = 0, \qquad (1.10.11)$$

tengliklar kelib chiqadi. (1.10.10) va (1.10.11) tengliklarni quyidagi ko'rinishda yozib olamiz:

$$[\theta(0,t)\cos\alpha + \theta'(0,t)\sin\alpha] + B(t)[\psi(0)\cos\alpha + \psi'(0)\sin\alpha] = 0,$$

$$[\theta(\pi,t)\cos\beta + \theta'(\pi,t)\sin\beta] + D(t)[\psi(\pi)\cos\beta + \psi'(\pi)\sin\beta] = 0.$$

Bulardan

$$B(t) = 0, \tag{1.10.12}$$

$$D(t) = -\frac{\theta(\pi, t)\cos\beta + \theta'(\pi, t)\sin\beta}{\psi(\pi)\cos\beta + \psi'(\pi)\sin\beta}, \tag{1.10.13}$$

kelib chiqadi.

Umumlashgan Grin funksiyasi ta'rifining birinchi shartiga ko'ra u uzluksiz bo'ladi, demak, quyidagi tenglik bajariladi:

$$A(t)\varphi_0(t) + B(t)\psi(t) - C(t)\varphi_0(t) - D(t)\psi(t) = 0.$$

Umumlashgan Grin funksiyasi ta'rifining uchinchi shartiga ko'ra

$$A(t)\varphi_0'(t) + B(t)\psi'(t) - C(t)\varphi_0'(t) - D(t)\psi'(t) = -1,$$

bo'ladi. Bulardan ushbu

$$\begin{cases} (A - C)\varphi_0(t) + (B - D)\psi(t) = 0, \\ (A - C)\varphi_0'(t) + (B - D)\psi'(t) = -1; \end{cases} \tag{1.10.14}$$

sistema hosil bo'ladi. (1.10.14) sistemani Kramer qoidasi yordamida yechamiz:

$$\Delta = \begin{vmatrix} \varphi_0(t) & \psi(t) \\ \varphi_0'(t) & \psi'(t) \end{vmatrix} = \omega, \quad \Delta_1 = \begin{vmatrix} 0 & \psi(t) \\ -1 & \psi'(t) \end{vmatrix} = -\psi(t),$$

$$\Delta_2 = \begin{vmatrix} \varphi_0(t) & 0 \\ \varphi_0'(t) & -1 \end{vmatrix} = \varphi_0(t).$$

Demak,

$$A - C = -\frac{\psi(t)}{\omega}, \tag{1.10.15}$$

$$B - D = \frac{\varphi_0(t)}{\omega}, \tag{1.10.16}$$

bo'ladi. B va D uchun topilgan (1.10.12) va (1.10.13) qiymatlar (1.10.16) tenglikni qanoatlantiradi. Haqiqatan ham, Grin ayniyatiga ko'ra ushbu

$$\int_0^{\pi}[\varphi_0(x)L\theta(x,t)-\theta(x,t)L\varphi_0(x)]dx=$$

$$=[\theta(x,t)\varphi_0'(x)-\varphi_0(x)\theta'(x,t)]|_0^{\pi},$$

$$\int_0^{\pi}\varphi_0^2(x)\varphi_0(t)dx=\theta(\pi,t)\varphi_0'(\pi)-\varphi_0(\pi)\theta'(\pi,t),$$

$$\varphi_0(t)=\theta(\pi,t)\varphi_0'(\pi)-\varphi_0(\pi)\theta'(\pi,t), \qquad (1.10.17)$$

tengliklar o'rinli bo'ladi. (1.10.12), (1.10.13) ifodalarni, (1.10.2) chegaraviy shartlarni va (1.10.17) tengliklarni hisobga olsak, quyidagi

$$B-D=-\frac{\theta(\pi,t)\cos\beta+\theta'(\pi,t)\sin\beta}{\psi(\pi)\cos\beta+\psi'(\pi)\sin\beta}=$$

$$=-\frac{\theta(\pi,t)\operatorname{ctg}\beta+\theta'(\pi,t)}{\psi(\pi)\operatorname{ctg}\beta+\psi'(\pi)}=-\frac{\theta(\pi,t)\left(-\frac{\varphi_0'(\pi)}{\varphi_0(\pi)}\right)+\theta'(\pi,t)}{\psi(\pi)\left(-\frac{\varphi_0'(\pi)}{\varphi_0(\pi)}\right)+\psi'(\pi)}=$$

$$=-\frac{\theta'(\pi,t)\varphi_0(\pi)-\theta(\pi,t)\varphi_0'(\pi)}{\psi'(\pi)\varphi_0(\pi)-\psi(\pi)\varphi_0'(\pi)}=\frac{\varphi_0(t)}{\omega},$$

tenglik hosil bo'ladi.

Bundan va (1.10.15) tenglikdan

$$A=-\frac{\psi(t)}{\omega}+C, \quad D=-\frac{\varphi_0(t)}{\omega} \qquad (1.10.18)$$

kelib chiqadi. (1.10.12) va (1.10.18) ifodalarni (1.10.9) tenglikka qo'yib, ushbu

$$H(x,t)=\begin{cases}\theta(x,t)+C(t)\varphi_0(x)-\dfrac{\psi(t)\varphi_0(x)}{\omega}, & x\le t,\\ \theta(x,t)+C(t)\varphi_0(x)-\dfrac{\varphi_0(t)\psi(x)}{\omega}, & x\ge t,\end{cases}$$

formulani olamiz, ya'ni

$$H(x,t)=\theta(x,t)+C(t)\varphi_0(x)-\begin{cases}\frac{\psi(t)\varphi_0(x)}{\omega}, & x\le t,\\ \frac{\varphi_0(t)\psi(x)}{\omega}, & x\ge t\,.\end{cases} \qquad (1.10.19)$$

19) ifodani umumlashgan Grin funksiyasining beshinchi
ıga qo'yamiz:

$$\int_0^\pi \theta(x,t)\varphi_0(x)dx + C(t) + \frac{\psi(t)}{\omega}\int_0^t \varphi_0^2(x)dx +$$

$$-\frac{\varphi_0(t)}{\omega}\int_t^\pi \psi(x)\varphi_0(x)dx = 0,$$

$$C(t) = -\int_0^\pi \theta(x,t)\varphi_0(x)dx - \frac{\psi(t)}{\omega}\int_0^t \varphi_0^2(x)dx -$$

$$+\frac{\varphi_0(t)}{\omega}\int_t^\pi \psi(x)\varphi_0(x)dx. \qquad (1.10.20)$$

(1.10.12), (1.10.13), (1.10.20), (1.10.18) tengliklardan mos ravishda $B(t)$, $D(t)$, $C(t)$, $A(t)$ noma'lumlarning qiymatlari topiladi. Bundan, (1.10.9) tenglikka muvofiq, $H(x,t)$ umumlashgan Grin funksiyasining ifodasi bir qiymatli aniqlanadi.■

Teorema 1.10.2. *Agar (1.10.1)+(1.10.2) chegaraviy masalada*

$$\int_0^\pi f(x)\varphi_0(x)dx = 0,$$

bo'lsa, u holda (1.10.1)+(1.10.2) chegaraviy masalaning (1.10.8) shartni qanoatlantiruvchi yechimi mavjud va yagona bo'lib, u ushbu

$$y(x) = \int_0^\pi H(x,t)f(t)dt, \qquad (1.10.21)$$

formula yordamida topiladi.

Isbot. Dastlab (1.10.21) formulani quyidagi ko'rinishda yozib olamiz:

$$y(x) = \int_0^x H(x,t)f(t)dt + \int_x^\pi H(x,t)f(t)dt. \qquad (1.10.22)$$

(1.10.22) tenglikning ikkala tomononi x o'zgaruvchi bo'yicha differensiallaymiz. Natijada ushbu

$$y'(x) = H(x,x)f(x) + \int_0^x H'_x(x,t)f(t)dt -$$

$$-H(x,x)f(x) + \int_x^\pi H'_x(x,t)f(t)dt,$$

$$y'(x) = \int_0^x H'_x(x,t)f(t)dt + \int_x^\pi H'_x(x,t)f(t)dt,$$

$$y''(x) = H'_x(x,x-0)f(x) + \int_0^x H''_{xx}(x,t)f(t)dt -$$

$$-H'_x(x,x+0)f(x) + \int_x^\pi H''_{xx}(x,t)f(t)dt, \qquad (1.10.23)$$

tengliklarni hosil qilamiz.

Umumlashgan Grin funksiyasi ta'rifining uchinchi va ikkinchi bandlaridan mos ravishda foydalanib, quyidagi tengliklarni topamiz:

$$H'_x(x,x-0) - H'_x(x,x+0) = -1, \qquad (1.10.24)$$

$$H_{xx} = q(x)H - \varphi_0(x)\varphi_0(t). \qquad (1.10.25)$$

(1.10.24) va (1.10.25) tengliklarni (1.10.23) formulaga qo'yamiz:

$$y''(x) = -f(x) + \int_0^x [q(x)H(x,t) - \varphi_0(x)\varphi_0(t)]f(t)dt +$$

$$+\int_x^\pi [q(x)H(x,t) - \varphi_0(x)\varphi_0(t)]f(t)dt =$$

$$= -f(x) + q(x)\int_0^\pi H(x,t)f(t)dt - \varphi_0(x)\int_0^\pi \varphi_0(t)f(t)dt.$$

Teorema shartiga ko'ra

$$y'' = -f(x) + q(x)y,$$

kelib chiqadi, ya'ni (1.10.21) formula bilan beriladigan funksiya (1.10.1) tenglamani qanoatlantiradi. Chegaraviy shartlar bajarilishi ravshan.■

Misol. Ushbu

$$\begin{cases} -y'' = f(x), \\ y'(a) = 0, \\ y'(b) = 0, \end{cases} \tag{1.10.26}$$

chegaraviy masalaning yechimini umumlashgan Grin funksiyasi yordamida topamiz. Berilgan chegaraviy masalaning yechimini topishdan oldin, unga mos bo'lgan bir jinsli

$$\begin{cases} -y'' = 0, \\ y'(a) = 0, \\ y'(b) = 0, \end{cases} \tag{1.10.27}$$

chegaraviy masalani yechamiz. Ko'rinib turibdiki, ushbu

$$\varphi_0(x) = \frac{1}{\sqrt{b-a}},$$

funksiya (1.10.27) chegaraviy masalaning normallangan yechimi bo'lib, qolgan yechimlar $\varphi_0(x)$ yechimga proporsional bo'ladi. Demak, (1.10.27) chegaraviy masalaning odatdagi Grin funksiyasi mavjud bo'lmaydi. Shuning uchun, uning umumlashgan Grin funksiyasini tuzishga tug'ri keladi. Avvalo ushbu

$$-y'' = \frac{1}{b-a}, \tag{1.10.28}$$

bir jinsli bo'lmagan tenglamaning umumiy yechimini topamiz:

$$y(x) = -\frac{1}{2(b-a)}x^2 + c_1x + c_2. \qquad (1.10.29)$$

Bundan foydalanib, (1.10.27) chegaraviy masalaning umumlashgan Grin funksiyasini ushbu

$$H(x,t) = \begin{cases} \dfrac{-1}{2(b-a)}x^2 + A_1x + B_1, & x \le t, \\ \dfrac{-1}{2(b-a)}x^2 + A_2x + B_2, & x \ge t, \end{cases} \qquad (1.10.30)$$

ko'rinishda izlaymiz. Umumlashgan Grin funksiyasining uzluksizligidan va berilgan chegaraviy shartlardan foydalanib, quyidagi

$$\begin{cases} A_1 = \dfrac{a}{b-a}, \\ A_2 = \dfrac{b}{b-a}, \\ A_1 - A_2 = -1, \\ B_1 - B_2 = t(A_2 - A_1), \end{cases} \qquad (1.10.31)$$

tenglamalar sistemasini hosil qilamiz. Bu sistemadan

$$B_1 = B_2 + t, \qquad (1.10.32)$$

ekanligini topamiz. Bularni (1.10.30) formulaga qo'yib, ushbu

$$H(x,t) = \begin{cases} \dfrac{-1}{2(b-a)}x^2 + \dfrac{a}{b-a}x + t + B_2, & x \le t, \\ \dfrac{-1}{2(b-a)}x^2 + \dfrac{b}{b-a}x + B_2, & x \ge t, \end{cases} \qquad (1.10.33)$$

tenglikni hosil qilamiz. Oxirgi tenglikdagi B_2 ning qiymatini

$$\int_a^b H(x,t)dx = 0,$$

ortogonallik shartidan foydalanib topamiz:

$$0 = \int_a^b H(x,t)dx = \int_a^t \left[\frac{-1}{2(b-a)}x^2 + \frac{a}{b-a}x + t + B_2\right] dx +$$

$$+\int_t^b \left[\frac{-1}{2(b-a)}x^2 + \frac{b}{b-a}x + B_2\right]dx =$$

$$= \frac{1}{2}t^2 - ta + \frac{a^2+ab+b^2}{3} + B_2(b-a).$$

Demak,

$$B_2 = \frac{-1}{2(b-a)}t^2 + \frac{a}{b-a}t - \frac{a^2+ab+b^2}{3(b-a)}.$$

Nihoyat,

$$H(x,t) = \begin{cases} \dfrac{-1}{2(b-a)}\left[(x-a)^2 + (t-b)^2 - \dfrac{1}{3}(a-b)^2\right], & x \le t, \\ \dfrac{-1}{2(b-a)}\left[(t-a)^2 + (x-b)^2 - \dfrac{1}{3}(a-b)^2\right], & x \ge t, \end{cases}$$

ekanligi kelib chiqadi. Shunday qilib,

$$\int_a^b f(x)dx = 0, \tag{1.10.34}$$

bo'lgan holda, berilgan (1.10.26) chegaraviy masalaning ushbu

$$\int_a^b y(x)dx = 0, \tag{1.10.35}$$

shartni qanoatlantiruvchi yechimi yagona bo'lib, u quyidagi

$$y(x) = \int_a^b H(x,t)f(t)dt, \tag{1.10.36}$$

formula yordamida topiladi.

Qaralayotgan chegaraviy masalaning umumiy yechimi

$$y(x) = \int_a^b H(x,t)f(t)dt + c, \qquad c = \text{const} \tag{1.10.37}$$

tenglik bilan beriladi. Agar (1.10.34) shart bajarilmasa, (1.10.26) chegaraviy masala yechimga ega emas.

Izoh 1.10.1. Ayrim hollarda qaralayotgan chegaraviy masalalarga mos bo'lgan bir jinsli chegaraviy masalalarning ikkita chiziqli erkli yechimi bo'lishi mumkin. Bu hollarda umumlashgan Grin funksiyasining ta'rifiga o'zgartirishlar kiritish kerak bo'ladi.

Quyidagi chegaraviy masalani ko'rib chiqamiz

$$-y'' + q(x)y = f(x), \quad a \le x \le b, \tag{1.10.38}$$

$$\begin{cases} y(a) = y(b), \\ y'(a) = y'(b) \,. \end{cases} \tag{1.10.39}$$

Bu chegaraviy masalaga mos keluvchi bir jinsli chegaraviy masalaning ikkita $\varphi_1(x)$ va $\varphi_2(x)$ chiziqli erkli ortonormallangan yechimlari mavjud bo'lsin. U holda bir jinsli bo'lmagan (1.10.38)+(1.10.39) chegaraviy masala yechimi mavjudligining zaruriy sharti

$$\int_a^b f(x)\varphi_j(x)dx = 0, \quad j = 1, 2 \tag{1.10.40}$$

ko'rinishda bo'ladi. Qaralayotgan holda ushbu

$$-y'' + q(x)y = f(x), \quad a \le x \le b, \tag{1.10.41}$$

$$\begin{cases} y(a) = y(b), \\ y'(a) = y'(b), \\ \int_a^b y(x)\varphi_j(x)dx = 0, \ j = 1, 2, \end{cases} \tag{1.10.42}$$

bir jinsli bo'lmagan chegaraviy masala yagona yechimga ega bo'ladi.

Ta'rif 1.10.1. Quyidagi shartlarni qanoatlantiruvchi $H(x,t)$ funksiyaga (1.10.38)+(1.10.39) chegaraviy masalaning umumlashgan Grin funksiyasi deyiladi:

1) $H(x,t)$ funksiya $[a,b]\times[a,b]$ to'plamda uzluksiz;

2) $H(x,t)$ funksiya $[a,t)$ va $(t,b]$ oraliqlarda ushbu

$$-H_{xx}+q(x)H=\varphi_1(x)\varphi_1(t)+\varphi_2(x)\varphi_2(t),$$

tenglamani qanoatlantiradi;

3) $H_x|_{x=t+0}-H_x|_{x=t-0}=-1;$

4) $H(x,t)$ funksiya (1.10.39) chegaraviy shartlarni qanoatlantiradi;

5) $\int\limits_0^\pi H(x,t)\varphi_j(x)dx=0,\quad j=1,\,2.$

Mustaqil yechish uchun mashqlar

1. Quyidagi chegaraviy masalalarning yechimlarini umumlashgan Grin funksiyasi yordamida toping.

$a)\begin{cases}-y''=f(x),\\ y'(0)=0,\\ y'(\pi)=0,\end{cases}\qquad b)\begin{cases}-y''-y=f(x),\\ y(0)=0,\\ y(\pi)=0,\end{cases}$

$c)\begin{cases}-y''-\frac{1}{4}y=f(x),\\ y'(0)=0,\\ y(\pi)=0,\end{cases}\qquad d)\begin{cases}-y''-y=f(x),\\ y'(0)=0,\\ y'(\pi)=0,\end{cases}$

$e)\begin{cases}-y''=f(x),\\ y(0)=y(\pi),\\ y'(0)=y'(\pi),\end{cases}\qquad f)\begin{cases}-y''-4y=f(x),\\ y(0)=y(\pi),\\ y'(0)=y'(\pi),\end{cases}$

$g)\begin{cases}-y''-y=f(x),\\ y(0)=-y(\pi),\\ y'(0)=-y'(\pi),\end{cases}\qquad h)\begin{cases}-y''=f(x),\\ y(-1)=y(1),\\ y'(-1)=y'(1),\end{cases}$

$i)\begin{cases}-y''=f(x),\\ y'(0)=-y(0),\\ y(1)=0,\end{cases}\qquad j)\begin{cases}-y''=f(x),\\ y(0)=0,\\ y'(1)=y(1),\end{cases}$

$$k)\begin{cases} -((1-x^2)y')' = f(x), \\ |y(-1)| < \infty, \\ |y(+1)| < \infty. \end{cases}$$

11-§. Yoyilma teoremasi. Kompleks analiz usuli

Mazkur paragrafda Shturm-Liuvill chegaraviy masalasi xos funksiyalarining $L^2[0,\pi]$ fazoda to'laligi ko'rsatiladi. Bu teorema ilk bor XIX asrning oxirlarida V.A.Steklov tomonidan isbotlangan.

Ushbu

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad 0 < x < \pi, \tag{1.11.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.11.2}$$

Shturm-Liuvill chegaraviy masalasi berilgan bo'lsin. Bu yerda λ - spektral parametr, h va H chekli haqiqiy sonlar, $q(x) \in C[0,\pi]$ haqiqiy funksiya.

(1.11.1) tenglamaning quyidagi

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = h \quad \text{va} \quad \psi(\pi,\lambda) = 1, \quad \psi'(\pi,\lambda) = -H,$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini $\varphi(x,\lambda)$ va $\psi(x,\lambda)$ orqali belgilaymiz. Bu yechimlar yordamida tuzilgan Vronskiy determinantini

$$\omega(\lambda) = W\{\varphi(x,\lambda),\, \psi(x,\lambda)\} \equiv \varphi\psi' - \psi\varphi'$$

orqali belgilaymiz. U holda

$$\omega(\lambda) = \begin{vmatrix} \varphi(0,\lambda) & \psi(0,\lambda) \\ \varphi'(0,\lambda) & \psi'(0,\lambda) \end{vmatrix} = \begin{vmatrix} 1 & \psi(0,\lambda) \\ h & \psi'(0,\lambda) \end{vmatrix} =$$

$$= \psi'(0,\lambda) - h\psi(0,\lambda) \equiv \tilde{\Delta}(\lambda),$$

$$\omega(\lambda) = \begin{vmatrix} \varphi(\pi,\lambda) & \psi(\pi,\lambda) \\ \varphi'(\pi,\lambda) & \psi'(\pi,\lambda) \end{vmatrix} = \begin{vmatrix} \varphi(\pi,\lambda) & 1 \\ \varphi'(\pi,\lambda) & -H \end{vmatrix} =$$

$$= -[H\varphi(\pi, \lambda) + \varphi'(\pi, \lambda)] \equiv -\Delta(\lambda),$$

ya'ni

$$\omega(\lambda) = \tilde{\Delta}(\lambda) = -\Delta(\lambda), \qquad (1.11.3)$$

tengliklar o'rinli bo'ladi.

Yuqoridagi paragraflarda $\omega(\lambda)$ butun funksiya ekanligi ko'rsatilgan edi. Bundan tashqari $\omega(\lambda)$ funksiyaning nollari $\{\lambda_n\}_0^\infty$ xos qiymatlardan iboratligi ham ko'rsatilgan edi. Xususan, $\omega(\lambda_n) = 0, \;\; n = 0, 1, 2, ...$ ekanligidan

$$\psi(x, \lambda_n) = C_n\varphi(x, \lambda_n), \quad C_n \neq 0, \quad n = 0,\ 1,\ 2,\ ..., \qquad (1.11.4)$$

tengliklar o'rinli bo'lishi kelib chiqadi.

Quyidagi

$$C_n\alpha_n^2 = \dot{\omega}(\lambda_n), \qquad (1.11.5)$$

tenglik bajariladi. Bu yerda

$$\alpha_n = \sqrt{\int_0^\pi \varphi^2(x, \lambda_n)dx}.$$

Haqiqatan ham, ushbu

$$-\varphi''(x, \lambda_n) + q(x)\varphi(x, \lambda_n) = \lambda_n\varphi(x, \lambda_n),$$
$$-\psi''(x, \lambda) + q(x)\psi(x, \lambda) = \lambda\psi(x, \lambda),$$

tengliklarni mos ravishda $-\psi(x, \lambda)$ va $\varphi(x, \lambda)$ funksiyalarga ko'paytirib, bir-biriga qo'shsak,

$$\varphi''(x, \lambda_n)\psi(x, \lambda) - \psi''(x, \lambda)\varphi(x, \lambda_n) = (\lambda - \lambda_n)\psi(x, \lambda)\varphi(x, \lambda_n),$$

ya'ni

$$(\lambda - \lambda_n)\psi(x, \lambda)\varphi(x, \lambda_n) = (\varphi'(x, \lambda_n)\psi(x, \lambda) - \psi'(x, \lambda)\varphi(x, \lambda_n))',$$

tenglik kelib chiqadi. Bunga ko'ra

$$(\lambda - \lambda_n)\int_0^\pi \psi(x, \lambda)\varphi(x, \lambda_n)dx =$$

$$= [\varphi'(x,\lambda_n)\psi(x,\lambda) - \psi'(x,\lambda)\varphi(x,\lambda_n)]|_0^\pi =$$

$$= [\varphi'(\pi,\lambda_n) + H\varphi(\pi,\lambda_n)] - [h\psi(0,\lambda) - \psi'(0,\lambda)] =$$

$$\Delta(\lambda_n) + \bar{\Delta}(\lambda) = \omega(\lambda) - \omega(\lambda_n),$$

ya'ni

$$\int_0^\pi \psi(x,\lambda)\varphi(x,\lambda_n)dx = \frac{\omega(\lambda) - \omega(\lambda_n)}{\lambda - \lambda_n},$$

ayniyat hosil bo'ladi. Bu yerda $\lambda \to \lambda_n$ bo'lganda limitga o'tsak, ushbu

$$\omega(\lambda_n) = \int_0^\pi \psi(x,\lambda_n)\varphi(x,\lambda_n)dx = C_n \int_0^\pi \varphi^2(x,\lambda_n)dx = C_n\alpha_n^2,$$

tenglik kelib chiqadi.

Teorema 1.11.1. *Shturm-Liuvill chegaraviy masalasining $\{\varphi(x,\lambda_n)\}_{n=0}^{\infty}$ xos funksiyalar sistemasi $L^2[0,\pi]$ fazoda to'la bo'ladi, ya'ni biror $f(x) \in L_2(0,\pi)$ funksiya uchun*

$$\int_0^\pi f(x)\varphi(x,\lambda_n)dx = 0, \quad n = 0,\ 1,\ 2,\ \ldots, \tag{1.11.6}$$

bo'lsa, $f(x) \equiv 0$ bo'ladi.

Isbot. Berilgan Shturm-Liuvill chegaraviy masalasining

$$G(x,t,\lambda) = -\frac{1}{\omega(\lambda)} \cdot \begin{cases} \varphi(x,\lambda)\psi(t,\lambda), & x \le t, \\ \varphi(t,\lambda)\psi(x,\lambda), & x \ge t, \end{cases} \tag{1.11.7}$$

Grin funksiyasini tuzib olamiz. Agar λ son (1.11.1)+(1.11.2) bir jinsli chegaraviy masalaning xos qiymati bo'lmasa, ya'ni $\omega(\lambda) \neq 0$ bo'lsa, u holda ushbu

$$Ly = \lambda y + f(x), \tag{1.11.8}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.11.9}$$

chegaraviy masalaning yechimini

$$y(x,\lambda) = \int_0^\pi G(x,t,\lambda)f(t)dt =$$

$$= -\frac{1}{\omega(\lambda)}\left(\psi(x,\lambda)\int_0^x \varphi(t,\lambda)f(t)dt + \varphi(x,\lambda)\int_x^\pi \psi(t,\lambda)f(t)dt\right), \tag{1.11.10}$$

formula bo'yicha topamiz. (1.11.1)+(1.11.2) chegaraviy masalaning λ_n xos qiymatlari uchun $\omega(\lambda_n) = 0$ va $\dot{\omega}(\lambda_n) \neq 0$ ni inobatga olib, (1.11.10) tenglik yordamida aniqlangan $y(x,\lambda)$ funksiyaning $\lambda = \lambda_n$ qutb maxsus nuqtalardagi chegirmasini hisoblaymiz:

$$\operatorname*{res}_{\lambda=\lambda_n} y(x,\lambda) = -\frac{1}{\dot{\omega}(\lambda_n)}\left(\psi(x,\lambda_n)\int_0^x \varphi(t,\lambda_n)f(t)dt + \right.$$

$$\left. + \varphi(x,\lambda_n)\int_x^\pi \psi(t,\lambda_n)f(t)dt\right) = -\frac{C_n}{\dot{\omega}(\lambda_n)}\varphi(x,\lambda_n)\int_0^\pi f(t)\varphi(t,\lambda_n)dt. \tag{1.11.11}$$

Agar biz (1.11.5) tenglikni ishlatsak,

$$\operatorname*{res}_{\lambda=\lambda_n} y(x,\lambda) = -\frac{1}{\alpha_n^2}\varphi(x,\lambda_n)\int_0^\pi f(t)\varphi(t,\lambda_n)dt =$$

$$= -u_n(x)\int_0^\pi f(t)u_n(t)dt,$$

kelib chiqadi. Bu yerda

$$u_n(x) = \frac{1}{\alpha_n}\varphi(x,\lambda_n),$$

ortonormallangan xos funksiya.

(1.11.6) tengliklarga asosan

$$\operatorname*{res}_{\lambda=\lambda_n} y(x,\lambda) = 0, \quad n = 0, 1, 2, \ldots,$$

kelib chiqadi. Bundan esa $y(x,\lambda)$ funksiya har bir tayinlangan $x \in [0,\pi]$ da $\lambda \in \mathbb{C}$ o'zgaruvchining butun funksiyasi ekanligi kelib chiqadi. Bu butun funksiyaning chegaralanganligini ko'rsatish uchun quyidagi lemmadan foydalanamiz.

Lemma 1.11.1. *Agar*

$$z = x + iy \in G_\delta = C \setminus \bigcup_{n=-\infty}^{\infty} \{|x - \pi n| < \delta, \ \ |y| < \delta\}$$

bo'lsa, ushbu

$$|\sin z| \geq \frac{1}{2} e^{-2\delta} \sin 2\delta \, e^{|\operatorname{Im} z|} \quad , \qquad (1.11.12)$$

baholash o'rinli bo'ladi. Bu yerda $0 < \delta < \frac{\pi}{2}$.

Isbot. $z = x + iy \in G_\delta$ bo'lib, $y \geq 0$ bo'lsin. Bu holda

$$A = e^{-y}|\sin z| = \left| e^{-y} \frac{e^{iz} - e^{-iz}}{2i} \right| = \frac{1}{2} \left| e^{-y+ix-y} - e^{-y-ix+y} \right| =$$

$$= \frac{1}{2} \left| e^{-ix}(e^{2ix-2y} - 1) \right| = \frac{1}{2} \sqrt{(1 - e^{-2y}\cos 2x)^2 + (e^{-2y}\sin 2x)^2}.$$

Agar $y \geq \delta$ bo'lsa, u holda

$$A \geq \frac{1}{2}\left(1 - e^{-2y}\cos 2x\right) \geq \frac{1}{2}\left(1 - e^{-2\delta}\right),$$

bo'ladi. Agar $0 \leq y \leq \delta$ bo'lib, $|x - \pi n| \geq \delta$, $n \in Z$ bo'lsa, $x - \pi n = \tilde{x}$ desak, $\delta \leq \tilde{x} \leq \pi - \delta$ bo'ladi. Quyidagi uchta holni ko'rib chiqamiz:

agar $2\delta \leq 2\tilde{x} \leq \frac{\pi}{2}$ bo'lsa,

$$A \geq \frac{1}{2} e^{-2y} |\sin 2x| = \frac{1}{2} e^{-2y} |\sin(2\pi n + 2\tilde{x})| =$$

$$= \frac{1}{2} e^{-2y} |\sin 2\tilde{x}| \geq \frac{1}{2} e^{-2\delta} \sin 2\delta;$$

agar $\frac{\pi}{2} \leq 2\tilde{x} \leq \frac{3\pi}{2}$ bo'lsa,

$$A \geq \frac{1}{2}(1 - e^{-2y}\cos 2\tilde{x}) \geq \frac{1}{2};$$

agar $\dfrac{3\pi}{2} \le 2\bar{x} \le 2\pi - 2\delta$ bo'lsa,

$$A \ge \frac{1}{2}e^{-2y}|\sin 2\bar{x}| \ge \frac{1}{2}e^{-2\delta}|\sin(2\pi - 2\delta)| = \frac{1}{2}e^{-2\delta}\sin 2\delta.$$

Demak, agar $z = x + iy \in G_\delta$ bo'lib, $y \ge 0$ bo'lsa,

$$A \ge \min\left\{\frac{1}{2}(1 - e^{-2\delta}),\ \frac{1}{2}e^{-2\delta}\sin 2\delta,\ \frac{1}{2}\right\},$$

bo'lar ekan. Quyidagi tengsizliklar bajarilishi ravshan:

$$\frac{1}{2} > \frac{1}{2}(1 - e^{-2\delta}) \ge \frac{1}{2}e^{-2\delta}2\delta \ge \frac{1}{2}e^{-2\delta}\sin 2\delta.$$

Bunga ko'ra ushbu

$$A \ge \frac{1}{2}e^{-2\delta}\sin 2\delta \qquad (1.11.13)$$

baholash o'rinli bo'ladi.

Endi $z = x + iy \in G_\delta$ bo'lib, $y \le 0$ bo'lgan holni ko'rib chiqamiz. Bu holda

$$B = e^y|\sin z| = \left|e^y\frac{e^{iz} - e^{-iz}}{2i}\right| =$$

$$= \frac{1}{2}\left|e^{y+ix-y} - e^{y-ix+y}\right| = \frac{1}{2}\left|e^{ix}(1 - e^{-2ix+2y})\right| =$$

$$= \frac{1}{2}\left|(1 - e^{2y}\cos 2x) + ie^{2y}\sin 2x\right| =$$

$$= \frac{1}{2}\sqrt{(1 - e^{2y}\cos 2x)^2 + (e^{2y}\sin 2x)^2},$$

bo'ladi. Agar $y \le -\delta$ bo'lib $\cos 2x \ge 0$ bo'lsa, u holda

$$B \ge \frac{1}{2}\left(1 - e^{-2y}\cos 2x\right) \ge \frac{1}{2}\left(1 - e^{-2\delta}\right),$$

bo'ladi. Agar $y \le -\delta$ bo'lib $\cos 2x \le 0$ bo'lsa, u holda

$$B \ge \frac{1}{2}\left(1 - e^{2y}\cos 2x\right) \ge \frac{1}{2},$$

bo'ladi.

Agar $-\delta \le y \le 0$ bo'lib, $|x-\pi n| \ge \delta$, $n \in Z$ bo'lsa, $x-\pi n = \tilde{x}$ desak, $\delta \le \tilde{x} \le \pi - \delta$ bo'ladi. Quyidagi uchta holni ko'rib chiqamiz:

agar $2\delta \le 2\tilde{x} \le \frac{\pi}{2}$ bo'lsa,

$$B \ge \frac{1}{2}e^{2y}|\sin 2x| = \frac{1}{2}e^{2y}|\sin(2\pi n + 2\tilde{x})| =$$

$$= \frac{1}{2}e^{2y}|\sin 2\tilde{x}| \ge \frac{1}{2}e^{-2\delta}\sin 2\delta;$$

agar $\frac{\pi}{2} \le 2\tilde{x} \le \frac{3\pi}{2}$ bo'lsa,

$$B \ge \frac{1}{2}(1 - e^{2y}\cos 2\tilde{x}) \ge \frac{1}{2};$$

agar $\frac{3\pi}{2} \le 2\tilde{x} \le 2\pi - 2\delta$ bo'lsa,

$$B \ge \frac{1}{2}e^{2y}|\sin 2\tilde{x}| \ge \frac{1}{2}e^{-2\delta}|\sin(2\pi - 2\delta)| = \frac{1}{2}e^{-2\delta}\sin 2\delta.$$

Demak, agar $z = x + iy \in G_\delta$ bo'lib, $y \le 0$ bo'lsa,

$$B \ge \frac{1}{2}e^{-2\delta}\sin 2\delta, \tag{1.11.14}$$

baholash o'rinli bo'ladi.

(1.11.13) va (1.11.14) tengsizliklarni birlashtirib, (1.11.12) baholashni hosil qilamiz. Lemma isbotlandi.

Agar (1.11.12) tengsizlikda $z = \pi\rho = \pi(\sigma + i\tau)$ desak,

$$\rho \in G_\delta = C \setminus \bigcup_{n=-\infty}^{\infty} \left\{ |\sigma - n| < \frac{\delta}{\pi}, \ |\tau| < \frac{\delta}{\pi} \right\},$$

bo'lganda

$$|\sin \rho\pi| \ge \frac{1}{2}e^{-2\delta}\sin 2\delta \cdot e^{\pi|\operatorname{Im}\rho|}, \tag{1.11.15}$$

bo'lishi kelib chiqadi.

(1.11.15) tengsizlik yordamida $|\omega(\lambda)|$ funksiyani $\rho = \sqrt{\lambda} \in G_\delta$ sohada quyidan baholaymiz. Shu maqsadda $x \in [0, \pi]$, $\rho = \sqrt{\lambda} =$

$\sigma + i\tau$, $|\rho| > 2\int\limits_0^\pi |q(t)|dt$ bo'lganda o'rinli bo'lgan ushbu

$$|c'(x,\lambda) + \rho\sin\rho x| \leq 2\int\limits_0^\pi |q(t)|dt \cdot e^{|\tau|x},$$

$$|s'(x,\lambda)| < 2e^{|\tau|x},$$

$$|c(x,\lambda)| < 2e^{|\tau|x},$$

$$|s(x,\lambda)| < \frac{2}{|\rho|} \cdot e^{|\tau|x},$$

baholashlardan foydalanamiz. Quyidagi tengsizliklar o'rinli:

$$|\omega(\lambda)| = |\rho\sin\rho\pi + (\omega(\lambda) - \rho\sin\rho\pi)| \geq |\rho\sin\rho\pi| - |\omega(\lambda) - \rho\sin\rho\pi|,$$

$$|\omega(\lambda) - \rho\sin\rho\pi| = |-[\varphi'(\pi,\lambda) + H\varphi(\pi,\lambda)] - \rho\sin\rho\pi| =$$

$$= |(c'(\pi,\lambda) + \rho\sin\rho\pi) + hs'(\pi,\lambda) + Hc(\pi,\lambda) + hHs(\pi,\lambda)| <$$

$$< 2\int\limits_0^\pi |q(t)|dt \cdot e^{|\tau|\pi} + 2|h| \cdot e^{|\tau|\pi} + 2|H| \cdot e^{|\tau|\pi} + \frac{2|hH|}{|\rho|} \cdot e^{|\tau|\pi} =$$

$$= 2\left\{\int\limits_0^\pi |q(t)|dt + |h| + |H| + \frac{|hH|}{|\rho|}\right\} \cdot e^{|\tau|\pi}.$$

Demak, $\rho \in G_\delta$ bo'lib, $|\rho| > 2\int\limits_0^\pi |q(t)|dt$ bo'lsa, (1.11.15) tengsizlikka ko'ra ushbu

$$|\omega(\lambda)| \geq \frac{1}{2}|\rho|e^{-2\delta+\pi|\tau|}\sin 2\delta -$$

$$-2\left\{\int\limits_0^\pi |q(t)|dt + |h| + |H| + \frac{|hH|}{|\rho|}\right\} \cdot e^{|\tau|\pi} =$$

$$= |\rho|e^{|\tau|\pi}\left\{\frac{1}{2}e^{-2\delta}\sin 2\delta - \frac{2}{|\rho|} \cdot \left(\int\limits_0^\pi |q(t)|dt + |h| + |H| + \frac{|hH|}{|\rho|}\right)\right\},$$

baholash bajariladi. Shunday $R^* > 0$ son topiladiki, bunda $\rho \in G_\delta$ bo'lib, $|\rho| \geq R^*$ bo'lganida

$$|\omega(\lambda)| \geq |\rho| e^{\pi|\tau|} \cdot C_\delta,$$

baholash o'rinli bo'ladi. Bu yerda

$$C_\delta = \frac{1}{4} e^{-2\delta} \sin 2\delta = \text{const} > 0.$$

$\rho \in G_\delta$ bo'lib, $|\rho| \geq R^*$ bo'lganida (1.11.10) tenglik bilan berilgan $y(x, \lambda)$ funksiyani baholaymiz:

$$|y(x,\lambda)| \leq \frac{1}{|\rho| e^{\pi|\tau|} \cdot C_\delta} \left\{ \left(2e^{|\tau|(\pi - x)} + \frac{2|H|}{|\rho|} \cdot e^{|\tau|(\pi - x)} \right) \times \right.$$

$$\times \int_0^x \left(2e^{|\tau| x} + \frac{2|h|}{|\rho|} \cdot e^{|\tau| x} \right) \cdot |f(t)| dt +$$

$$\left. + \left(2e^{|\tau| x} + \frac{2h}{|\rho|} e^{|\tau| x} \right) \cdot \int_x^\pi \left(2e^{|\tau|(\pi - x)} + \frac{2|H|}{|\rho|} \cdot e^{|\tau|(\pi - x)} \right) |f(t)| dt \right\} =$$

$$= \frac{4}{|\rho| \cdot C_\delta} \left(1 + \frac{|h|}{|\rho|} \right) \cdot \left(1 + \frac{|H|}{|\rho|} \right) \int_0^\pi |f(t)| dt. \qquad (1.11.16)$$

Bu tengsizlikka asosan $y(x, \lambda)$ butun funksiya chegaralangan bo'ladi va $\rho \in G_\delta$ bo'lganda

$$\max_{0 \leq x \leq \pi} |y(x, \lambda)| \to 0, \quad (|\lambda| \to \infty).$$

Liuvill teoremasiga ko'ra $y(x, \lambda) \equiv 0$ kelib chiqadi. Buni (1.11.8) tenglamaga qo'yib, $f(x) \equiv 0$ bo'lishini ko'ramiz. ■

Teorema 1.11.2. *Ixtiyoriy $f(x) \in AC[0, \pi]$ - absolyut uzluksiz funksiya uchun ushbu*

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi(x, \lambda_n), \qquad (1.11.17)$$

funksional qator tekis yaqinlashuvchi bo'ladi. Bu yerda

$$a_n = \frac{1}{\alpha_n^2}\int\limits_0^\pi f(t)\varphi(t,\lambda_n)dt,$$

$$\alpha_n = \sqrt{\int\limits_0^\pi \varphi^2(x,\lambda_n)dx}, \ n = 0, 1, 2, \dots . \tag{1.11.18}$$

Isbot. (1.11.10) tenglik bilan berilgan $y(x,\lambda)$ funksiyani quyidagi ko'rinishda yozib olamiz:

$$y(x,\lambda) = -\frac{1}{\lambda\omega(\lambda)}\left(\psi(x,\lambda)\int\limits_0^x (-\varphi''(t,\lambda) + q(t)\varphi(t,\lambda))f(t)dt+\right.$$

$$\left.+\varphi(x,\lambda)\int\limits_x^\pi (-\psi''(t,\lambda) + q(t)\psi(t,\lambda))f(t)dt\right).$$

Bu tenglikda bo'laklab integrallash amalini bajaramiz:

$$y(x,\lambda) = \frac{1}{\lambda\omega(\lambda)}\left(\psi(x,\lambda)\int\limits_0^x f(t)d\varphi'(t,\lambda) + \varphi(x,\lambda)\int\limits_x^\pi f(t)d\psi'(t,\lambda)\right) -$$

$$-\frac{1}{\lambda\omega(\lambda)}\left(\psi(x,\lambda)\int\limits_0^x q(t)\varphi(t,\lambda)f(t)dt + \varphi(x,\lambda)\int\limits_x^\pi q(t)f(t)\psi(t,\lambda)dt\right),$$

$$y(x,\lambda) = \frac{1}{\lambda\omega(\lambda)}(\psi(x,\lambda)f(x)\varphi'(x,\lambda) - \psi(x,\lambda)f(0)\varphi'(0,\lambda) -$$

$$-\psi(x,\lambda)\int\limits_0^x f'(t)\varphi'(t,\lambda)dt + \psi'(\pi,\lambda)f(\pi)\varphi(x,\lambda) -$$

$$-\psi'(x,\lambda)f(x)\varphi(x,\lambda) - \varphi(x,\lambda)\int\limits_x^\pi f'(t)\psi'(t,\lambda)dt) -$$

$$-\frac{1}{\lambda\omega(\lambda)}\left(\psi(x,\lambda)\int\limits_0^x q(t)\varphi(t,\lambda)f(t)dt + \varphi(x,\lambda)\int\limits_x^\pi q(t)f(t)\psi(t,\lambda)dt\right),$$

$$y(x,\lambda) = \frac{1}{\lambda\omega(\lambda)}f(x)[\psi(x,\lambda)\varphi'(x,\lambda) - \psi'(x,\lambda)\varphi(x,\lambda)] -$$

$$-\frac{1}{\lambda\omega(\lambda)}\left(\psi(x,\lambda)\int\limits_0^x g(t)\varphi'(t,\lambda)dt + \varphi(x,\lambda)\int\limits_x^\pi g(t)\psi'(t,\lambda)dt\right) -$$

$$-\frac{1}{\lambda\omega(\lambda)}(hf(0)\psi(x,\lambda) + Hf(\pi)\varphi(x,\lambda)) -$$

$$-\frac{1}{\lambda\omega(\lambda)}\left(\psi(x,\lambda)\int_0^x q(t)\varphi(t,\lambda)f(t)dt+\varphi(x,\lambda)\int_x^\pi q(t)f(t)\psi(t,\lambda)dt\right),$$

$$y(x,\lambda)=-\frac{f(x)}{\lambda}-\frac{1}{\lambda}\left(z_1(x,\lambda)+z_2(x,\lambda)\right). \tag{1.11.19}$$

Bu yerda $g(t)=f'(t)$ bo'lib,

$$z_1(x,\lambda)=\frac{1}{\omega(\lambda)}\left(\psi(x,\lambda)\int_0^x g(t)\varphi'(t,\lambda)dt+\right.$$

$$\left.+\varphi(x,\lambda)\int_x^\pi g(t)\psi'(t,\lambda)dt\right), \tag{1.11.20}$$

$$z_2(x,\lambda)=\frac{1}{\omega(\lambda)}\left(\quad hf(0)\psi(x,\lambda)+Hf(\pi)\varphi(x,\lambda)+\right.$$

$$\left.+\psi(x,\lambda)\int_0^x q(t)\varphi(t,\lambda)f(t)dt+\varphi(x,\lambda)\int_x^\pi q(t)\psi(t,\lambda)f(t)dt\right). \tag{1.11.21}$$

Yechimlar asimptotikasidan foydalanib $z_2(x,\lambda)$ funksiyani baholaymiz. $\rho\in G_\delta$, $|\rho|\geq R^*$ bo'lsin, u holda ushbu

$$|z_2(x,\lambda)|\leq\frac{3}{|\rho|e^{\pi|\tau|}\cdot C_\delta}\left(|h|\cdot|f(0)|e^{|\tau|\pi}+|H|\cdot|f(\pi)|e^{|\tau|\pi}\right)+$$

$$+\frac{9}{|\rho|e^{\pi|\tau|}\cdot C_\delta}\left(e^{|\tau|(\pi-x)}\int_0^x|q(t)|\cdot|f(t)|\cdot e^{|\tau|x}dt+\right.$$

$$\left.+e^{|\tau|x}\int_x^\pi|q(t)|\cdot|f(t)|\cdot e^{|\tau|(\pi-x)}dt\right)=$$

$$=\frac{1}{|\rho|C_\delta}\left(3|h||f(0)|+3|H||f(\pi)|+9\int_0^\pi|q(t)||f(t)|dt\right)=\frac{A_\delta}{|\rho|},$$

baholash o'rinli bo'ladi. Bu baholashga asosan

$$\lim_{\substack{|\rho|\to\infty \\ \rho\in G_\delta}} \max_{0\le x\le\pi} |z_2(x,\lambda)| = 0, \tag{1.11.22}$$

bo'ladi. Agar $g(t) \in AC[0,\pi]$ bo'lsa, u holda (1.11.20) ifodada bo'laklab integrallash amalini qo'llab, uni quyidagi ko'rinishda yozamiz:

$$z_1(x,\lambda) = \frac{1}{\omega(\lambda)}\left(\psi(x,\lambda)\int_0^x g(t)d\varphi(t,\lambda) + \varphi(x,\lambda)\int_x^\pi g(t)d\psi(t,\lambda)\right) =$$

$$= \frac{1}{\omega(\lambda)}\big(\psi(x,\lambda)[g(x)\varphi(x,\lambda) - g(0)\varphi(0,\lambda)] -$$

$$-\psi(x,\lambda)\int_0^x g'(t)\varphi(t,\lambda)dt + \varphi(x,\lambda)[g(\pi)\psi(\pi,\lambda) -$$

$$\left. - g(x)\psi(x,\lambda)] - \varphi(x,\lambda)\int_x^\pi g'(t)\psi(t,\lambda)dt\right) =$$

$$= \frac{1}{\omega(\lambda)}\left(\varphi(x,\lambda)g(\pi) - \psi(x,\lambda)g(0) - \psi(x,\lambda)\int_0^x g'(t)\varphi(t,\lambda)dt - \right.$$

$$\left. - \varphi(x,\lambda)\int_x^\pi g'(t)\psi(t,\lambda)dt\right). \tag{1.11.23}$$

Yechimlar asimptotikasidan foydalanib, $z_1(x,\lambda)$ funksiyani baholaymiz. $\rho \in G_\delta$, $|\rho| \ge R^*$ bo'lsin, u holda ushbu

$$|z_1(x,\lambda)| \le \frac{3}{|\rho|e^{\pi|\tau|}C_\delta}\left(|g(\pi)|e^{|\tau|\pi} + |g(0)|e^{|\tau|\pi}\right) +$$

$$+\frac{9}{|\rho|e^{\pi|\tau|}C_\delta}\left(e^{|\tau|(\pi-x)}\int_0^x |g'(t)|e^{|\tau|x}dt + e^{|\tau|x}\int_x^\pi |g'(t)|e^{|\tau|(\pi-x)}dt\right) =$$

$$= \frac{1}{|\rho|C_\delta}\left(3|g(\pi)| + 3|g(0)| + 9\int_0^\pi |g'(t)|dt\right) = \frac{B_\delta}{|\rho|}, \qquad (1.11.24)$$

baholash o'rinli bo'ladi. Bu baholashga asosan ushbu

$$\lim_{\substack{|\rho|\to\infty \\ \rho\in G_\delta}} \max_{0\le x\le\pi} |z_1(x,\lambda)| = 0, \qquad (1.11.25)$$

tenglik bajariladi. (1.11.25) tenglik $g(t) \in L_2(0,\pi)$ bo'lganida ham bajarilishini ko'rsatamiz. Ixtiyoriy $\varepsilon > 0$ son berilgan bo'lsin. U holda shunday $g_\varepsilon(t) \in AC[0,\pi]$ funksiya topiladiki, bunda

$$\int_0^\pi |g(t) - g_\varepsilon(t)|dt < \frac{\varepsilon \cdot C_\delta}{18}, \qquad (1.11.26)$$

tengsizlik bajariladi.

$\rho \in G_\delta$, $|\rho| \ge R^*$ bo'lsin. Quyidagi baholashlarni bajaramiz:

$$|z_1(x,\lambda, g - g_\varepsilon)| \le \frac{9}{|\rho|e^{\pi|\tau|}C_\delta}\left(e^{|\tau|(\pi-x)}\int_0^x |g(t) - g_\varepsilon(t)||\rho|e^{|\tau|x}dt + \right.$$

$$\left. + e^{|\tau|x}\int_x^\pi |g(t) - g_\varepsilon(t)||\rho|e^{|\tau|(\pi-x)}dt\right) =$$

$$= \frac{9}{C_\delta}\int_0^\pi |g(t) - g_\varepsilon(t)|dt < \frac{\varepsilon}{2}. \qquad (1.11.27)$$

(1.11.24) baholashni $g_\varepsilon(t)$ uchun qo'llasak, bunga ko'ra shunday $\tilde{R}_\varepsilon \ge R^*$ son topiladiki, $\rho \in G_\delta$, $|\rho| \ge \tilde{R}_\varepsilon$ bo'lganida ushbu

$$|z_1(x,\lambda,g_\varepsilon)| < \frac{\varepsilon}{2}, \qquad (1.11.28)$$

tengsizlik bajariladi.

Demak, $\rho \in G_\delta$, $|\rho| \ge \tilde{R}_\varepsilon$ bo'lsa,

$$|z_1(x,\lambda,g)| \le |z_1(x,\lambda,g-g_\varepsilon)| + |z_1(x,\lambda,g_\varepsilon)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

bo'ladi, ya'ni (1.11.25) tenglik $g(t) \in L^2(0,\pi)$ bo'lganida ham bajarilar ekan.

Endi ushbu

$$I_N(x) = \frac{1}{2i\pi}\int\limits_{\Gamma_N} y(x,\lambda)d\lambda,$$

integralni qaraymiz. Bu yerda $\Gamma_N = \left\{\lambda : \ |\lambda| = \left(N+\frac{1}{2}\right)^2\right\}$.

Chegirmalar haqidagi Koshi teoremasiga asosan quyidagi tenglik o'rinli:

$$\frac{1}{2i\pi}\int\limits_{\Gamma_N}\left[-\frac{f(x)}{\lambda} - \frac{1}{\lambda}\left(z_1(x,\lambda)+z_2(x,\lambda)\right)\right]d\lambda = \sum_{n=0}^{N}\operatorname*{res}_{\lambda=\lambda_n} y(x,\lambda),$$

$$-f(x) - \frac{1}{2i\pi}\int\limits_{\Gamma_N}\left[\frac{1}{\lambda}\left(z_1(x,\lambda)+z_2(x,\lambda)\right)\right]d\lambda = -\sum_{n=0}^{N} a_n\varphi(x,\lambda_n). \tag{1.11.29}$$

Quyidagi integral limitini hisoblaymiz:

$$\varepsilon_N(x) = \frac{1}{2i\pi}\int\limits_{\Gamma_N}\left[\frac{1}{\lambda}\left(z_1(x,\lambda)+z_2(x,\lambda)\right)\right]d\lambda =$$

$$= \left\{\lambda = \left(N+\frac{1}{2}\right)^2\cdot e^{it},\ 0\le t\le 2\pi\right\} =$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi}\left[z_1\left(x,(N+\frac{1}{2})^2e^{it}\right) + z_2\left(x,(N+\frac{1}{2})^2e^{it}\right)\right]dt.$$

Demak,

$$\lim_{N\to\infty}\max_{0\le x\le\pi}|\varepsilon_N(x)| = 0. \tag{1.11.30}$$

ekan. Bunga ko'ra (1.11.29) tenglikda $N\to\infty$ bo'lganda limitga o'tsak

$$f(x) = \sum_{n=0}^{\infty} a_n\varphi(x,\lambda_n), \tag{1.11.31}$$

kelib chiqadi. (1.11.30) ga ko'ra (1.11.31) qator $x \in [0, \pi]$ da tekis yaqinlashadi.■

12-§. Shturm-Liuvill chegaraviy masalasi uchun yoyilma teoremasi va Parseval tengligi

Teorema 1.12.1. *Nol soni ushbu*

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.12.1}$$

Shturm-Liuvill masalasining xos qiymati bo'lmasa, quyidagi chegaraviy masala

$$\begin{cases} -y'' + q(x)y = \lambda y + f(x), \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.12.2}$$

ushbu

$$y(x) = \lambda \int_0^\pi G(x,t)y(t)dt + \int_0^\pi G(x,t)f(t)dt, \tag{1.12.3}$$

integral tenglamaga ekvivalent bo'ladi. Bu yerda $G(x,t)$ funksiya (1.12.1) masalaning $\lambda = 0$ qiymatga mos keluvchi Grin funksiyasi.

Isbot. Nol soni (1.12.1) masalaning xos qiymati bo'lmagani uchun $G(x,t)$ Grin funksiyasi mavjud va yagona bo'ladi. (1.12.2) masalada ushbu

$$F(x) = \lambda y(x) + f(x) \tag{1.12.4}$$

belgilashni kiritsak, hosil bo'lgan masalaning yechimi quyidagi

$$y(x) = \int_0^\pi G(x,t)F(t)dt, \tag{1.12.5}$$

formula bilan beriladi. (1.12.4) belgilashni hisobga olsak, bu formuladan (1.12.3) integral tenglama kelib chiqadi. ■

(1.12.3) tenglama Fredgolmning ikkinchi turdagi integral tenglamasidir.

Natija 1.12.1. Nol soni ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.12.6}$$

Shturm-Liuvill masalasining xos qiymati bo'lmasa, bu chegaraviy masala quyidagi

$$y(x) = \lambda \int_0^\pi G(x,t)y(t)dt, \tag{1.12.7}$$

integral tenglamaga ekvivalentdir.

Lemma 1.12.1. *Agar $H(x,t)$ funksiya $[0,\pi]\times[0,\pi]$ kvadratda haqiqiy uzluksiz, simmetrik va noldan farqli bo'lsa, ushbu*

$$u(x) = \lambda \int_0^\pi H(x,t)u(t)dt, \tag{1.12.8}$$

integral tenglama xos qiymatga ega, ya'ni λ parametrning shunday λ_0 qiymati mavjudki, bunda (1.12.8) integral tenglama noldan farqli yechimga ega bo'ladi.

Isbot. $L^2[0,\pi]$ Gilbert fazosida quyidagi

$$Au(x) = \int_0^\pi H(x,t)u(t)dt,$$

integral operatorni ko'rib chiqamiz. Bu yerdagi integral operatorning yadrosi $H(x,t)$ ushbu $[0,\pi]\times[0,\pi]$ kvadratda uzluksiz va simmetrik $H(x,t) = \overline{H(t,x)}$ funksiya bo'lgani uchun A o'z-o'ziga qo'shma va kompakt operator bo'ladi. A operator o'z-o'ziga qo'shma bo'lgani uchun uning normasi quyidagi formula orqali topiladi:

$$\|A\| = \sup_{\|u\|=1} |(Au,u)|\,.$$

Aniq yuqori chegaraning ta'rifiga asosan, shunday $\{u_n(x)\}$ ketma-ketlik mavjudki, bunda $\|u_n\| = 1$ bo'lib,

$$(Au_n, u_n) \to \mu_0, \quad (n \to \infty),$$

bo'ladi. Bu yerda $\mu_0 = \|A\|$ yoki $\mu_0 = -\|A\|$.

Quyidagilar o'rinli:

$$\|Au_n - \mu_0 u_n\|^2 = (Au_n - \mu_0 u_n, Au_n - \mu_0 u_n) =$$
$$= \|Au_n\|^2 - 2\mu_0(Au_n, u_n) + \mu_0^2\|u_n\| \leq$$
$$\leq \|A\|^2\|u_n\|^2 - 2\mu_0(Au_n, u_n) + \mu_0^2\|u_n\| = 2\mu_0^2 - 2\mu_0(Au_n, u_n).$$

Bundan

$$Au_n - \mu_0 u_n \to 0, \quad (n \to \infty), \tag{1.12.9}$$

kelib chiqadi. A kompakt operator bo'lgani uchun $\{Au_n\}$ ketma-ketlikdan yaqinlashuvchi $\{Au_{n_k}\}$ qismiy ketma-ketlik ajratib olish mumkin, ya'ni

$$Au_{n_k} \to v_0\,, \quad (k \to \infty). \tag{1.12.10}$$

(1.12.9) va (1.12.10) ga asosan

$$\mu_0 u_{n_k} \to v_0\,, \quad (k \to \infty),$$

va

$$Au_{n_k} \to \frac{1}{\mu_0} Av_0\,, \quad (k \to \infty), \tag{1.12.11}$$

kelib chiqadi. (1.12.10) va (1.12.11) ga ko'ra

$$Av_0 = \mu_0 v_0,$$

tenglikka ega bo'lamiz. Bu yerda $v_0 \neq 0$, chunki

$$\|v_0\| = \lim_{k\to\infty} \|\mu_0 u_{n_k}\| = |\mu_0| = \|A\| \neq 0.$$

Shunday qilib, $\mu_0 \neq 0$ son A operatorning xos qiymati va $v_0(x)$ funksiya bu xos qiymatga mos keluvchi xos funksiya ekan:

$$\int_0^\pi H(x,t)v_0(t)dt = \mu_0 v_0(x).$$

Bunga ko'ra $\lambda_0 = \frac{1}{\mu_0}$ son (1.12.8) integral tenglamaning xos qiymati bo'ladi, $v_0(x)$ funksiya esa unga mos keluvchi xos funksiya bo'ladi. ■

Teorema 1.12.2. *Nol soni (1.12.1) chegaraviy masalaning xos qiymati bo'lmasa, ushbu*

$$u(x) = \lambda \int_0^{\pi} G(x,t)u(t)dt,$$

integral tenglama cheksiz ko'p xos qiymatlarga ega bo'ladi.

Isbot. $L^2[0,\pi]$ Gilbert fazosida quyidagi

$$Ru(x) = \int_0^{\pi} G(x,t)u(t)dt,$$

integral operatorni ko'rib chiqamiz. Bu yerdagi integral operatorning $G(x,t)$ yadrosi ushbu $[0,\pi] \times [0,\pi]$ kvadratda haqiqiy, uzluksiz va simmetrik $G(x,t) = G(t,x)$ funksiya bo'lgani uchun R o'z-o'ziga qo'shma va kompakt operator bo'ladi. Lemma 1.12.1 ga asosan R operatorning $\mu_0 \neq 0$ xos qiymati mavjud va unga $v_0(x)$ xos funksiya mos keladi. Bundan tashqari $|\mu_0| = \|R\|$, $\|v_0(x)\| = 1$ deb hisoblaymiz.

Endi

$$R_1u(x) = \int_0^{\pi} G_1(x,t)u(t)dt, \tag{1.12.12}$$

operatorni ko'rib chiqamiz. Bu yerda

$$G_1(x,t) = G(x,t) - \mu_0 v_0(x)v_0(t).$$

(1.12.12) tenglik yordamida aniqlangan R_1 operator ham o'z-o'ziga qo'shma va kompakt operator bo'ladi. Lemma 1.12.1 ga asosan R_1 operatorning $\mu_1 \neq 0$ xos qiymati va $v_1(x) \in L^2[0,\pi]$ xos funksiyasi mavjud bo'lishini, hamda ushbu

$$|\mu_1| = \|R_1\|,$$

tenglik bajarilishini ko'rsatish mumkin. $\|v_1(x)\| = 1$ deb hisoblaymiz.

Topilgan $v_0(x)$, $v_1(x) \in L^2[0,\pi]$ funksiyalar o'zaro ortogonal bo'ladi. Haqiqatan ham, ixtiyoriy $u(x) \in L^2[0,\pi]$ funksiya uchun

$$(R_1u, v_0) = \int_0^\pi \left(\int_0^\pi G_1(t,s)u(s)ds \right) v_0(t)dt =$$

$$= \int_0^\pi \int_0^\pi G(t,s)u(s)v_0(t)dsdt - \mu_0 \int_0^\pi \int_0^\pi v_0^2(t)v_0(s)u(s)dsdt =$$

$$= \int_0^\pi u(s) \left(\int_0^\pi G(s,t)v_0(t)dt \right) ds -$$

$$-\mu_0 \left(\int_0^\pi v_0^2(t)dt \right) \left(\int_0^\pi u(s)v_0(s)ds \right) =$$

$$= (u, Rv_0) - \mu_0(u, v_0) = (u, \mu_0 v_0) - \mu_0(u, v_0) = 0,$$

bo'lishidan, xususiy holda $(v_1, v_0) = 0$ kelib chiqadi. Shuning uchun

$$Rv_1 = R_1v_1 + \mu_0 v_0(x) \int_0^\pi v_0(t)v_1(t)dt = R_1v_1 = \mu_1 v_1,$$

bo'ladi, ya'ni μ_1 son R operator uchun ham xos qiymat bo'ladi va unga $v_1(x)$ xos funksiya mos keladi. Topilgan xos qiymatlar uchun ushbu

$$|\mu_1| = |(R_1v_1, v_1)| = |(Rv_1, v_1)| \leq \|R\| = |\mu_0|,$$

ya'ni

$$|\mu_0| \geq |\mu_1|$$

tengsizlik o‘rinli bo‘ladi. Shu jarayonni yanada davom qildiramiz. Buning uchun quyidagi integral operatorni tuzib olamiz:

$$R_2u(x) = \int_0^{\pi} G_2(x,t)u(t)dt.$$

Bu yerda $G_2(x,t) = G_1(x,t) - \mu_1 v_1(x)v_1(t)$. Yuqorida ta’kidlaganimiz kabi R_2 ham o‘z-o‘ziga qo‘shma kompakt operator bo‘ladi. Shuning uchun shunday $v_2(x) \in L^2[0,\pi]$ funksiya topilib,

$$|\mu_2| = \|R_2\|, \quad \|v_2(x)\| = 1.$$

O‘z navbatida $v_2(x)$ funksiya R operator uchun ham xos funksiya bo‘ladi, ya’ni

$$Rv_2 = R_2v_2 = \mu_2 v_2.$$

Topilgan xos qiymatlar uchun

$$|\mu_0| \geq |\mu_1| \geq |\mu_2|,$$

tengsizliklar bajarilishini ko‘rsatish mumkin. $v_0(x)$, $v_1(x)$ va $v_2(x)$ xos funksiyalar esa, o‘zaro ortoganal bo‘ladi.

Agar $\|R_m\| \neq 0$, $m \in N$ bo‘lsa, bu jarayonni cheksiz davom qildirish mumkin. Natijada, $\{v_n(x)\}$ ortonormallangan xos funksiyalar mavjudligi va ularga mos keluvchi xos qiymatlar uchun

$$|\mu_0| \geq |\mu_1| \geq |\mu_2| \geq ...,$$

tengsizliklar bajarilishi kelib chiqadi.

Endi $\|R_m\| \neq 0$, $m \in N$ ekanini isbotlaymiz. Buning uchun $\|R_m\| = 0$ deb faraz qilamiz. Bu holda R_m operatorning yadrosi

$$G_m(x,t) = G(x,t) - \sum_{n=0}^{m} \mu_n v_n(x)v_n(t),$$

ko‘rinishda bo‘ladi. Bu tengliklarning ikkala tomonini $f(t) \in L^2[0,\pi]$ funksiyaga ko‘paytirib, $[0,\pi]$ oraliqda integrallasak,

$$R_m f(x) = Rf(x) - \sum_{n=0}^{m} \mu_n v_n(x)(f, v_n),$$

hosil bo'ladi. Bu tengliklarning ikkala tarafiga L operatorni ta'sir qildirib,

$$L(Rf) = f, \quad R = L^{-1} \quad \text{va} \quad Lv_n = \frac{1}{\mu_n} v_n$$

ekanini inobatga olsak,

$$0 = f(x) - \sum_{n=0}^{m} \mu_n L v_n(x)(f, v_n),$$

ya'ni

$$f(x) = \sum_{n=0}^{m} v_n(x)(f, v_n),$$

bo'lishini topamiz. Bu esa $f(x) \in L^2[0, \pi]$ ixtiyoriy funksiya ekanligiga ziddir.

Shunday qilib, R operatorning

$$|\mu_0| \geq |\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_m| \geq \dots,$$

cheksiz ko'p xos qiymatlari va

$$v_0(x),\ v_1(x), \dots, v_m(x), \dots,$$

cheksiz ko'p ortonormallangan xos funksiyalari mavjud ekan. ■

Yuqoridagi mulohazalardan (1.12.1) Shturm-Liuvill chegaraviy masalasining cheksiz ko'p $\lambda_n = \dfrac{1}{\mu_n}$ xos qiymatlari mavjud bo'lib, ular uchun ushbu

$$|\lambda_0| \leq |\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_m| \leq \dots,$$

tengsizlikning bajarilishi kelib chiqadi.

Teorema 1.12.3. *Nol soni (1.12.1) chegaraviy masalaning xos qiymati bo'lmasa, uning Grin funksiyasi uchun ushbu*

$$G(x, t) = \sum_{n=0}^{\infty} \frac{u_n(x) u_n(t)}{\lambda_n}, \qquad (1.12.13)$$

tasvir o'rinli bo'ladi. Bu yerda λ_n orqali (1.12.1) masalaning xos qiymatlari va $u_n(x)$ orqali ularga mos keluvchi ortonormallangan xos funksiyalari belgilangan.

Isbot. Quyidagi yordamchi funksiyani kiritib olamiz:

$$H(x,t) = G(x,t) - \sum_{n=0}^{\infty} \frac{u_n(x)u_n(t)}{\lambda_n}. \qquad (1.12.14)$$

Bu yerdagi funksional qator Veyershtrass alomatiga ko'ra tekis va absolyut yaqinlashadi, chunki ortonormallangan xos funksiyalar n ga bog'liq bo'lmagan o'zgarmas bilan chegaralangan va xos qiymatlar uchun ushbu

$$\lambda_n = n^2 + c_0 + \gamma_n, \ \{\gamma_n\} \in l_2,$$

asimptotik formula o'rinli. Bu qatorning har bir hadi uzluksiz funksiya bo'lganligi uchun uning yig'indisi ham uzluksiz funksiya bo'ladi. Demak, $H(x,t)$ funksiya uzluksiz ekan. $H(x,t)$ simmetrik bo'lishi Grin funksiyasining simmetrikligidan kelib chiqadi.

$H(x,t) \equiv 0$ ekanligini isbotlaymiz. Buning uchun teskarisini faraz qilamiz, u holda yuqoridagi lemmaga ko'ra shunday $\tilde{\lambda}$ son mavjudki, bunda

$$u(x) = \tilde{\lambda} \int_0^{\pi} H(x,t)u(t)dt,$$

integral tenglama noldan farqli $u(x) \not\equiv 0$ yechimga ega bo'ladi. (1.12.1) masalaning xos funksiyalari uchun ((1.12.7) integral tenglamaga ko'ra) ushbu

$$\int_0^{\pi} G(x,t)u_n(t)dt = \frac{u_n(x)}{\lambda_n}, \qquad (1.12.15)$$

tenglik bajariladi. Bunga asosan

$$\int_0^{\pi} H(x,t)u_n(t)dt = \int_0^{\pi} \left\{ G(x,t) - \sum_{k=0}^{\infty} \frac{u_k(x)u_k(t)}{\lambda_k} \right\} u_n(t)dt =$$

$$= \int_0^\pi G(x,t)u_n(t)dt - \sum_{k=0}^{\infty} \frac{u_k(x)}{\lambda_k} \int_0^\pi u_k(t)u_n(t)dt =$$

$$= \frac{1}{\lambda_n}u_n(x) - \frac{1}{\lambda_n}u_n(x) = 0, \qquad (1.12.16)$$

bo'ladi.

(1.12.16) tengliklardan foydalanib, $u(x)$ funksiya $u_n(x)$, $n = 0, 1, 2, ...$ xos funksiyalarga ortogonal bo'lishini ko'rsatamiz:

$$\int_0^\pi u(x)u_n(x)dx = \int_0^\pi \left\{ \tilde{\lambda} \int_0^\pi H(x,t)u(t)dt \right\} u_n(x)dx =$$

$$= \tilde{\lambda} \int_0^\pi \left\{ \int_0^\pi H(x,t)u_n(x)dx \right\} u(t)dt = 0. \qquad (1.12.17)$$

Demak, $u(x)$ funksiya $u_n(x)$, $n = 0, 1, 2, ...$ funksiyalarga ortogonal ekan.

Endi $u(x)$ funksiyaning o'zi ham (1.12.1) chegaraviy masalaning xos funksiyasi bo'lishini ko'rsatamiz. Buning uchun $u(x)$ funksiya ushbu

$$u(x) = \tilde{\lambda} \int_0^\pi G(x,t)u(t)dt,$$

integral tenglamani qanoatlantirishini ko'rsatish kifoya:

$$u(x) = \tilde{\lambda} \int_0^\pi H(x,t)u(t)dt =$$

$$= \tilde{\lambda} \int_0^\pi \left\{ G(x,t) - \sum_{n=0}^{\infty} \frac{u_n(x)u_n(t)}{\lambda_n} \right\} u(t)dt =$$

$$= \tilde{\lambda} \int_0^\pi G(x,t)u(t)dt - \tilde{\lambda} \sum_{n=0}^{\infty} \frac{u_n(x)}{\lambda_n} \int_0^\pi u_n(t)u(t)dt =$$

$$= \tilde{\lambda} \int_0^\pi G(x,t)u(t)dt.$$

$u(x)$ funksiya (1.12.1) chegaraviy masalaning xos funksiyasi bo'lgani uchun, u $u_n(x)$, $n = 0, 1, 2, ...$ xos funksiyalardan biriga proporsional, ya'ni $u(x) = Cu_{n_0}(x)$ bo'ladi. $u(x)$ funksiya $u_{n_0}(x)$ xos funksiyaga ortogonal bo'lgani uchun

$$\int_0^\pi u(x)u_{n_0}(x)dx = 0, \quad C\int_0^\pi u_{n_0}^2(x)dx = 0, \quad C = 0, \quad u(x) \equiv 0,$$

bo'ladi, bu esa farazimizga ziddir. Demak, $H(x,t) \equiv 0$ ekan. Bundan

$$G(x,t) = \sum_{n=0}^{\infty} \frac{u_n(x)u_n(t)}{\lambda_n},$$

kelib chiqadi. ■

Izoh. (1.12.13) funksional qator absolyut va tekis yaqinlashuvchi bo'lib, uning yig'indisi $[0,\pi] \times [0,\pi]$ kvadratda uzluksiz funksiya bo'lgani uchun, ushbu

$$G(x,x) = \sum_{n=0}^{\infty} \frac{u_n^2(x)}{\lambda_n},$$

tenglik o'rinli bo'ladi. Oxirgi tenglikning ikkala tarafini $[0,\pi]$ oraliqda integrallab,

$$\int_0^\pi G(x,x)dx = \sum_{n=0}^{\infty} \frac{1}{\lambda_n}, \tag{1.12.17'}$$

bo'lishini topamiz. Bu formulaga $G(x,t)$ – Grin yadrosining izi deyiladi.

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$$Ly \equiv -y'' = \lambda y,$$
$$y(0) = 0, \; y(\pi) = 0,$$

chegaraviy masalaning Grin yadrosining izini hisoblang.

Berilgan chegaraviy masalaning xos qiymatlari $\lambda_n = n^2, n = 1, 2, 3, \ldots$ va ortonormallangan xos funksiyalari

$$u_n(x) = \sqrt{\frac{2}{\pi}} \sin nx,$$

topilgan edi. To'qqizinchi paragrafda bayon qilingan algoritm yordamida, berilgan Dirixle chegaraviy masalasining $\lambda = 0$ bo'lgan holdagi Grin funksiyasini tuzishimiz mumkin:

$$G(x,t) = \frac{1}{\pi} \begin{cases} x(\pi - t), & x \le t, \\ t(\pi - x), & x \ge t. \end{cases}$$

Endi Grin funksiyasining izi uchun topilgan (1.12.17′) formulani tekshiramiz:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^{\pi} G(x,x)dx = \frac{1}{\pi} \int_0^{\pi} x(\pi - x)dx = \frac{\pi^2}{6}.$$

Shunday qilib (1.12.17′) formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

ko'rinishni oladi.

Teorema 1.12.4. (*Yoyilma haqida*). *Agar $f(x) \in C^2[0, \pi]$ ushbu*

$$\begin{cases} f(0)\cos\alpha + f'(0)\sin\alpha = 0, \\ f(\pi)\cos\beta + f'(\pi)\sin\beta = 0, \end{cases} \tag{1.12.18}$$

chegaraviy shartlarni qanoatlantiruvchi ixtiyoriy funksiya bo'lsa, u holda quyidagi

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x), \tag{1.12.19}$$

tasvir o'rinli bo'ladi. Bu yerda $u_n(x)$ funksiyalar (1.12.1) chegaraviy masalaning ortonormallangan xos funksiyalari bo'lib, a_n ko-

effitsiyentlar ushbu

$$a_n = \int_0^\pi f(t)\, u_n(t)\, dt, \qquad (1.12.20)$$

tenglik bilan aniqlanadi. (1.12.19) qator tekis va absolyut yaqinlashuvchi bo'ladi.

Isbot. Quyidagi belgilashni kiritib olamiz:

$$-f''(x) + q(x)f(x) = g(x). \qquad (1.12.21)$$

Grin funksiyasining xossasiga ko'ra (1.12.21)+(1.12.18) chegaraviy masalaning yechimi ushbu

$$f(x) = \int_0^\pi G(x,t)g(t)dt, \qquad (1.12.22)$$

tenglik bilan beriladi. Grin funksiyasi uchun teorema 1.12.3 da olingan yoyilmani (1.12.22) tenglikka qo'yamiz:

$$f(x) = \int_0^\pi \left\{ \sum_{n=0}^{\infty} \frac{u_n(x)u_n(t)}{\lambda_n} \right\} g(t)dt =$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{\lambda_n} \int_0^\pi g(t)u_n(t)dt \right\} u_n(x). \qquad (1.12.23)$$

$$a_n = \frac{1}{\lambda_n} \int_0^\pi g(t)u_n(t)dt = \frac{1}{\lambda_n} \int_0^\pi Lf(t)u_n(t)dt =$$

$$= \frac{1}{\lambda_n} \int_0^\pi f(t)Lu_n(t)dt = \int_0^\pi f(t)u_n(t)dt.$$

(1.12.19) qatorning tekis va absolyut yaqinlashishi uning (1.12.23) ko'rinishda yozilishidan va xos qiymatlar asimptotikasidan kelib chiqadi.■

(1.12.20) tengliklar bilan aniqlangan a_n, $n = 0, 1, 2, \dots$ sonlarga $f(x)$ funksiyaning Furye koeffitsiyentlari deyiladi.

Teorema 1.12.5.(*Parseval tengligi*).*Ixtiyoriy* $f(x) \in L^2[0, \pi]$ *funksiya uchun ushbu*

$$\int_0^\pi f^2(x)dx = \sum_{n=0}^\infty a_n^2, \tag{1.12.24}$$

tenglik o'rinli bo'ladi. Bu yerda, a_n *koeffitsiyentlar ushbu*

$$a_n = \int_0^\pi f(t)\, u_n(t)\, dt, \tag{1.12.25}$$

tenglik bilan aniqlanib, $u_n(x)$ *funksiyalar (1.12.1) chegaraviy masalaning ortonormallangan xos funksiyalaridir.*

Isbot. 1) $f(x) \in C^2[0, \pi]$ bo'lsin va u (1.12.18) chegaraviy shartlarni qanoatlantirsin. U holda yoyilma haqidagi teoremaga ko'ra

$$f(x) = \sum_{n=0}^\infty a_n u_n(x),$$

bo'ladi. Bu tenglikning ikkala tomonini ham $f(x)$ funksiyaga ko'paytirib, $[0, \pi]$ oraliqda integrallasak,

$$\int_0^\pi f^2(x)dx = \sum_{n=0}^\infty a_n \int_0^\pi f(x)u_n(x)dx = \sum_{n=0}^\infty a_n^2,$$

bo'ladi.

2) $f(x) \in L^2[0, \pi]$ ixtiyoriy funksiya bo'lsin, u holda quyidagi shartlarni qanoatlantiruvchi $f_n(x)$ funksiyalar mavjud:

$a)f_n(x) \in C^2[0, \pi]$,

$b)f_n(x)$ chegaraviy shartlarni qanoatlantiradi,

$c)$ $\lim\limits_{n\to\infty} \int_0^\pi (f(x) - f_n(x))^2\, dx = 0.$

Bu fikr funksional analiz kursida isbot qilinadi.

$f_n(x)$ funksiyalar uchun birinchi bandga binoan Parseval tengligi bajariladi:

$$\int_0^{\pi} f_n^2(x)dx = \sum_{k=0}^{\infty} \left(a_k^{(n)}\right)^2. \qquad (1.12.25')$$

Bu yerda

$$a_k^{(n)} = \int_0^{\pi} f_n(x)u_k(x)dx.$$

Xususan $f_n(x) - f_m(x)$ funksiyalar uchun ham Parseval tengligi bajariladi, ya'ni

$$\int_0^{\pi} [f_n(x) - f_m(x)]^2 dx = \sum_{k=0}^{\infty} \left(a_k^{(n)} - a_k^{(m)}\right)^2. \qquad (1.12.26)$$

Quyidagi belgilashni kiritib olamiz:

$$a^{(n)} = (a_0^{(n)},\ a_1^{(n)},\ ...), \quad n = 1, 2, \dots .$$

$f_n(x)$ ketma-ketlikning $L^2[0,\ \pi]$ fazoda fundamental ekanligidan, (1.12.26) tenglikka asosan

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \left\|a^{(n)} - a^{(m)}\right\| = \lim_{\substack{m\to\infty \\ n\to\infty}} \|f_n - f_m\|_{L^2[0,\pi]} = 0,$$

bo'ladi. Demak, $a^{(n)}$ vektorlar ketma-ketligi l_2 fazoda fundamental ekan. Bu yerda ushbu $\|\cdot\|$ belgi l_2 fazodagi normani bildiradi. l_2 fazo to'la bo'lgani uchun $a^{(n)}$ ketma-ketlik biror $a = (a_0,\ a_1,\ \dots) \in l_2$ vektorga yaqinlashadi.

Normaning $\big|\|x\| - \|y\|\big| \le \|x - y\|$ xossasidan foydalanib, quyidagi baholashlarga ega bo'lamiz:

$$\left|\sqrt{\int_0^{\pi} f_n^2(x)\,dx} - \sqrt{\int_0^{\pi} f^2(x)\,dx}\right| \le \sqrt{\int_0^{\pi} [f_n(x) - f(x)]^2 dx},$$

$$\left|\sqrt{\sum_{k=0}^{\infty} \left(a_k^{(n)}\right)^2} - \sqrt{\sum_{k=0}^{\infty} a_k^2}\right| \le \sqrt{\sum_{k=0}^{\infty} [a_k^{(n)} - a_k]^2}.$$

Bu baholashlarga asoslanib, $f_n(x)$ funksiyalar uchun yozilgan (1.12.25') Parseval tengligida $n \to \infty$ da limitga o'tsak, $f(x)$ uchun (1.12.24) Parseval tengligi kelib chiqadi. ■

Natija 1.12.1. Agar $f(x), g(x) \in L^2[0,\pi]$ ixtiyoriy funksiyalar bo'lsa, u holda $f(x)+g(x)$ va $f(x)-g(x)$ funksiyalar uchun Parseval tengligi

$$\int_0^\pi [f(x)+g(x)]^2 dx = \sum_{k=0}^{\infty} [a_k + b_k]^2,$$

$$\int_0^\pi [f(x)-g(x)]^2 dx = \sum_{k=0}^{\infty} [a_k - b_k]^2,$$

ko'rinishda bo'ladi. Bularni bir-biridan ayirib, 4 ga bo'lsak, ushbu

$$\int_0^\pi f(x)g(x)dx = \sum_{k=0}^{\infty} a_k b_k, \tag{1.12.27}$$

tenglik kelib chiqadi. Bu tenglikka Parseval tengligining umumlashmasi deyiladi.

Ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = -\sin\alpha, \\ y'(0) = \cos\alpha, \end{cases}$$

Koshi masalasining yechimini $\varphi(x,\lambda)$ orqali belgilaylik.

Quyidagi monoton o'suvchi

$$\rho(\lambda) = \begin{cases} 0, & \lambda = 0, \\ -\sum\limits_{\lambda \le \lambda_n < 0} \dfrac{1}{\alpha_n^2}, & \lambda < 0, \\ \sum\limits_{0 < \lambda_n < \lambda} \dfrac{1}{\alpha_n^2}, & \lambda > 0, \end{cases}$$

funksiyaga (1.12.1) chegaraviy masalaning spektral funksiyasi deyiladi. Bu yerda

$$\alpha_n^2 = \int_0^\pi \varphi^2(x, \lambda_n)\, dx.$$

(1.12.24) Parseval tengligini spektral funksiya va Stiltes integrali yordamida, ushbu

$$\int_0^\pi f^2(x)dx = \int_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda),$$

ko'rinishda yozish mumkin. Bu yerda

$$F(\lambda) = \int_0^\pi f(t)\varphi(t, \lambda)dt.$$

Parseval tengligining umumlashmasi esa, spektral funksiya orqali quyidagi tarzda yoziladi:

$$\int_0^\pi f(x)g(x)dx = \int_{-\infty}^{\infty} F(\lambda)G(\lambda)d\rho(\lambda),$$

bu yerda

$$G(\lambda) = \int_0^\pi g(t)\varphi(t, \lambda)dt.$$

Teorema 1.12.6. (*Rezolventa uchun yoyilma formulasi*). *(1.12.1) chegaraviy masalaning ortonormallangan xos funksiyalari $u_n(x)$, $n = 0, 1, 2, \ldots$ bo'lib, ushbu*

$$a_n = \int_0^\pi f(t)\, u_n(t)\, dt, \quad n = 0, 1, 2, \ldots,$$

sonlar $f(x)$ funksiyaning Furye koeffitsiyentlari bo'lsin. Agar λ son (1.12.1) chegaraviy masalaning xos qiymati bo'lmasa, u holda *quyidagi*

$$\begin{cases} -y'' + q(x)y = \lambda y + f(x), \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.12.28}$$

chegaraviy masalaning yechimi uchun ushbu

$$y(x,\lambda) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n - \lambda} u_n(x), \tag{1.12.29}$$

tasvir o'rinli bo'ladi.

Isbot. λ son xos qiymat bo'lmagani uchun (1.12.28) chegaraviy masalaning yechimi ushbu

$$y(x,\lambda) = \int_0^{\pi} G(x,t,\lambda) f(t)dt, \tag{1.12.30}$$

ko'rinishda bo'ladi. (1.12.28) chegaraviy masala yechimining yoyilmasi quyidagi ko'rinishda bo'lsin:

$$y(x,\lambda) = \sum_{n=0}^{\infty} b_n u_n(x). \tag{1.12.31}$$

U holda

$$b_n = \int_0^{\pi} y(x,\lambda)u_n(x)dx = \frac{1}{\lambda_n}\int_0^{\pi} y(x,\lambda)Lu_n(x)dx =$$

$$= \frac{1}{\lambda_n}\int_0^{\pi} Ly(x,\lambda)u_n(x)dx =$$

$$= \frac{1}{\lambda_n}\int_0^{\pi} \left[-y''(x,\lambda) + q(x)y(x,\lambda)\right] u_n(x)\,dx =$$

$$= \frac{1}{\lambda_n}\int_0^{\pi} \left[\lambda y(x,\lambda) + f(x)\right] u_n(x)\,dx = \frac{\lambda}{\lambda_n} b_n + \frac{1}{\lambda_n} a_n,$$

bo'ladi. Bundan

$$\lambda_n b_n = \lambda b_n + a_n, \quad b_n = \frac{a_n}{\lambda_n - \lambda},$$

kelib chiqadi. Oxirgi ifodani (1.12.31) tenglikka qo'yib, (1.12.29) tasvirni hosil qilamiz.■

Natija 1.12.2. (1.12.29) yoyilmaga a_n koeffitsiyentning (1.12.25) formuladagi ifodasini qo'ysak, ushbu

$$y(x,\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n - \lambda} \left\{ \int_0^{\pi} f(t) u_n(t) dt \right\} u_n(x) =$$

$$= \int_0^{\pi} \left\{ \sum_{n=0}^{\infty} \frac{u_n(x) u_n(t)}{\lambda_n - \lambda} \right\} f(t) dt, \tag{1.12.32}$$

tenglik kelib chiqadi. Bu tenglikni

$$y(x,\lambda) = \int_0^{\pi} G(x,t,\lambda) f(t) dt,$$

formula bilan tenglashtirib, $f(t)$ funksiyaning ixtiyoriyligini inobatga olsak, quyidagi yoyilma hosil bo'ladi:

$$G(x,t,\lambda) = \sum_{n=0}^{\infty} \frac{u_n(x) u_n(t)}{\lambda_n - \lambda}. \tag{1.12.33}$$

Izoh 1.12.1. Teorema 1.12.3 dagi (1.12.13) yoyilma, (1.12.33) yoyilmaning xususiy holidir. Shunday bo'lsa ham, (1.12.33) yoyilmani (1.12.13) formuladan keltirib chiqarish mumkin.

(1.12.33) yoyilmada $t = x$ deb, hosil bo'lgan tenglikni $[0,\ \pi]$ oraliqda integrallasak va xos funksiyalarning normallanganligini e'tiborga olsak, Grin yadrosining izi uchun ushbu

$$\int_0^{\pi} G(x,x,z) dx = \sum_{n=0}^{\infty} \frac{1}{\lambda_n - z}, \tag{1.12.34}$$

tenglik kelib chiqadi. Quyidagi

$$N(\lambda) = \sum_{\lambda_n \le \lambda} 1,$$

funksiyani kiritib olamiz. $N(\lambda)$ ning qiymati λ sondan oshmaydigan xos qiymatlar sonini bildiradi. Bu funksiya yordamida (1.12.34) tenglikni ushbu

$$\int_0^{\pi} G(x, x, z)dx = \int_{-\infty}^{\infty} \frac{dN(\lambda)}{\lambda - z},$$

ko'rinishda yozish mumkin. Bu formulaga Karleman formulasi deyiladi.

Izoh 1.12.2. (1.12.33) formulaga Parseval tengligini qo'llasak

$$\int_0^{\pi} |G(x, t, \lambda)|^2 dt = \sum_{n=0}^{\infty} \frac{u_n^2(x)}{(\lambda_n - \lambda)^2},$$

kelib chiqadi. Bu tenglikning ikkala tarafini $[0, \pi]$ oraliq bo'yicha integrallab

$$\int_0^{\pi}\int_0^{\pi} |G(x, t, \lambda)|^2 dt dx = \sum_{n=0}^{\infty} \frac{1}{(\lambda_n - \lambda)^2}, \qquad (1.12.35)$$

tenglikni topamiz. Bundan ko'rinadiki, Shturm-Liuvill differensial operatoriga teskari bo'lgan integral operatorning yadrosi Gilbert-Shmidt shartini qanoatlantirar ekan.

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$$Ly \equiv -y'' = \lambda y,$$
$$y(0) = 0, \quad y(\pi) = 0,$$

chegaraviy masalaning Grin funksiyasi uchun (1.12.35) tenglikning bajarilishini tekshiring.

Berilgan Dirixle chegaraviy masalasining barcha xos qiymatlari

$$\lambda_n = n^2, \quad n = 1, 2, 3, \ldots,$$

ko'rinishda bo'lib, $\lambda = 0$ nuqta xos qiymat bo'lmaydi. Shuning uchun $\lambda = 0$ bo'lganda (1.12.35) tenglik quyidagi ko'rinishni oladi

$$\int_0^\pi \int_0^\pi |G(x,t)|^2 dxdt = \sum_{n=1}^\infty \frac{1}{\lambda_n^2}, \tag{1.12.36}$$

Bu yerda

$$G(x,t) = \frac{1}{\pi}\begin{cases} x(\pi - t), & x \le t, \\ t(\pi - x), & x \ge t. \end{cases}$$

Endi (1.12.36) tenglikni tekshiramiz:

$$\sum_{n=1}^\infty \frac{1}{\lambda_n^2} = \sum_{n=1}^\infty \frac{1}{n^4} = \int_0^\pi \int_0^\pi |G(x,t)|^2 dxdt =$$

$$= \int_0^\pi \left[\int_0^x |G(x,t)|^2 dt + \int_x^\pi |G(x,t)|^2 dt \right] dx =$$

$$= \frac{1}{\pi^2} \int_0^\pi \left[\int_0^x t^2(\pi - x)^2 dt + \int_x^\pi x^2(\pi - t)^2 dt \right] dx = \frac{\pi^4}{90}$$

Demak, (1.12.36) tenglik quyidagi ko'rinishni oladi:

$$\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Izoh 1.12.3. Ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.12.37}$$

Shturm-Liuvill chegaraviy masalasini qaraylik. Bu yerda $q(x) \in C[0,\pi]$ haqiqiy uzluksiz funksiya bo'lib,

$$\int_0^\pi q(x)dx = 0,$$

shartni qanoatlantirsin. Bu yerda h va H chekli haqiqiy sonlar.

(1.12.37) chegaraviy masalaning Grin funksiyasini $G(x,t,\lambda)$ va xos qiymatlarini $\{\lambda_n\}_{n=0}^{\infty}$ orqali belgilaylik. (1.12.37) chegaraviy masalada $q(x) \equiv 0$ bo'lsa, u holda ushbu

$$\begin{cases} -y'' = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.12.38}$$

chegaraviy masalaning Grin funksiyasini $G_0(x,t,\lambda)$ va xos qiymatlari ketma-ketligini $\{\lambda_n^0\}_{n=0}^{\infty}$ deb belgilasak, quyidagi lemma o'rinli bo'ladi.

Lemma 1.12.1 (*Gelfand-Levitan*). *Quyidagi*

$$\sum_{n=0}^{\infty}(\lambda_n - \lambda_n^0) = \lim_{\lambda\to+\infty}\lambda^2\left[\sum_{n=0}^{\infty}\left(\frac{1}{\lambda+\lambda_n^0} - \frac{1}{\lambda+\lambda_n}\right)\right], \tag{1.12.39}$$

$$\sum_{n=0}^{\infty}(\lambda_n - \lambda_n^0) = \lim_{\lambda\to+\infty}\lambda^2\left[\int_0^{\pi} G_0(x,x,\lambda)dx - \int_0^{\pi} G(x,x,\lambda)dx\right], \tag{1.12.40}$$

tengliklar o'rinli bo'ladi.

Isbot. (1.12.39) tenglik bajarilishi o'z-o'zidan ravshan. Endi (1.12.34) formulani (1.12.37) va (1.12.38) chegaraviy masalalar $G(x,t,\lambda)$ va $G_0(x,t,\lambda)$ Grin funksiyalarining izlari uchun yozib olamiz:

$$\int_0^{\pi} G(x,x,\lambda)dx = \sum_{n=0}^{\infty}\frac{1}{\lambda_n - \lambda}, \tag{1.12.41}$$

$$\int_0^{\pi} G_0(x,x,\lambda)dx = \sum_{n=0}^{\infty}\frac{1}{\lambda_n^0 - \lambda}. \tag{1.12.42}$$

Quyidagi limitni hisoblaymiz:

$$\lim_{\lambda\to+\infty}\lambda^2\left[\int_0^\pi G_0(x,x,\lambda)dx-\int_0^\pi G(x,x,\lambda)dx\right]=$$

$$=\lim_{\lambda\to+\infty}\lambda^2\left[\sum_{n=0}^{\infty}\frac{1}{\lambda_n^0-\lambda}-\sum_{n=0}^{\infty}\frac{1}{\lambda_n-\lambda}\right]=$$

$$=\sum_{n=0}^{\infty}\left[\lim_{\lambda\to+\infty}\lambda^2\left(\frac{1}{\lambda_n^0-\lambda}-\frac{1}{\lambda_n-\lambda}\right)\right]=$$

$$=\sum_{n=0}^{\infty}\left[\lim_{\lambda\to+\infty}\frac{\lambda_n-\lambda_n^0}{\left(\frac{\lambda_n^0}{\lambda}-1\right)\left(\frac{\lambda_n}{\lambda}-1\right)}\right]=\sum_{n=0}^{\infty}(\lambda_n-\lambda_n^0).\blacksquare$$

Mustaqil yechish uchun mashqlar

1. Quyidagi o'zgarmas potensialli Shturm-Liuvill chegaraviy masalalari uchun yoyilma teoremasini va Parseval tengligini yozing $(0\le x\le\pi)$:

$$a)\begin{cases}-y''=\lambda y,\\ y(0)=0,\\ y(\pi)=0,\end{cases}\quad b)\begin{cases}-y''=\lambda y,\\ y'(0)=0,\\ y'(\pi)=0,\end{cases}\quad c)\begin{cases}-y''=\lambda y,\\ y'(0)=0,\\ y(\pi)=0,\end{cases}$$

$$d)\begin{cases}-y''=\lambda y,\\ y(0)=0,\\ y'(\pi)=0,\end{cases}\quad e)\begin{cases}-y''=\lambda y,\\ y'(0)-hy(0)=0,\\ y'(\pi)-hy(\pi)=0,\end{cases}$$

2. Quyidagi o'zgaruvchan potensialli Shturm-Liuvill chegaraviy masalalari uchun yoyilma teoremasini va Parseval tengligini yozing $(0\le x\le\pi)$:

$$a)\ \begin{cases} -y'' + \dfrac{2}{(x+1)^2}y = \lambda y, \\ y'(0) + y(0) = 0, \\ y'(\pi) + \dfrac{1}{\pi+1}y(\pi) = 0, \end{cases}$$

$$b)\ \begin{cases} -y'' + \dfrac{2h^2}{(hx-1)^2}y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + \dfrac{h}{\pi h - 1}y(\pi) = 0, \dfrac{1}{h} \notin [0,\pi] \end{cases}$$

$$c)\ \begin{cases} -y'' - \dfrac{2}{\operatorname{ch}^2 x}y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + \operatorname{th}\pi y(\pi) = 0, \end{cases}$$

$$d)\ \begin{cases} -y'' - \dfrac{2}{\operatorname{ch}^2 x}y = \lambda y, \\ y'(0) = 0, \\ y'(\pi) + \operatorname{th}\pi y(\pi) = 0, \end{cases}$$

$$e)\ \begin{cases} -y'' - \dfrac{2m^2}{ch^2 mx}y = \lambda y, \\ y'(0) = 0, \\ y'(\pi) + mth(m\pi)y(\pi) = 0, \end{cases}$$

$$f)\ \begin{cases} -y'' - \dfrac{2m^2}{ch^2(mx+\alpha)}y = \lambda y, \\ y'(0) + m\operatorname{th}\alpha\, y(0) = 0, \\ y'(\pi) + m\operatorname{th}(m\pi+\alpha)y(\pi) = 0. \end{cases}$$

3. Quyidagi simvolik tengliklar bajarilishini ko'rsating:

$$a)\quad \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0}\cos nx \cos nt = \delta(x-t), \quad x,t \in (0,\pi),$$

bu yerda

$$\alpha_n^0 = \begin{cases} \pi, & n=0 \\ \dfrac{\pi}{2}, & n \geq 1. \end{cases}$$

b) $$\sum_{n=0}^{\infty} \frac{1}{\alpha_n} \varphi(x,\lambda_n)\varphi(t,\lambda_n) = \delta(x-t), \quad x,t \in (0,\pi),$$

bu yerda

$$\alpha_n = \int_0^{\pi} \varphi^2(x,\lambda_n)dx, \quad n \geq 0$$

bo'lib, $\varphi(x,\lambda_n)$ orqali (1.12.1) chegaraviy masalaning $\lambda = \lambda_n$ xos qiymatiga mos keluvchi xos funksiyasi belgilangan.

13-§. Teng yaqinlashish haqida teorema

Bu paragrafda $f(x)$ funksiyaga mos keluvchi xos funksiyalardan tuzilgan qator hamda $\cos nx$ funksiyalar bo'yicha tuzilgan qator bir xil ma'noda yaqinlashishini B.M.Levitan usulida ko'rsatamiz.

Ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.13.1}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C^1[0,\ \pi]$ haqiqiy qiymatli funksiya.

$f(x) \in L^1[0,\pi]$ ixtiyoriy funksiya bo'lsin. Quyidagi belgilashlarni kiritib olamiz:

$$\sigma_n(x) = \int_0^{\pi} f(t) \left\{ \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^{n} \cos kx \, \cos kt \right\} dt,$$

$$s_n(x) = \int_0^{\pi} f(t) \left\{ \sum_{k=0}^{n} u_k(x)\, u_k(t) \right\} dt.$$

Bu yerda $u_k(x)$, $k = 0, 1, 2, \ldots$ orqali (1.13.1) chegaraviy masalaning ortonormallangan xos funksiyalari belgilangan. Agar

$$\Phi_n(x, t) = \sum_{k=0}^{n} u_k(x)\, u_k(t) - \left\{ \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^{n} \cos kx \, \cos kt \right\},$$

belgilash kiritsak, u holda

$$s_n(x) - \sigma_n(x) = \int_0^{\pi} \Phi_n(x, t) f(t) dt,$$

bo'ladi.

Lemma 1.13.1. *Shunday o'zgarmas $M > 0$ soni mavjudki, n ning ixtiyoriy qiymatlari uchun ushbu*

$$|\Phi_n(x, t)| < M, \quad x, t \in [0, \pi], \tag{1.13.2}$$

baholash o'rinli bo'ladi.

Isbot. Ortonormallangan xos funksiyalarning quyidagi

$$u_n(x) = \sqrt{\frac{2}{\pi}} \left\{ \cos nx + a(x) \frac{\sin nx}{n} \right\} + \underline{\underline{O}}\left(\frac{1}{n^2} \right), \qquad n \to \infty,$$

$$a(x) = -c_0 x + h + \frac{1}{2} \int_0^x q(t) dt, \quad c_0 = \frac{h + H}{\pi} + \frac{1}{2\pi} \int_0^{\pi} q(t) dt$$

asimptotikasiga asosan

$$u_k(x) u_k(t) - \frac{2}{\pi} \cos kx \, \cos kt = \frac{2}{\pi} [a(t) \cos kx \, \sin kt +$$

$$+ a(x) \cos kt \, \sin kx] + \underline{\underline{O}}\left(\frac{1}{k^2} \right) = \frac{1}{2\pi} \cdot [a(x) + a(t)] \frac{\sin k(x+t)}{k} +$$

$$+ \frac{1}{2\pi} \cdot [a(x) - a(t)] \frac{\sin k(x-t)}{k} + \underline{\underline{O}}\left(\frac{1}{k^2} \right), \tag{1.13.3}$$

bo'lishi kelib chiqadi. $a(x)$ funksiyaning $[0, \pi]$ kesmada chegaralangan ekanligi ravshan.

Endi ushbu

$$\sum_{k=1}^{n} \frac{\sin kt}{k},$$

yig'indining $[0,\ \pi]$ intervalda chegaralanganligini isbot qilamiz. Quyidagi

$$\sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t - \sin\frac{t}{2}}{2\sin\frac{t}{2}},$$

formulaga asosan

$$\tau_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k} = \int_0^x \left(\sum_{k=1}^{n} \cos kt\right) dt =$$

$$= \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t - \sin\frac{t}{2}}{2\sin\frac{t}{2}} dt =$$

$$= \int_0^x \frac{1}{t} \sin\left(n+\frac{1}{2}\right) t\, dt + \int_0^x \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right) \sin\left(n+\frac{1}{2}\right) t dt - \frac{1}{2}x =$$

$$= \int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin u}{u} du + \int_0^x \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right) \sin\left(n+\frac{1}{2}\right) t\, dt - \frac{1}{2}x,$$

bo'ladi. Ushbu

$$J(x) = \int_0^x \frac{\sin u}{u} du,$$

funksiya $(0,\ \infty)$ oraliqda chegaralangan va $x = \pi$ nuqtada o'zining eng katta qiymatiga erishadi. Bundan $0 \le x \le \pi$ oraliqda ushbu

$$|\tau_n(x)| \le \int_0^{\pi} \frac{\sin t}{t} dt + \int_0^{\pi} \left|\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right| dt + \frac{1}{2}\pi, \qquad (1.13.4)$$

tengsizlik o'rinli bo'lishi kelib chiqadi. Bu esa $\tau_n(x)$ funksiyalar ketma-ketligining tekis chegaralanganligini bildiradi. (1.13.3) tenglik va (1.13.4) tengsizlikdan (1.13.2) baholash kelib chiqadi.■

Lemma 1.13.2. *Agar $g(x)$ funksiya $[0,\ \pi]$ oraliqda ikkinchi tartibli uzluksiz hosilaga ega bo'lib, ushbu*

$$\begin{cases} g'(0) - hg(0) = 0, \\ g'(\pi) + Hg(\pi) = 0, \end{cases} \tag{1.13.5}$$

chegaraviy shartlarni qanoatlantirsa, u holda $n \to \infty$ da quyidagi

$$\int_0^\pi \Phi_n(x,t)g(t)dt,$$

integral nolga tekis yaqinlashadi.

Isbot. Yoyilma haqidagi va Furye qatorlari haqidagi teoremalarga asosan, $g(x)$ funksiya uchun yozilgan $s_n(x)$ va $\sigma_n(x)$ funksiyalarning $g(x)$ funksiyaga tekis yaqinlashishi kelib chiqadi. Demak, ushbu

$$s_n(x) - \sigma_n(x) = \int_0^\pi \Phi_n(x,\ t)g(t)dt,$$

ayirma nolga tekis yaqinlashadi.

Teorema[1] 1.13.1. (*B.M.Levitan*). *Ixtiyoriy $f(x) \in L^1[0,\pi]$ funksiya uchun ushbu $s_n(x) - \sigma_n(x)$ ayirma $[0,\ \pi]$ oraliqda nolga tekis yaqinlashadi.*

Isbot. $f(x) \in L^1[0,\pi]$ ixtiyoriy funksiya bo'lsin. U holda ixtiyoriy $\varepsilon > 0$ soni uchun (1.13.5) chegaraviy shartlarni qanoatlantiruvchi shunday $g_\varepsilon(x) \in C^2[0,\ \pi]$ funksiya topiladiki, bunda ushbu

$$\int_0^\pi |f(x) - g_\varepsilon(x)|dx < \frac{\varepsilon}{2M},$$

shart bajariladi. Bu yerda $M > 0$ ixtiyoriy son.

[1] Bu teorema $f(x) \in C[0,\pi]$ holida E.Aynsning "Обыкновенные дифференциальные уравнения"(M.: 2005, 340-342 betlar) kitobida isbotlangan

$s_n(x) - \sigma_n(x)$ ayirmani quyidagi tarzda yozib olamiz:

$$s_n(x) - \sigma_n(x) = \int_0^\pi \Phi_n(x,\, t)\,[f(t) - g_\varepsilon(t)]\, dt +$$

$$+ \int_0^\pi \Phi_n(x,\, t) g_\varepsilon(t)\, dt. \qquad (1.13.6)$$

Lemma 1.13.1 ga asosan, quyidagi

$$\left| \int_0^\pi \Phi_n(x,\, t)\,[f(t) - g_\varepsilon(t)]\, dt \right| \le$$

$$\le M \int_0^\pi |f(t) - g_\varepsilon(t)|\, dt < M \cdot \frac{\varepsilon}{2M} = \frac{\varepsilon}{2}, \qquad (1.13.7)$$

tengsizlik o'rinli bo'ladi. Lemma 1.13.2 ga ko'ra, tayinlangan $\varepsilon > 0$ son uchun shunday $N = N(\varepsilon) > 0$ topilib, $n > N$ bo'lganda ushbu

$$\left| \int_0^\pi \Phi_n(x,\, t) g_\varepsilon(t) dt \right| < \frac{\varepsilon}{2}, \quad x \in [0,\, \pi], \qquad (1.13.8)$$

baholash o'rinli bo'ladi. (1.13.6) tenglik va (1.13.7), (1.13.8) baholashlarga ko'ra

$$|s_n(x) - \sigma_n(x)| < \varepsilon, \quad x \in [0,\, \pi]$$

bo'ladi. ■

Izoh 1.13.1. Quyidagi

$$1. \begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases}$$

$$2. \begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases}$$

$$3.\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases}$$

Shturm-Liuvill masalalari uchun teng yaqinlashish haqidagi teoremani isbotlashda $\cos nx$, $n = 0, 1, 2, \ldots$ funksiyalar o'rniga birinchi va ikkinchi holda $\cos\left(n + \frac{1}{2}\right)x$, $n = 0, 1, 2, \ldots$ va $\sin\left(n + \frac{1}{2}\right)x$, $n = 0, 1, 2, \ldots$ funksiyalar, uchinchi holda esa, $\sin nx$, $n = 1, 2, \ldots$ funksiyalar olinadi.

14-§. Shturm teoremasi va undan kelib chiqadigan natijalar

Ushbu mavzuni quyidagi

$$\begin{cases} -y'' = \lambda y, \quad 0 \le x \le \pi, \\ y'(0) = 0, \\ y'(\pi) = 0, \end{cases}$$

oddiy chegaraviy masala xos funksiyalarining ildizlarini o'rganishdan boshlaymiz. Bu masalaning xos qiymatlari

$$\lambda_n = n^2, \quad n = 0, 1, 2, \ldots,$$

va xos funksiyalari

$$\varphi_0(x) = 1, \; \varphi_1(x) = \cos x, \; \varphi_2(x) = \cos 2x, \; \ldots, \; \varphi_n(x) = \cos nx, \; \ldots,$$

bo'lishi ravshan. Bu yerda xos qiymatlar o'sish tartibida joylashtirilgan bo'lib, ularni nomerlash noldan boshlangan. $\varphi_n(x)$ va $\varphi_{n+1}(x)$ funksiyalarning $(0, \pi)$ oraliqqa tegishli ildizlarini topamiz:

$$\varphi_n(x) = 0, \quad \cos nx = 0, \quad nx = -\frac{\pi}{2} + \pi k, \quad x = \frac{\pi(2k-1)}{2n},$$

$$x_1^{(n)} = \frac{\pi}{2n}, \quad x_2^{(n)} = \frac{3\pi}{2n}, \quad x_3^{(n)} = \frac{5\pi}{2n}, \quad \ldots, \quad x_n^{(n)} = \frac{\pi(2n-1)}{2n}. \tag{1.14.1}$$

Xuddi shunday

$$x_1^{(n+1)} = \frac{\pi}{2(n+1)}, \qquad x_2^{(n+1)} = \frac{3\pi}{2(n+1)},$$
$$x_3^{(n+1)} = \frac{5\pi}{2(n+1)}, \quad \dots, \; x_{n+1}^{(n+1)} = \frac{\pi(2n+1)}{2(n+1)}, \tag{1.14.2}$$

bo'ladi. (1.14.1) va (1.14.2) tengliklardan quyidagi xulosalar kelib chiqadi:

1) $\varphi_n(x)$ xos funksiyaning $(0, \pi)$ oraliqdagi ildizlari soni n ta bo'ladi;

2) $\varphi_n(x)$ va $\varphi_{n+1}(x)$ xos funksiyalarning $(0, \pi)$ oraliqdagi ildizlari almashinib kelishadi

$$x_1^{(n+1)} < x_1^{(n)} < x_2^{(n+1)} < x_2^{(n)} < \dots < x_n^{(n)} < x_{n+1}^{(n+1)},$$

ya'ni $\varphi_n(x)$ xos funksiyaning ketma-ket kelgan ixtiyoriy ikkita ildizi orasida $\varphi_{n+1}(x)$ xos funksiyaning faqat bitta ildizi bor.

Keltirilgan bu ikkita tasdiq umumiy holda ham o'rinli bo'lar ekan. Bu tasdiqlarni isbot qilishda quyidagi Shturm teoremasi muhim o'rin egallaydi.

Teorema 1.14.1 (*Shturm*). *Quyidagi ikkita*

$$-y'' + a(x)y = 0, \quad (0 < x < \pi), \tag{1.14.3}$$

$$-y'' + \tilde{a}(x)y = 0, \quad (0 < x < \pi), \tag{1.14.4}$$

tenglama berilgan bo'lib, $a(x) > \tilde{a}(x)$, $x \in [0, \pi]$ bo'lsin, hamda $\varphi(x)$ va $\tilde{\varphi}(x)$ funksiyalar mos ravishda (1.14.3) va (1.14.4) tenglamalarning noldan farqli ixtiyoriy yechimlari bo'lsin. U holda $\varphi(x)$ funksiyaning ixtiyoriy ikkita ildizi orasida $\tilde{\varphi}(x)$ funksiyaning kamida bitta ildizi bo'ladi.

Isbot. Ushbu

$$-\varphi'' + a(x)\varphi = 0, \tag{1.14.5}$$

$$-\tilde{\varphi}'' + \tilde{a}(x)\tilde{\varphi} = 0, \tag{1.14.6}$$

ayniyatlarni mos ravishda $\tilde{\varphi}(x)$ va $\varphi(x)$ funksiyalarga ko'paytirib, birinchisidan ikkinchisini ayirsak,

$$\tilde{\varphi}''\varphi - \varphi''\tilde{\varphi} + [a(x) - \tilde{a}(x)]\varphi\tilde{\varphi} = 0,$$

$$\varphi''\tilde{\varphi} - \tilde{\varphi}''\varphi = [a(x) - \tilde{a}(x)]\varphi\tilde{\varphi},$$

$$(\varphi'\tilde{\varphi} - \tilde{\varphi}'\varphi)' = [a(x) - \tilde{a}(x)]\varphi\tilde{\varphi}, \qquad (1.14.7)$$

kelib chiqadi.

x_1 va x_2 orqali $\varphi(x)$ funksiyaning ketma-ket kelgan ixtiyoriy ikkita ildizini belgilaymiz. (1.14.7) ayniyatni $[x_1, x_2]$ oraliqda integrallasak,

$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) = \int\limits_{x_1}^{x_2} [a(x) - \tilde{a}(x)]\varphi(x)\tilde{\varphi}(x)dx, \qquad (1.14.8)$$

tenglik hosil bo'ladi.

$\tilde{\varphi}(x)$ funksiya (x_1, x_2) oraliqda ildizga ega emas deb faraz qilaylik. Qulaylik uchun (x_1, x_2) oraliqda $\varphi(x) > 0$, $\tilde{\varphi}(x) > 0$ deb hisoblashimiz mumkin. U holda (1.14.8) tenglikning o'ng tomoni musbat bo'ladi. $\varphi(x_1) = 0$, $\varphi(x_2) = 0$, $\varphi(x) > 0$, $x \in (x_1, x_2)$ bo'lgani uchun $\varphi'(x_1) \geq 0$, $\varphi'(x_2) \leq 0$ bo'ladi.

Agar $\varphi'(x_1) = 0$ yoki $\varphi'(x_2) = 0$ bo'lsa, yagonalik teoremasiga ko'ra $\varphi(x) \equiv 0$ bo'ladi. Demak, $\varphi'(x_1) > 0$ va $\varphi'(x_2) < 0$ bo'lar ekan. Bu tengsizliklardan quyidagi

$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) \leq 0,$$

baholash kelib chiqadi. Oxirgi tengsizlikdan ziddiyat kelib chiqadi, chunki (1.14.8) tenglikka asosan ushbu

$$\varphi'(x_2)\tilde{\varphi}(x_2) - \varphi'(x_1)\tilde{\varphi}(x_1) > 0,$$

tengsizlik o'rinli bo'ladi. ■

Natija 1.14.1. $a(x) > m^2 > 0$ bo'lsin. U holda ushbu

$$-y'' + a(x)y = 0, \quad 0 < x < \pi,$$

tenglamaning noldan farqli ixtiyoriy $\varphi(x)$ yechimi $(0,\pi)$ oraliqda ko‘pi bilan bitta ildizga ega bo‘lishi mumkin. Haqiqatan ham, Shturm teoremasida $\bar{a}(x) \equiv m^2$ deb olsak, ikkinchi tenglama

$$-y'' + m^2 y = 0,$$

ko‘rinishda bo‘ladi. Bu tenglama $y = e^{mx}$ yechimga ega. $\varphi(x)$ funksiyaning ildizlari soni bittadan ko‘p deb faraz qilaylik. U holda $\varphi(x)$ funksiyaning ketma-ket kelgan ikkita ildizi orasida $y = e^{mx}$ funksiyaning kamida bitta ildizi bo‘ladi. Ziddiyat, chunki $y = e^{mx}$ yechim ildizga ega emas.

Teorema 1.14.2 (*Taqqoslash teoremasi*). *Quyidagi ikkita*

$$-y'' + a(x)y = 0, \quad (0 < x < \pi), \tag{1.14.9}$$

$$-y'' + \bar{a}(x)y = 0, \quad (0 < x < \pi), \tag{1.14.10}$$

tenglama berilgan bo‘lib, $a(x) > \bar{a}(x)$, $x \in [0,\pi]$ *bo‘lsin.* $\varphi(x)$ *va* $\bar{\varphi}(x)$ *mos ravishda (1.14.9) va (1.14.10) tenglamalarning ushbu*

$$\begin{cases} \varphi(0) = -\sin\alpha, \\ \varphi'(0) = \cos\alpha, \end{cases} \quad \begin{cases} \bar{\varphi}(0) = -\sin\alpha, \\ \bar{\varphi}'(0) = \cos\alpha, \end{cases}$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimlari bo‘lsin. U holda

1) agar $\varphi(x)$ *yechimning* $(0,\pi]$ *yarim oraliqda* m *ta ildizi bo‘lsa,* $\bar{\varphi}(x)$ *yechimning* $(0,\pi]$ *yarim oraliqdagi ildizlari soni* m *tadan kam bo‘lmaydi;*

2) $\bar{\varphi}(x)$ *funksiyaning* k *nomerli ildizi* $\varphi(x)$ *funksiyaning* k *nomerli ildizidan kichik bo‘ladi.*

Isbot. Mavjudlik va yagonalik teoremasining shartlari bajarilganda, ikkinchi tartibli bir jinsli differensial tenglama noldan farqli ixtiyoriy yechimining ildizlari chekli limitik nuqtaga ega bo‘lmasligini ko‘rsatamiz. Teskarisini faraz qilaylik, ya’ni $y(x_n) =$

0 bo'lib, $x_n \to a$, $(n \to \infty)$ bo'lsin. U holda $y(a) = 0$ va

$$y'(a) = \lim_{x_n \to a} \frac{y(x_n) - y(a)}{x_n - a} = 0,$$

bo'ladi. Mavjudlik va yagonalik teoremasiga ko'ra $y(x) \equiv 0$ bo'ladi, bu esa $y(x) \not\equiv 0$ shartga zid.

Demak, $y(x)$ yechimning chekli oraliqdagi ildizlari orasida eng kattasi va eng kichigi hamisha mavjud bo'ladi.

$\varphi(x)$ yechimning $(0, \pi]$ yarim oraliqdagi eng kichik ildizini x_1 orqali belgilaylik. $\tilde{\varphi}(x)$ yechim $(0, x_1]$ oraliqda kamida bitta ildizga ega ekanligini ko'rsatamiz. Teskarisini faraz qilamiz, ya'ni $\tilde{\varphi}(x)$ yechim $(0, x_1]$ oraliqda ildizga ega emas deb faraz qilamiz. Umumiylikni buzmasdan, qulaylik uchun $(0, x_1]$ oraliqda $\varphi(x) > 0$ va $\tilde{\varphi}(x) > 0$ deb hisoblaymiz. $\varphi(x_1) = 0$ bo'lgani uchun x_1 nuqtaning yetarlicha kichik atrofida $\varphi(x)$ kamayuvchi bo'ladi, bundan $\varphi'(x_1) \leq 0$ kelib chiqadi. Ushbu

$$(\varphi'\tilde{\varphi} - \tilde{\varphi}'\varphi)' = [a(x) - \tilde{a}(x)]\varphi\tilde{\varphi},$$

ayniyatni $[0, x_1]$ oraliqda integrallaymiz:

$$[\varphi'(x_1)\tilde{\varphi}(x_1) - \tilde{\varphi}'(x_1)\varphi(x_1)] - [\varphi'(0)\tilde{\varphi}(0) - \tilde{\varphi}'(0)\varphi(0)] =$$

$$= \int_0^{x_1} [a(x) - \tilde{a}(x)]\varphi(x)\tilde{\varphi}(x)dx. \qquad (1.14.11)$$

Boshlang'ich shartlarni va $\varphi(x_1) = 0$ tenglikni e'tiborga olsak,

$$\varphi'(x_1)\tilde{\varphi}(x_1) = \int_0^{x_1} [a(x) - \tilde{a}(x)]\varphi(x)\tilde{\varphi}(x)dx, \qquad (1.14.12)$$

kelib chiqadi. Ziddiyat, chunki tenglikning chap tomoni noldan oshmaydi, o'ng tomoni esa noldan katta.

Demak, $\tilde{\varphi}(x)$ funksiya $(0, x_1)$ oraliqda kamida bitta ildizga ega ekan. Bu ildizlardan bittasini $\tilde{x}_1$ bilan belgilaylik, $\varphi(x)$ funksiyaning $(0,\pi]$ yarim oraliqdagi ildizlarini

$$x_1 < x_2 < ... < x_m,$$

orqali belgilaymiz.

Shturm teoremasiga ko'ra

$$(x_1, x_2),\ (x_2, x_3),\ ...,\ (x_{m-1}, x_m),$$

oraliqlarning har birida $\tilde{\varphi}(x)$ funksiya kamida bitta ildizga ega. Har bir oraliqdan bittadan ildiz olib, ularni

$$\tilde{x}_1 < \tilde{x}_2 < ... < \tilde{x}_m,$$

orqali belgilaymiz. U holda quyidagi tengsizliklar kelib chiqadi:

$$0 < \tilde{x}_1 < x_1 < \tilde{x}_2 < x_2 < ... < x_{m-1} < \tilde{x}_m < x_m \le \pi.$$

Bu tengsizliklardan $\tilde{\varphi}(x)$ funksiyaning ildizlari soni m tadan kam emasligi va $\tilde{\varphi}(x)$ funksiyaning k nomerli ildizi $\varphi(x)$ funksiyaning k nomerli ildizidan kichik bo'lishi kelib chiqadi.■

Teorema 1.14.3. *Quyidagi*

$$-y'' + a(x)y = 0, \quad (0 < x < \pi), \tag{1.14.13}$$

tenglama berilgan bo'lib, $\varphi(x)$ va $\tilde{\varphi}(x)$ uning ikkita chiziqli erkli yechimlari bo'lsin. U holda agar $\varphi(x)$ funksiya $(0,\pi)$ oraliqda kamida ikkita ildizga ega bo'lsa, $\varphi(x)$ funksiyaning ketma-ket kelgan ixtiyoriy ikkita ildizi orasida $\tilde{\varphi}(x)$ funksiyaning yagona ildizi bo'ladi, ya'ni (1.14.13) tenglama chiziqli erkli yechimlarining ildizlari almashinib keladi.

Isbot. x_1 va x_2 orqali $\varphi(x)$ funksiyaning ketma-ket kelgan ikkita ildizini belgilaymiz va $\tilde{\varphi}(x)$ funksiya (x_1, x_2) oraliqda ildizga ega emas deb faraz qilamiz. U holda

$$f(x) = \frac{\varphi(x)}{\tilde{\varphi}(x)},$$

funksiya (x_1, x_2) oraliqda uzluksiz bo'lib, uning chetlarida nolga aylanadi. Chekli orttirmalar haqidagi Lagranj teoremasiga ko'ra shunday $x_0 \in (x_1, x_2)$ topiladiki, bunda $f'(x_0) = 0$ bo'ladi, ya'ni

$$f'(x_0) = \frac{\varphi'(x_0)\tilde{\varphi}(x_0) - \varphi(x_0)\tilde{\varphi}'(x_0)}{\tilde{\varphi}^2(x_0)} = \frac{W\{\tilde{\varphi}(x_0), \varphi(x_0)\}}{\tilde{\varphi}^2(x_0)} = 0.$$

Bu esa ziddiyat, chunki $\varphi(x)$ va $\tilde{\varphi}(x)$ chiziqli erkli yechimlardir.■

15-§. Ossillyatsion teoremalar

$\varphi(x, \lambda)$ orqali ushbu

$$-y'' + q(x)y = \lambda y, \quad 0 < x < \pi, \tag{1.15.1}$$

tenglamaning quyidagi

$$\begin{cases} \varphi(0, \lambda) = -\sin\alpha, \\ \varphi(0, \lambda) = \cos\alpha, \end{cases} \tag{1.15.2}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. Ushbu

$$\varphi(x, \lambda) = 0, \quad (0 \le x \le \pi),$$

tenglamaning ildizlari λ parametrga nisbatan funksiya bo'lishi ravshan. Bu funksiyalar λ parametrga nisbatan uzluksiz bo'lishini isbotlaymiz.

Teorema 1.15.1. *Agar $x_0 \in (0, \pi)$ son $\varphi(x, \lambda_0)$ funksiyaning ildizi bo'lsa, ixtiyoriy $\varepsilon > 0$ musbat son uchun shunday $\delta > 0$ son topiladiki, bunda $|\lambda - \lambda_0| < \delta$ bo'lganda $\varphi(x, \lambda)$ funksiya $|x - x_0| < \varepsilon$ oraliqda faqat bitta ildizga ega bo'ladi.*

Isbot. Agar x_0 son $\varphi(x, \lambda_0)$ funksiyaning karrali ildizi bo'lsa, ya'ni $\varphi'(x_0, \lambda_0) = 0$ bo'lsa, mavjudlik va yagonalik teoremasiga asosan $\varphi(x, \lambda_0) \equiv 0$ bo'ladi. Boshlang'ich shartlarga ko'ra bunday bo'lishi mumkin emas. Demak, $\varphi'(x_0, \lambda_0) \neq 0$ ekan. Aniqlik uchun

$\varphi'(x_0, \lambda_0) > 0$ deb hisoblaymiz. Uzluksiz funksiyaning xossalariga ko'ra bu tengsizlik x_0 nuqtaning yetarlicha kichik atrofida ham bajariladi, ya'ni $\varepsilon > 0$ yetarlicha kichik bo'lib, $|x - x_0| \le \varepsilon$ bo'lsa, $\varphi'(x, \lambda_0) > 0$ bo'ladi. Bundan $\varphi(x_0 - \varepsilon, \lambda_0) < 0$, $\varphi(x_0 + \varepsilon, \lambda_0) > 0$ bo'lishi kelib chiqadi. $\varphi'(x, \lambda)$ funksiya λ ga nisbatan uzluksiz bo'lgani uchun (u hatto λ ga nisbatan butun funksiya bo'lishini bilamiz) shunday $\delta > 0$ son topiladiki, $|\lambda - \lambda_0| \le \delta$ bo'lib, $|x - x_0| \le \varepsilon$ bo'lsa, $\varphi'(x, \lambda) > 0$ bo'ladi. Demak, $|\lambda - \lambda_0| \le \delta$ bo'lsa, $\varphi(x, \lambda)$ funksiya $|x - x_0| \le \varepsilon$ oraliqda monoton o'suvchi funksiya ekan. Bundan $|\lambda - \lambda_0| \le \delta$ bo'lganda $\varphi(x, \lambda)$ funksiya $|x - x_0| \le \varepsilon$ oraliqda ko'pi bilan bitta ildizga ega bo'lishi mumkinligi kelib chiqadi. $\varphi(x_0 - \varepsilon, \lambda)$ va $\varphi(x_0 + \varepsilon, \lambda)$ funksiyalar uzluksiz bo'lib, $\varphi(x_0 - \varepsilon, \lambda_0) < 0$, $\varphi(x_0 + \varepsilon, \lambda_0) > 0$ bo'lgani uchun λ_0 sonning shunday atrofi topiladiki, bunda $\varphi(x_0 - \varepsilon, \lambda) < 0$ va $\varphi(x_0 + \varepsilon, \lambda) > 0$ bo'ladi. Bu fikrdan esa, $|\lambda - \lambda_0| < \delta$ bo'lganda, $\varphi(x, \lambda)$ funksiya $|x - x_0| < \varepsilon$ oraliqda kamida bitta ildizga ega bo'lishi kelib chiqadi.■

Natija 1.15.1. Agar $\lambda_1 < \lambda_2$ bo'lsa, $\varphi(x, \lambda_1)$ yechim ildizlarining soni $\varphi(x, \lambda_2)$ yechim ildizlarining sonidan oshmaydi.

Haqiqatan ham, bu holda $a(x) = q(x) - \lambda_1$, $\tilde{a}(x) = q(x) - \lambda_2$ deb olsak, $\lambda_1 < \lambda_2$ bo'lgani uchun $a(x) > \tilde{a}(x)$ bo'ladi. Taqqoslash teoremasiga ko'ra, $\varphi(x, \lambda_2)$ funksiyaning $(0, \pi]$ yarim oraliqdagi ildizlari soni $\varphi(x, \lambda_1)$ funksiyaning $(0, \pi]$ yarim oraliqdagi ildizlari sonidan kam emas. Bu fikrdan λ oshgani sari $\varphi(x, \lambda)$ yechimning $(0, \pi]$ oraliqdagi ildizlari soni kamaymasligi (ortib borishi) kelib chiqadi.

λ parametr ortgani sari $\varphi(x, \lambda)$ yechimning $(0, \pi]$ oraliqdagi ildizlari nol nuqtaga tomon harakatlanadi, ya'ni yangi ildiz π nuqta orqali kirib keladi.

Misol. Ushbu

$$-y'' = \lambda y,$$

tenglama $\varphi(x,\lambda) = \dfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$ yechimining ildizlarini topamiz.

Agar $\lambda < 0$ bo'lsa,

$$\varphi(x,\lambda) = \frac{sh\sqrt{|\lambda|}x}{\sqrt{|\lambda|}},$$

bo'ladi va u $(0,\infty)$ oraliqda ildizga ega bo'lmaydi, agar $\lambda = 0$ bo'lsa,

$$\varphi(x,\lambda) = x,$$

bo'ladi va u ham $(0,\infty)$ oraliqda ildizga ega bo'lmaydi. Agar $\lambda > 0$ bo'lsa,

$$\varphi(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

yechimning $(0,\infty)$ oraliqdagi ildizlari

$$x_n(\lambda) = \frac{\pi n}{\sqrt{\lambda}}, \quad n = 1,\ 2,\ \ldots,$$

bo'ladi. λ parametr ortgani sari bu ildizlar nolga tomon harakat qilishadi va $(0,\pi]$ oraliqdagi ildizlar soni ortib boradi.

Quyidagi

$$-y'' + q(x)y = \lambda y, \qquad 0 < x < \pi, \tag{1.15.3}$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \end{cases} \tag{1.15.4}$$

Shturm-Liuvill masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0,\pi]$ haqiqiy funksiya, α, β haqiqiy sonlar.

Teorema 1.15.2(*Ossillyatsiya teoremasi*).*(1.15.3)+(1.15.4) Shturm-Liuvill chegaraviy masalasi cheksizta* $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ *xos qiymatlarga ega. Bundan tashqari* λ_n *xos qiymatga mos keluvchi* $y_n(x)$ *xos funksiya* $(0,\pi)$ *oraliqda* n *ta ildizga ega bo'ladi.*

Isbot. $\varphi(x,\lambda)$ orqali (1.15.3) tenglamaning (1.15.2) boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaylik. $|q(x)| < m$, $x \in [0,\pi]$ bo'lsin. Taqqoslash teoremasida

$$a(x) = q(x) - \lambda, \quad \tilde{a}(x) = -m - \lambda,$$

deymiz. U holda ushbu

$$-y'' + (-m - \lambda)y = 0,$$

tenglamaning (1.15.2) boshlang'ich shartlarni qanoatlantiruvchi yechimi

$$\tilde{\varphi}(x,\lambda) = -\sin\alpha \cdot ch\sqrt{-m-\lambda}x + \cos\alpha \cdot \frac{sh\sqrt{-m-\lambda}x}{\sqrt{-m-\lambda}},$$

formula bilan beriladi. $\tilde{\varphi}(x,\lambda)$ yechimning aniqlanishiga ko'ra shunday $M > 0$ son mavjudki $\lambda < -M$ bo'lganda $\tilde{\varphi}(x,\lambda)$ yechim $(0,\pi]$ oraliqda ildizga ega bo'lmaydi. Bundan, taqqoslash teoremasiga muvofiq $\lambda < -M$ bo'lganda $\varphi(x,\lambda)$ yechimning ham ildizi yo'q ekanligi kelib chiqadi. Chunki $\tilde{\varphi}(x,\lambda)$ funksiyaning ildizlari soni $\varphi(x,\lambda)$ funksiyaning ildizlari sonidan kam bo'lmaydi.

Endi taqqoslash teoremasida $\tilde{a}(x) = q(x) - \lambda$, $a(x) = m - \lambda$ deb olamiz. U holda

$$-y'' + (m - \lambda)y = 0,$$

tenglamaning (1.15.2) boshlang'ich shartlarni qanoatlantiruvchi yechimi ushbu

$$\varphi(x,\lambda) = -\sin\alpha \cdot \cos\sqrt{\lambda - m}x + \cos\alpha \cdot \frac{\sin\sqrt{\lambda - m}x}{\sqrt{\lambda - m}},$$

formula bilan beriladi. λ parametr cheksiz oshgani sari bu yechimning $(0,\pi]$ oraliqdagi ildizlari soni ortib boraveradi. Taqqoslash teoremasiga asosan λ parametr cheksiz oshgani sari

$$-y'' + [q(x) - \lambda]y = 0,$$

tenglamaning (1.15.2) boshlang'ich shartlarni qanoatlantiruvchi $\tilde{\varphi}(x,\lambda)$ yechimining ham $(0,\pi]$ oraliqdagi ildizlar soni ortib boraveradi.

Ushbu $\tilde{\varphi}(x,\lambda)=0$ tenglamani ko'rib chiqamiz. Teorema 1.15.1 ga asosan bu tenglamaning ildizlari λ parametrga uzluksiz bog'liq bo'ladi. $\tilde{\varphi}(x,\lambda) = 0$ tenglamaning $(0,\infty)$ oraliqdagi ildizlarini $x_n(\lambda)$ orqali belgilaymiz. Ikkinchi tomondan λ parametr ortgani sari $\tilde{\varphi}(x,\lambda)$ yechimning ildizlari nol nuqta tomon harakat qiladi, ammo nol nuqta orqali chiqib ketmaydi (chunki ular musbat). $(0,\pi]$ oraliqqa yangi ildizlar π nuqta orqali kirib keladi. Boshqacha qilib aytganda λ parametrning shunday μ_n qiymati borki, bunda $x_n(\mu_n)=\pi$ bo'ladi.

Demak, μ_0, μ_1, μ_2, ... sonlar uchun $\tilde{\varphi}(\pi,\mu_n) = 0$ bo'ladi. $\tilde{\varphi}(x,\mu_n)$ funksiyaning $(0,\pi)$ oraliqdagi ildizlari

$$x_0(\mu_n),\ x_1(\mu_n),\ \ldots,\ x_{n-1}(\mu_n),$$

bo'ladi, ya'ni $\tilde{\varphi}(x,\mu_n)$ funksiya $(0,\pi)$ oraliqda n ta ildizga ega.

Agar (1.15.3)+(1.15.4) chegaraviy masalada $\sin\beta = 0$ bo'lsa, $\tilde{\varphi}(x,\mu_n)$ funksiya xos funksiya va μ_n xos qiymat bo'ladi. Bu xos funksiya $(0,\pi)$ oraliqda n ta ildizga ega bo'lishini yuqorida ko'rsatdik. Demak, $\sin\beta = 0$ bo'lgan holda teoremadagi tasdiqlar isbotlandi.

Endi $\sin\beta \neq 0$ bo'lsin. U holda

$$u(x) = \tilde{\varphi}(x,\bar{\lambda}_1), \quad v(x) = \tilde{\varphi}(x,\bar{\lambda}_2)$$

funksiyalarni kiritib olamiz. Bu yerda $\mu_n < \bar{\lambda}_1 < \bar{\lambda}_2 < \mu_{n+1}$.

Taqqoslash teoremasiga asosan $u(x)$ va $v(x)$ funksiyalarning har biri $(0,\pi]$ yarim oraliqda $n+1$ ta ildizga ega bo'lishi kelib chiqadi.

$\tilde{\varphi}(\pi,\lambda)$ funksiyaning $\mu_0,\mu_1,\mu_2,\ldots$ sonlardan boshqa ildizi bo'lmagani uchun $u(\pi)\neq 0$, $v(\pi)\neq 0$ bo'ladi.

$u(x)$ funksiyaning $(0,\pi]$ oraliqdagi eng katta ildizini x_n bilan belgilaymiz. $v(x)$ funksiya $[x_n,\pi]$ kesmada ildizga ega emasligini ko'rsatamiz. Taqqoslash teoremasiga ko'ra $v(x)$ funksiyaning $(n+1)$ - ildizi x_n dan kichik bo'ladi, ya'ni $v(x)$ funksiya $(0,x_n)$ oraliqda $n+1$ ta ildizga ega. Agar $v(x)$ funksiyaning $[x_n,\pi]$ kesmada ildizi bor bo'lsa, uning $(0,\pi]$ yarim oraliqdagi ildizlari soni $n+1$ dan oshadi. Ziddiyat, chunki $v(x)$ funksiya $(0,\pi]$ oraliqda $n+1$ ta ildizga ega.

Quyidagi tengsizlik bajarilishi ravshan:

$$\frac{d}{dx}\left\{u^2\left(\frac{u'}{u}-\frac{v'}{v}\right)\right\}=$$

$$=2uu'\left(\frac{u'}{u}-\frac{v'}{v}\right)+u^2\left(\frac{u''}{u}-\frac{v''}{v}\right)-u^2\left(\frac{u'^2}{u^2}-\frac{v'^2}{v^2}\right)=$$

$$=\frac{(u'v-uv')^2}{v^2}+u^2(\tilde{\lambda}_2-\tilde{\lambda}_1)\geq 0. \qquad (1.15.5)$$

Bu tengsizlikni $[x_n,\pi]$ kesmada integrallasak, quyidagi

$$u^2(\pi)\left\{\frac{u'(\pi)}{u(\pi)}-\frac{v'(\pi)}{v(\pi)}\right\}-u^2(x_n)\left\{\frac{u'(x_n)}{u(x_n)}-\frac{v'(x_n)}{v(x_n)}\right\}>0,$$

$$u^2(\pi)\left\{\frac{u'(\pi)}{u(\pi)}-\frac{v'(\pi)}{v(\pi)}\right\}>0,$$

$$\frac{u'(\pi)}{u(\pi)}>\frac{v'(\pi)}{v(\pi)},$$

tengsizlik kelib chiqadi.

Demak, agar $\mu_n<\tilde{\lambda}_1<\tilde{\lambda}_2<\mu_{n+1}$ bo'lsa, ushbu

$$\frac{\tilde{\varphi}'(\pi,\tilde{\lambda}_1)}{\tilde{\varphi}(\pi,\tilde{\lambda}_1)}>\frac{\tilde{\varphi}'(\pi,\tilde{\lambda}_2)}{\tilde{\varphi}(\pi,\tilde{\lambda}_2)},$$

tengsizlik o'rinli ekan, ya'ni $\dfrac{\tilde{\varphi}'(\pi,\lambda)}{\tilde{\varphi}(\pi,\lambda)}$ funksiya (μ_n,μ_{n+1}) oraliqda kamayuvchi funksiya ekan. Bunga ko'ra $\tilde{\varphi}(\pi,\mu_n)=0$ va

$\tilde{\varphi}(\pi, \mu_{n+1}) = 0$ tengliklarga asosan ushbu

$$\frac{\tilde{\varphi}'(\pi, \lambda)}{\tilde{\varphi}(\pi, \lambda)} \to -\infty, \qquad (\lambda \to \mu_{n+1}),$$

$$\frac{\tilde{\varphi}'(\pi, \lambda)}{\tilde{\varphi}(\pi, \lambda)} \to +\infty, \qquad (\lambda \to \mu_n),$$

munosabatlar o'rinli bo'ladi.

Demak, (μ_n, μ_{n+1}) oraliqda shunday yagona λ_n son topiladiki, bunda quyidagi

$$\frac{\tilde{\varphi}'(\pi, \lambda_n)}{\tilde{\varphi}(\pi, \lambda_n)} = -\operatorname{ctg}\beta, \tag{1.15.6}$$

tenglik bajariladi. (1.15.6) tenglik (1.15.4) chegaraviy shartlarning ikkinchisi bajarilishini ko'rsatadi, ya'ni $\tilde{\varphi}(x, \lambda_n)$ funksiya (1.15.3)+(1.15.4) masalaning xos funksiyasi va λ_n son xos qiymati bo'ladi. $(0, \pi)$ oraliqda $\tilde{\varphi}(x, \lambda_n)$ va $\tilde{\varphi}(x, \mu_n)$ funksiyalar ildizlarining soni bir xil bo'lgani uchun $\tilde{\varphi}(x, \lambda_n)$ xos funksiya $(0, \pi)$ oraliqda n ta ildizga ega.■

16-§. Bitta chegaraviy sharti bilan farq qiluvchi Shturm-Liuvill masalalari

$\varphi(x, \lambda)$ orqali ushbu

$$-y'' + q(x)y = \lambda y, \quad 0 < x < \pi, \tag{1.16.1}$$

tenglamaning quyidagi

$$\begin{cases} \varphi(0, \lambda) = 1, \\ \varphi(0, \lambda) = h, \end{cases} \tag{1.16.2}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy funksiya va h chekli haqiqiy son.

Teorema 1.16.1. *Ushbu*

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.16.3}$$

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + \tilde{H}y(\pi) = 0, \end{cases} \tag{1.16.4}$$

chegaraviy masalalarning xos qiymatlari mos ravishda $\{\lambda_n\}_{n=0}^{\infty}$ *va* $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ *bo'lsin. Agar* $H > \tilde{H}$ *bo'lsa, u holda* $\{\lambda_n\}_{n=0}^{\infty}$ *va* $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ *ketma-ketliklarning hadlari almashinib keladi, ya'ni quyidagi*

$$\tilde{\lambda}_0 < \lambda_0 < \tilde{\lambda}_1 < \lambda_1 < \tilde{\lambda}_2 < \lambda_2 < ... < \tilde{\lambda}_n < \lambda_n < \tilde{\lambda}_{n+1} < ..., \tag{1.16.5}$$

tengsizliklar bajariladi.

Isbot. Ossillyatsiya teoremasining isbotida ushbu $\dfrac{\varphi'(\pi,\lambda)}{\varphi(\pi,\lambda)}$ funksiya (μ_n, μ_{n+1}) oraliqlarning har birida kamayuvchi bo'lishi ko'rsatildi. Bu yerda $\mu_0, \mu_1, \mu_2, ...$ orqali $\varphi(\pi,\lambda) = 0$ tenglamaning ildizlari belgilangan. (1.16.3) va (1.16.4) masalalarning xos qiymatlari mos ravishda ushbu

$$\frac{\varphi'(\pi,\lambda)}{\varphi(\pi,\lambda)} = -H, \tag{1.16.6}$$

$$\frac{\varphi'(\pi,\lambda)}{\varphi(\pi,\lambda)} = -\tilde{H}, \tag{1.16.7}$$

tenglamalardan aniqlanadi. $H > \tilde{H}$ bo'lgani uchun $\mu_n < \tilde{\lambda}_n < \lambda_n < \mu_{n+1}$ bo'ladi.■

Teorema 1.16.2. *Ushbu*

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.16.8}$$

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - \tilde{h}y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.16.9}$$

chegaraviy masalalarning xos qiymatlari mos ravishda $\{\lambda_n\}_{n=0}^{\infty}$ *va* $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ *bo'lsin. Agar* $h > \tilde{h}$ *bo'lsa, u holda* $\{\lambda_n\}_{n=0}^{\infty}$ *va* $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$

ketma-ketliklarning hadlari almashinib keladi, ya'ni quyidagi

$$\tilde{\lambda}_0 < \lambda_0 < \tilde{\lambda}_1 < \lambda_1 < \tilde{\lambda}_2 < \lambda_2 < \ldots < \tilde{\lambda}_n < \lambda_n < \tilde{\lambda}_{n+1} < \ldots \quad , \tag{1.16.10}$$

tengsizliklar bajariladi.

Bu teorema xuddi oldingi teorema kabi isbot qilinadi.

$\tilde{\varphi}(x, \lambda)$ orqali (1.16.1) tenglamaning quyidagi

$$\begin{cases} \tilde{\varphi}(0, \lambda) = 1, \\ \tilde{\varphi}(0, \lambda) = \tilde{h}, \end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. Bu yerda $\tilde{h} \neq h$. Qulaylik uchun $\tilde{h} < h$ deymiz.

Teorema 1.16.3. *(1.16.8) va (1.16.9) chegaraviy masalalar uchun ushbu*

$$\frac{1}{\pi}(\tilde{h} - h) = \lim_{n\to\infty} n(\sqrt{\tilde{\lambda}_n} - \sqrt{\lambda_n}),$$

$$\alpha_n^2 = \frac{\tilde{h} - h}{\tilde{\lambda}_n - \lambda_n} \cdot \prod_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{\lambda_k - \lambda_n}{\tilde{\lambda}_k - \lambda_n}, \quad n = 0,\ 1,\ 2,\ \ldots,$$

formulalar o'rinli. Bu yerda α_n, $n = 0, 1, 2, \ldots$ sonlar (1.16.8) masalaning normallovchi o'zgarmaslarini bildiradi.

Isbot. Quyidagi integralni hisoblaymiz:

$$\alpha_n^2 = \int_0^{\pi} \varphi^2(x, \lambda_n)dx. \tag{1.16.11}$$

Buning uchun (1.16.1) tenglamaning ushbu

$$y(x, \lambda) = \tilde{\varphi}(x, \lambda) + M(\lambda)\varphi(x, \lambda), \tag{1.16.12}$$

ko'rinishdagi

$$y'(\pi, \lambda) + Hy(\pi, \lambda) = 0, \tag{1.16.13}$$

chegaraviy shartni qanoatlantiruvchi yechimini kiritib olamiz.

(1.16.12) ifodani (1.16.13) tenglikka qo'yib $M(\lambda)$ funksiyani topamiz:

$$M(\lambda) = -\frac{\tilde{\varphi}'(\pi, \lambda) + H\tilde{\varphi}(\pi, \lambda)}{\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda)}. \qquad (1.16.14)$$

$M(\lambda)$ funksiyaning qutb maxsus nuqtalari va nollari mos ravishda (1.16.8) va (1.16.9) chegaraviy masalalarning xos qiymatlaridan iborat bo'lishi ravshan.

Quyidagi hisoblashlarni bajaramiz:

$$(\lambda - \lambda_n)\int_0^\pi y(x, \lambda)\varphi(x, \lambda_n)dx =$$

$$= \int_0^\pi \{\lambda y(x, \lambda) \cdot \varphi(x, \lambda_n) - y(x, \lambda) \cdot \lambda_n \varphi(x, \lambda_n)\}dx =$$

$$= \int_0^\pi \{[-y''(x, \lambda) + q(x)y(x, \lambda)] \cdot \varphi(x, \lambda_n) - y(x, \lambda)\left[-\varphi''(x, \lambda_n) + \right.$$

$$\left. + q(x)\varphi(x, \lambda_n)\right]\} dx = \int_0^\pi \{\varphi''(x, \lambda_n)y(x, \lambda) - y''(x, \lambda)\varphi(x, \lambda_n)\}dx =$$

$$= \int_0^\pi \{\varphi'(x, \lambda_n)y(x, \lambda) - y'(x, \lambda)\varphi(x, \lambda_n)\}'dx =$$

$$= -[\varphi'(0, \lambda_n)y(0, \lambda) - y'(0, \lambda)\varphi(0, \lambda_n)] =$$

$$= -h[1 + M(\lambda)] + [\tilde{h} + M(\lambda)h] = \tilde{h} - h,$$

ya'ni ushbu

$$(\lambda - \lambda_n)\int_0^\pi y(x, \lambda)\varphi(x, \lambda_n)dx = \tilde{h} - h,$$

tengik o'rinli ekan. Bundan

$$(\lambda-\lambda_n)\int_0^\pi \tilde{\varphi}(x,\lambda)\varphi(x,\lambda_n)dx+$$

$$+(\lambda-\lambda_n)M(\lambda)\int_0^\pi \varphi(x,\lambda)\varphi(x,\lambda_n)dx=\tilde{h}-h, \qquad (1.16.15)$$

kelib chiqadi. (1.16.15) tenglikda $\lambda\to\lambda_n$ limitga o'tsak, quyidagi

$$-\frac{\tilde{\Delta}(\lambda_n)}{\Delta'(\lambda_n)}\int_0^\pi \varphi^2(x,\lambda_n)dx=\tilde{h}-h, \qquad (1.16.16)$$

formula hosil bo'ladi. Bunga ko'ra

$$\alpha_n^2=-(\tilde{h}-h)\frac{\Delta'(\lambda_n)}{\tilde{\Delta}(\lambda_n)}, \qquad (1.16.17)$$

bo'ladi. Oldingi paragraflarda biz ushbu

$$\sqrt{\lambda_n}=n+\frac{1}{n}\left\{\frac{h+H}{\pi}+\frac{1}{2\pi}\int_0^\pi q(t)dt\right\}+\frac{\gamma_n}{n},\ \{\gamma_n\}\in l_2, \qquad (1.16.18)$$

$$\sqrt{\tilde{\lambda}_n}=n+\frac{1}{n}\left\{\frac{\tilde{h}+H}{\pi}+\frac{1}{2\pi}\int_0^\pi q(t)dt\right\}+\frac{\tilde{\gamma}_n}{n},\ \{\tilde{\gamma}_n\}\in l_2 \qquad (1.16.19)$$

asimptotikalarni va ushbu

$$\Delta\lambda)=\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda)=\pi(\lambda_0-\lambda)\prod_{k=1}^\infty\frac{\lambda_k-\lambda}{k^2}, \qquad (1.16.20)$$

$$\tilde{\Delta}(\lambda)=\tilde{\varphi}'(\pi,\lambda)+H\tilde{\varphi}(\pi,\lambda)=\pi(\tilde{\lambda}_0-\lambda)\prod_{k=1}^\infty\frac{\tilde{\lambda}_k-\lambda}{k^2}, \qquad (1.16.21)$$

yoyilmalarni keltirib chiqargan edik. (1.16.18) va (1.16.19) formulalardan

$$\frac{1}{\pi}(\tilde{h}-h)=\lim_{n\to\infty}n(\sqrt{\tilde{\lambda}_n}-\sqrt{\lambda_n}), \qquad (1.16.22)$$

kelib chiqadi. (1.16.20) formuladan esa

$$\Delta'(\lambda_0) = -\pi\prod_{k=1}^{\infty}\frac{\lambda_k - \lambda_0}{k^2}, \tag{1.16.23}$$

va

$$\Delta'(\lambda_n) = -\pi\frac{\lambda_0 - \lambda_n}{n^2}\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{\lambda_k - \lambda_n}{k^2}, \quad (n \geq 1), \tag{1.16.24}$$

hosil bo'ladi. (1.16.23), (1.16.24) va (1.16.21) ifodalarni (1.16.17) tenglikka qo'yamiz:

$$\alpha_0^2 = (\tilde{h} - h)\frac{1}{\tilde{\lambda}_0 - \lambda_0}\prod_{k=1}^{\infty}\frac{\lambda_k - \lambda_0}{\tilde{\lambda}_k - \lambda_0};$$

$$\alpha_n^2 = (\tilde{h} - h)\frac{\lambda_0 - \lambda_n}{\tilde{\lambda}_0 - \lambda_n}\cdot\frac{1}{\tilde{\lambda}_n - \lambda_n}\cdot\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{\lambda_k - \lambda_n}{\tilde{\lambda}_k - \lambda_n}, \quad (n \geq 1).$$

Bu ikkala tenglikni birlashtirib yozish mumkin:

$$\alpha_n^2 = \frac{\tilde{h} - h}{\tilde{\lambda}_n - \lambda_n}\cdot\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\lambda_k - \lambda_n}{\tilde{\lambda}_k - \lambda_n}, \quad n = 0,\ 1,\ 2,\ \ldots \ \blacksquare \tag{1.16.25}$$

17-§. Krum almashtirishi

Mazkur paragrafda Shturm-Liuvill chegaraviy masalasi xos qiymatlari va xos funksiyalarining ayrim xossalari bilan tanishamiz.

Ushbu

$$-y'' + q(x)y = \lambda y, \quad 0 < x < 1, \tag{1.17.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(1) - Hy(1) = 0, \end{cases} \tag{1.17.2}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda h, H haqiqiy sonlar, λ kompleks parametr. (1.17.1)+(1.17.2) masalaning xos qiymatlarini

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots,$$

orqali, ularga mos keluvchi xos funksiyalarni

$$\varphi_0(x), \varphi_1(x), \varphi_2(x), ..., \varphi_n(x), ...,$$

orqali belgilaymiz. $q(x) \in C^\infty(0, 1)$ bo'lsin, u holda $\varphi_k(x)$, xos funksiyalar ham cheksiz marta differensiallanuvchi bo'ladi.

Quyidagi funksiyalarni kiritib olamiz.

$$W_n(x) = W\{\varphi_0(x), \varphi_1(x), \varphi_2(x), ..., \varphi_{n-1}(x)\},$$

$$W_{n,k}(x) = W\{\varphi_0(x), \varphi_1(x), \varphi_2(x), ..., \varphi_{n-1}(x), \varphi_k(x)\},$$

$$\varphi_{n,k}(x) = \frac{W_{n,k}(x)}{W_n(x)}, \quad n = 1, 2, 3, ...$$

Teorema 1.17.1. (*Krum M.M.*)*Agar $k \geq n$ bo'lsa, $\varphi_{n,k}(x)$ quyidagi Shturm-Liuvill masalasining λ_k xos qiymatiga mos keluvchi xos funksiyasi bo'ladi:*

$$y'' + [\lambda - q_n(x)]y = 0, \quad 0 < x < 1, \tag{1.17.3}$$

$$\begin{cases} y(0) = 0, \\ y(1) = 0. \end{cases} \tag{1.17.4}$$

Bu yerda

$$q_n(x) = q(x) - 2(\ln W_n(x))''. \tag{1.17.5}$$

Agar $n = 1$ bo'lsa, (1.17.3)+(1.17.4) m*asala regulyar bo'ladi. Agar $n > 1$ bo'lsa,*

$$q_n(x) \sim \begin{cases} \dfrac{n(n-1)}{x^2}, & x \to 0, \\ \dfrac{n(n-1)}{(1-x)^2}, & x \to 1, \end{cases}$$

$q_n(x) \in C(0, 1)$ *bo'ladi. Agar $k < n$ bo'lsa, $\varphi_{n,k}(x) \equiv 0$ bo'ladi. Agar $k > n$ bo'lsa $\varphi_{n,k}(x)$ funksiya* $(0, 1)$ *oraliqda $k - n$ ta nolga ega bo'ladi. Bundan tashqari $\varphi_{n,k}(x)$, $k \geq n$ funksiyalar sistemasi $L^2(0, 1)$ fazoda to'la bo'ladi.*

Izoh.1.17.1. Ayrim adabiyotlarda $n = 1$ bo'lganda Krum almashtirishi Darbu almashtirishi deb yuritiladi.

Isbot. $n = 1$ bo'lgan hol. Bu holda

$$W_1(x) = \varphi_0(x),\; W_{1,k}(x) = W\{\varphi_0(x), \varphi_k(x)\},$$

$$\varphi_{1,k}(x) = \frac{W_{1,k}(x)}{W_1(x)} = \frac{\begin{vmatrix} \varphi_0(x) & \varphi_k(x) \\ \varphi_0'(x) & \varphi_k'(x) \end{vmatrix}}{\varphi_0(x)} =$$

$$= \frac{\varphi_0(x)\varphi_k'(x) - \varphi_0'(x)\varphi_k(x)}{\varphi_0(x)} =$$

$$= \varphi_k'(x) - \frac{\varphi_0'(x)}{\varphi_0(x)} \cdot \varphi_k(x) = \varphi_k'(x) - v(x)\varphi_k(x) \qquad (1.17.6)$$

bo'ladi. Bu yerda

$$v(x) = \frac{\varphi_0'(x)}{\varphi_0(x)}.$$

Ravshanki,

$$v'(x) + v^2(x) = \frac{\varphi''{}_0(x)\varphi_0(x) - \varphi'^2_0(x)}{\varphi_0^2(x)} + \frac{\varphi'^2_0(x)}{\varphi_0^2(x)} = \frac{\varphi''{}_0(x)}{\varphi_0(x)}, \qquad (1.17.7)$$

$$\varphi_0''(x) + [\lambda_0 - q(x)]\varphi_0(x) = 0,$$

$$\varphi_0''(x) = [q(x) - \lambda_0]\varphi_0(x),$$

$$\frac{\varphi_0''(x)}{\varphi_0(x)} = q(x) - \lambda_0. \qquad (1.17.8)$$

(1.17.7) va (1.17.8) tengliklarga ko'ra

$$v'(x) + v^2(x) = q(x) - \lambda_0, \qquad (1.17.9)$$

o'rinli bo'ladi.

Quyidagi hosilani hisoblaymiz:

$$(\varphi_0(x)\varphi_{1,k}(x))' = \varphi_0'(x)\varphi_{1,k}(x) + \varphi_0(x)\varphi_{1,k}'(x) =$$

$$= \varphi_0'(x)[\varphi_k'(x) - v(x)\varphi_k(x)] +$$

$$+ \varphi_0(x)[\varphi_k''(x) - v'(x)\varphi_k(x) - v(x)\varphi_k'(x)] =$$

$$= \varphi_0'(x)\varphi_k'(x) - v(x)\varphi_0'(x)\varphi_k(x) +$$

$$+\varphi_0(x)\varphi_k''(x)-v'(x)\varphi_0(x)\varphi_k(x)-v(x)\varphi_0(x)\varphi_k'(x)=\varphi_0(x)\varphi_k''(x)-$$

$$-\{v'(x)\varphi_0(x)\varphi_k(x)+v(x)\varphi_0'(x)\varphi_k(x)+v(x)\varphi_0(x)\varphi_k'(x)-$$

$$-\varphi_0'(x)\varphi_k'(x)\}=\varphi_0(x)\varphi_k''(x)-J, \qquad (1.17.10)$$

$$J=[q(x)-\lambda_0-v^2(x)]\varphi_0(x)\varphi_k(x)+v(x)\varphi_0'(x)\varphi_k(x)+$$

$$+v(x)\varphi_0(x)\varphi_k'(x)-\varphi'_0(x)\varphi'_k(x)=$$

$$=(q(x)-\lambda_0)\varphi_0(x)\varphi_k(x)-\frac{\varphi'^2_0(x)}{\varphi_0^2(x)}\varphi_0(x)\varphi_k(x)+$$

$$+\frac{\varphi'^2_0(x)}{\varphi_0(x)}\varphi_k(x)=(q(x)-\lambda_0)\varphi_0(x)\varphi_k(x). \qquad (1.17.11)$$

(1.17.10) va (1.17.11) tengliklarga ko‘ra

$$(\varphi_0(x)\varphi_{1,k}(x))'=\varphi_0(x)[q(x)-\lambda_k]\varphi_k(x)-\varphi_0(x)[q(x)-\lambda_0]\varphi_k(x)=$$

$$=(\lambda_0-\lambda_k)\varphi_0(x)\varphi_k(x),$$

o‘rinli bo‘ladi. Demak,

$$(\varphi_0(x)\varphi_{1,k}(x))'=(\lambda_0-\lambda_k)\varphi_0(x)\varphi_k(x), \qquad (1.17.12)$$

o‘rinli. Bundan tashqari

$$\varphi_{1,k}(0)=\left.\left(\varphi_k'(x)-\frac{\varphi_0'(x)}{\varphi_0(x)}\cdot\varphi_k(x)\right)\right|_{x=0}=$$

$$=\varphi_k'(0)-\frac{\varphi_0'(0)\,\varphi_k(0)}{\varphi_0(0)}=h\varphi_k(0)-$$

$$-\frac{h\varphi_0(0)\,\varphi_k(0)}{\varphi_0(0)}=h\varphi_k(0)-h\varphi_k(0)=0,$$

$$\varphi_{1,k}(1)=\left.\left(\varphi_k'(x)-\frac{\varphi_0'(x)}{\varphi_0(x)}\cdot\varphi_k(x)\right)\right|_{x=1}=$$

$$=\varphi_k'(1)-\frac{\varphi_0'(1)\,\varphi_k(1)}{\varphi_0(1)}=H\varphi_k(1)-$$

$$-\frac{H\varphi_0(1)\,\varphi_k(1)}{\varphi_0(1)}=H\varphi_k(1)-H\varphi_k(1)=0,$$

ya'ni

$$\begin{cases} \varphi_{1,k}(0) = 0, \\ \varphi_{1,k}(1) = 0. \end{cases} \qquad (1.17.13)$$

(1.17.12) tenglikni $[0, x]$ oraliqda integrallaymiz:

$$\varphi_0(x)\,\varphi_{1,k}(x) - \varphi_0(0)\,\varphi_{1,k}(0) = (\lambda_0 - \lambda_k)\int_0^x \varphi_0(t)\,\varphi_k(t)\,dt.$$

Bu yerda (1.17.13) ni hisobga olsak,

$$\varphi_0(x)\,\varphi_{1,k}(x) = (\lambda_0 - \lambda_k)\int_0^x \varphi_0(t)\,\varphi_k(t)\,dt, \qquad (1.17.14)$$

kelib chiqadi. (1.17.12) tenglikni $[x, 1]$ oraliqda integrallaymiz:

$$\varphi_0(1)\,\varphi_{1,k}(1) - \varphi_0(x)\,\varphi_{1,k}(x) = (\lambda_0 - \lambda_k)\int_x^1 \varphi_0(t)\,\varphi_k(t)\,dt.$$

Bu yerda (1.17.13) ni hisobga olsak,

$$\varphi_0(x)\,\varphi_{1,k}(x) = -(\lambda_0 - \lambda_k)\int_x^1 \varphi_0(t)\,\varphi_k(t)\,dt, \qquad (1.17.15)$$

kelib chiqadi.

Quyidagi hisoblashlarni bajaramiz:

$$\varphi'_{1,k}(x) + v(x)\,\varphi_{1,k}(x) = (\varphi'_k(x) - v(x)\,\varphi_k(x))' + v(x)\,\varphi_{1,k}(x) =$$

$$= \varphi''_k(x) - v'(x)\,\varphi_k(x) - v(x)\,\varphi'_k(x) + v(x)\,\varphi'_k(x) - v^2(x)\,\varphi_k(x) =$$

$$= \varphi''_k(x) - \left(v'(x) + v^2(x)\right)\varphi_k(x) =$$

$$= [q(x) - \lambda_k]\varphi_k(x) - [q(x) - \lambda_0]\varphi_k(x) = (\lambda_0 - \lambda_k)\,\varphi_k(x),$$

ya'ni

$$\varphi'_{1,k}(x) = (\lambda_0 - \lambda_k)\,\varphi_k(x) - v(x)\,\varphi_{1,k}(x). \qquad (1.17.16)$$

(1.17.16) dan hosila olamiz:

$$\varphi''_{1,k}(x) = (\lambda_0 - \lambda_k)\varphi'_k(x) - v'(x)\varphi_{1,k}(x) - v(x)\varphi'_{1,k}(x) =$$

$$= (\lambda_0 - \lambda_k)\varphi'_k(x) - v'(x)\varphi_{1,k}(x) -$$

$$-v(x)\left((\lambda_0 - \lambda_k)\varphi_k(x) - v(x)\varphi_{1,k}(x)\right) =$$

$$= (\lambda_0 - \lambda_k)\varphi'_k(x) - v'(x)\varphi_{1,k}(x) -$$

$$-(\lambda_0 - \lambda_k)v(x)\varphi_k(x) + v^2(x)\varphi_{1,k}(x) =$$

$$= (\lambda_0 - \lambda_k)\cdot(\varphi'_k(x) - v(x)\varphi_k(x)) + \left(v^2(x) - v'(x)\right)\cdot\varphi_{1,k}(x) =$$

$$= (\lambda_0 - \lambda_k)\varphi_{1,k}(x) + \left(v^2(x) - v'(x)\right)\varphi_{1,k}(x) =$$

$$= \left[\lambda_0 - \lambda_k + v^2(x) - v'(x)\right]\cdot\varphi_{1,k}(x) =$$

$$= \left[\left(\lambda_0 - v'(x) + v^2(x)\right) - \lambda_k\right]\cdot\varphi_{1,k}(x) = \left[q_1(x) - \lambda_k\right]\cdot\varphi_{1,k}(x). \tag{1.17.17}$$

Bu yerda

$$q_1(x) = \lambda_0 - v'(x) + v^2(x). \tag{1.17.18}$$

(1.17.9) tenglikka ko'ra (1.17.18) ayniyatni quyidagicha yozamiz:

$$q_1(x) = q(x) - 2v'(x). \tag{1.17.19}$$

Agar $v(x) = (\ln\varphi_0(x))' = (\ln W_1(x))'$ tenglikni hisobga olsak, (1.17.19) tenglik ushbu

$$q_1(x) = q(x) - 2\left(\ln W_1(x)\right)'', \tag{1.17.20}$$

ko'rinishni oladi. Agar (1.17.6) tenglikni hisobga olsak,

$$\frac{\varphi_{1,k}(x)}{\varphi_0(x)} = \frac{\varphi'_k(x) - v(x)\varphi_k(x)}{\varphi_0(x)} = \frac{\varphi'_k(x) - \frac{\varphi'_0(x)}{\varphi_0(x)}\varphi_k(x)}{\varphi_0(x)} =$$

$$= \frac{\varphi'_k(x)\varphi_0(x) - \varphi_k(x)\varphi'_0(x)}{\varphi_0^2(x)} = \left(\frac{\varphi_k(x)}{\varphi_0(x)}\right)', \tag{1.17.21}$$

kelib chiqadi.

Ma'lumki $\varphi_k(x)$ xos funksiya (0,1) oraliqda k ta ildizga ega. Agar $f(x) = \dfrac{\varphi_k(x)}{\varphi_0(x)}$ funksiyani tuzib olsak, u (0,1) oraliqda k

ta ildizga ega. Roll teoremasiga ko'ra, har bir (x_m, x_{m+1}), $m = 1, 2, ..., k-1$ oraliqda $f'(x)$ funksiyaning kamida bitta ildizi bor. Demak, $\varphi_{1,k}(x)$ funksiya (0,1) oraliqda kamida $k-1$ ta ildizga ega. Agar $\varphi_{1,k}(x)$ funksiyaning (0,1) oraliqdagi ildizlar soni $k-1$ tadan ko'p bo'lsa, masalan k ta bo'lsa, (1.17.13) ga asosan $[0,1]$ kesmada $\varphi_{1,k}(x)$ funksiyaning ildizlar soni $k+2$ ta bo'ladi. (1.17.12) ga asosan Roll teoremasiga muvofiq $\varphi_k(x)$ funksiyaning (0,1) oraliqda kamida $k+1$ ta ildizi bo'ladi. Ziddiyat, chunki $\varphi_k(x)$ funksiyaning (0,1) oraliqda aniq k ta ildizi bor. Demak, $\varphi_{1,k}(x)$ funksiyaning (0,1) oraliqda aniq $k-1$ ta ildizi bor ekan. Bu fikrdan $\varphi_{1,k}(x)$, $k = 1, 2, ...,$ xos funksiyalar (1.17.3)+(1.17.4) masalaning barcha xos funksiyalaridan iborat bo'lishi kelib chiqadi.

Agar $\lambda \neq \lambda_0$ bo'lib, $y(x)$ funksiya (1.17.1) tenglamaning ixtiyoriy yechimi bo'lsa, ushbu

$$y_1(x) = \frac{W\{\varphi_0(x), y(x)\}}{W_1(x)} = \frac{\varphi_0(x)y'(x) - \varphi_0'(x)y(x)}{\varphi_0(x)} =$$

$$= y'(x) - \frac{\varphi_0'(x)}{\varphi_0(x)} \cdot y(x),$$

funksiya (1.17.3) tenglamaning yechimi bo'ladi.

Agar $\lambda = \lambda_0$ bo'lsa, $W\{\varphi_0(x), y(x)\} = \text{const}$ bo'ladi, Bu yerda $y(x)$ funksiya (1.17.1) tenglamaning ixtiyoriy yechimi. Bundan, (1.17.3) tenglamaning bitta yechimi $\dfrac{1}{\varphi_0(x)}$ bo'lishi kelib chiqadi. Quyidagi funksiyalar (1.17.3) tenglamaning $\lambda = \lambda_0$ bo'lgandagi chiziqli erkli yechimlari bo'ladi:

$$\frac{1}{\varphi_0(x)} \int_0^x \varphi_0^2(t)dt, \quad \frac{1}{\varphi_0(x)} \int_x^1 \varphi_0^2(t)dt.$$

(1.17.3)+(1.17.4) masalaning $\varphi_{1,k}(x)$, $k = 1, 2, ...,$ xos funksiyalaridan boshqa xos funksiyasi yo'q ekanligini, (1.17.3) tenglamaning umumiy yechimi haqidagi fikrlardan foydalanib tekshirib ko'rish mumkin.

Endi teoremadagi fikrni $n > 1$ holda isbotlash maqsadida $q_{n-1}(x)$ uchun isbot qilindi deb olib, bu fikr $q_n(x)$ uchun ham o'rinli ekanligini ko'rsatamiz.

$W_{n,k}(x)$ determinant uchun, ushbu

$$W_{n,k}(x)W_{n-1}(x) = W_n(x)W'_{n-1,k}(x) - W_{n-1,k}(x)W'_n(x), \tag{1.17.22}$$

ayniyat o'rinli bo'ladi. Bunga ko'ra,

$$\varphi_{n,k}(x) = \frac{W_{n,k}(x)}{W_n(x)} = \frac{W'_{n-1,k}(x)}{W_{n-1}(x)} - \frac{W_{n-1,k}(x)}{W_{n-1}(x)}\frac{W'_n(x)}{W_n(x)} =$$

$$= \frac{1}{W_{n-1}(x)}(\varphi_{n-1,k}(x)W_{n-1})' - \varphi_{n-1,k}(x)\frac{W'_n(x)}{W_n(x)} =$$

$$= \frac{1}{W_{n-1}(x)}(\varphi'_{n-1,k}(x)W_{n-1}\varphi_{n-1,k}(x)W'_{n-1}) - \varphi_{n-1,k}(x)\frac{W'_n(x)}{W_n(x)} =$$

$$= \varphi'_{n-1,k}(x) - \varphi_{n-1,k}(x)\left[\frac{W'_n(x)}{W_n(x)} - \frac{W'_{n-1}(x)}{W_{n-1}(x)}\right] =$$

$$= \varphi'_{n-1,k}(x) - v_{n-1}(x)\varphi_{n-1,k}(x). \tag{1.17.23}$$

Bu yerda

$$v_{n-1}(x) = \frac{W'_n(x)}{W_n(x)} - \frac{W'_{n-1}(x)}{W_{n-1}(x)}. \tag{1.17.24}$$

(1.17.23) tenglikdan

$$\varphi_{n,k}(x) = \frac{1}{\varphi_{n-1,n-1}(x)}\left[\varphi'_{n-1,k}(x)\varphi_{n-1,n-1}(x) - \right.$$

$$\left. - \varphi_{n-1,k}(x)v_{n-1}(x)\varphi_{n-1,n-1}(x)\right], \tag{1.17.25}$$

kelib chiqadi. Quyidagi ifodani

$$\varphi'_{n-1,n-1}(x) = \left(\frac{W_n(x)}{W_{n-1}(x)}\right)' = \frac{W'_n(x)W_{n-1}(x) - W'_{n-1}(x)W_n(x)}{W^2_{n-1}(x)} =$$

$$= \frac{W'_n(x)}{W_{n-1}(x)} - \frac{W'_{n-1}(x)W_n(x)}{W^2_{n-1}(x)} =$$

$$= \frac{W_n(x)}{W_{n-1}(x)}\left[\frac{W_n'(x)}{W_n(x)} - \frac{W_{n-1}'(x)}{W_{n-1}(x)}\right] = \varphi_{n-1,n-1}(x)v_{n-1}(x).$$

(1.17.25) tenglikka qo'yamiz:

$$\varphi_{n,k}(x) = \frac{1}{\varphi_{n-1,n-1}(x)}W\{\varphi_{n-1,n-1}(x),\ \varphi_{n-1,k}(x)\}. \qquad (1.17.26)$$

Ushbu

$$v_n(x) = \frac{\varphi_{n,n}'(x)}{\varphi_{n,n}(x)}, \qquad (1.17.27)$$

belgilashni kiritamiz. U holda

$$v_n'(x) + v_n^2(x) = \left(\frac{\varphi_{nn}'(x)}{\varphi_{nn}(x)}\right)' + \left(\frac{\varphi_{nn}'(x)}{\varphi_{nn}(x)}\right)^2 =$$

$$= \frac{\varphi_{nn}''(x)\varphi_{nn}(x) - \varphi_{nn}'^2(x)}{\varphi_{nn}^2(x)} + \left(\frac{\varphi_{nn}'(x)}{\varphi_{nn}(x)}\right)^2 = \frac{\varphi_{nn}''(x)}{\varphi_{nn}(x)}, \qquad (1.17.28)$$

bo'ladi. (1.17.26) formulaga ko'ra

$$(\varphi_{n-1,n-1}(x)\varphi_{n,k}(x))' = W'\{\varphi_{n-1,n-1}(x), \varphi_{n-1,k}(x)\} =$$

$$= \begin{vmatrix} \varphi_{n-1,n-1}(x) & \varphi_{n-1,k}(x) \\ [q_{n-1}(x) - \lambda_{n-1}]\varphi_{n-1,n-1}(x) & [q_{n-1}(x) - \lambda_k]\varphi_{n-1,k}(x) \end{vmatrix} =$$

$$= (\lambda_{n-1} - \lambda_k)\varphi_{n-1,n-1}(x)\varphi_{n-1,k}(x), \qquad (1.17.29)$$

$$\varphi_{n,k}'(x) = -\frac{\varphi_{n-1,n-1}'(x)}{\varphi_{n-1,n-1}^2(x)}W\{\varphi_{n-1,n-1}(x), \varphi_{n-1,k}(x)\} +$$

$$+(\lambda_{n-1} - \lambda_k)\varphi_{n-1,k}(x),$$

$$\varphi''_{n,k}(x) = -\frac{\varphi''_{n-1,n-1}(x)\varphi_{n-1,n-1}(x) - 2\varphi'^2_{n-1,n-1}(x)}{\varphi_{n-1,n-1}^3(x)} \times$$

$$\times W\{\varphi_{n-1,n-1}(x), \varphi_{n-1,k}(x)\} -$$

$$-\frac{\varphi'_{n-1,n-1}(x)}{\varphi_{n-1,n-1}(x)}(\lambda_{n-1} - \lambda_k)\varphi_{n-1,k}(x) + (\lambda_{n-1} - \lambda_k)\varphi'_{n-1,k}(x) =$$

$$= \frac{2\varphi'^2_{n-1,n-1}(x) - [q_{n-1}(x) - \lambda_{n-1}]\varphi_{n-1,n-1}^2(x)}{\varphi_{n-1,n-1}^2(x)} \times$$

$$\times\left(\frac{\varphi_{n-1,n-1}(x)\varphi'_{n-1,k}(x)-\varphi'_{n-1,n-1}(x)\varphi_{n-1,k}(x)}{\varphi_{n-1,n-1}(x)}\right)-$$

$$-(\lambda_{n-1}-\lambda_k)\frac{\varphi'_{n-1,n-1}(x)\varphi_{n-1,k}(x)-\varphi_{n-1,n-1}(x)\varphi'_{n-1,k}(x)}{\varphi_{n-1,n-1}(x)}=$$

$$=\varphi_{n,k}(x)\left\{2\left(\frac{\varphi'_{n-1,n-1}(x)}{\varphi_{n-1,n-1}(x)}\right)^2-q_{n-1}(x)+2\lambda_{n-1}-\lambda_k\right\},\tag{1.17.30}$$

bo'ladi. (1.17.24) formulaga ko'ra

$$v'_{n-1}(x)=\left(\frac{W'_n(x)}{W_n(x)}-\frac{W'_{n-1}(x)}{W_{n-1}(x)}\right)'=(\ln W_n(x)-\ln W_{n-1}(x))''=$$

$$=\left(\ln\frac{W_n(x)}{W_{n-1}(x)}\right)''=(\ln\varphi_{n-1,n-1}(x))''=\left(\frac{\varphi'_{n-1,n-1}(x)}{\varphi_{n-1,n-1}(x)}\right)'=$$

$$=\frac{\varphi''_{n-1,n-1}(x)\varphi_{n-1,n-1}(x)-\varphi'^2_{n-1,n-1}(x)}{\varphi^2_{n-1,n-1}(x)}=[q_{n-1}(x)-\lambda_{n-1}]-$$

$$\left(\frac{\varphi'_{n-1,n-1}(x)}{\varphi_{n-1,n-1}(x)}\right)^2=q_{n-1}(x)-\lambda_{n-1}-v^2_{n-1}(x).\tag{1.17.31}$$

(1.17.31) tenglikni (1.17.30) ayniyatga qo'yamiz:

$$\varphi''_{n,k}(x)=\varphi_{n,k}(x)\cdot\{2[q_{n-1}-\lambda_{n-1}-v'_{n-1}]-q_{n-1}+2\lambda_{n-1}-\lambda_k\}=$$

$$=\varphi_{n,k}(x)\cdot\{q_{n-1}(x)-2v'_{n-1}(x)-\lambda_k\}=\varphi_{n,k}(x)\cdot[q_n(x)-\lambda_k].\tag{1.17.32}$$

Bu yerda

$$q_n(x)=q_{n-1}(x)-2v'_{n-1}(x).\tag{1.17.33}$$

(1.17.5) tenglikka ko'ra

$q_{n-1}(x)=q(x)-2(\ln W_{n-1}(x))''$, bo'ladi. Bunga asosan (1.17.33) ayniyatdan

$$q_n(x)=q_{n-1}(x)-2(\ln W_n(x)-\ln W_{n-1}(x))''=$$

$$=q(x)-2(\ln W_{n-1}(x))''-2(\ln W_n(x))''+2(\ln W_{n-1}(x))'',$$

ya'ni

$$q_n(x) = q(x) - 2(\ln W_n(x))'', \tag{1.17.34}$$

kelib chiqadi. (1.17.32) tenglikda $k = n$ deb, (1.17.28) ayniyatga qo'ysak,

$$v_n'(x) + v_n^2(x) = q_n(x) - \lambda_n, \tag{1.17.35}$$

kelib chiqadi. $\varphi_{n,k}(x)$ funksiyalar uchun

$$\varphi_{n,k}(0) = 0, \ \ \varphi_{n,k}(1) = 0$$

chegaraviy shartlar bajarilishini ko'rsatamiz. Buning uchun quyidagi asimptotikalarni induksiya usulida isbot qilamiz:

$$\varphi_{n,k}(x) = C_{n,k}\prod_{i=0}^{n-1}(\lambda_i - \lambda_k)x^n[1 + \underline{\underline{O}}(x^2)], \ x \to 0, \ \ (C_{n,k} \neq 0), \tag{1.17.36}$$

$$\varphi_{n,k}'(x) = n \cdot \frac{\varphi_{n,k}(x)}{x} \cdot [1 + \underline{\underline{O}}(x^2)], \ x \to 0, \tag{1.17.37}$$

$$v_n(x) = n \cdot \frac{1}{x} \cdot [1 + \underline{\underline{O}}(x^2)], \ x \to 0. \tag{1.17.38}$$

$n = 1$ bo'lsin. U holda $\varphi_{1,k}(0) = 0$ va

$$\varphi_{1,k}(x) = \frac{\lambda_0 - \lambda_k}{\varphi_0(x)}\int_0^x \varphi_0(t)\varphi_k(t)dt,$$

bo'ladi. Bu holda $\varphi_0(x) \neq 0, \ \ x \in [0,1]$. Bularga ko'ra

$$\lim_{x\to 0}\frac{\varphi_{1,k}(x)}{x} = \frac{\lambda_0 - \lambda_k}{\varphi_0(0)} \cdot \lim_{x\to 0}\{\frac{1}{x}\int_0^x \varphi_0(t)\varphi_k(t)dt\} = (\lambda_0 - \lambda_k)\varphi_k(0),$$

ya'ni

$$\varphi_{1,k}(x) \sim (\lambda_0 - \lambda_k)\varphi_k(0)x, \ x \to 0, \ \varphi_k(0) \neq 0. \tag{1.17.39}$$

(1.17.12) formulaga ko'ra

$$\varphi_0'(x)\varphi_{1,k}(x) + \varphi_0(x)\varphi_{1,k}'(x) = (\lambda_0 - \lambda_k)\varphi_0(x)\varphi_k(x),$$

$$\varphi_{1,k}'(x) = (\lambda_0 - \lambda_k)\varphi_k(x) - \frac{\varphi_0'(x)}{\varphi_0(x)}\varphi_{1,k}(x) \sim$$
$$\sim (\lambda_0 - \lambda_k)[\varphi_k(0) + xh\varphi_k(0) + \underline{\underline{O}}(x^2)] -$$
$$-(h + \underline{\underline{O}}(x))(\lambda_0 - \lambda_k)\varphi_k(0)x + \underline{\underline{O}}(x^2) \sim$$
$$\sim (\lambda_0 - \lambda_k)\varphi_k(0)[1 + \underline{\underline{O}}(x^2)] \sim \frac{1}{x}\varphi_{1,k}(x)[1 + \underline{\underline{O}}(x^2)],$$

ya'ni

$$\varphi_{1,k}'(x) \sim \frac{1}{x}\varphi_{1,k}(x) \cdot [1 + \underline{\underline{O}}(x^2)], \tag{1.17.40}$$

o'rinli bo'ladi. (1.17.39) va (1.17.40) asimptotikaga ko'ra

$$v_1(x) = \frac{\varphi_{1,1}'(x)}{\varphi_{1,1}(x)} \sim \frac{\frac{1}{x}\varphi_{1,1}(x)[1 + \underline{\underline{O}}(x^2)]}{\varphi_{1,1}(x)} = \frac{1}{x}[1 + \underline{\underline{O}}(x^2)], \tag{1.17.41}$$

o'rinli bo'ladi. Demak, $n = 1$ bo'lgan holda (1.17.36), (1.17.37), (1.17.38) asimptotikalar isbotlandi. (1.17.36)-(1.17.38) asimptotikalarni n uchun to'g'ri deb olib, ularni $n+1$ uchun ham to'g'ri bo'lishini ko'rsatamiz. (1.17.26) formulaga ko'ra:

$$\varphi_{n+1,k}(x) = \varphi_{n,k}'(x) - v_n(x)\varphi_{n,k}(x) =$$
$$= \frac{n\varphi_{n,k}(x)}{x}[1 + \underline{\underline{O}}(x^2)] - \frac{n}{x}[1 + \underline{\underline{O}}(x^2)]\varphi_{n,k}(x) = \bar{\bar{o}}(1), \; x \to 0. \tag{1.17.42}$$

(1.17.29) ayniyatga ko'ra:

$$\varphi_{n,n}(x)\varphi_{n+1,k}(x) - \varphi_{n,n}(0)\varphi_{n+1,k}(0) = (\lambda_n - \lambda_k)\int_0^x \varphi_{n,n}(t)\varphi_{n,k}(t)dt,$$

bo'ladi. (1.17.42) ga muvofiq

$$\varphi_{n,n}(x)\varphi_{n+1,k}(x) = (\lambda_n - \lambda_k)\int_0^x \varphi_{n,n}(t)\varphi_{n,k}(t)dt,$$

$$\varphi_{n+1,k}(x) = (\lambda_n - \lambda_k) \cdot \frac{1}{\varphi_{n,n}(x)} \cdot \int_0^x \varphi_{n,n}(t)\varphi_{n,k}(t)dt. \tag{1.17.42'}$$

Bunga ko‘ra $x \to 0$ da

$$\varphi_{n+1,k}(x) \sim \frac{\lambda_n - \lambda_k}{x^n[1+\underline{\underline{O}}(x^2)]}\int\limits_0^x t^{2n}[1+\underline{\underline{O}}(t^2)]C_{n,k}\prod_{i=0}^{n-1}(\lambda_i - \lambda_k)dt =$$

$$= C_{n,k}\prod_{i=0}^{n}(\lambda_i - \lambda_k)\cdot\frac{\frac{x^{2n+1}}{2n+1}[1+\underline{\underline{O}}(x^2)]}{x^n[1+\underline{\underline{O}}(x^2)]} =$$

$$= C_{n+1,k}\prod_{i=0}^{n}(\lambda_i - \lambda_k)x^{n+1}[1+\underline{\underline{O}}(x^2)]. \qquad (1.17.43)$$

Bu yerda $C_{n+1,k} = \dfrac{C_{n,k}}{2n+1} \neq 0$.

(1.17.42′) dan hosila olsak,

$$\frac{\varphi'_{n+1,k}(x)}{\varphi_{n+1,k}(x)} = -\frac{\varphi'_{n,n}(x)}{\varphi_{n,n}(x)} + (\lambda_n - \lambda_k)\frac{\varphi_{n,k}(x)}{\varphi_{n+1,k}(x)} \sim -n\cdot\frac{1}{x}\cdot[1+\underline{\underline{O}}(x^2)]+$$

$$+(\lambda_n-\lambda_k)\cdot\frac{C_{n,k}\prod\limits_{i=0}^{n-1}(\lambda_i-\lambda_k)x^n[1+\underline{\underline{O}}(x^2)]}{C_{n+1,k}\prod\limits_{i=0}^{n}(\lambda_i-\lambda_k)x^{n+1}[1+\underline{\underline{O}}(x^2)]} = -\frac{n}{x}\cdot[1+\underline{\underline{O}}(x^2)]+$$

$$+\frac{2n+1}{x}\cdot[1+\underline{\underline{O}}(x^2)] = \frac{n+1}{x}\cdot[1+\underline{\underline{O}}(x^2)], \quad x \to 0, \qquad (1.17.44)$$

kelib chiqadi. (1.17.44) ga asosan ushbu

$$v_{n+1}(x) = \frac{\varphi'_{n+1,n+1}(x)}{\varphi_{n+1,n+1}(x)} = \frac{n+1}{x}\cdot[1+\underline{\underline{O}}(x^2)], \quad x \to 0, \qquad (1.17.45)$$

asimptotikani topamiz. Shunday qilib, (1.17.36), (1.17.37), (1.17.38) asimptotikalar isbotlandi. Xuddi shunday usul bilan shunga o‘xshash asimptotikalarni $x \to 1$ bo‘lgan holda ham isbot qilish mumkin, bunda x o‘rnida $1-x$ lar turadi. Bu asimptotikalarga ko‘ra $\varphi_{n,k}(0) = 0$, $\varphi_{n,k}(1) = 0$ bo‘ladi.

Endi ushbu

$$q_n(x) = \frac{n(n-1)}{x^2} + \underline{\underline{O}}(1), \qquad (1.17.46)$$

asimptotikani keltirib chiqaramiz. $n = 1$ bo'lgan holda $\varphi_0(x)$ funksiya $[0, 1]$ kesmada ildizga ega emasligidan

$$q_1(x) = q(x) - 2\left(\frac{\varphi_0'(x)}{\varphi_0(x)}\right)'$$

formulaga ko'ra $q_1(x) \in C[0, 1]$ bo'lishi kelib chiqadi ($q(x)$ yetarlicha silliq). (1.17.46) asimptotika n uchun isbot qilindi deb olib, $n + 1$ uchun to'g'ri bo'lishini ko'rsatamiz:

$$q_{n+1}(x) = q_n(x) - 2v_n'(x) = q_n(x) - 2(q_n(x) - \lambda_n - v_n^2(x)) =$$

$$= 2\lambda_n + 2v_n^2(x) - q_n(x) =$$

$$= 2\lambda_n + \frac{2n^2}{x^2}[1 + \underline{\underline{O}}(x^2)] - \frac{n(n-1)}{x^2} - \underline{\underline{O}}(1) =$$

$$= \frac{n(n+1)}{x^2} + \underline{\underline{O}}(1), \ x \to 0.$$

Shunga o'xshash fikr $x \to 1$ bo'lganda ham o'rinli bo'ladi. ■

Krum almashtirishiga doir misollar keltiramiz.

Misol 1. Quyidagi

$$-y'' + \frac{2h^2}{(hx-1)^2}y = \lambda y, \tag{1.17.47}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + \dfrac{h}{\pi h - 1}y(\pi) = 0, \end{cases} \tag{1.17.48}$$

Shturm-Liuvill chegaraviy masalasining xos qiymatlarini va xos funksiyalarini topamiz va unga Krum almashtirishini qo'llaymiz.

Dastlab (1.17.47) tenglamaning (1.17.48) chegaraviy shartlardan birinchisini qanoatlantiruvchi noldan farqli yechimini topib olamiz. Uni quyidagi ko'rinishda izlaymiz:

$$\varphi(x, \lambda) = \cos\sqrt{\lambda}x + a(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}. \tag{1.17.49}$$

(1.17.49) funksiyaning hosilalarini topib, (1.17.47) tenglamaga qo'yamiz:

$$\varphi'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + a'(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + a(x)\cos\sqrt{\lambda}x,$$

$$\varphi''(x,\lambda) = -\lambda\cos\sqrt{\lambda}x + a''(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} +$$
$$+2a'(x)\cos\sqrt{\lambda}x - a(x)\sqrt{\lambda}\sin\sqrt{\lambda}x,$$

$$\varphi''(x,\lambda) + \lambda\varphi(x,\lambda) = 2a'(x)\cos\sqrt{\lambda}x + a''(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} =$$
$$= 2a'(x)\left(\cos\sqrt{\lambda}x + \frac{a''(x)}{2a'(x)}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right),$$

oxirgi tenglikka ko'ra

$$a'(x) = \frac{h^2}{(hx-1)^2}, \quad \frac{a''(x)}{2a'(x)} = a(x).$$

Bu yerdagi birinchi tenglikdan

$$a(x) = -\frac{h}{hx-1} + c,$$

bo'lishi kelib chiqadi. Buni ikkinchi tenglikka qo'ysak,

$$a(x) = \frac{a''(x)}{2a'(x)} = -\frac{2h^3}{(hx-1)^3}\cdot\frac{(hx-1)^2}{2h^2} = -\frac{h}{hx-1},$$

hosil bo'ladi. Demak,

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x - \frac{h}{hx-1}\cdot\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}. \qquad (1.17.50)$$

Bu yechim uchun $\varphi(0,\lambda) = 1$ bo'lishi ravshan.

Endi $\varphi'(x,\lambda)$ ni topamiz:

$$\varphi'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \frac{h^2}{(hx-1)^2}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{h}{hx-1}\cos\sqrt{\lambda}x,$$

$$\varphi'(0,\lambda) = h.$$

Demak, $\varphi(x,\lambda)$ yechim chegaraviy shartlardan birinchisini qanoatlantiradi. Buni ikkinchi chegaraviy shartga qo'ysak, ushbu

$$-\sqrt{\lambda}\sin\sqrt{\lambda}\pi=0,$$

xarakteristik tenglama kelib chiqadi. Bu tenglamaning ildizlari quyidagi sonlardan iborat:

$$\lambda_k=k^2,\quad k=0,1,\dots.$$

Bu xos qiymatlarga quyidagi xos funksiyalar mos keladi:

$$\varphi_k(x)=\varphi(x,\lambda_k)=\cos kx-\frac{h}{hx-1}\cdot\frac{\sin kx}{k},\quad k=0,1,\dots,$$

ya'ni

$$\varphi_0(x)=-\frac{1}{hx-1},\quad \varphi_1(x)=\cos x-\frac{h}{hx-1}\sin x,\ \dots.$$

Krum teoremasida $n=1$ bo'lgan holda quyidagi tengliklar o'rinli bo'ladi:

$$W_1(x)=\varphi_0(x)=-\frac{1}{hx-1},\quad W_1'(x)=\frac{h}{(hx-1)^2}$$

$$\frac{W_1'(x)}{W_1(x)}=-\frac{h}{hx-1},$$

$$q_1(x)=q(x)-2\left(\frac{W_1'(x)}{W_1(x)}\right)'=\frac{2h^2}{(hx-1)^2}-2\frac{h^2}{(hx-1)^2}=0.$$

Shunday qilib, ushbu

$$\begin{cases}-y''=\lambda y,\\ y(0)=0,\\ y(\pi)=0,\end{cases}$$

chegaraviy masalani topamiz. Bu masalaning xos qiymatlari $\lambda_k=k^2,\ k=1,2,\dots,$ bo'ladi.

Krum teoremasida $n=2$ bo'lgan holda

$$W_2(x)=\begin{vmatrix}\varphi_0 & \varphi_1\\ \varphi_0' & \varphi_1'\end{vmatrix}=$$

$$= \begin{vmatrix} -\frac{1}{hx-1} & \cos x - \frac{h}{hx-1}\sin x \\ \frac{h}{(hx-1)^2} & -\sin x + \frac{h^2}{(hx-1)^2}\sin x - \frac{h}{hx-1}\cos x \end{vmatrix} = \frac{\sin x}{hx-1}, \tag{1.17.51}$$

bo'ladi. Bundan

$$W'_2(x) = \frac{\cos x \cdot (hx-1) - h\sin x}{(hx-1)^2},$$

$$\frac{W_2'(x)}{W_2(x)} = \frac{(hx-1)\cos x - h\sin x}{(hx-1)\sin x} = \operatorname{ctg} x - \frac{h}{hx-1},$$

$$\left(\frac{W_2'(x)}{W_2(x)}\right)' = -\frac{1}{\sin^2 x} + \frac{h^2}{(hx-1)^2},$$

$$q_2(x) = \frac{2h^2}{(hx-1)^2} + \frac{2}{\sin^2 x} - \frac{2h^2}{(hx-1)^2} = \frac{2}{\sin^2 x},$$

kelib chiqadi, ya'ni ushbu

$$\begin{cases} -y'' + \dfrac{2}{\sin^2 x} y = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases}$$

masala hosil bo'lar ekan. Krum teoremasiga asosan bu masalaning xos qiymatlari $\lambda_k = k^2,\ k = 2, 3, \ldots$ bo'ladi.

Krum teoremada $n = 3$ bo'lgan holda

$$W_3(x) = \begin{vmatrix} \varphi_0(x) & \varphi_1(x) & \varphi_2(x) \\ \varphi_0'(x) & \varphi_1'(x) & \varphi_2'(x) \\ \varphi_0''(x) & \varphi_1''(x) & \varphi_2''(x) \end{vmatrix} = \varphi_0''(x) \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} -$$

$$-\varphi_1''(x) \begin{vmatrix} \varphi_0(x) & \varphi_2(x) \\ \varphi_0'(x) & \varphi_2'(x) \end{vmatrix} + \varphi_2''(x) \begin{vmatrix} \varphi_0(x) & \varphi_1(x) \\ \varphi_0'(x) & \varphi_1'(x) \end{vmatrix}. \tag{1.17.52}$$

bo'ladi. Quyidagi determinatlarni hisoblaymiz:

$$\begin{vmatrix} \varphi_0(x) & \varphi_2(x) \\ \varphi_0'(x) & \varphi_2'(x) \end{vmatrix} = \frac{2\sin 2x}{hx-1}, \tag{1.17.53}$$

$$\begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} = -2\cos x \sin 2x + \sin x \cos 2x +$$

$$+\frac{3h}{2(hx-1)}\sin x \sin 2x. \tag{1.17.54}$$

(1.17.51), (1.17.53), (1.17.54) larni (1.17.52) tenglikka qo'ysak

$$W_3(x) = \frac{4\sin^3 x}{hx-1}, \tag{1.17.55}$$

tenglik xosil bo'ladi. Bunga ko'ra

$$W_3'(x) = \frac{12(hx-1)\sin^2 x \cos x - 4h\sin^3 x}{(hx-1)^2}, \tag{1.17.56}$$

topamiz. Endi quyidagilarni hisoblab olamiz:

$$\frac{W_3'(x)}{W_3(x)} = 3\operatorname{ctg} x - \frac{h}{hx-1},$$

$$q_3(x) = \frac{2h^2}{(hx-1)^2} - 2\left(-\frac{3}{\sin^2 x} + \frac{h^2}{(hx-1)^2}\right) = \frac{6}{\sin^2 x}.$$

Demak, bu holda ushbu

$$\begin{cases} -y'' + \dfrac{6}{\sin^2 x}y = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases}$$

masala hosil bo'lar ekan. Krum teoremasiga asosan bu masalaning xos qiymatlari $\lambda_k = k^2$, $k = 3, 4, ...$, sonlardan iborat.

Izoh 1.17.1. (1.17.1)+(1.17.2) chegaraviy masala yordamida (1.17.3)+ (1.17.4) chegaraviy masala bir qiymatli tuziladi, ammo (1.17.3)+(1.17.4) chegaraviy masala yordamida (1.17.1)+(1.17.2) ga qaytadigan bo'lsak, $\lambda_0 < \lambda_1 < ... < \lambda_{k-1}$ sonlarni istalgancha o'zgartirib, (1.17.1)+(1.17.2) ko'rinishdagi cheksiz ko'p masalani hosil qilishmiz mumkin. Masalan, ushbu

$$\begin{cases} -y'' + q_1(x)y = \lambda y, \quad 0 < x < \pi, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{1.17.57}$$

chegaraviy masala berilgan bo'lib, uning xos qiymatlari $\lambda_1 < \lambda_2 < ... < \lambda_n < ...$ bo'lsin va bu xos qiymatlarga $\varphi_{1,1}(x)$, $\varphi_{1,2}(x)$, ... xos funksiyalar mos kelsin. $\lambda_0 < \lambda_1$ ixtiyoriy son bo'lsin. (1.17.1)+(1.17.2) ko'rinishdagi chegaraviy masalani quyidagicha tuzamiz. Avvalo ushbu

$$\lambda_0 - v' + v^2 = q_1(x), \tag{1.17.58}$$

Rikkati tenglamasining biror $v(x)$ yechimini topamiz. So'ngra

$$q(x) = q_1(x) + 2v', \quad h = v(0), \quad H = v(\pi),$$

deymiz. Bu holda, xosil bo'lgan masalaning xos funksiyalari ushbu

$$\varphi_0(x) = \exp\left\{\int_0^x v(t)dt\right\},$$

$$\varphi_k(x) = \frac{1}{(\lambda_0 - \lambda_k)\varphi_0(x)} \cdot (\varphi_0(x)\varphi_{1,k}(x))',$$

formulalar orqali topiladi.

Misol 2. $q_1(x) \equiv 0$ bo'lsin. Bu holda

$$\begin{cases} -y'' + q_1(x)y = \lambda y, & 0 < x < \pi, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases}$$

chegaraviy masalaning xos qiymatlari $\lambda_k = k^2$, $k = 1, 2, ...$, va ularga mos keluvchi xos fuksiyalar $\varphi_{1,k}(x) = \sin kx$, $k = 1, 2, ...$ bo'ladi. (1.17.58) tenglamani yechamiz:

$$v' = \lambda_0 + v^2.$$

$\lambda_0 = -a^2 < 0$ deb olaylik. U holda

$$v = \frac{a(1 + c_2 e^{2ax})}{1 - c_2 e^{2ax}},$$

$$q(x) = q_1(x) + 2v'(x) = \frac{2a^2 c_2}{\left(\dfrac{c_2 e^{ax} - e^{-ax}}{2}\right)^2},$$

bo'ladi. $h = v(0)$, $H = v(\pi)$ formulalardan

$$h = a, \quad H = \frac{a(1 + c_2 e^{2\pi a})}{1 - c_2 e^{2\pi a}},$$

kelib chiqadi. $c_2 = -1$ deb olsak, bu holda quyidagi

$$\begin{cases} -y'' - \dfrac{2a^2}{ch^2 ax} y = \lambda y, \\ y'(0) = ay(0), \\ y'(\pi) = -ath\pi a \cdot y(\pi), \end{cases} \tag{1.17.59}$$

chegaraviy masala hosil bo'ladi. (1.17.59) chegaraviy masalaning xos qiymatlari $\lambda_0 = -a^2$, $\lambda_1 = 1^2$, $\lambda_2 = 2^2, \ldots$ sonlardan iborat. $\lambda_0 = 0$ va $\lambda_0 = a^2$, $a > 0$ bo'lgan hollar ham shu tarzda ko'rib chiqiladi.

18-§. Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izini hisoblashning P.Laks usuli

Ushbu

$$\begin{cases} Ly \equiv -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.18.1}$$

Shturm-Liuvill masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C^1[0, \pi]$ haqiqiy funksiya va h, H – chekli haqiqiy sonlar. Bu chegaraviy masalaning xos qiymatlari $\{\lambda_n\}_{n=0}^{\infty}$ bo'lsin. Quyidagi asimptotik formula o'rinli bo'lishi bizga ma'lum:

$$\lambda_n = n^2 + \frac{2h + 2H}{\pi} + c_0 + \frac{\gamma_n}{n}, \ \{\gamma_n\} \in l_2. \tag{1.18.2}$$

Bu yerda

$$c_0 = \frac{1}{\pi} \int_0^{\pi} q(x) dx. \tag{1.18.3}$$

Agar biz xos qiymatlardan ushbu

$$\sum_{n=0}^{\infty} \lambda_n,$$

qatorni tuzib olsak, (1.18.2) asimptotik formulaga ko'ra bu qator uzoqlashuvchi bo'ladi, ya'ni Shturm-Liuvill operatorining oddiy ma'nodagi izi mavjud emas.

Agar biz ushbu

$$\sum_{n=0}^{\infty} [\lambda_n - n^2 - \frac{2h + 2H}{\pi} - c_0], \tag{1.18.4}$$

sonli qatorni qaraydigan bo'lsak, (1.18.2) asimptotik formulaga ko'ra bu qator yaqinlashuvchi bo'ladi.

Ta'rif 1.18.1. (1.18.4) qatorning yig'indisiga (1.18.1) Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izi deyiladi.

(1.18.1) chegaraviy masalaning regulyarlashtirilgan izi, ilk bor, 1953 yilda I.M.Gelfand va B.M.Levitan tomonidan hisoblangan. Mazkur paragrafda biz Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izini hisoblashning P.Laks usuli bilan tanishamiz.

$q(x) \equiv 0$ bo'lgan holda, (1.18.1) chegaraviy masalaning xos qiymatlarini $\{\lambda_n(0)\}_{n=0}^{\infty}$ orqali, ortonormallangan xos funksiyalarini $y_n(x, 0)$ orqali belgilaymiz.

Teorema 1.18.1. (*P.Laks*). *(1.18.1) Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izi uchun quyidagi formula o'rinli:*

$$\sum_{n=0}^{\infty} [\lambda_n - \lambda_n(0) - c_0] = \int_0^{\pi} q(x)S(x)dx. \tag{1.18.5}$$

Bu yerda

$$S(x) = \sum_{n=0}^{\infty} \left(y_n^2(x, 0) - \frac{1}{\pi} \right). \tag{1.18.6}$$

Isbot. Quyidagi Shturm-Liuvill operatorlari oilasini ko'rib chiqamiz:

$$\begin{cases} Ly \equiv -y'' + t \cdot q(x)y = \lambda(t)y, \quad 0 \le x \le \pi, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0. \end{cases} \tag{1.18.7}$$

Bu yerda $t \in [0,1]$ parametr. (1.18.7) chegaraviy masalaning xos qiymatlarini $\lambda_n(t)$ orqali va ularga mos keluvchi ortonormallangan xos funksiyalarni $y_n(x,t)$ orqali belgilaymiz. Ushbu

$$L(t)y_n = \lambda_n(t)y_n,$$

tenglikni y_n funksiyaga skalyar ko'paytiramiz

$$(L(t)y_n, y_n) = \lambda_n(t).$$

Bundan t bo'yicha hosila olamiz

$$((L(t)y_n)^{\cdot}, y_n) + (L(t)y_n, \dot{y}_n) = \dot{\lambda}_n(t), \tag{1.18.8}$$

$$(L(t)y_n)^{\cdot} = -\dot{y}_n'' + tq(x)\dot{y}_n + q(x)y_n = L(t)\dot{y}_n + q(x)y_n. \tag{1.18.9}$$

(1.18.9) ifodani (1.18.8) tenglikka qo'yamiz:

$$(L(t)\dot{y}_n, y_n) + (q(x)y_n, y_n) + (L(t)y_n, \dot{y}_n) = \dot{\lambda}_n(t). \tag{1.18.10}$$

$L(t)$ operatorning simmetrikligidan foydalanib, (1.18.10) tenglikni ushbu

$$(\dot{y}_n, L(t)y_n) + (q(x)y_n, y_n) + (L(t)y_n, \dot{y}_n) = \dot{\lambda}_n(t), \tag{1.18.11}$$

tarzda yozib olamiz. $L(t)y_n = \lambda_n(t)y_n$ tenglikka asosan (1.18.11) ayniyat quyidagi ko'rinishni oladi:

$$\lambda_n(t)[(\dot{y}_n, y_n) + (y_n, \dot{y}_n)] + (q(x)y_n, y_n) = \dot{\lambda}_n(t),$$

$$\lambda_n(t)(y_n, y_n)^{\cdot} + (q(x)y_n, y_n) = \dot{\lambda}_n(t),$$

$$(q(x)y_n, y_n) = \dot{\lambda}_n(t),$$

ya'ni

$$\int_0^{\pi} q(x)y_n^2(x,t)dx = \dot{\lambda}_n(t). \qquad (1.18.12)$$

(1.18.12) tenglikni t bo'yicha $[0,1]$ kesmada integrallaymiz:

$$\lambda_n(1) - \lambda_n(0) = \int_0^{\pi} q(x) \int_0^1 y_n^2(x,t)dtdx, \qquad (1.18.13)$$

va (1.18.13) tenglikdan (1.18.3) tenglikni ayiramiz:

$$\lambda_n(1)-\lambda_n(0)-c_0 = \int_0^{\pi} q(x)\left\{\int_0^1 \left[y_n^2(x,t) - \frac{1}{\pi}\right]dt\right\}dx. \quad (1.18.14)$$

(1.18.14) tenglikdan

$$\sum_{n=0}^{\infty}[\lambda_n(1) - \lambda_n(0) - c_0] = \int_0^{\pi} q(x)\int_0^1 \sum_{n=0}^{\infty}\left[y_n^2(x,t) - \frac{1}{\pi}\right]dtdx, \qquad (1.18.15)$$

kelib chiqadi.

Ta'rif 1.18.2. Agar ushbu

$$\sum_{n=0}^{\infty}\int_0^{\pi} f_n(x)\varphi(x)dx,$$

sonli qator cheksiz marta differensiallanuvchi, 0 va π nuqtalarning biror atrofida nolga aylanuvchi ixtiyoriy $\varphi(x)$ funksiya uchun yaqinlashuvchi bo'lsa, u holda ushbu

$$\sum_{n=0}^{\infty} f_n(x),$$

funksional qator umumlashgan ma'noda yaqinlashuvchi deyiladi.

Lemma 1.18.1 (*P.Laks*). *Ushbu*

$$\sum_{n=0}^{\infty}\left[y_n^2(x,t) - \frac{1}{\pi}\right], \qquad (1.18.16)$$

qator umumlashgan ma'noda yaqinlashuvchi bo'ladi va uning yig'indisi t parametrga bog'liq bo'lmaydi.

Isbot. Ta'rif 1.18.2 ga ko'ra ushbu

$$\sum_{n=0}^{\infty} \cos 2nx,$$

qator umumlashgan ma'noda yaqinlashuvchi bo'ladi.

(1.18.16) qatorning umumlashgan ma'noda yaqinlashuvchi bo'lishi, ortonormallangan xos funksiyalarning ushbu

$$y_n(x,t) = \sqrt{\frac{2}{\pi}}\left\{\cos nx + a(x,t)\frac{\sin nx}{n}\right\} + \underline{\underline{O}}\left(\frac{1}{n^2}\right), \quad n \to \infty,$$

$$a(x,t) = -\frac{h+H}{\pi}x + h - \frac{xt}{2\pi}\int_0^{\pi} q(s)ds + \frac{t}{2}\int_0^{x} q(s)ds,$$

asimptotikasidan kelib chiqadi. (1.18.16) qator yig'indisini $S(x,t)$ bilan belgilaymiz

$$S(x,t) = \sum_{n=0}^{\infty}\left[y_n^2(x,t) - \frac{1}{\pi}\right],$$

va bu funksiyaning t parametrga bog'liq emasligini ko'rsatamiz. Buning uchun t bo'yicha olingan hosila nolga teng bo'lishini ko'rsatish kifoya:

$$\dot{S}(x,t) = 2\sum_{n=0}^{\infty} y_n(x,t)\dot{y}_n(x,t), \tag{1.18.17}$$

$$\dot{y}_n(x,t) = \sum_{m=0}^{\infty} a_{m,n} y_m(x,t), \tag{1.18.18}$$

$$a_{m,n} = (\dot{y}_n, y_m). \tag{1.18.19}$$

Xos funksiyalarning ortonormallanganligini, ya'ni

$$(y_n, y_m) = \delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

bo'lishini e'tiborga olib, ushbu

$$(\dot{y}_n, y_m) + (y_n, \dot{y}_m) = 0,$$

tenglikni hosil qilamiz. (1.18.19) formulaga asosan

$$a_{m,n} + a_{n,m} = 0, \tag{1.18.20}$$

bo'ladi.

(1.18.18) ifodani (1.18.17) tenglikka qo'yamiz:

$$\dot{S}(x,t) = 2\sum_{n,m=0}^{\infty} a_{m,n} y_n(x,t) y_m(x,t). \tag{1.18.21}$$

(1.18.20) tenglikka ko'ra (1.18.21) qatorning yig'indisi uchun $\dot{S}(x,t) = 0$ bajariladi, ya'ni $S(x,t)$ funksiya t parametrga bog'liq emas. *Lemma 1.18.1 isbotlandi.*■

Bu lemmaga asosan quyidagi tenglik o'rinli:

$$S(x,t) = S(x,0) = \sum_{n=0}^{\infty}\left[y_n^2(x,0) - \frac{1}{\pi}\right].$$

(1.18.15) tenglikning o'ng tarafidagi qator o'rniga $S(x,0)$ ni qo'yib (1.18.5) tenglikni hosil qilamiz:

$$\sum_{n=0}^{\infty}[\lambda_n(1) - \lambda_n(0) - c_0] = \int_0^{\pi} q(x)S(x,0)dx.$$

Teorema 1.18.1 isbotlandi.■

(1.18.1) chegaraviy masalaning regulyarlashtirilgan izini hisoblashdan oldin, quyidagi

$$\begin{aligned} &-y'' + q(x)y = \lambda y, \ \ 0 \le x \le \pi, \\ &y(0) = 0, \ \ y(\pi) = 0, \end{aligned} \tag{1.18.22}$$

yordamchi Dirixle masalasining izini hisoblaymiz.

(1.18.22) chegaraviy masalaning xos qiymatlarini $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n, ...$ orqali belgilaymiz. Bu holda λ_n xos qiymatlar uchun quyidagi

$$\lambda_n = n^2 + c_0 + \frac{\beta_n}{n}, \ \{\beta_n\} \in l_2 \tag{1.18.23}$$

asimptotik formula o'rinli bo'lishi bizga ma'lum. Bu yerda

$$c_0 = \frac{1}{\pi}\int_0^{\pi} q(x)dx. \tag{1.18.24}$$

Ushbu

$$\sum_{n=1}^{\infty} [\lambda_n - n^2 - c_0], \tag{1.18.25}$$

sonli qator (1.18.23) asimptotik formulaga ko'ra absolyut yaqinlashuvchi bo'ladi. Shuning uchun (1.18.25) sonli qator yagona yig'indiga ega bo'ladi.

Ta'rif 1.18.3. (1.18.25) sonli qatorning yig'indisiga (1.18.22) Dirixle masalasining izi deyiladi.

Teorema 1.18.2 (*I.M.Gelfand, B.M.Levitan*). *(1.18.22) Dirixle chegaraviy masalasining regulyarlashtirilgan izi uchun quyidagi*

$$\sum_{n=1}^{\infty} [\lambda_n - n^2 - c_0] = \frac{1}{2}c_0 - \frac{q(0) + q(\pi)}{4}, \tag{1.18.26}$$

formula o'rinli.

Isbot. (1.18.26) formulani P.Laks usulidan foydalanib isbotlaymiz. Buning uchun avvalo $q(x) \equiv 0$ holda hosil bo'lgan

$$-y'' = \lambda y,$$

$$y(0) = 0, \ \ y(\pi) = 0,$$

chegaraviy masalaning xos qiymatlarini va ortonormallangan xos funksiyalarini topib olamiz:

$$\lambda_n(0) = n^2, \ \ n = 1, 2, 3, ...,$$

$$y_n(x,0)=\sqrt{\frac{2}{\pi}}\sin nx,\ \ n=1,2,3,\dots .$$

So'ngra Laks teoremasidagi

$$S(x,0)=\sum_{n=1}^{\infty}\left[y_n^2(x,0)-\frac{1}{\pi}\right]=\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\sin^2 nx-\frac{1}{\pi}\right]=$$

$$=-\frac{1}{\pi}\sum_{n=1}^{\infty}\cos 2nx, \tag{1.18.27}$$

funksional qatorning yig'indisini topamiz. (1.18.27) qator odatdagi ma'noda uzoqlashuvchi, chunki qator yaqinlashishining zaruriy sharti bajarilmaydi. Ammo bu qator umumlashgan ma'noda yaqinlashadi. Umumlashgan funksiyalar kursidan ([21]) bizga quyidagi tengliklar ma'lum:

$$\sum_{n=-\infty}^{\infty}\cos nx=\pi\sum_{n=-\infty}^{\infty}\delta(x-2\pi n),\ \ x\in(-\infty,\infty), \tag{1.18.28}$$

$$\delta(cx)=\frac{1}{|c|}\delta(x). \tag{1.18.29}$$

(1.18.28) tenglikni $[0,\ 2\pi]$ oraliqda qarasak,

$$\sum_{n=-\infty}^{\infty}\cos nx=\pi[\delta(x)+\delta(x-2\pi)], \tag{1.18.30}$$

bo'ladi. (1.18.30) tenglikda x o'rniga $2x$ ni qo'yamiz:

$$\sum_{n=-\infty}^{\infty}\cos 2nx=\pi[\delta(2x)+\delta(2(x-\pi))],\ \ x\in[0,\ \pi]. \tag{1.18.31}$$

(1.18.29) formuladan foydalanib, (1.18.31) tenglikni quyidagi ko'rinishda yozamiz:

$$1+2\sum_{n=1}^{\infty}\cos 2nx=\frac{\pi}{2}[\delta(x)+\delta(x-\pi)],\ \ x\in[0,\ \pi].$$

Demak, ushbu

$$\sum_{n=1}^{\infty}\cos 2nx=-\frac{1}{2}+\frac{\pi}{4}[\delta(x)+\delta(x-\pi)],\ \ x\in[0,\ \pi],$$

formula o'rinli bo'lar ekan. Bundan foydalanib (1.18.27) tenglikni quyidagi ko'rinishda yozish mumkin:

$$S(x,0) = \frac{1}{2\pi} - \frac{1}{4}[\delta(x) + \delta(x-\pi)].$$

(1.18.5) tenglikdan quyidagi

$$\sum_{n=1}^{\infty}[\lambda_n - n^2 - c_0] = \int_0^{\pi} q(x)S(x,0)dx =$$

$$= \int_0^{\pi} q(x)\left[\frac{1}{2\pi} - \frac{1}{4}[\delta(x) + \delta(x-\pi)]\right]dx =$$

$$= \frac{1}{2\pi}\int_0^{\pi} q(x)dx - \frac{1}{4}\int_0^{\pi} q(x)\delta(x)dx - \frac{1}{4}\int_0^{\pi} q(x)\delta(x-\pi)dx =$$

$$= \frac{1}{2\pi}\int_0^{\pi} q(x)dx - \frac{1}{4}[q(0) + q(\pi)] = \frac{1}{2}c_0 - \frac{1}{4}[q(0) + q(\pi)],$$

tenglik kelib chiqadi.■

Endi (1.18.1) chegaraviy masalani regulyarlashtirilgan izini Krum almashtirishidan foydalanib hisoblashimiz mumkin.

Teorema 1.18.3 (*B.M.Levitan*). *(1.18.1) Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izi uchun ushbu*

$$\sum_{n=0}^{\infty}[\lambda_n - n^2 - c_0] = -\frac{1}{2}c_0 - \frac{h^2 + H^2}{2} + \frac{1}{4}[q(0) + q(\pi)]$$

formula o'rinli. Bu yerda

$$c_0 = \frac{1}{\pi}\int_0^{\pi} q(x)dx + \frac{2}{\pi}(H + h).$$

Isbot. Krum almashtirishi yordamida (1.18.1) chegaraviy masalani quyidagi

$$\begin{cases} -y'' + [\lambda - q_1(x)]y = 0, \quad 0 \le x \le \pi, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{1.18.32}$$

Dirixle masalasiga keltiramiz. Bu yerda

$$q_1(x) = q(x) - 2v'(x), \quad v(x) = \frac{\varphi_0'(x)}{\varphi_0(x)}, \tag{1.18.33}$$

$\varphi_0(x)$ funksiya (1.18.1) chegaraviy masalaning λ_0 xos qiymatiga mos keluvchi xos funksiyasi. $v(x)$ funksiya esa quyidagi shartlarni qanoatlantiradi:

$$v'(x) = q(x) - \lambda_0 - v^2(x), \tag{1.18.34}$$

$$v(0) = h, \quad v(\pi) = -H. \tag{1.18.35}$$

Agar (1.18.1) chegaraviy masalaning xos qiymatlari $\lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...$ bo'lsa, u holda (1.18.32) masalaning xos qiymatlari $\lambda_1 < \lambda_2 < ... < \lambda_n < ...$ bo'ladi. (1.18.33) va (1.18.34) tengliklardan ushbu

$$q_1(x) = 2\lambda_0 + 2v^2(x) - q(x), \tag{1.18.36}$$

formula kelib chiqadi. (1.18.32) chegaraviy masala uchun regulyarlashtirilgan izlar formulasini yozamiz:

$$\sum_{n=1}^{\infty} [\lambda_n - n^2 - c_1] = \frac{1}{2}c_1 - \frac{q_1(0) + q_1(\pi)}{4}. \tag{1.18.37}$$

Bu yerda

$$c_1 = \frac{1}{\pi}\int_0^{\pi} q_1(x)dx. \tag{1.18.38}$$

(1.18.33) ifodani (1.18.38) tenglikka qo'yib, (1.18.35) formulalarni inobatga olsak,

$$c_1 = \frac{1}{\pi}\int_0^{\pi} [q(x) - 2v'(x)]dx = \frac{1}{\pi}\int_0^{\pi} q(x)dx - \frac{2}{\pi}(v(\pi) - v(0)) =$$

$$= \frac{1}{\pi}\int_0^\pi q(x)dx + \frac{2}{\pi}(H+h) = c_0,$$

kelib chiqadi. Endi (1.18.36) tenglikni (1.18.37) formulaga qo'yib, ushbu

$$\sum_{n=1}^{\infty}[\lambda_n - n^2 - c_0] = \frac{1}{2}c_0 - \lambda_0 - \frac{1}{2}(h^2 + H^2) + \frac{q(0)+q(\pi)}{4},$$

$$\sum_{n=0}^{\infty}[\lambda_n - n^2 - c_0] = -\frac{1}{2}c_0 - \frac{1}{2}(h^2 + H^2) + \frac{q(0)+q(\pi)}{4},$$

izlar formulasini topamiz. ■

Mustaqil yechish uchun mashqlar

1. Quyidagi Shturm-Liuvill chegaraviy masalalarining izini P.Laks usuli yordamida hisoblang:

$$a)\ \begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases} \qquad b)\ \begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) = 0, \\ y'(\pi) = 0, \end{cases}$$

$$c)\ \begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) = 0, \\ y(\pi) = 0, \end{cases} \qquad d)\ \begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(\pi) = 0, \end{cases}$$

$$e)\ \begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = y(\pi), \\ y'(0) = y'(\pi), \end{cases} \qquad f)\ \begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = -y(\pi), \\ y'(0) = -y'(\pi), \end{cases}$$

$$g)\ \begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) - hy(\pi) = 0. \end{cases}$$

2. Potentsialida maxsusligi bo'lgan quyidagi Shturm-Liuvill chegaraviy masalalarining regulyarlashtirilgan izini P.Laks usuli yordamida hisoblang:

$$a)\ \begin{cases} -y'' + \dfrac{2}{x^2}y + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + \dfrac{1}{\pi}y(\pi) = 0, \end{cases}$$

$$b)\ \begin{cases} -y'' + \dfrac{2h^2}{(hx-1)^2}y + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + \dfrac{h}{h\pi - 1}y(\pi) = 0, \end{cases}$$

$$c)\ \begin{cases} -y'' + \dfrac{2}{\sin^2 x}y + q(x)y = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases}$$

$$d)\ \begin{cases} -y'' + \dfrac{2}{2\sin^2 \dfrac{x}{2}}y + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(\pi) = 0, \end{cases}$$

19-§. Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izini B.M.Levitan usulida hisoblash

Quyidagi Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz:

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.19.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0. \end{cases} \tag{1.19.2}$$

Bu yerda $q(x) \in C^1[0,\pi]$ haqiqiy funksiya va h, H chekli haqiqiy sonlar.

(1.19.1)+(1.19.2) chegaraviy masalaning xos qiymatlarini λ_n, $n = 0, 1, 2, ...$ orqali belgilaymiz. Ushbu

$$\lambda_n = n^2 + c_0 + \frac{\gamma_n}{n}, \ \{\gamma_n\} \in l_2$$

asimptotik formula o'rinli bo'lishini oldingi paragraflarda ko'rsatgan edik. Bu yerda

$$c_0 = \frac{2}{\pi}(h + H) + \frac{1}{\pi}\int_0^\pi q(x)dx. \tag{1.19.3}$$

Bunga ko'ra ushbu

$$\sum_{n=0}^{\infty} (\lambda_n - n^2 - c_0) < \infty, \tag{1.19.4}$$

sonli qator absolyut yaqinlashuvchi bo'ladi.

Ta'rif 1.19.1.(1.19.4) qatorning yig'indisiga (1.19.1)–(1.19.2) Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izi deyiladi.

Teorema 1.19.1 (*B.M.Levitan*). *(1.19.1)+(1.19.2) Shturm-Liuvill chegaraviy masalasining regulyarlashtirilgan izi uchun quyidagi formula o'rinli:*

$$\sum_{n=0}^{\infty}(\lambda_n - n^2 - c_0) = \frac{1}{4}[q(0) + q(\pi)] -$$

$$-\frac{1}{2\pi}\int_0^\pi q(t)dt - \frac{h+H}{\pi} - \frac{h^2+H^2}{2}.$$

Isbot. $\varphi(x,\lambda)$ orqali (1.19.1) tenglamaning

$$\varphi(0,\lambda)=1, \quad \varphi'(0,\lambda)=h,$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

Ushbu

$$\varphi'(\pi,\lambda)+H\,\varphi(\pi,\lambda)=\pi(\lambda_0-\lambda)\prod_{n=1}^{\infty}\frac{\lambda_n-\lambda}{n^2}, \qquad (1.19.5)$$

yoyilma bizga ma'lum. $\lambda \to -\infty$ bo'lganda bu tenglik ikkala tomonining asimptotikasini o'rganamiz va ularni taqqoslaymiz.

Qulaylik uchun $\lambda=-\mu$ va

$$F(\mu)=\pi(\lambda_0+\mu)\prod_{n=1}^{\infty}\frac{\lambda_n+\mu}{n^2},$$

belgilashlarni kiritib olamiz. Ushbu

$$\frac{sh\pi\sqrt{\mu}}{\pi\sqrt{\mu}}=\prod_{n=1}^{\infty}\frac{n^2+\mu}{n^2},$$

formula yordamida quyidagi

$$\Delta(\mu)=\frac{\pi(\lambda_0+\mu)\prod\limits_{n=1}^{\infty}\frac{\lambda_n+\mu}{n^2}}{\prod\limits_{n=1}^{\infty}\frac{n^2+\mu}{n^2}}\cdot\frac{sh\pi\sqrt{\mu}}{\pi\sqrt{\mu}}=$$

$$=\left(\frac{\lambda_0}{\sqrt{\mu}}+\sqrt{\mu}\right)\cdot sh\pi\sqrt{\mu}\cdot\prod_{n=1}^{\infty}\frac{\lambda_n+\mu}{n^2+\mu}=$$

$$=\left(\frac{\lambda_0}{\sqrt{\mu}}+\sqrt{\mu}\right)\cdot sh\pi\sqrt{\mu}\cdot\prod_{n=1}^{\infty}\left(1-\frac{n^2-\lambda_n}{n^2+\mu}\right), \qquad (1.19.6)$$

tenglikni hosil qilamiz va

$$\psi(\mu)=\prod_{n=1}^{\infty}\left(1-\frac{n^2-\lambda_n}{n^2+\mu}\right),$$

funksiyaning $\mu \to +\infty$ dagi asimptotikasini o'rganamiz. Buning uchun ushbu

$$\ln \psi(\mu) = \sum_{n=1}^{\infty} \ln\left(1 - \frac{n^2 - \lambda_n}{n^2 + \mu}\right) = -\sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(\frac{n^2 - \lambda_n}{n^2 + \mu}\right)^k, \tag{1.19.7}$$

tenglik va quyidagi lemmadan foydalanamiz:

Lemma 1.19.1. *Agar* $|n^2 - \lambda_n| \le a$ *bo'lsa, u holda ushbu*

$$\sum_{n=1}^{\infty} \frac{|n^2 - \lambda_n|^k}{(n^2 + \mu)^k} \le \frac{\pi}{2} \cdot \frac{a^k}{\mu^{k-\frac{1}{2}}}, \quad k = 1, 2, 3, ..., \tag{1.19.8}$$

baholashlar o'rinli bo'ladi.

Isbot. Quyidagicha baholashlarni bajaramiz:

$$\sum_{n=1}^{\infty} \frac{|n^2 - \lambda_n|^k}{(n^2 + \mu)^k} \le a^k \sum_{n=1}^{\infty} \frac{1}{(n^2 + \mu)^k} \le a^k \int_0^{\infty} \frac{dx}{(x^2 + \mu)^k} =$$

$$= \frac{a^k}{\mu^{k-\frac{1}{2}}} \int_0^{\infty} \frac{dt}{(1 + t^2)^k} \le \frac{a^k}{\mu^{k-\frac{1}{2}}} \int_0^{\infty} \frac{dt}{1 + t^2} = \frac{\pi}{2} \cdot \frac{a^k}{\mu^{k-\frac{1}{2}}}. \blacksquare$$

(1.19.8) baholashga asosan

$$\sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \frac{|n^2 - \lambda_n|^k}{(n^2 + \mu)^k} \le \frac{\pi}{2} \sum_{k=2}^{\infty} \frac{a^k}{\mu^{k-\frac{1}{2}}} = \frac{\pi}{2} \frac{a^2}{\mu^{\frac{3}{2}}} \sum_{k=0}^{\infty} \left(\frac{a}{\mu}\right)^k = \underline{\underline{O}}\left(\mu^{-\frac{3}{2}}\right), \tag{1.19.9}$$

bo'ladi. Ushbu

$$-\sum_{n=1}^{\infty} \frac{n^2 - \lambda_n}{n^2 + \mu} = \sum_{n=1}^{\infty} \frac{\lambda_n - n^2}{\mu + n^2} = \sum_{n=1}^{\infty} (\lambda_n - n^2 - c_0) \cdot \frac{1}{\mu + n^2} +$$

$$+ c_0 \sum_{n=1}^{\infty} \frac{1}{\mu + n^2} = c_0 \sum_{n=1}^{\infty} \frac{1}{\mu + n^2} +$$

$$+ \frac{1}{\mu} \sum_{n=1}^{\infty} (\lambda_n - n^2 - c_0) - \frac{1}{\mu} \sum_{n=1}^{\infty} (\lambda_n - n^2 - c_0) n^2 \cdot \frac{1}{\mu + n^2}, \tag{1.19.10}$$

ayniyat bajarilishi ravshan.

Quyidagi

$$\sum_{n=1}^{\infty}\frac{1}{\mu+n^2}=\frac{\pi cth\pi\sqrt{\mu}}{2\sqrt{\mu}}-\frac{1}{2\mu}=\frac{\pi}{2\sqrt{\mu}}-\frac{1}{2\mu}+\underline{\underline{O}}(e^{-2\pi\sqrt{\mu}}), \tag{1.19.11}$$

tenglik o'rinli bo'lishi ma'lum. Quyidagi

$$\sup_n\left|(\lambda_n-n^2-c_0)n^2\right|<\infty,$$

baholash o'rinli bo'lganligi uchun (1.19.11) tenglikdan

$$\frac{1}{\mu}\sum_{n=1}^{\infty}(\lambda_n-n^2-c_0)n^2\cdot\frac{1}{\mu+n^2}=\underline{\underline{O}}\left(\mu^{-\frac{3}{2}}\right), \tag{1.19.12}$$

kelib chiqadi.

(1.19.9), (1.19.10), (1.19.12) va (1.19.7) tengliklardan

$$\ln\psi(\mu)=\frac{c_0\pi}{2\sqrt{\mu}}+\frac{1}{\mu}\left(S_\lambda-\lambda_0+\frac{c_0}{2}\right)+\underline{\underline{O}}\left(\mu^{-\frac{3}{2}}\right), \tag{1.19.13}$$

hosil bo'ladi. Bu yerda

$$S_\lambda=\sum_{n=0}^{\infty}(\lambda_n-n^2-c_0), \tag{1.19.14}$$

belgilash kiritildi. (1.19.13) tenglikdan esa

$$\psi(\mu)=\exp\left\{\frac{c_0\pi}{2\sqrt{\mu}}+\frac{1}{\mu}\left(S_\lambda-\lambda_0+\frac{c_0}{2}\right)+\underline{\underline{O}}\left(\mu^{-\frac{3}{2}}\right)\right\}=$$

$$=1+\frac{c_0\pi}{2\sqrt{\mu}}+\frac{1}{\mu}\left(S_\lambda-\lambda_0+\frac{c_0}{2}+\frac{c_0^2\pi^2}{8}\right)+\underline{\underline{O}}\left(\mu^{-\frac{3}{2}}\right), \tag{1.19.15}$$

kelib chiqadi. Bunga ko'ra (1.19.6) funksiya uchun ushbu

$$F(\mu)=\left(\frac{\lambda_0}{\sqrt{\mu}}+\sqrt{\mu}\right)\times$$

$$\times sh\pi\sqrt{\mu}\left\{1+\frac{c_0\pi}{2\sqrt{\mu}}+\frac{1}{\mu}\left(S_\lambda-\lambda_0+\frac{c_0}{2}+\frac{c_0^2\pi^2}{8}\right)++\underline{\underline{O}}\left(\mu^{-\frac{3}{2}}\right)\right\}=$$

$$=\frac{1}{2}e^{\pi\sqrt{\mu}}\left\{\sqrt{\mu}+\frac{c_0\pi}{2}+\frac{1}{\sqrt{\mu}}\left(S_\lambda+\frac{c_0}{2}+\frac{c_0^2\pi^2}{8}\right)+\underline{\underline{O}}\left(\mu^{-1}\right)\right\},\tag{1.19.16}$$

asimptotik formula o'rinli bo'ladi.

Endi $\varphi'(\pi,-\mu)+H\varphi(\pi,-\mu)$ funksiyaning $\mu\to+\infty$ dagi asimptotikasini o'rganamiz.

Quyidagi formulalar bizga ma'lum:

$$\varphi(x,-\mu)=ch\sqrt{\mu}x+h\frac{sh\sqrt{\mu}x}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}\int_0^x sh\sqrt{\mu}(x-t)q(t)\varphi(t,-\mu)dt,$$

$$\varphi'(x,-\mu)=\sqrt{\mu}sh\sqrt{\mu}x+hch\sqrt{\mu}x+\int_0^x ch\sqrt{\mu}(x-t)q(t)\varphi(t,-\mu)dt.$$

Bulardan

$$\varphi(\pi,-\mu)=\frac{1}{2}e^{\pi\sqrt{\mu}}\left\{1+\frac{h}{\sqrt{\mu}}+\right.$$

$$\left.+\frac{1}{\sqrt{\mu}}\int_0^\pi sh\sqrt{\mu}(\pi-t)ch\sqrt{\mu}tq(t)dt+\underline{\underline{O}}\left(\frac{1}{\mu}\right)\right\}=$$

$$=\frac{1}{2}e^{\pi\sqrt{\mu}}\left\{1+\frac{h}{\sqrt{\mu}}+\frac{1}{2\sqrt{\mu}}\int_0^\pi q(t)dt+\underline{\underline{O}}\left(\frac{1}{\mu}\right)\right\},\tag{1.19.17}$$

va

$$\varphi'(\pi,-\mu)=\sqrt{\mu}sh\sqrt{\mu}\pi+hch\sqrt{\mu}\pi+\int_0^\pi ch\sqrt{\mu}(\pi-t)q(t)\times$$

$$\times\left\{ch\sqrt{\mu}t+h\frac{sh\sqrt{\mu}t}{\sqrt{\mu}}+\frac{1}{\sqrt{\mu}}\int_0^t sh\sqrt{\mu}(t-s)q(s)ch\sqrt{\mu}sds\right\}dt+$$

$$+\underline{\underline{O}}\left(\frac{e^{\pi\sqrt{\mu}}}{\mu}\right)=\frac{1}{2}e^{\pi\sqrt{\mu}}\left\{\sqrt{\mu}+h+\frac{1}{2}\int_0^\pi q(t)dt+\right.$$

$$+\frac{1}{\sqrt{\mu}}\left[\frac{1}{4}q(0)+\frac{1}{4}q(\pi)+\frac{h}{2}\int_0^\pi q(t)dt+\frac{1}{8}\left(\int_0^\pi q(t)dt\right)^2\right]+\underline{\underline{O}}\left(\frac{1}{\mu}\right)\Bigg\} \tag{1.19.18}$$

tengliklar hosil bo'ladi.

(1.19.17) va (1.19.18) tengliklarga ko'ra ushbu

$$\varphi'(\pi,-\mu)+H\varphi(\pi,-\mu)=\frac{1}{2}e^{\pi\sqrt{\mu}}\Bigg\{\sqrt{\mu}+h+\frac{1}{2}\int_0^\pi q(t)dt+H+$$

$$+\frac{1}{\sqrt{\mu}}\Bigg[\frac{1}{4}q(0)+\frac{1}{4}q(\pi)+\frac{h}{2}\int_0^\pi q(t)dt+\frac{1}{8}\left(\int_0^\pi q(t)dt\right)^2+$$

$$+hH+\frac{H}{2}\int_0^\pi q(t)dt\Bigg]+\underline{\underline{O}}\left(\frac{1}{\mu}\right)\Bigg\} \tag{1.19.19}$$

formula o'rinli bo'ladi.

(1.19.16) va (1.19.19) tengliklardan

$$S_\lambda+\frac{c_0}{2}+\frac{c_0^2\pi^2}{8}=$$

$$=\frac{1}{4}q(0)+\frac{1}{4}q(\pi)+hH+\frac{h+H}{2}\int_0^\pi q(t)dt+\frac{1}{8}\left(\int_0^\pi q(t)dt\right)^2,$$

ya'ni ushbu

$$\sum_{n=0}^{\infty}(\lambda_n-n^2-c_0)=\frac{1}{4}[q(0)+q(\pi)]-$$

$$-\frac{1}{2\pi}\int_0^\pi q(t)dt-\frac{h+H}{\pi}-\frac{h^2+H^2}{2},$$

formula kelib chiqadi. ■

Mustaqil yechish uchun mashqlar

1. Quyidagi Shturm-Liuvill chegaraviy masalalarining regulyarlashtirilgan izini B.M.Levitan usuli yordamida hisoblang:

$$a)\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \qquad b)\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases}$$

$$c)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \qquad d)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y(\pi) = 0, \end{cases}$$

$$e)\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) = 0, \\ y'(\pi) = 0, \end{cases} \qquad f)\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) = 0, \\ y(\pi) = 0, \end{cases}$$

$$g)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 0, \\ y'(\pi) = 0, \end{cases} \qquad h)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = y(\pi), \\ y'(0) = y'(\pi), \end{cases}$$

$$j)\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = -y(\pi), \\ y'(0) = -y'(\pi), \end{cases} \qquad k)\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) - hy(\pi) = 0. \end{cases}$$

20-§. Almashtirish operatori

Shturm-Liuvill operatorlari spektral nazariyasining teskari masalalarini o'rganishda almashtirish operatorlari muhim rol o'ynaydi. Ular ikkita har xil Shturm-Liuvill tenglamasining yechimlarini o'zaro bog'laydi. Almashtirish operatorlari ilk bor B.M.Levitan va J.Delsartlarning ilmiy ishlarida vujudga kelgan. Bu operator ixtiyoriy Shturm-Liuvill tenglamasi uchun A.Povzner tomonidan qurilgan. Spektral analizning teskari masalasini yechishda almashtirish operatorlari I.M.Gelfand, B.M.Levitan va V.A.Marchenkolar tomonidan foydalanilgan.

Quyidagi

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.20.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{1.20.2}$$

Shturm-Liuvill chegaraviy masalasini qaraylik. Bu yerda λ spektral parametr, $q(x) \in C[0,\pi]$ haqiqiy funksiya va h, H chekli haqiqiy sonlar.

$c(x,\lambda)$, $s(x,\lambda)$, $\varphi(x,\lambda)$, $\psi(x,\lambda)$ funksiyalar orqali (1.20.1) tenglamaning mos ravishda quyidagi

$$c(0,\lambda) = 1, \ c'(0,\lambda) = 0; \quad s(0,\lambda) = 0, \ s'(0,\lambda) = 1;$$
$$\varphi(0,\lambda) = 1, \ \varphi'(0,\lambda) = h; \ \psi(\pi,\lambda) = 1, \ \psi'(\pi,\lambda) = -H$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini belgilaymiz.

Teorema 1.20.1. *Shturm-Liuvill tenglamasining $c(x,\lambda)$ yechimi uchun ushbu*

$$c(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t\,dt, \tag{1.20.3}$$

integral tasvir o'rinli. Bu yerda $K(x,t)$ haqiqiy uzluksiz funksiya bo'lib,

$$K(x,x) = \frac{1}{2}\int_0^x q(t)dt \tag{1.20.4}$$

shartni qanoatlantiradi.

Isbot. Yuqoridagi paragraflarda $c(x, \lambda)$ funksiya uchun ushbu

$$c(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x \frac{\sin\sqrt{\lambda}(x-\tau)}{\sqrt{\lambda}} q(\tau)c(\tau,\lambda)d\tau, \tag{1.20.5}$$

integral tenglama olingan edi. Bu integral tenglamada

$$\frac{\sin\sqrt{\lambda}(x-\tau)}{\sqrt{\lambda}} = \int_\tau^x \cos\sqrt{\lambda}(t-\tau)dt,$$

formuladan foydalansak, (1.20.5) tenglama quyidigi ko'rinishni oladi:

$$c(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x q(\tau)c(\tau,\lambda)\left(\int_\tau^x \cos\sqrt{\lambda}(t-\tau)dt\right)d\tau.$$

Integrallash tartibini almashtirib, oxirgi tenglamani

$$c(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x \left(\int_0^t q(\tau)c(\tau,\lambda)\cos\sqrt{\lambda}(t-\tau)d\tau\right)dt,$$

ko'rinishida yozib olamiz. Hosil bo'lgan ikkinchi tur Volterra integral tenglamasini ketma-ket yaqinlashish usulidan foydalanib yechamiz. Buning uchun $c(x, \lambda)$ yechimni

$$c(x,\lambda) = \sum_{n=0}^{\infty} c_n(x,\lambda), \tag{1.20.6}$$

qator ko'rinishida izlaymiz. Bu yerda

$$c_0(x,\lambda) = \cos\sqrt{\lambda}x,$$

$$c_{n+1}(x,\lambda)=\int_0^x\left(\int_0^t q(\tau)c_n(\tau,\lambda)\cos\sqrt{\lambda}(t-\tau)d\tau\right)dt. \quad (1.20.7)$$

Endi, matematik induksiya usulidan foydalanib, $c_n(x,\lambda)$, $n=1,2,\ldots$ funksiyalarni ushbu

$$c_n(x,\lambda)=\int_0^x K_n(x,t)\cos\sqrt{\lambda}\,tdt, \quad (1.20.8)$$

ko'rinishida tasvirlanishini ko'rsatamiz. Bu yerda $K_n(x,t)$ funksiya λ spektral parametrga bog'liq emas.

Dastavval $c_1(x,\lambda)$ funksiyani ushbu

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)],$$

tenglikdan foydalanib hisoblaymiz:

$$c_1(x,\lambda)=\int_0^x\left(\int_0^t q(\tau)\cos\sqrt{\lambda}\tau\cos\sqrt{\lambda}(t-\tau)d\tau\right)dt=$$

$$=\frac{1}{2}\int_0^x\cos\sqrt{\lambda}t\left(\int_0^t q(\tau)d\tau\right)dt+$$

$$+\frac{1}{2}\int_0^x\left(\int_0^t q(\tau)\cos\sqrt{\lambda}(t-2\tau)d\tau\right)dt.$$

Ikkinchi integralda $t-2\tau=s$ almashtirish bajarib,

$$c_1(x,\lambda)=\frac{1}{2}\int_0^x\cos\sqrt{\lambda}t\left(\int_0^t q(\tau)d\tau\right)dt+$$

$$+\frac{1}{4}\int_0^x\left(\int_{-t}^t q\left(\frac{t-s}{2}\right)\cos\sqrt{\lambda}sds\right)dt,$$

bo'lishini topamiz. Bu tenglikning o'ng tomonidagi ikkinchi integralda integrallash tartibini almashtiramiz. Natijada quyidagi

$$c_1(x,\lambda)=\frac{1}{2}\int_0^x \cos\sqrt{\lambda}t\left(\int_0^t q(\tau)d\tau\right)dt+$$

$$+\frac{1}{4}\int_0^x \cos\sqrt{\lambda}s\left(\int_s^x q\left(\frac{t-s}{2}\right)dt\right)ds+$$

$$+\frac{1}{4}\int_{-x}^0 \cos\sqrt{\lambda}s\left(\int_{-s}^x q\left(\frac{t-s}{2}\right)dt\right)ds=$$

$$=\frac{1}{2}\int_0^x \cos\sqrt{\lambda}t\left(\int_0^t q(\tau)\,d\tau\right)dt+$$

$$+\frac{1}{4}\int_0^x \cos\sqrt{\lambda}s\left(\int_s^x \left[q\left(\frac{t-s}{2}\right)+q\left(\frac{t+s}{2}\right)\right]dt\right)ds,$$

tenglikni hosil qilamiz. Shunday qilib, (1.20.8) tenglik $n=1$ holda to'g'riligiga ishonch hosil qildik:

$$c_1(x,\lambda)=\int_0^x K_1(x,t)\cos\sqrt{\lambda}\,t\,dt,$$

$$K_1(x,t)=\frac{1}{2}\int_0^t q(\tau)d\tau+\frac{1}{4}\int_t^x\left[q\left(\frac{s-t}{2}\right)+q\left(\frac{s+t}{2}\right)\right]ds=$$

$$=\frac{1}{2}\int_0^{\frac{x+t}{2}} q(\xi)d\xi+\frac{1}{2}\int_0^{\frac{x-t}{2}} q(\xi)d\xi,\quad t\le x. \qquad (1.20.9)$$

Faraz qilaylik (1.20.8) tenglik biror $n\ge 1$ holda o'rinli bo'lsin.

U holda (1.20.8) tenglikni (1.20.7) formulaga qo‘yib,

$$c_{n+1}(x,\lambda)=\int_0^x\int_0^t q(\tau)\cos\sqrt{\lambda}(t-\tau)\int_0^\tau K_n(\tau,s)\cos\sqrt{\lambda}s\,ds\,d\tau\,dt=$$

$$=\frac{1}{2}\int_0^x\int_0^t q(\tau)\int_0^\tau K_n(\tau,s)\Big[\cos\sqrt{\lambda}(s+t-\tau)+$$

$$+\cos\sqrt{\lambda}(s-t+\tau)\Big]\,ds\,d\tau\,dt,$$

tenglikni hosil qilamiz. Bu yerda avvalo integralni ikkiga ajratib so‘ngra ushbu $s+t-\tau=\xi$ va $s-t+\tau=\xi$, almashtirishlarni mos ravishda bajarsak, quyidagi tenglikka kelamiz:

$$c_{n+1}(x,\lambda)=\frac{1}{2}\int_0^x\int_0^t q(\tau)\int_{t-\tau}^t K_n(\tau,\xi+\tau-t)\cos\sqrt{\lambda}\xi\,d\xi\,d\tau\,dt+$$

$$+\frac{1}{2}\int_0^x\int_0^t q(\tau)\int_{\tau-t}^{2\tau-t} K_n(\tau,\xi+t-\tau)\cos\sqrt{\lambda}\xi\,d\xi\,d\tau\,dt.$$

Oxirgi tenglikda integrallar tartibini o‘zgartirib,

$$c_{n+1}(x,\lambda)=\int_0^x K_{n+1}(x,t)\cos\sqrt{\lambda}\,t\,dt,$$

ekanini topamiz. Bu yerda

$$K_{n+1}(x,t)=\frac{1}{2}\int_t^x\Bigg[\int_{\xi-t}^{\xi} q(\tau)K_n(\tau,t+\tau-\xi)d\tau+$$

$$+\int_{\frac{\xi+t}{2}}^{\xi} q(\tau)K_n(\tau,t-\tau+\xi)d\tau+\int_{\frac{\xi-t}{2}}^{\xi-t} q(\tau)K_n(\tau,-t-\tau+\xi)d\tau\Bigg]d\xi. \quad (1.20.10)$$

Endi (1.20.8) tenglikni (1.20.6) formulaga qo'yib, (1.20.3) tenglikni hosil qilamiz. Bu yerda

$$K(x,t) = \sum_{n=1}^{\infty} K_n(x,t). \qquad (1.20.11)$$

(1.20.9) va (1.20.10) tengliklardan foydalanib, ushbu

$$|K_n(x,t)| \le (Q(x))^n \frac{x^{n-1}}{(n-1)!}, \quad Q(x) = \int_0^x |q(\xi)| d\xi, \qquad (1.20.12)$$

tengsizlikni hosil qilamiz.

Haqiqatan ham, (1.20.9) tenglikdan $t \le x$ bo'lganda

$$|K_1(x,t)| \le \frac{1}{2}\int_0^{\frac{x+t}{2}} |q(\xi)| d\xi + \frac{1}{2}\int_0^{\frac{x-t}{2}} |q(\xi)| d\xi \le \int_0^x |q(\xi)| d\xi = Q(x),$$

kelib chiqadi.

Agar biror $n \ge 1$ uchun (1.20.12) tengsizlik bajarilsa, u holda (1.20.10) tenglikdan foydalanib,

$$|K_{n+1}(x,t)| \le \frac{1}{2}\int_t^x \left[\int_{\frac{\xi+t}{2}}^{\xi} |q(\tau)|(Q(\tau))^n \frac{\tau^{n-1}}{(n-1)!} d\tau + \right.$$

$$\left. \int_{\frac{\xi-t}{2}}^{\xi} |q(\tau)|(Q(\tau))^n \frac{\tau^{n-1}}{(n-1)!} d\tau \right] d\xi \le$$

$$\le \int_0^x \int_0^{\xi} |q(\tau)|(Q(\tau))^n \frac{\tau^{n-1}}{(n-1)!} d\tau d\xi \le$$

$$\le \int_0^x (Q(\xi))^{n+1} \frac{\xi^{n-1}}{(n-1)!} d\xi \le (Q(x))^{n+1} \frac{x^n}{n!},$$

baholashga ega bo'lamiz. (1.20.12) tengliklardan foydalanib, (1.20.11) funksional qatorning $0 \leq t \leq x \leq \pi$ sohada absolyut va tekis yaqinlashishini payqash qiyinchilik tug'dirmaydi. (1.20.9)-(1.20.11) formulalardan $K(x,t)$ funksiyaning silliqlik darajasi, ushbu

$$\int_0^x q(t)dt,$$

funksiyaning silliqlik darajasi bilan bir xil ekanligi kelib chiqadi. (1.20.9) va (1.20.10) tengliklarga ko'ra

$$K_1(x,x) = \frac{1}{2}\int_0^x q(t)dt, \quad K_{n+1}(x,x) = 0, \quad n \geq 1,$$

bo'lgani uchun

$$K(x,x) = \frac{1}{2}\int_0^x q(t)dt,$$

bo'ladi.

Xuddi shuningdek $s(x,\lambda)$ va $\varphi(x,\lambda)$ yechimlar uchun ham almashtirish operatorlarining ko'rinishini topish mumkin. ∎

Teorema 1.20.2. *Shturm-Liuvill tenglamasining $s(x,\lambda)$ va $\varphi(x,\lambda)$ yechimlari uchun*

$$s(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x P(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt, \tag{1.20.13}$$

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x G(x,t)\cos\sqrt{\lambda}t dt, \tag{1.20.14}$$

integral tasvirlar o'rinli. Bu yerda $P(x,t)$ va $G(x,t)$ haqiqiy uzluksiz funksiyalar bo'lib, ularning silliqligi ushbu

$$\int_0^x q(t)dt,$$

funksiyaning silliqligi bilan bir xil bo'ladi va ushbu

$$G(x,x) = h + \frac{1}{2}\int_0^x q(t)dt, \tag{1.20.15}$$

$$P(x,x) = \frac{1}{2}\int_0^x q(t)dt, \tag{1.20.16}$$

tengliklar bajariladi.

Isbot. $s(x,\lambda)$ funksiya quyidagi

$$s(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(t)s(x,\lambda)dt,$$

integral tenglamani qanoatlantiradi. Bu integral tenglamani quyidagi ko'rinishda yozib olamiz:

$$s(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x\int_0^t q(\tau)s(\tau,\lambda)\cos\sqrt{\lambda}(t-\tau)d\tau dt.$$

Oxirgi integral tenglamaga ketma-ket yaqinlashish usulini qo'llaymiz:

$$s(x,\lambda) = \sum_{n=0}^{\infty} s_n(x,\lambda), \tag{1.20.17}$$

$$s_0(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$s_{n+1}(x,\lambda) = \int_0^x\int_0^t q(\tau)s_n(\tau,\lambda)\cos\sqrt{\lambda}(t-\tau)d\tau dt.$$

Xuddi teorema 1.20.1 da qo'llanilgan usuldan foydalanib, ushbu

$$s_n(x,\lambda) = \int_0^x P_n(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt,\ n \geq 1$$

tasvirni olamiz. Bu yerda

$$P_1(x,t)=\frac{1}{2}\int\limits_0^{\frac{x+t}{2}} q(\xi)d\xi-\frac{1}{2}\int\limits_0^{\frac{x-t}{2}} q(\xi)d\xi,$$

$$P_{n+1}(x,t)=\frac{1}{2}\int\limits_t^x\left[\int\limits_{\xi-t}^{\xi} q(\tau)P_n(\tau,t+\tau-\xi)d\tau+\right.$$

$$\left.+\int\limits_{\frac{\xi+t}{2}}^{\xi} q(\tau)P_n(\tau,t-\tau+\xi)d\tau-\int\limits_{\frac{\xi-t}{2}}^{\xi-t} q(\tau)P_n(\tau,-t-\tau+\xi)d\tau\right]d\xi,$$

$$|P_n(x,t)|\le (Q(x))^n\frac{x^{n-1}}{(n-1)!},\quad Q(x)=\int\limits_0^x|q(\xi)|d\xi.$$

Bu baholashlardan (1.20.17) qator $0\le t\le x\le\pi$ to'plamda absolyut va tekis yaqinlashishi kelib chiqadi. Demak,

$$P_1(x,x)=\frac{1}{2}\int\limits_0^x q(t)dt,\quad P_{n+1}(x,x)=0,\ \ n\ge 1,$$

$$P(x,t)=\sum_{n=1}^{\infty}P_n(x,t).$$

Bularga asosan (1.20.13) va (1.20.16) tengliklarga ega bo'lamiz.■

21-§. Almashtirish operatorining umumiy ko'rinishi

E chiziqli normallangan fazo bo'lib, A va B uning E_1 va E_2 qism fazolarida aniqlangan chiziqli operatorlar bo'lsin.

Ta'rif 1.21.1. O'zi va teskarisi uzluksiz bo'lgan, $XA = BX$ yoki $A = X^{-1}BX$ shartni qanoatlantiruvchi $X : E_1 \to E_2$ chiziqli operatorga A va B operatorlar uchun almashtirish operatori deyiladi.

Lemma 1.21.1. *Agar $\varphi \in E_1$ vektor A operatorning λ xos qiymatiga mos keluvchi xos vektori bo'lsa, $\psi = X\varphi$ vektor B operatorning xuddi shu λ xos qiymatiga mos keluvchi xos vektori bo'ladi.*

Isbot. $B\psi = BX\varphi = XA\varphi = X\lambda\varphi = \lambda X\varphi = \lambda\psi$.■

Lemma 1.21.2. *A, B, C operatorlar mos ravishda E fazoning E_1, E_2, E_3 qism fazolarida aniqlangan operatorlar bo'lib, A va B operatorlar uchun almashtirish operatori X, B va C operatorlar uchun almashtirish operatori Y bo'lsa, A va C operatorlar uchun almashtirish operatori YX bo'ladi.*

Isbot. Almashtirish operatorining ta'rifiga ko'ra $XA = BX$, $YB = CY$ bo'ladi. Ikkinchi tenglikdan $B = Y^{-1}CY$ ifodani topib, birinchisiga qo'ysak, ushbu $XA = Y^{-1}CYX$ tenglikka ega bo'lamiz, ya'ni $(YX)A = C(YX)$ tenglik kelib chiqadi.■

Endi yarim o'qda berilgan Shturm-Liuvill operatori uchun almashtirish operatorining ko'rinishini topish bilan shug'ullanamiz.

$E = C^1[0,\infty)$ bo'lib, A va B operatorlar quyidagi

$$A = -\frac{d^2}{dx^2} + q_1(x), \qquad 0 \le x < \infty, \tag{1.21.1}$$

$$B = -\frac{d^2}{dx^2} + q_2(x), \qquad 0 \le x < \infty, \tag{1.21.2}$$

ko'rinishga ega bo'lsin. Bu yerda $q_1(x)$, $q_2(x)$ funksiyalar $[0, \infty)$ oraliqda berilgan uzluksiz funksiyalardir.

E_k orqali E fazodagi $f'(0) = h_k f(0)$ $(k = 1, 2)$ shartni qanoatlantiruvchi, ikki marta uzluksiz differensiallanuvchi funksiyalar to'plamini belgilaylik. Bu yerda h_1 va h_2 chekli haqiqiy sonlar.

Teorema 1.21.1. *A va B operatorlarining $X : E_1 \to E_2$ almashtirish operatori mavjud bo'lib, u uchun quyidagi tasvir o'rinli:*

$$Xf(x) = f(x) + \int_0^x K(x,t)f(t)dt. \tag{1.21.3}$$

Bu yerda $K(x,t)$ yadro quyidagi

$$\frac{\partial^2 K}{\partial x^2} - q_2(x)K = \frac{\partial^2 K}{\partial t^2} - q_1(t)K, \tag{1.21.4}$$

tenglamani va

$$K(x,x) = h_2 - h_1 + \frac{1}{2}\int_0^x [q_2(s) - q_1(s)]ds, \tag{1.21.5}$$

$$\left(\frac{\partial K}{\partial t} - h_1 K\right)\bigg|_{t=0} = 0, \tag{1.21.6}$$

shartlarni qanoatlantiradi. Aksincha, $K(x,t)$ funksiya (1.21.4) tenglamaning (1.21.5), (1.21.6) shartlarni qanoatlantiruvchi yechimi bo'lsa, (1.21.3) tenglik bilan berilgan X operator A va B chiziqli operatorlar uchun almashtirish operatori bo'ladi.

Isbot. I. $Xf(x) \in E_2$ bo'lgani uchun

$$(Xf)'|_{x=0} = h_2(Xf)|_{x=0} =$$

$$= h_2\left(f(x) + \int_0^x K(x,t)f(t)dt\right)\Bigg|_{x=0} = h_2 f(0),$$

tenglik o'rinli bo'ladi. Ikkinchi tomondan, (1.21.3) tenglikdan hosila olib, ushbu

$$(Xf)' = f'(x) + K(x,x)f(x) + \int_0^x \frac{\partial K}{\partial x} f(t)dt, \tag{1.21.7}$$

tenglikda $x = 0$ desak, quyidagi

$$(Xf)'|_{x=0} = f'(0) + K(0,0)f(0) = h_1 f(0) + K(0,0)f(0) =$$
$$= (h_1 + K(0,0))\, f(0),$$

ifoda hosil bo'ladi. $f \in E_1$ funksiyaning ixtiyoriy ekanligidan ushbu

$$K(0,0) = h_2 - h_1, \tag{1.21.8}$$

tenglik kelib chiqadi.

Endi $B(Xf) = X(Af)$ shartni ishlatamiz. (1.21.7) tenglikdan hosila olib,

$$(Xf)'' = f''(x) + f(x)\frac{d}{dx}K(x,x) + K(x,x)f'(x)+$$
$$+ \left.\frac{\partial K}{\partial x}\right|_{t=x} f(x) + \int_0^x \frac{\partial^2 K}{\partial x^2} f(t)dt,$$

ushbu

$$B(Xf) = -(Xf)'' + q_2(x)(Xf) = -f''(x) - f(x)\frac{d}{dx}K(x,x)-$$
$$-K(x,x)f'(x) - \left.\frac{\partial K}{\partial x}\right|_{t=x} f(x)+$$
$$+q_2(x)f(x) - \int_0^x \left(\frac{\partial^2 K}{\partial x^2} - q_2(x)K\right) f(t)dt\,, \tag{1.21.9}$$

tenglikni hosil qilamiz. Ushbu

$$\int_0^x K(x,t)f''(t)dt = \int_0^x K(x,t)df'(t) =$$

$$= K(x,x)f'(x) - K(x,0)f'(0) - \int_0^x \frac{\partial K(x,t)}{\partial t} df(t) =$$

$$= K(x,x)f'(x) - h_1 K(x,0)f(0) - \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x} f(x)+$$

$$+\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=0} f(0)+\int_0^x \frac{\partial^2 K(x,t)}{\partial t^2} f(t)dt,$$

bo'laklab integrallashni ishlatsak, quyidagi

$$X(Af)=[-f''(x)+q_1(x)f(x)]+$$

$$+\int_0^x K(x,t)[-f''(t)+q_1(t)f(t)]dt=$$

$$=-f''(x)+q_1(x)f(x)+\left.\left(h_1K(x,t)-\frac{\partial K(x,t)}{\partial t}\right)\right|_{t=0} f(0)-$$

$$-K(x,x)f'(x)+\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x} f(x)-\int_0^x \left(\frac{\partial^2 K}{\partial t^2}-q_1(t)K\right) f(t)dt, \qquad (1.21.10)$$

tenglik kelib chiqadi.

(1.21.9) va (1.21.10) ifodalarni bir-biriga tenglab, ushbu

$$-f(x)\frac{d}{dx}K(x,x)-\left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x} f(x)+q_2(x)f(x)-$$

$$-\int_0^x \left(\frac{\partial^2 K}{\partial x^2}-q_2(x)K\right) f(t)dt=\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x} f(x)+q_1(x)f(x)+$$

$$+\left.\left(h_1K(x,t)-\frac{\partial K(x,t)}{\partial t}\right)\right|_{t=0} f(0)-\int_0^x \left(\frac{\partial^2 K}{\partial t^2}-q_1(t)K\right) f(t)dt, \qquad (1.21.11)$$

tenglikka ega bo'lamiz. (1.21.11) tenglikda quyidagi

$$\frac{dK(x,x)}{dx}=\left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x}+\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x},$$

formuladan va $f(x)$ funksiyaning ixtiyoriy ekanligidan foydalansak, hamda $K(0,0)=h_2-h_1$ tenglikni e'tiborga olsak, (1.21.4), (1.21.5), (1.21.6) tengliklar kelib chiqadi.

Biz A va B chiziqli operatorlar uchun almashtirish operatori (1.21.3) ko'rinishda izlansa, uning yadrosi (1.21.4), (1.21.5), (1.21.6) shartlarni qanoatlantirishini isbotladik.

$K(x,t)$ funksiya (1.21.4), (1.21.5), (1.21.6) shartlarni qanoatlantirishidan (1.21.3) tenglik bilan beriladigan operator A va B operatorlarning almashtirish operatori bo'lishini ko'rsatish uchun qilingan ishlarni teskari tartibda bajarish yetarli.

II. Endi esa, (1.21.4)+(1.21.5)+(1.21.6) masala yechimining mavjudligini va yagonaligini isbotlaymiz. Buning uchun avvalo almashtirish operatorining ikkinchi xossasidan foydalanib, bu masala o'rniga soddaroq masalani qaraymiz, ya'ni umumiylikni buzmagan holda $q_2(x) \equiv 0$, $h_1 = 0$ yoki $h_2 = 0$ deb hisoblash mumkin ekanini ko'rsatamiz:

$$A = -\frac{d^2}{dx^2} + q(x), \quad B = -\frac{d^2}{dx^2}, \quad C = -\frac{d^2}{dx^2} + r(x),$$

bo'lib,

$$E_1 = \{f(x) \in E| \quad f'(0) = h_1 f(0)\},$$

$$E_2 = \{f(x) \in E| \quad f'(0) = 0\},$$

$$E_3 = \{f(x) \in E| \quad f'(0) = h_3 f(0)\},$$

bo'lsin. Bundan tashqari A va B operatorlar uchun almashtirish operatori ushbu

$$Xf(x) = f(x) + \int_0^x K_1(x,t)f(t)dt, \tag{1.21.12}$$

ko'rinishda, B va C operatorlar uchun almashtirish operatori esa quyidagi

$$Yf(x) = f(x) + \int_0^x K_2(x,t)f(t)dt, \tag{1.21.13}$$

ko'rinishda bo'lsin. U holda A va C operatorlar uchun almashtirish operatori ushbu

$$(YX)f(x) = Y\left\{ f(x) + \int_0^x K_1(x,t)f(t)dt \right\} =$$

$$= f(x) + \int_0^x K_1(x,t)f(t)dt +$$

$$+ \int_0^x K_2(x,t)\left[f(t) + \int_0^t K_1(t,s)f(s)ds \right] dt =$$

$$= f(x) + \int_0^x \left[K_1(x,t) + K_2(x,t) + \int_t^x K_2(x,s)K_1(s,t)ds \right] f(t)dt =$$

$$= f(x) + \int_0^x K_3(x,t)f(t)dt.$$

ko'rinishda bo'ladi. Bu yerda

$$K_3(x,t) = K_1(x,t) + K_2(x,t) + \int_t^x K_2(x,s)K_1(s,t)ds\,.$$

III. Demak, (1.21.4)+(1.21.5)+(1.21.6) masalani tekshirishni $q_2(x) \equiv 0$ va h_1 yoki h_2 nolga teng bo'lgan holda olib borish yetarli. Biz $h_2 = 0$ bo'lgan holni ko'rib chiqamiz, $h_1 = 0$ bo'lgan holda ham xuddi shunday bo'ladi. Ushbu

$$\frac{\partial^2 K}{\partial x^2} = \frac{\partial^2 K}{\partial t^2} - q(t)K, \qquad (1.21.14)$$

$$K|_{t=x} = -h - \frac{1}{2}\int_0^x q(s)ds, \qquad (1.21.15)$$

$$\left(\frac{\partial K(x,t)}{\partial t} - hK(x,t) \right)\Bigg|_{t=0} = 0, \qquad (1.21.16)$$

masalaning yechimi mavjudligini va yagonaligini ko'rsatishimiz kerak.

Agar bu masalada ushbu $\xi = x + t, \quad \eta = x - t$ almashtirishni bajarsak, quyidagi

$$K_x = K_\xi + K_\eta,$$

$$K_{xx} = (K_\xi + K_\eta)_\xi + (K_\xi + K_\eta)_\eta = K_{\xi\xi} + 2K_{\xi\eta} + K_{\eta\eta},$$

$$K_t = K_\xi - K_\eta,$$

$$K_{tt} = (K_\xi - K_\eta)_\xi - (K_\xi - K_\eta)_\eta = K_{\xi\xi} - 2K_{\xi\eta} + K_{\eta\eta},$$

formulalardan, ushbu

$$K_{\xi\eta} = -\frac{1}{4}q\left(\frac{\xi - \eta}{2}\right)K,$$

$$K|_{\eta=0} = -h - \frac{1}{2}\int_0^{\frac{\xi}{2}} q(s)ds,$$

$$(K_\xi - K_\eta - hK)|_{\eta=\xi} = 0,$$

tengliklarni hosil qilamiz.

Endi esa oxirgi masalaga ekvivalent bo'lgan integral tenglama tuzamiz. Buning uchun oxirgi masalada quyidagi

$$K(x,t) = K\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right) = A(\xi, \eta),$$

belgilashni kiritib, uni

$$A_{\alpha\beta} = -\frac{1}{4}q\left(\frac{\alpha - \beta}{2}\right)A, \tag{1.21.17}$$

$$A(\alpha, 0) = -h - \frac{1}{2}\int_0^{\frac{\alpha}{2}} q(s)ds, \tag{1.21.18}$$

$$(A_\alpha - A_\beta - hA)|_{\beta=\alpha} = 0, \tag{1.21.19}$$

ko'rinishda yozib olamiz, hamda (1.21.17) tenglikni $[0, \eta]$ oraliqda β bo'yicha integrallaymiz:

$$A_\alpha(\alpha, \eta) - A_\alpha(\alpha, 0) = -\frac{1}{4}\int_0^\eta q\left(\frac{\alpha-\beta}{2}\right) A(\alpha, \beta) d\beta. \quad (1.21.20)$$

(1.21.18) tenglikka asosan $A_\alpha(\alpha, 0) = -\frac{1}{4} q\left(\frac{\alpha}{2}\right)$ bo'lgani uchun ushbu

$$A_\alpha(\alpha, \eta) = -\frac{1}{4} q\left(\frac{\alpha}{2}\right) - \frac{1}{4}\int_0^\eta q\left(\frac{\alpha-\beta}{2}\right) A(\alpha, \beta) d\beta, \quad (1.21.21)$$

tenglik o'rinli bo'ladi. (1.21.21) tenglikni $[\eta, \xi]$ oraliqda α bo'yicha integrallab, ushbu

$$A(\xi, \eta) - A(\eta, \eta) =$$
$$= -\frac{1}{4}\int_\eta^\xi q\left(\frac{\alpha}{2}\right) d\alpha - \frac{1}{4}\int_\eta^\xi \left\{ \int_0^\eta q\left(\frac{\alpha-\beta}{2}\right) A(\alpha, \beta) d\beta \right\} d\alpha$$
(1.21.22)

ayniyatga ega bo'lamiz.

Endi $A(\eta, \eta)$ ni hisoblaymiz. (1.21.19) tenglikka ko'ra ushbu

$$2A_\alpha|_{\beta=\alpha} = A_\alpha|_{\beta=\alpha} + A_\alpha|_{\beta=\alpha} = A_\alpha|_{\beta=\alpha} +$$
$$+(A_\beta + hA)|_{\beta=\alpha} = (A_\alpha + A_\beta + hA)|_{\beta=\alpha} = \frac{dA(\alpha, \alpha)}{d\alpha} +$$
$$+hA(\alpha, \alpha) = e^{-h\alpha}(e^{h\alpha} A(\alpha, \alpha))', \quad (1.21.23)$$

ayniyat bajariladi. (1.21.21) va (1.21.23) tengliklardan quyidagi

$$(e^{h\alpha} A(\alpha, \alpha))' = -\frac{1}{2} e^{h\alpha} \left\{ q\left(\frac{\alpha}{2}\right) + \int_0^\alpha q\left(\frac{\alpha-\beta}{2}\right) A(\alpha, \beta) d\beta \right\},$$
(1.21.24)

formula kelib chiqadi. (1.21.24) ayniyatni $[0, \eta]$ oraliqda α bo'yicha integrallaymiz va (1.21.18) shartdan foydalanib, quyidagi

$$A(\eta, \eta) = -he^{-h\eta} -$$

$$-\frac{1}{2}e^{-h\eta}\int_0^\eta e^{h\alpha}\left\{q\left(\frac{\alpha}{2}\right)+\int_0^\alpha q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)d\beta\right\}d\alpha, \tag{1.21.25}$$

tenglikni keltirib chiqaramiz. (1.21.25) ifodani (1.21.22) tenglikka qo'yib, ushbu

$$A(\xi,\eta)=-he^{-h\eta}-\frac{1}{4}\int_\eta^\xi q\left(\frac{\alpha}{2}\right)d\alpha-\frac{1}{2}e^{-h\eta}\int_0^\eta e^{h\alpha}q\left(\frac{\alpha}{2}\right)d\alpha-$$

$$-\frac{1}{2}e^{-h\eta}\int_0^\eta e^{h\alpha}\left\{\int_0^\alpha q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)d\beta\right\}d\alpha-$$

$$-\frac{1}{4}\int_\eta^\xi\left\{\int_0^\eta q\left(\frac{\alpha-\beta}{2}\right)A(\alpha,\beta)d\beta\right\}d\alpha, \tag{1.21.26}$$

ayniyatga ega bo'lamiz, ya'ni $A(\alpha,\beta)$ funksiya (1.21.26) integral tenglamani qanoatlantirar ekan. Aksincha, $A(\alpha,\beta)$ funksiya (1.21.26) integral tenglamani qanoatlantirsa va $q(x)$ funksiya uzluksiz differensiallanuvchi bo'lsa, (1.21.26) integral tenglamadan foydalanib, $A(\alpha,\beta)$ funksiya (1.21.17) +(1.21.18)+(1.21.19) masalaning yechimi bo'lishini to'g'ridan-to'g'ri tekshirib ko'rish mumkin.

(1.21.26) tenglama Volterra turidagi integral tenglama bo'lgani uchun uning yechimi mavjud va yagonadir. Buni ketma-ket yaqinlashishlar usuli bilan ko'rsatish mumkin.

$q(x)$ funksiya uzluksiz differensiallanuvchi bo'lmasa, uni uzluksiz differensiallanuvchi $q_n(x)$ funksiyalar bilan yaqinlashtirib, $K_n(x,t)$ funksiyalar ketma-ketligini hosil qilamiz, bu ketma-ketlikning limiti $K(x,t)$ almashtirish operatorining yadrosi bo'ladi.■

Izoh 1.21.1. Shuni aytib o'tish kerakki, $K(x,t)$ yadro λ parametrga bog'liq emas.

Misol 1. Agar teorema 1.21.1 ning shartida A va B operatorlar quyidagi

$$A = -\frac{d^2}{dx^2}, \qquad h_1 = 0,$$

va

$$B = -\frac{d^2}{dx^2} + q(x), \qquad h_2 = h,$$

ko'rinishda berilgan bo'lsa, almashtirish operatori ushbu

$$Xf(x) = f(x) + \int_0^x K(x,t)f(t)dt,$$

ko'rinishda bo'ladi. Bu yerda $K(x,t)$ yadro ushbu

$$\begin{cases} K_{xx} - q(x)K = K_{tt}\,, \\ K(x,x) = h + \frac{1}{2}\int\limits_0^x q(s)ds\,, \\ K_t|_{t=0} = 0\,, \end{cases} \qquad (1.21.27)$$

masalaning yechimidir.

$\varphi(x,\lambda)$ va $\varphi_0(x,\lambda)$ orqali mos ravishda quyidagi

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y(0) = 1\,, \\ y'(0) = h\,, \end{cases} \quad \text{va} \quad \begin{cases} -y'' = \lambda y\,, \\ y(0) = 1\,, \\ y'(0) = 0\,, \end{cases}$$

Koshi masalalarining yechimlarini belgilaylik. Bizga ma'lumki

$$\varphi_0(x,\lambda) = \cos\sqrt{\lambda}x$$

bo'ladi. Almashtirish operatorining hossasiga ko'ra ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt, \qquad (1.21.28)$$

tenglik o'rinli bo'ladi.

Misol 2. Agar teorema 1.21.1 ning shartida A va B operatorlar ushbu

$$A = -\frac{d^2}{dx^2} + q(x), \quad h_1 = h,$$

va

$$B = -\frac{d^2}{dx^2}, \quad h_2 = 0,$$

ko'rinishda berilgan bo'lsin desak, almashtirish operatori quyidagi

$$Xf(x) = f(x) + \int_0^x H(x,t)f(t)dt,$$

ko'rinishda bo'ladi. Bu yerda $H(x,t)$ yadro ushbu

$$\begin{cases} H_{xx} = H_{tt} - q(t)H\,, \\ H(x,x) = -h - \frac{1}{2}\int_0^x q(s)ds\,, \\ (H_t - hH)|_{t=0} = 0\,, \end{cases} \qquad (1.21.29)$$

masalaning yechimidir.

Almashtirish operatorining hossasiga ko'ra ushbu

$$\cos\sqrt{\lambda}x = \varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)dt, \qquad (1.21.30)$$

tenglik bajariladi.

Izoh 1.21.2. Bu yerda ham $H(x,t)$ yadro λ parametrga bog'liq emas.

Keyinchalik

$$q(x) = 2\frac{dK(x,x)}{dx}, \qquad (1.21.31)$$

$$q(x) = -2\frac{dH(x,x)}{dx}, \qquad (1.21.32)$$

va

$$h = K(0,0) = -H(0,0), \qquad (1.21.33)$$

bog'lanishlar muhim ahamiyat kasb etadi.

22-§. Teskari masalaning qo'yilishi. Yagonalik teoremalari

Mazkur paragrafda spektral analiz teskari masalasining qo'yilishi va yagonalik teoremalarini isbotlash usullari bilan tanishamiz. Bu usullarning tatbiqiy ahamiyati juda keng bo'lgani uchun ulardan spektral analizning har xil turdagi teskari masalalarini o'rganishda foydalanish mumkin.

Shturm-Liuvill operatori spektral nazariyasining teskari masalasi ilk bor V.A.Ambarsumyan [6] tomonidan o'rganilgan.

Ushbu

$$-y'' + q(x)y = \lambda y, \quad y'(0) = 0, \quad y'(\pi) = 0, \tag{1.22.1}$$

Shturm-Liuvill chegaraviy masalasini qaraymiz. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya.

Agar (1.22.1) chegaraviy masalada $q(x) \equiv 0$, $x \in [0, \pi]$ bo'lsa, u holda $\lambda_n = n^2$, $n \geq 0$ bo'lishi ravshan.

Teorema 1.22.1 *(V.A. Ambarsumyan). Agar (1.22.1) chegaraviy masalaning xos qiymatlari uchun, ushbu*

$$\lambda_n = n^2, \quad n \geq 0, \tag{1.22.2}$$

tenglik bajarilsa, u holda $q(x) \equiv 0$, $x \in [0, \pi]$ bo'ladi.

Isbot. (1.22.1) chegaraviy masalaning xos qiymatlari uchun

$$\sqrt{\lambda_n} = n + \frac{c_0}{n} + \frac{\gamma_n}{n}, \quad \{\gamma_n\} \in l_2, \tag{1.22.3}$$

asimptotik formulaning o'rinli bo'lishi mazkur bobning beshinchi paragrafida ko'rsatilgan edi. Bu yerda

$$c_0 = \frac{1}{2\pi} \int_0^{\pi} q(x)dx. \tag{1.22.4}$$

(1.22.2) va (1.22.3) tengliklarni tenglashtirib,

$$c_0 = 0, \quad \int_0^{\pi} q(x)dx = 0, \tag{1.22.5}$$

ekanini topamiz. (1.22.1) chegaraviy masalaning $\lambda_0 = 0$ eng kichik xos qiymatiga mos keluvchi xos funksiya $y_0(x)$ bo'lsa u holda

$$y_0'' + q(x)y_0 = 0, \quad y_0'(0) = 0, \quad y_0'(\pi) = 0, \tag{1.22.6}$$

bo'ladi. Ossilyatsiya teoremasiga asosan $y_0(x)$ funksiya $(0, \pi)$ oraliqda nolga ega emas. Agar $y_0(0) = 0$ yoki $y_0(\pi) = 0$ bo'lsa chegaraviy shartlardan $y_0(x) \equiv 0$ ziddiyat kelib chiqadi. Demak, $y_0(x) \neq 0$, $x \in [0, \pi]$.

Ushbu

$$0 = \int_0^{\pi} q(x)dx = \int_0^{\pi} \frac{y_0''(x)}{y_0(x)}dx = \int_0^{\pi} \left(\frac{y_0'(x)}{y_0(x)}\right)^2 dx + \int_0^{\pi} \left(\frac{y_0'(x)}{y_0(x)}\right)' dx =$$

$$= \int_0^{\pi} \left(\frac{y_0'(x)}{y_0(x)}\right)^2 dx + \frac{y_0'(x)}{y_0(x)}\bigg|_{x=0}^{x=\pi} =$$

$$= \int_0^{\pi} \left(\frac{y_0'(x)}{y_0(x)}\right)^2 dx + \frac{y_0'(\pi)}{y_0(\pi)} - \frac{y_0'(0)}{y_0(0)} = \int_0^{\pi} \left(\frac{y_0'(x)}{y_0(x)}\right)^2 dx,$$

tenglikdan

$$\int_0^{\pi} \left(\frac{y_0'(x)}{y_0(x)}\right)^2 dx = 0,$$

hosil bo'ladi. Oxirgi tenglikdan $y_0'(x) = 0$, ya'ni $y_0(x) = \text{const}$ kelib chiqadi. (1.22.6) tenglamadan

$$y_0''(x) - q(x)y_0(x) = 0,$$

$q(x) = 0$ ekanligi kelib chiqadi.■

Umuman olganda $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ xos qiymatlarning, ya'ni spektrning berilishi Shturm-Liuvill chegaraviy masalasini yagona aniqlamaydi. Chunki, ikkita har-xil Shturm-Liuvill chegaraviy masalasi bir xil spektrga ega bo'lishi mumkin.

Masalan, ushbu[2]

$$-y'' = \lambda y, \qquad y'(0) = y'(\pi) = 0,$$

va

$$-y''+\frac{2}{(1+x)^2}y = \lambda y, \qquad y'(0)+y(0) = 0, \quad y'(\pi)+\frac{1}{\pi+1}y(\pi) = 0$$

Shturm-Liuvill chegaraviy masalalari $\lambda_n = n^2$, $n \geq 0$ bir xil spektrga ega.

Quyidagi

$$-y'' + q(x)y = \lambda y, \quad x \in [0, \pi], \tag{1.22.7}$$

$$y'(0) - hy(0) = 0, \tag{1.22.8}$$

$$y'(\pi) + Hy(\pi) = 0, \tag{1.22.9}$$

Shturm-Liuvill chegaraviy masalasi berilgan bo'lsin. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya, h, H - chekli haqiqiy sonlar.

$\varphi(x, \lambda)$ orqali (1.22.7) tenglamaning

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h,$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

(1.22.7)-(1.22.9) Shturm-Liuvill chegaraviy masalasining xos qiymatlar ketma-ketligini λ_n, $n \geq 0$ orqali belgilaylik. Bu xos qiymatlarga quyidagi $\varphi(x, \lambda_n)$, $n \geq 0$ xos funksiyalar mos keladi. Normallovchi o'zgarmaslar ketma-ketligini

$$\alpha_n = \sqrt{\int_0^\pi \varphi^2(x, \lambda_n)dx}, \quad n \geq 0,$$

[2]Bu misolda $H + h < 0$. Ambarsumyan teoremasini, ushbu $H + h \geq 0$ shartni qanoatlantiruvchi Shturm-Liuvill chegaraviy masalasi uchun ham isbotlash mumkin.

orqali belgilaymiz.

Ta'rif 1.22.1. Ushbu $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ ketma-ketliklar juftligiga Shturm-Liuvill chegaraviy masalasining spektral xarakteristikalari yoki spektral berilganlari deyiladi.

Ta'rif 1.22.2. $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalar yordamida $q(x)$ funksiyani va h, H sonlarni topish masalasiga teskari spektral masala deyiladi.

Yuqoridagi masalani atroflicha o'rganish maqsadida ikkinchi

$$-y'' + \tilde{q}(x)y = \lambda y, \quad x \in [0, \pi], \tag{1.22.10}$$

$$y'(0) - \tilde{h}y(0) = 0, \tag{1.22.11}$$

$$y'(\pi) + \tilde{H}y(\pi) = 0, \tag{1.22.12}$$

Shturm-Liuvill chegaraviy masalasini qaraymiz. Bu yerda $\tilde{q}(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya, $\tilde{h}$, $\tilde{H}$ - chekli haqiqiy sonlar.

$\tilde{\varphi}(x, \lambda)$ orqali (1.22.10) tenglamaning

$$\tilde{\varphi}(0, \lambda) = 1, \quad \tilde{\varphi}'(0, \lambda) = \tilde{h},$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

(1.22.10)-(1.22.12) chegaraviy masalasining xos qiymatlar ketma-ketligini $\tilde{\lambda}_n$, $n \geq 0$ orqali, normallovchi o'zgarmaslar ketma-ketligini

$$\tilde{\alpha}_n = \sqrt{\int_0^{\pi} \tilde{\varphi}^2(x, \tilde{\lambda}_n)dx}, \quad n \geq 0,$$

orqali belgilaymiz.

Teorema 1.22.2 *(V.A.Marchenko). Agar $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$, $n \geq 0$ bo'lsa, u holda ushbu*

$$h = \tilde{h}, \quad H = \tilde{H}, \quad q(x) = \tilde{q}(x), \quad x \in [0, \pi],$$

tengliklar o'rinli bo'ladi.

Isbot. Almashtirish operatori haqidagi teorema 1.21.1 ga asosan

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \int_0^x K(x,t)\varphi(t,\lambda)dt, \qquad (1.22.13)$$

tenglik o'rinli bo'ladi.

Ixtiyoriy $f(x) \in L^2(0,\pi)$ funksiyaning $(f,\tilde{\varphi})$ Furye koeffitsiyentini (1.22.13) tenglikdan foydalanib hisoblaymiz:

$$\int_0^\pi f(x)\tilde{\varphi}(x,\lambda)dx = \int_0^\pi f(x)\left[\varphi(x,\lambda) + \int_0^x K(x,t)\varphi(t,\lambda)dt\right]dx =$$

$$= \int_0^\pi f(x)\varphi(x,\lambda)dx + \int_0^\pi f(x)\left[\int_0^x K(x,t)\varphi(t,\lambda)dt\right]dx =$$

$$= \int_0^\pi f(x)\varphi(x,\lambda)dx + \int_0^\pi \left[\int_x^\pi K(t,x)f(t)dt\right]\varphi(x,\lambda)dx =$$

$$= \int_0^\pi \left[f(x) + \int_x^\pi K(t,x)f(t)dt\right]\varphi(x,\lambda)dx = \int_0^\pi g(x)\varphi(x,\lambda)dx. \qquad (1.22.14)$$

Bu yerda

$$g(x) = f(x) + \int_x^\pi K(t,x)f(t)dt. \qquad (1.22.15)$$

(1.22.14) tenglikda $\lambda = \tilde{\lambda}_n$, $n = 0, 1, 2, \ldots$ deb olsak, u holda ushbu

$$\tilde{a}_n = \frac{1}{\tilde{\alpha}_n}\int_0^\pi f(x)\tilde{\varphi}(x,\tilde{\lambda}_n)dx = \frac{1}{\alpha_n}\int_0^\pi f(x)\tilde{\varphi}(x,\lambda_n)dx,$$

$$a_n = \frac{1}{\alpha_n}\int_0^\pi g(x)\varphi(x,\lambda_n)dx$$

Furye koeffitsiyentlari uchun $\tilde{a}_n = a_n$, $n = 0, 1, 2, \ldots$ tenglik bajariladi. Parseval tengligiga asosan

$$\int_0^\pi |f(x)|^2 dx = \sum_{n=0}^{\infty} |\tilde{a}_n|^2 = \sum_{n=0}^{\infty} |a_n|^2 = \int_0^\pi |g(x)|^2 dx,$$

kelib chiqadi, ya'ni

$$\|f\|_{L^2} = \|g\|_{L^2}. \tag{1.22.16}$$

Ushbu

$$Af(x) = f(x) + \int_x^\pi K(t, x) f(t) dt,$$

operatorni qaraylik. U holda (1.22.15) ga asosan

$$Af(x) = g(x),$$

bo'ladi. (1.22.16) tenglikdan esa

$$\|Af\|_{L^2} = \|f\|_{L^2},$$

kelib chiqadi. Bu esa A operatorning $L^2(0, \pi)$ fazoda unitarligini bildiradi. Unitar operatorlar uchun $A^*A = I$ tenglik o'rinli bo'ladi.

A^* operatorni topish qiyin emas:

$$A^*h(x) = h(x) + \int_0^x K(x, t) h(t) dt.$$

$A^*\{Af(x)\}$ ning aniq ifodasini topamiz:

$$A^*\{Af(x)\} = Af(x) + \int_0^x K(x, t)\{Af(t)\} dt =$$

$$= f(x) + \int_x^\pi K(t, x) f(t) dt +$$

$$+\int_0^x K(x,t)\left\{f(t)+\int_t^\pi K(s,t)f(s)ds\right\}dt=$$

$$=f(x)+\int_x^\pi K(t,x)f(t)dt+\int_0^x K(x,t)f(t)dt+$$

$$+\int_0^x\left\{\int_t^\pi K(x,t)K(s,t)f(s)ds\right\}dt.$$

Bu yerda integrallash tartibini almashtirib quyidagi

$$A^*\{Af(x)\}=f(x)+\int_0^x\left\{K(x,t)+\int_0^t K(x,s)K(t,s)ds\right\}f(t)dt+$$

$$+\int_x^\pi\left\{K(t,x)+\int_0^x K(x,s)K(t,s)ds\right\}f(t)dt$$

formulaga ega bo'lamiz.

$A^*\{Af(x)\}=f(x)$ ayniyatdan foydalansak

$$\int_0^x\left\{K(x,t)+\int_0^t K(x,s)K(t,s)ds\right\}f(t)dt+$$

$$+\int_x^\pi\left\{K(t,x)+\int_0^x K(x,s)K(t,s)ds\right\}f(t)dt=0$$

tenglik keib chiqadi. Bu yerda

$$f(t)=\begin{cases}K(x,t)+\int\limits_0^t K(x,s)K(t,s)ds, & t\in[0,x],\\ 0, & t\in(x,\pi],\end{cases}$$

deb olsak, ushbu

$$\int_0^x\left\{K(x,t)+\int_0^t K(x,s)K(t,s)ds\right\}^2dt=0$$

tengik kelib chiqadi. Bunga ko‘ra

$$K(x,t)+\int_0^t K(x,s)K(t,s)ds=0.$$

Oxirgi tenglik x ning har bir tayinlangan qiymatida $K(x,t)$ funksiyaga nisbatan bir jinsli Volterra integral tenglamasidir. Bunday tenglama faqat nol yechimga ega bo‘lishidan $K(x,t)\equiv 0$ $(t\leq x)$ kelib chiqadi. Buni (1.22.13) formulaga qo‘ysak

$$\varphi(x,\lambda)=\tilde{\varphi}(x,\lambda),$$

ayniyat xosil bo‘ladi. Boshlang‘ich va chegaraviy shartlarga ko‘ra

$$h=\tilde{h} \text{ va } H=\tilde{H}.$$

Ushbu

$$-\varphi''+q(x)\varphi=\lambda\varphi,$$
$$-\varphi''+\tilde{q}(x)\varphi=\lambda\varphi,$$

tengliklardan foydalanib,

$$[q(x)-\tilde{q}(x)]\varphi(x,\lambda)=0,$$

bo‘lishini topamiz. Bundan va $q(x)$, $\tilde{q}(x)$ funksiyalarning uzluksizligini hamda $\varphi(x,\lambda)$ funksiyaning nollarini ajralganligini e’tiborga olsak, $q(x)=\tilde{q}(x)$ ayniyat kelib chiqadi. ■

II BOB. TESKARI MASALA YECHISHNING GELFAND-LEVITAN USULI

Chiziqli differensial operatorlar spektral nazariyasining teskari masalalari zamonaviy matematik fizikaning muhim sohalaridan biridir. Oldingi bobda Shturm-Liuvill operatori $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalari uchun asimptotikalar topilgan edi. Bu bobda spektral xarakteristikalar yordamida Shturm-Liuvill tenglamasining koeffitsiyentini va chegaraviy shartlarini topish, ya'ni teskari masalani yechish bilan shug'ullanamiz.

Teskari masalani yechishning bir nechta usullari bor. Bu usullar orasida Gelfand-Levitan usuli muhim o'rin egallaydi. Usulning asosiy bosqichlaridan biri, almashtirish operatorining yadrosiga nisbatan olingan chiziqli integral tenglamadir. Bu integral tenglama teskari masalaning asosiy integral tenglamasi yoki Gelfand-Levitan integral tenglamasi deb yuritiladi. Teskari masalani yechish usulini bayon qilishdan oldin Shturm-Liuvill masalasi bilan bog'liq bo'lgan qatorlar va integral tenglamalarga oid ayrim zaruriy ma'lumotlarni keltiramiz.

1-§. Zaruriy ma'lumotlar

Quyidagi Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz:

$$Ly \equiv -y'' + q(x)y = \lambda y,\ 0 \le x \le \pi, \tag{2.1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0. \end{cases} \tag{2.1.2}$$

Bu yerda $q(x) \in L^2(0,\pi)$ haqiqiy funksiya bo'lib, h, H – chekli haqiqiy sonlar. Oldingi bobda (2.1.1)+(2.1.2) chegaraviy masalaning $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalari quyidagi

$$\begin{gathered}\sqrt{\lambda_n} \equiv k_n = n + \frac{\omega}{n\pi} + \frac{\chi_n}{n}, \quad \{\chi_n\} \in l_2, \\ \alpha_n = \frac{\pi}{2} + \frac{\tilde{\chi}_n}{n}, \quad \{\tilde{\chi}_n\} \in l_2, \\ \lambda_n \neq \lambda_m, \ \ n \neq m, \ \ \alpha_n > 0, \ \ n = 0, 1, 2, \dots, \end{gathered} \tag{2.1.3}$$

shartlarni qanoatlantirishini ko'rsatgan edik. Bu yerda

$$\alpha_n = \int_0^{\pi} \varphi^2(x, \lambda_n) dx \tag{2.1.4}$$

bo'lib, $\varphi(x, \lambda)$ orqali (2.1.1) tenglamaning

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi belgilangan.

Lemma 2.1.1. *Agar berilgan $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari (2.1.3) shartlarni qanoatlantirsa, u holda ushbu*

$$a(x) = \sum_{n=0}^{\infty} \left(\frac{\cos\sqrt{\lambda_n}x}{\alpha_n} - \frac{\cos nx}{\alpha_n^0} \right), \tag{2.1.5}$$

funksiya $W_2^1(0, 2\pi)$ Sobolev fazosiga qarashli bo'ladi. Bu yerda

$$\alpha_n^0 = \begin{cases} \dfrac{\pi}{2}, & n > 0, \\ \pi, & n = 0. \end{cases}$$

Isbot. (2.1.5) funksional qatorni quyidagi

$$a(x) = \sum_{n=0}^{\infty} \left[\frac{1}{\alpha_n^0} \left(\cos\sqrt{\lambda_n}x - \cos nx \right) + \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \cos\sqrt{\lambda_n}x \right],$$

ko'rinishda yozib olamiz. Ushbu

$$\delta_n = \sqrt{\lambda_n} - n,$$

belgilashni kiritamiz va quyidagi

$$\cos\sqrt{\lambda_n}x - \cos nx = \cos(n + \delta_n)x - \cos nx =$$

$$= -\sin \delta_n x \sin nx - 2\sin^2 \frac{\delta_n x}{2} \cos nx =$$

$$= -\delta_n x \sin nx - (\sin \delta_n x - \delta_n x) \sin nx - 2\sin^2 \frac{\delta_n x}{2} \cos nx$$

tenglikdan foydalanib, $a(x)$ uchun yozilgan qatorni ikki guruhga ajratamiz:

$$a(x) = A_1(x) + A_2(x).$$

Bu yerda

$$A_1(x) = -\frac{2\omega x}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin nx}{n} = -\frac{2\omega x}{\pi^2} \cdot \frac{\pi - x}{2}, \quad 0 < x < 2\pi,$$

$$A_2(x) = \frac{1}{\alpha_0} \cos \sqrt{\lambda_0} x - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \cos \sqrt{\lambda_n} x -$$

$$-\frac{2}{\pi} x \sum_{n=1}^{\infty} \chi_n \frac{\sin nx}{n} -$$

$$-\frac{2}{\pi} \sum_{n=1}^{\infty} (\sin \delta_n x - \delta_n x) \sin nx - \frac{4}{\pi} \sum_{n=1}^{\infty} \sin^2 \frac{\delta_n x}{2} \cos nx. \quad (2.1.6)$$

Ushbu

$$\delta_n = \underline{\underline{O}} \left(\frac{1}{n} \right), \quad \frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} = \frac{\gamma_n}{n}, \quad \{\gamma_n\} \in l_2,$$

munosabatlardan, (2.1.6) tenglikda ishtirok qilayotgan funksional qatorlarning $[0, 2\pi]$ oraliqda absolyut va tekis yaqinlashishi kelib chiqadi hamda $A_2(x) \in W_2^1(0, 2\pi)$. Demak, $a(x) \in W_2^1(0, 2\pi)$.∎

Xuddi shuningdek quyidagi lemmani ham isbotlash mumkin.

Lemma 2.1.2. *Agar berilgan $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari quyidagi*

$$\sqrt{\lambda_n} = n + \sum_{j=1}^{N+1} \frac{c_j}{n^j} + \frac{\chi_n}{n^{N+1}}, \quad c_{2p} = 0, \ p \geq 0,$$

$$\alpha_n = \frac{\pi}{2} + \sum_{j=1}^{N+1} \frac{a_j}{n^j} + \frac{\tilde{\chi}_n}{n^{N+1}}, \quad a_{2p+1} = 0, \quad p \geq 0,$$

$$\lambda_n \neq \lambda_m, \quad n \neq m, \quad \alpha_n > 0, \quad n \geq 0, \quad \{\chi_n\}, \{\tilde{\chi}_n\} \in l_2,$$

shartlarni qanoatlantirsa, u holda $a(x) \in W_2^{N+1}(0, 2\pi)$ *bo'ladi.*

Lemma 2.1.3. $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ *haqiqiy sonlar ketma-ketliklari (2.1.3) shartlarni qanoatlantirsin.* $c_0 > 0$ *tayinlangan son bo'lsin. Agar* $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ *va* $\{\tilde{\alpha}_n\}_{n=0}^{\infty}$, $\tilde{\alpha}_n > 0$ *haqiqiy sonlar ketma-ketliklari quyidagi*

$$\delta = \left(\sum_{n=0}^{\infty} ((n+1)\xi_n)^2\right)^{\frac{1}{2}} \leq c_0,$$

shartni qanoatlantirsa, u holda

$$\tilde{a}(x) = \sum_{n=0}^{\infty} \left(\frac{\cos\sqrt{\tilde{\lambda}_n}x}{\tilde{\alpha}_n} - \frac{\cos\sqrt{\lambda_n}x}{\alpha_n}\right) \in W_2^1(0, 2\pi), \qquad (2.1.7)$$

bo'lib,

$$\max_{0 \leq x \leq 2\pi} |\tilde{a}(x)| \leq c\sum_{n=0}^{\infty} \xi_n, \quad \|\tilde{a}(x)\|_{W_2^1} \leq c\delta,$$

bo'ladi. Bu yerda

$$\xi_n = \left|\sqrt{\tilde{\lambda}_n} - \sqrt{\lambda_n}\right| + |\tilde{\alpha}_n - \alpha_n|,$$

c o'zgarmas son $\{\lambda_n\}_{n=0}^{\infty}$ *va* $\{\alpha_n\}_{n=0}^{\infty}$ *ketma-ketliklarga va* c_0 *songa bog'liq.*

Isbot. Lemmaning shartiga ko'ra

$$\sum_{n=0}^{\infty} \xi_n = \sum_{n=0}^{\infty} (n+1)\xi_n \cdot \frac{1}{n+1} \leq$$

$$\leq \sqrt{\sum_{n=0}^{\infty} ((n+1)\xi_n)^2}\sqrt{\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}} \leq c\delta.$$

(2.1.7) ifodani quyidagi ko'rinishda yozib olamiz:

$$\tilde{a}(x) = \sum_{n=0}^{\infty} \left(\left(\frac{1}{\tilde{\alpha}_n} - \frac{1}{\alpha_n}\right)\cos\sqrt{\lambda_n}x + \right.$$

$$+\frac{1}{\tilde{\alpha}_n}\left(\cos\sqrt{\tilde{\lambda}_n}x-\cos\sqrt{\lambda_n}x\right)\Bigg)=$$

$$=\sum_{n=0}^{\infty}\left(\frac{\alpha_n-\tilde{\alpha}_n}{\tilde{\alpha}_n\alpha_n}\cos\sqrt{\tilde{\lambda}_n}x+\right.$$

$$\left.+\frac{2}{\tilde{\alpha}_n}\sin\frac{(\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n})x}{2}\sin\frac{(\sqrt{\lambda_n}+\sqrt{\tilde{\lambda}_n})x}{2}\right).$$

Bu qatorlar absolyut va tekis yaqinlashuvchi bo'ladi. Shuning uchun $\tilde{a}(x)$ uzluksiz funksiya bo'lib,

$$|\tilde{a}(x)|\leq c\sum_{n=0}^{\infty}\xi_n,$$

baholash o'rinli bo'ladi. (2.1.7) tenglikni differensiallab, $\tilde{a}'(x)$ hosilani hisoblaymiz:

$$\tilde{a}'(x)=-\sum_{n=0}^{\infty}\left(\frac{\sqrt{\tilde{\lambda}_n}\sin\sqrt{\tilde{\lambda}_n}x}{\tilde{\alpha}_n}-\frac{\sqrt{\lambda_n}\sin\sqrt{\lambda_n}x}{\alpha_n}\right)=$$

$$=-\sum_{n=0}^{\infty}\left[\left(\frac{\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n}-\frac{\sqrt{\lambda_n}}{\alpha_n}\right)\sin\sqrt{\lambda_n}x+\right.$$

$$+$$

$$\left.\frac{\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n}(\sin\sqrt{\tilde{\lambda}_n}x-\sin\sqrt{\lambda_n}x)\right]=$$

$$=\sum_{n=0}^{\infty}\left[\left(\frac{\tilde{\alpha}_n\sqrt{\lambda_n}-\alpha_n\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n\alpha_n}\right)\sin\sqrt{\lambda_n}x+\right.$$

$$+x\frac{\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n}(\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n})\cos\frac{\sqrt{\lambda_n}+\sqrt{\tilde{\lambda}_n}}{2}x+$$

$$\left.+\frac{\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n}\left(2\sin\frac{\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n}}{2}x-x(\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n})\right)\cos\frac{\sqrt{\lambda_n}+\sqrt{\tilde{\lambda}_n}}{2}x\right]=$$

$$=A_1(x)+A_2(x).$$

Bu yerda

$$A_1(x)=\sum_{n=0}^{\infty}\frac{\tilde{\alpha}_n\sqrt{\lambda_n}-\alpha_n\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n\alpha_n}\sin\sqrt{\lambda_n}x+$$

$$+x\sum_{n=0}^{\infty}\frac{\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n}\left(\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n}\right)\cos\sqrt{\lambda_n}x, \qquad (2.1.8)$$

$$A_2(x)=\sum_{n=0}^{\infty}\frac{\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n}\left(2\sin\frac{\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n}}{2}x-\right.$$

$$\left.-x(\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n})\right)\cos\frac{\sqrt{\lambda_n}+\sqrt{\tilde{\lambda}_n}}{2}x+$$

$$+x\sum_{n=0}^{\infty}\frac{\sqrt{\tilde{\lambda}_n}}{\tilde{\alpha}_n}(\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n})\left(\cos\frac{\sqrt{\lambda_n}+\sqrt{\tilde{\lambda}_n}}{2}x-\cos\sqrt{\lambda_n}x\right). \qquad (2.1.9)$$

(2.1.9) tenglikdagi qatorlar absolyut va tekis yaqinlashuvchi bo‘lib, ushbu

$$|A_2(x)|\le c\sum_{n=0}^{\infty}\left|\sqrt{\lambda_n}-\sqrt{\tilde{\lambda}_n}\right|,$$

baholash o‘rinli bo‘ladi. (2.1.8) tenglikdagi qatorlar $L^2(0,2\pi)$ fazoning normasi bo‘yicha yaqinlashuvchi bo‘lib, quyidagi

$$||A_1(x)||_{L^2}\le c\delta,$$

baholash o‘rinli bo‘ladi. Bu baholashlardan foydalanib, lemma 2.1.3 isbotlanadi.

Teorema 2.1.1. (*N. Levinson*). *Agar*

$$\int_1^v\frac{N(u)}{u}du>v-\frac{1}{8}\ln v-c,\quad c=\mathrm{const},$$

bo‘lsa, u holda $f(x)\in L^2(0,\pi)$ funksiya uchun ushbu

$$\int_0^{\pi}f(x)\cos\sqrt{\lambda_n}x\,dx=0,\quad n=0,1,2,...,$$

tengliklarning bajarilishidan $f(x) \equiv 0$ *ekanligi kelib chiqadi. Bu yerda*

$$N(u) = \sum_{\sqrt{\lambda_n} \leq u} 1.$$

Lemma 2.1.4. *Agar* $\{\lambda_n\}_{n=0}^{\infty}$ *haqiqiy sonlar ketma-ketligi (2.1.3) shartlarni qanoatlantirsa, u holda*

$$\left\{\cos\sqrt{\lambda_n}x\right\}_{n=0}^{\infty},$$

funksiyalar ketma-ketligi $L^2(0, \pi)$ *Gilbert fazosida to'la sistemani tashkil qiladi.*

Isbot. λ_n sonlar ketma-ketligi Levinson teoremasining shartlarini qanoatlantirishini ko'rsatamiz. Haqiqatan ham, yetarlicha katta n larda λ_n sonlar ketma-ketligi uchun

$$\sqrt{\lambda_n} = n + \underline{\underline{O}}\left(\frac{1}{n}\right),$$

bajariladi. Bundan esa $N(u) = [u] + 1$ bo'lib, ushbu

$$\frac{N(u)}{u} \geq 1,$$

tengsizlikning o'rinli bo'lishi kelib chiqadi. Shuning uchun Levinson teoremasining sharti bajariladi. Demak,

$$\left\{\cos\sqrt{\lambda_n}x\right\}_{n=0}^{\infty},$$

funksiyalar sistemasi $L^2(0, \pi)$ fazoda to'la bo'lar ekan. ■

Lemma 2.1.5. *Agar ushbu* $\{\lambda_n\}_{n=0}^{\infty}$ *sonlar ketma-ketligi (2.1.3) shartlarni qanoatlantirsa va ushbu*

$$\sum_{n=0}^{\infty} c_n \cos\sqrt{\lambda_n}x,$$

funksional qator $L^2(0, \pi)$ *Gilbert fazosida nolga yaqinlashsa, u holda* $c_n = 0$, $n = 0, 1, 2, \ldots$ *bo'ladi.*

Isbot. Quyidagi formula yordamida A chiziqli operatorni aniqlaymiz:

$$A(\cos nx) = \cos\sqrt{\lambda_n}x.$$

A operator

$$A = I + P,$$

ko'rinishga ega ekanini ko'rsatamiz. Bu yerda P kompakt operator. Buning uchun A operatorning $\cos nx$ bazisdagi matritsasini (a_{mn}) orqali belgilaymiz. U holda ushbu

$$a_{mn} = \frac{1}{\pi}\int_0^{\pi} \cos\sqrt{\lambda_n}x \cos mx dx =$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(\cos nx + \frac{\alpha_n(x)}{n}\right) \cos mx dx =$$

$$= \delta_{mn} + \frac{1}{n\pi}\int_0^{\pi} \alpha_n(x) \cos mx dx = \delta_{mn} + P_{mn},$$

tenglikka ega bo'lamiz. Bu yerda

$$P_{mn} = \frac{1}{n\pi}\int_0^{\pi} \alpha_n(x) \cos mx dx.$$

Parseval tengligidan ushbu

$$\sum_{m,n} P_{mn}^2 = \sum_n \frac{1}{2\pi n^2}\int_0^{\pi} \alpha_n^2(x) dx < \infty,$$

qatorning yaqinlashuvchiligi kelib chiqadi. Bu esa $P = (P_{mn})$ operatorning kompaktligini ko'rsatadi. Lemma 2.1.4 ga asosan A operator $L^2(0,\pi)$ fazoni $L^2(0,\pi)$ fazoga o'tkazadi. Shuning uchun, Fredgolm teoremasiga ko'ra $A^{-1} = B$ mavjud bo'lib, uning uchun

$$B = I + Q,$$

tasvir o'rinli va Q kompakt operator bo'ladi. Bundan esa

$$B(\cos\sqrt{\lambda_n}x) = \cos nx,$$

ekanligi kelib chiqadi.

Faraz qilaylik, ushbu

$$\sum_{n=0}^{\infty} c_n \cos\sqrt{\lambda_n}x,$$

funksional qator $L^2(0, \pi)$ fazoda nolga yaqinlashsin. U holda B operatorni qo'llab, $L^2(0, \pi)$ fazoda nolga yaqinlashuvchi

$$\sum_{n=0}^{\infty} c_n \cos nx,$$

qatorni hosil qilamiz. $\cos nx$, $n = 0, 1, 2, \ldots$ funksiyalarning ortogonalligidan $c_n = 0$, $n = 0, 1, 2, \ldots$ kelib chiqadi.■

Teorema 2.1.1. *B banax fazosida quyidagi*

$$(I + A)y = f,$$

$$(I + A_0)y_0 = f_0,$$

tenglamalarni qaraylik. Bu yerda A va A_0 operatorlar B banax fazosida berilgan chiziqli chegaralangan operatorlar, I esa birlik operator. Faraz qilaylik ushbu

$$R_0 = (I + A_0)^{-1},$$

chiziqli chegaralangan operator mavjud bo'lsin. Agar

$$\|A - A_0\| \leq (2\|R_0\|)^{-1},$$

bo'lsa, u holda

$$R = (I + A)^{-1},$$

chiziqli chegaralangan operator mavjud bo'lib,

$$R = R_0\left(I + \sum_{k=1}^{\infty}((A_0 - A)R_0)^k\right)$$

va

$$\|\mathrm{R} - \mathrm{R}_0\| \le 2\|\mathrm{R}_0\|^2 \cdot \|A - A_0\|,$$

bo'ladi. Bundan tashqari ushbu

$$\|y - y_0\| \le c_0(\|A - A_0\| + \|f - f_0\|)$$

baholash o'rinli bo'ladi. Bu yerda c_0 *o'zgarmas faqat* $\|\mathrm{R}_0\|$ *va* $\|f_0\|$ *ga bog'liq.*

Isbot. Ushbu

$$I + A = (I + A_0) + (A - A_0) = (I + (A - A_0)\mathrm{R}_0)(I + A_0),$$

tenglikdan foydalanamiz. Teorema shartiga ko'ra

$$\|(A - A_0)\mathrm{R}_0\| \le \frac{1}{2},$$

bo'lgani uchun, quyidagi chiziqli chegaralangan operator mavjud:

$$\mathrm{R} \equiv (I + A)^{-1} = \mathrm{R}_0(I + (A_0 - A)\mathrm{R}_0)^{-1} =$$

$$= \mathrm{R}_0\left(I + \sum_{k=1}^{\infty}((A_0 - A)\mathrm{R}_0)^k\right).$$

Oxirgi tenglikdan

$$\|\mathrm{R}\| \le 2\|\mathrm{R}_0\|,$$

$$\|\mathrm{R} - \mathrm{R}_0\| \le \|\mathrm{R}_0\| \cdot \frac{\|(A - A_0)\mathrm{R}_0\|}{1 - \|(A - A_0)\mathrm{R}_0\|} \le 2\|\mathrm{R}_0\|^2 \cdot \|A - A_0\|.$$

baholashlarni keltirib chiqaramiz.

Endi $y - y_0$ ni baholaymiz. Buning uchun ushbu

$$y - y_0 = \mathrm{R}f - \mathrm{R}_0 f_0 = (\mathrm{R} - \mathrm{R}_0)f_0 + \mathrm{R}(f - f_0),$$

tasvirdan foydalanamiz. Natijada

$$\|y - y_0\| \le 2\|\mathrm{R}_0\|^2 \cdot \|f_0\| \cdot \|A - A_0\| + 2\|\mathrm{R}_0\| \cdot \|f - f_0\|,$$

kelib chiqadi.■

Lemma 2.1.6. *Ushbu*

$$\psi(x,\mu)+\int_0^1 H(x,t,\mu)\psi(t,\mu)dt=f(x,\mu), \tag{2.1.10}$$

integral tenglama berilgan bo'lib, uning $H(x,t,\mu)$ yadrosi va $f(x,\mu)$ ozod hadi μ parametrga va x o'zgaruvchiga nisbatan uzluksiz bo'lsin. Agar ushbu

$$\psi(x,\mu)+\int_0^1 H(x,t,\mu)\psi(t,\mu)dt=0, \tag{2.1.11}$$

bir jinsli integral tenglama μ parametrning μ_0 qiymatida faqat nol yechimga ega bo'lsa, u holda μ_0 nuqtaning biror atrofida (2.1.10) integral tenglamaning yechimi mavjud hamda x va μ ga nisbatan uzluksiz funksiya bo'ladi. Agar $H(x,t,\mu)$ va $f(x,\mu)$ funksiyalar m marta μ parametr bo'yicha uzluksiz differensiallanuvchi bo'lsa, u holda (2.1.10) integral tenglamaning yechimi ham μ bo'yicha m marta uzluksiz differensiallanuvchi bo'ladi.

Isbot. I. (2.1.10) integral tenglamaning $H(x,t,\mu)$ yadrosi uzluksiz bo'lgani uchun, uni quyidagi

$$H(x,t,\mu)=H(x,t,\mu_0)+\tilde{H}(x,t,\mu)=H_0+\tilde{H},$$

ko'rinishda yozib olamiz. Bu yerda μ parametrning μ_0 nuqta atrofidagi qiymatlari uchun

$$\left|\tilde{H}(x,t,\mu)\right|<\varepsilon,$$

tengsizlik bajariladi. Qulaylik uchun (2.1.10) tenglamani

$$\psi+H_0\psi+\tilde{H}\psi=f, \tag{2.1.12}$$

simvolik ko'rinishda yozib olamiz. Fredgolm teoremasiga ko'ra $(I+H_0)^{-1}$ operator mavjud. (2.1.12) tenglikka $(I+H_0)^{-1}$ operatorni ta'sir qildiramiz:

$$\psi+(I+H_0)^{-1}\tilde{H}\psi=(I+H_0)^{-1}f. \tag{2.1.13}$$

$(I + H_0)^{-1}\tilde{H}$ operator normasini hohlagancha kichraytirish mumkin bo'lgani uchun (2.1.13) tenglamaga ketma-ket yaqinlashishlar usulini qo'llash mumkin. Bundan esa $\psi(x, \mu)$ yechimning μ_0 nuqtada uzluksizligi kelib chiqadi.

II. Differensiallanuvchiligini isbotlash uchun $H(x, t, \mu)$ funksiyani

$$H = H_m + \tilde{H}$$

ko'rinishda yozib olamiz. Bunda

$$H_m = H(x, t, \mu_0) + \left.\frac{\partial H}{\partial \mu}\right|_{\mu=\mu_0} \cdot (\mu - \mu_0) + \ldots + \frac{1}{m!} \left.\frac{\partial^m H}{\partial \mu^m}\right|_{\mu=\mu_0} \cdot (\mu - \mu_0)^m.$$

Bu yerda μ parametrning μ_0 nuqta atrofidagi qiymatlari uchun

$$\left|\tilde{H}(x, t, \mu)\right| < |\mu - \mu_0|\varepsilon,$$

tengsizlik bajariladi. Lemma isbotining birinchi qismiga ko'ra $(I + H_m)^{-1}$ operator mavjud. Demak, ushbu

$$\psi + (I + H_m)^{-1}\tilde{H}\psi = (I + H_m)^{-1}f,$$

tenglamaga ketma-ket yaqinlashishlar usulini qo'llash mumkin. Bundan esa $\psi(x, \mu)$ yechimning μ_0 nuqta atrofida μ bo'yicha m marta differensiallanuvchiligi kelib chiqadi.■

2-§. Gelfand-Levitan integral tenglamasi

Yuqoridagi paragraflarda $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari ushbu

$$-y'' + q(x)y = \lambda y, \quad q(x) \in L_2(0,\pi), \tag{2.2.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, & h \in R^1, \\ y'(\pi) + Hy(\pi) = 0, & H \in R^1, \end{cases} \tag{2.2.2}$$

chegaraviy masalaning spektral xarakteristikalari bo'lishi uchun quyidagi

$$\begin{gathered} \sqrt{\lambda_n} \equiv k_n = n + \frac{\omega}{n\pi} + \frac{\chi_n}{n}, \quad \omega = h + H + \frac{1}{2}\int_0^{\pi} q(t)dt, \\ \alpha_n = \frac{\pi}{2} + \frac{\tilde{\chi}_n}{n}, \quad \{\chi_n\} \in l_2, \quad \{\tilde{\chi}_n\} \in l_2, \\ \lambda_n \neq \lambda_m, \quad n \neq m, \quad \alpha_n > 0, \quad n = 0,1,2,\dots, \end{gathered} \tag{2.2.3}$$

shartlarning bajarilishi zarur ekanligi ko'rsatilgan edi.

Endi teskari masalani yechish bilan shug'ullanamiz, ya'ni (2.2.3) shartlarni qanoatlantiruvchi $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari yordamida (2.2.1)+(2.2.2) chegaraviy masalaning $q(x)$ potensialini va chegaraviy shartlardagi h, H sonlarni topish algoritmini keltiramiz. Buning uchun, ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t\,dt, \tag{2.2.4}$$

almashtirish operatoridan foydalanamiz. Bu yerda $K(x,t)$ haqiqiy uzluksiz funksiya bo'lib, uning silliqlik darajasi $\int_0^x q(t)dt$ funksiyaning silliqlik darajasi bilan bir xil bo'ladi va

$$K(x,x) = h + \frac{1}{2}\int_0^x q(t)dt, \tag{2.2.5}$$

shartni qanoatlantiradi. Oxirgi tenglikdan ko‘rinadiki, agar biz $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalar yordamida (2.2.4) almashtirish operatorining $K(x,t)$ yadrosini topsak, u holda $q(x)$ potentsialni va h, H sonlarni quyidagi

$$q(x) = 2\frac{d}{dx}K(x,x), \quad h = K(+0,+0), \quad H = \omega - h - \frac{1}{2}\int_0^{\pi} q(t)dt, \tag{2.2.6}$$

formulalardan topishimiz mumkin bo‘ladi.

Almashtirish operatorining $K(x,t)$ yadrosini topishdan oldin $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalar yordamida ushbu

$$F(x,t) = \sum_{n=0}^{\infty}\left(\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{\alpha_n} - \frac{\cos nx\cos nt}{\alpha_n^0}\right), \tag{2.2.7}$$

funksiyani tuzib olamiz. Bu yerda

$$\alpha_n^0 = \begin{cases} \frac{\pi}{2}, & n > 0, \\ \pi, & n = 0. \end{cases}$$

$F(x,t)$ funksiyani quyidagi tarzda yozish mumkin:

$$F(x,t) = \frac{1}{2}[a(x+t) + a(x-t)],$$

$$a(x) = \sum_{n=0}^{\infty}\left(\frac{\cos\sqrt{\lambda_n}x}{\alpha_n} - \frac{\cos nx}{\alpha_n^0}\right).$$

Bundan lemma 2.1.1 ga asosoan $F(x,t)$ uzluksiz funksiya bo‘lishi va

$$F'(x,x) \in L^2(0,\pi),$$

kelib chiqadi.

Teorema 2.2.1. *Har bir tayinlagan $x \in (0,\pi]$ uchun, almashtirish operatorining $K(x,t)$ yadrosi ushbu*

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad (0 < t < x), \tag{2.2.8}$$

chiziqli integral tenglamani qanoatlantiradi. Bu integral tenglama Gelfand-Levitan integral tenglamasi yoki teskari masalaning asosiy integral tenglamasi deb yuritiladi.

Isbot. (2.2.4) tenglikni $\cos\sqrt{\lambda}x$ ga nisbatan yechib,

$$\cos\sqrt{\lambda}x = \varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)dt, \qquad (2.2.9)$$

tasvirni hosil qilamiz. Bu yerda $H(x,t)$ uzluksiz funksiya. (2.2.4) va (2.2.9) formulalardan foydalanib, quyidagi yig'indini hisoblaymiz:

$$\sum_{n=0}^{N} \frac{\varphi(x,\lambda_n)\cos\sqrt{\lambda_n}t}{\alpha_n} = \sum_{n=0}^{N} \left(\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{\alpha_n} + \right.$$

$$\left. + \frac{\cos\sqrt{\lambda_n}t}{\alpha_n} \int_0^x K(x,s)\cos\sqrt{\lambda_n}s ds \right),$$

$$\sum_{n=0}^{N} \frac{\varphi(x,\lambda_n)\cos\sqrt{\lambda_n}t}{\alpha_n} =$$

$$= \sum_{n=0}^{N} \left(\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n} + \frac{\varphi(x,\lambda_n)}{\alpha_n} \int_0^t H(t,s)\varphi(s,\lambda_n)ds \right).$$

Oxirgi ikki tenglikning o'ng tomonlarini tenglashtirib, quyidagi

$$\Phi_N(x,t) = I_{1,N}(x,t) + I_{2,N}(x,t) + I_{3,N}(x,t) + I_{4,N}(x,t), \quad (2.2.10)$$

ayniyatni hosil qilamiz. Bu yerda

$$\Phi_N(x,t) = \sum_{n=0}^{N} \left(\frac{\varphi(x,\lambda_n)\varphi(t,\lambda_n)}{\alpha_n} - \frac{\cos nx \cos nt}{\alpha_n^0} \right),$$

$$I_{1,N}(x,t) = \sum_{n=0}^{N} \left(\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{\alpha_n} - \frac{\cos nx \cos nt}{\alpha_n^0} \right),$$

$$I_{2,N}(x,t)=\sum_{n=0}^{N}\frac{\cos nt}{\alpha_n^0}\int_0^x K(x,s)\cos ns ds,$$

$$I_{3,N}(x,t)=\sum_{n=0}^{N}\int_0^x K(x,s)\left(\frac{\cos\sqrt{\lambda_n}t\cos\sqrt{\lambda_n}s}{\alpha_n}-\frac{\cos nt\cos ns}{\alpha_n^0}\right)ds,$$

$$I_{4,N}(x,t)=-\sum_{n=0}^{N}\frac{\varphi(x,\lambda_n)}{\alpha_n}\int_0^t H(t,s)\varphi(s,\lambda_n)ds.$$

Agar $f(x)\in AC[0,\pi]$ absolyut uzluksiz funksiya bo'lsa, u holda yoyilma teoremasiga ko'ra

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\int_0^\pi f(t)\Phi_N(x,t)dt=0,$$

$$\lim_{N\to\infty}\int_0^\pi f(t)I_{1,N}(x,t)dt=\int_0^\pi f(t)F(x,t)dt,$$

$$\lim_{N\to\infty}\int_0^\pi f(t)I_{2,N}(x,t)dt=\int_0^\pi f(t)K(x,t)dt,$$

$$\lim_{N\to\infty}\int_0^\pi f(t)I_{3,N}(x,t)dt=\int_0^\pi f(t)\left(\int_0^x K(x,s)F(s,t)ds\right)dt,$$

$$\lim_{N\to\infty}\int_0^\pi f(t)I_{4,N}(x,t)dt=$$

$$=-\lim_{N\to\infty}\sum_{n=0}^{N}\frac{\varphi(x,\lambda_n)}{\alpha_n}\int_0^\pi\varphi(s,\lambda_n)\left[\int_s^\pi H(t,s)f(t)dt\right]ds=$$

$$=-\int_x^\pi f(t)H(t,x)dt,$$

kelib chiqadi. $x < t$ larda $K(x,t) = H(x,t) = 0$ deb aniqlaymiz. Endi (2.2.10) tenglikning ikkala tomonini $f(t)$ funksiyaga ko'paytirib, $[0,\pi]$ oraliqda integralaymiz va hosil bo'lgan ifodada $N \to \infty$ bo'lganda limitga o'tib, quyidagi

$$\int_0^\pi f(t)K(x,t)dt + \int_0^\pi f(t)F(x,t)dt +$$

$$+\int_0^\pi f(t)\left[\int_0^x K(x,s)F(s,t)ds\right]dt -$$

$$-\int_x^\pi f(t)H(t,x)dt = 0,$$

tenglikni hosil qilamiz. $f(x)$ funksiyaning ixtiyoriyligidan

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds - H(t,x) = 0,$$

kelib chiqadi. $t < x$ larda $H(t,x) = 0$ bo'lishini inobatga olsak (2.2.8) Gelfand-Levitan integral tenglamasi kelib chiqadi.■

Endi $\{\alpha_n\}_{n=0}^\infty$ va $\{\lambda_n\}_{n=0}^\infty$ haqiqiy sonlar ketma-ketliklari (2.2.3) shartlarni qanoatlantirsin deylik. U holda (2.2.7) formula yordamida $F(x,t)$ funksiyani tuzib olamiz va uni (2.2.8) Gelfand-Levitan integral tenglamasiga qo'yamiz. Gelfand-Levitan integral tenglamasi har bir tayinlangan $x \in (0,\pi]$ da ikkinchi tur Fredgolm integral tenglamasidir.

Lemma 2.2.1. *Har bir tayinlangan $x \in (0,\pi]$ da (2.2.8) Gelfand-Levitan integral tenglamasi $L^2(0,x)$ fazoda yagona yechimga ega.*

Isbot. (2.2.8) integral tenglama yechimining mavjudligini ko'rsatish uchun, unga mos keluvchi bir jinsli integral tenglama faqat nol yechimga ega ekanini ko'rsatish yetarli. Buning uchun ushbu

$$g(t)+\int_0^x F(s,t)g(s)ds=0, \qquad (2.2.11)$$

tenglama noldan farqli $g(t)$ yechimga ega bo'lsin deb faraz qilaylik. U holda

$$\int_0^x g^2(t)dt+\int_0^x\int_0^x F(s,t)g(s)g(t)dsdt=0,$$

yoki

$$\int_0^x g^2(t)dt+\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\left(\int_0^x g(t)\cos\sqrt{\lambda_n}\,tdt\right)^2-$$

$$-\sum_{n=0}^{\infty}\frac{1}{\alpha_n^0}\left(\int_0^x g(t)\cos ntdt\right)^2=0, \qquad (2.2.12)$$

tenglik o'rinli bo'ladi. Bu tenglikda $t>x$ larda $g(t)=0$ deb olamiz va ushbu

$$\int_0^x g^2(t)dt=\sum_{n=0}^{\infty}\frac{1}{\alpha_n^0}\left(\int_0^x g(t)\cos ntdt\right)^2,$$

Parseval tengligidan foydalanamiz. Natijada (2.2.12) tenglik quyidagi ko'rinishni oladi:

$$\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\left(\int_0^x g(t)\cos\sqrt{\lambda_n}tdt\right)^2=0.$$

Bu yerda $\alpha_n>0,\;\; n=0,1,2,...$ ekanligini e'tiborga olsak,

$$\int_0^x g(t)\cos\sqrt{\lambda_n}tdt=0,\quad n=0,1,2,...,$$

ya'ni

$$\int_0^\pi g(t)\cos\sqrt{\lambda_n}t dt = 0, \quad n = 0, 1, 2, ...,$$

kelib chiqadi. Lemma 2.1.4 ga asosan $\left\{\cos\sqrt{\lambda_n}x\right\}_{n=0}^{\infty}$ funksiyalar sistemasi $L^2(0, \pi)$ fazoda to'la bo'lgani uchun $g(t) \equiv 0$ bo'lishi kelib chiqadi.■

Shunday qilib (2.2.8) Gelfand-Levitan integral tenglamasining yechimi mavjudligi ko'rsatildi. Endi $K(x, t)$ yechimning silliqlik darajasini aniqlaymiz. (2.2.8) integral tenglamaning ko'rinishidan $K(x, t)$ funksiyaning t o'zgaruvchi bo'yicha silliqlik darajasi $F(x, t)$ funksiyaning t o'zgaruvchi bo'yicha silliqlik darajasi bilan bir xil ekani kelib chiqadi. Lekin $K(x, t)$ funksiyaning x o'zgaruvchi bo'yicha silliqlik darajasini to'g'ridan-to'gri aytish ancha qiyin.

Faraz qilaylik $K(x, t)$ funksiya (2.2.8) integral tenglamaning yechimi bo'lsin. U holda (2.2.8) tenglamada

$$t \to tx, \quad s \to sx,$$

o'zgaruvchilarni almashtirib, uni

$$K(x, xt) + F(x, xt) + x\int_0^1 K(x, xs)F(xs, xt)ds = 0, \ \ 0 \le t \le 1, \tag{2.2.13}$$

ko'rinishga keltiramiz. Quyidagi

$$K(x, xt) = \psi(t, x),$$

$$xF(xs, xt) = H(t, s, x),$$

$$-F(x, xt) = f(t, x),$$

belgilashlarni kiritsak, (2.2.8) tenglama

$$\psi(t,x) + \int_0^1 H(t,s,x)\psi(s,x)ds = f(t,x),$$

ko‘rinishni oladi. Bu yerda $x = \mu$ parametr kiritsak oxirgi tenglama ushbu

$$\psi(t,\mu) + \int_0^1 H(t,s,\mu)\psi(s,\mu)ds = f(t,\mu), \tag{2.2.14}$$

ko‘rinishga keladi. $H(t,s,\mu)$ yadro va $f(t,\mu)$ ozod had uzluksiz ekanligidan lemma 2.1.6 ga ko‘ra $\psi(t,\mu)$ funksiyaning uzluksiz bo‘lishi kelib chiqadi. Shuning uchun $K(x,t)$ funksiya x o‘zgaruvchiga nisbatan uzluksiz bo‘lib, uning silliqlik darajasi $F(x,t)$ funksiyaning silliqlik darajasi bilan bir xil bo‘ladi. Xususan,

$$\frac{d}{dx}K(x,x) \in L^2(0,\pi),$$

bo‘ladi.

3-§. Differensial tenglamani keltirib chiqarish

Yuqoridagi paragrafda (2.1.3) shartni qanoatlantiruvchi $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketligi yordamida $K(x,t)$ funksiyani topish algoritmi keltirilgan edi. Topilgan $K(x,t)$ funksiyani (2.2.4) tenglikka qo‘yib,

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt, \tag{2.3.1}$$

funksiyani aniqlaymiz.

Quyidagi teorema teskari masalani yechish jarayonidagi muhim bosqichlardan biri hisoblanadi.

Teorema 2.3.1. *Agar $K(x,t)$ funksiya (2.2.8) Gelfand-Levitan integral tenglamasining yechimi bo'lsa, u holda (2.3.1) formula orqali aniqlangan $\varphi(x,\lambda)$ funksiya ushbu*

$$-y'' + q(x)y = \lambda y, \quad 0 < x \le \pi, \tag{2.3.2}$$

tenglamani va

$$\varphi(0,\lambda) = 1, \ \varphi'(0,\lambda) = h, \ h = K(+0,+0) = -F(0,0), \tag{2.3.3}$$

boshlang'ich shartlarni qanoatlantiradi. Bu yerda

$$q(x) = 2\frac{d}{dx}K(x,x). \tag{2.3.3'}$$

Isbot. I. Dastlab $a(x) \in W_2^2(0,2\pi)$ bo'lsin deb faraz qilamiz. Bu yerda

$$a(x) = \sum_{n=0}^{\infty}\left(\frac{\cos\sqrt{\lambda_n}x}{\alpha_n} - \frac{\cos nx}{\alpha_n^0}\right), \quad \alpha_n^0 = \begin{cases} \frac{\pi}{2}, & n > 0, \\ \pi, & n = 0. \end{cases}$$

Ushbu

$$J(x,t) \equiv K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \tag{2.3.4}$$

$$F(x,t) = \frac{1}{2}[a(x+t) + a(x-t)], \tag{2.3.5}$$

ayniyatni differensiallab, $J_t(x,t)$, $J_{tt}(x,t)$ va $J_{xx}(x,t)$ hosilalarni hisoblaymiz:

$$J_t(x,t) \equiv K_t(x,t) + F_t(x,t) + \int_0^x K(x,s)F_t(s,t)ds = 0, \tag{2.3.6}$$

$$J_{tt}(x,t) \equiv K_{tt}(x,t) + F_{tt}(x,t) + \int_0^x K(x,s)F_{tt}(s,t)ds = 0, \tag{2.3.7}$$

$$J_{xx}(x,t) \equiv K_{xx}(x,t) + F_{xx}(x,t) + \frac{dK(x,x)}{dx}F(x,t) + K(x,x)F_x(x,t) +$$

$$+\left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x} F(x,t) + \int_0^x K_{xx}(x,s)F(s,t)ds = 0. \qquad (2.3.8)$$

(2.3.5) tenglikdan

$$F_{tt}(s,t) = F_{ss}(s,t) \quad \text{va} \quad F_t(x,t)|_{t=0} = 0,$$

kelib chiqadi. (2.3.6) tenglikda $t = 0$ desak,

$$\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=0} = 0 \qquad (2.3.9)$$

bo'ladi. Endi (2.3.7) tenglikda bo'laklab integrallash qoidasidan foydalanib,ushbu

$$J_{tt}(x,t) \equiv K_{tt}(x,t) + F_{tt}(x,t) + K(x,x)\left.\frac{\partial F(s,t)}{\partial s}\right|_{s=x} -$$

$$-\left.\frac{\partial K(x,s)}{\partial s}\right|_{s=x} F(x,t) + \int_0^x K_{ss}(x,s)F(s,t)ds = 0, \qquad (2.3.10)$$

tenglikka ega bo'lamiz. (2.3.4), (2.3.8), (2.3.10) tengliklardan va ushbu

$$J_{xx}(x,t) - J_{tt}(x,t) - q(x)J(x,t) \equiv 0,$$

ayniyatdan foydalanib,

$$[K_{xx}(x,t) - K_{tt}(x,t) - q(x)K(x,t)] +$$

$$+\int_0^x [K_{xx}(x,s) - K_{ss}(x,s) - q(x)K(x,s)]F(s,t)ds \equiv 0,$$

tenglikni topamiz. Bu esa bir jinsli Gelfand-Levitan integral tenglamasi bo'lib, u faqat nol yechimga ega bo'lgani uchun

$$K_{xx}(x,t) - K_{tt}(x,t) - q(x)K(x,t) = 0, \;\; 0 < t < x, \qquad (2.3.11)$$

hosil bo'ladi.

(2.3.1) tenglikning ikkala tarafini ikki marta differensiallab $\varphi'(x,\lambda)$ va $\varphi''(x,\lambda)$ hosilalarni hisoblaymiz:

$$\varphi'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + K(x,x)\cos\sqrt{\lambda}x + \int_0^x K_x(x,t)\cos\sqrt{\lambda}t dt,$$

$$\tag{2.3.12}$$

$$\varphi''(x,\lambda) = -\lambda\cos\sqrt{\lambda}x - \sqrt{\lambda}K(x,x)\sin\sqrt{\lambda}x +$$

$$+\left(\frac{dK(x,x)}{dx} + \left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x}\right)\cos\sqrt{\lambda}x + \int_0^x K_{xx}(x,t)\cos\sqrt{\lambda}t dt.$$

$$\tag{2.3.12'}$$

Endi (2.3.1) tenglikda ikki marta bo'laklab integrallash qoidasini qo'llab quyidagi tenglikni hosil qilamiz:

$$\lambda\varphi(x,\lambda) = \lambda\cos\sqrt{\lambda}x + \lambda\int_0^x K(x,t)\cos\sqrt{\lambda}t dt =$$

$$= \lambda\cos\sqrt{\lambda}x + K(x,x)\sqrt{\lambda}\sin\sqrt{\lambda}x +$$

$$+\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x}\cos\sqrt{\lambda}x - \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=0} - \int_0^x K_{tt}(x,t)\cos\sqrt{\lambda}t dt.$$

(2.3.1) va (2.3.12') tengliklardan foydalanib,

$$\varphi''(x,\lambda)+\lambda\varphi(x,\lambda)-q(x)\varphi(x,\lambda) = \left(2\frac{dK(x,x)}{dx} - q(x)\right)\cos\sqrt{\lambda}x -$$

$$-\left.\frac{\partial K(x,t)}{\partial t}\right|_{t=0} + \int_0^x [K_{xx}(x,t) - K_{tt}(x,t) - q(t)K(x,t)]\cos\sqrt{\lambda}t dt,$$

ayniyatga ega bo'lamiz. Bu yerda (2.2.3'), (2.3.9) va (2.3.11) tengliklarni inobatga olsak, $\varphi(x,\lambda)$ funksiya (2.3.2) tenglamani qanoatlantirishi kelib chiqadi. Endi (2.3.1) va (2.3.12) tengliklarda $x = 0$ deb (2.3.3) boshlang'ich shartlarni topamiz.

II. Umumiy hol, ya'ni $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikalar (2.2.3) shartlarni qanoatlantirganda teoremani isbotlaymiz. Bu holda lemma 2.1.1 ga asosan

$$a(x) \in W_2^1(0, 2\pi),$$

bo'ladi. $\tilde{\varphi}(x, \lambda)$ orqali ushbu

$$\begin{gathered} -\tilde{\varphi}''(x, \lambda) + q(x)\tilde{\varphi}(x, \lambda) = \lambda\tilde{\varphi}(x, \lambda), \\ \tilde{\varphi}(0, \lambda) = 1, \ \tilde{\varphi}'(0, \lambda) = h\,, \end{gathered} \tag{2.3.13}$$

Koshi masalasining yechimini belgilaymiz. U holda $\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda)$ ekanligini ko'rsatamiz. Buning uchun ushbu

$$\sqrt{\lambda_{n,(j)}} = n + \frac{\omega}{n\pi} + \frac{\chi_{n,(j)}}{n^2}, \quad \{\chi_{n,(j)}\} \in l_2,$$

$$\alpha_{n,(j)} = \frac{\pi}{2} + \frac{\omega_{n,(j)}}{n^2} + \frac{\tilde{\chi}_{n,(j)}}{n^2}, \quad \{\tilde{\chi}_{n,(j)}\} \in l_2,$$

ko'rinishidagi $\left\{\lambda_{n,(j)}\right\}_{n=0}^{\infty}$ va $\left\{\alpha_{n,(j)}\right\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklarini shunday tanlaymizki, natijada ular quyidagi shartni qanoatlantirsinlar:

$$\delta_j = \left(\sum_{n=0}^{\infty} \left|(n+1)\xi_{n,(j)}\right|^2\right)^{\frac{1}{2}} \to 0, \ j \to \infty.$$

Bu yerda

$$\xi_{n,(j)} = \left|\sqrt{\lambda_{n,(j)}} - \sqrt{\lambda_n}\right| + \left|\alpha_{n,(j)} - \alpha_n\right|.$$

Quyidagi belgilashni kiritib olamiz:

$$a_j(x) = \sum_{n=0}^{\infty} \left[\frac{\cos\sqrt{\lambda_{n,(j)}}x}{\alpha_{n,(j)}} - \frac{\cos nx}{\alpha_n^0}\right], \quad j \geq 1.$$

Lemma 2.1.2 ga ko'ra $a_j(x) \in W_2^2(0,\, 2\pi)$, $j \geq 1$ bo'ladi.

$K_j(x, t)$ orqali

$$K_j(x,t) + F_j(x,t) + \int_0^x K_j(x,s)F_j(s,t)ds = 0, \quad 0 < t < x,$$

Gelfand-Levitan integral tenglamasining yechimini belgilaylik. Bu yerda

$$F_j(x,t) = \frac{1}{2}[a_j(x+t) + a_j(x-t)].$$

Quyidagi

$$q_j(x) = 2\frac{d}{dx}K_j(x,x), \quad h_j = K_j(+0,\ +0),$$

$$\varphi_j(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K_j(x,t)\cos\sqrt{\lambda}\,tdt, \tag{2.3.14}$$

formulalar yordamida $q_j(x)$,h_j, $\varphi_j(x,\lambda)$, $j \geq 1$ ketma-ketliklarni tuzib olamiz. Teorema isbotini birinchi bandiga asosan ushbu

$$-\varphi_j''(x,\ \lambda) + q_j(x)\varphi_j(x,\ \lambda) = \lambda\varphi_j(x,\ \lambda),$$

$$\varphi_j(0,\lambda) = 1, \quad \varphi_j(0,\lambda) = h_j,$$

tengliklar o'rinli bo'ladi. Lemma 2.1.3 ga asosan

$$\lim_{j\to\infty}\|a_j(x) - a(x)\|_{W_2^1} = 0. \tag{2.3.15}$$

Bundan foydalanib,

$$\lim_{j\to\infty}\|F_j(x,t) - F(x,t)\|_{W_2^1} = 0,$$

tenglikni topamiz. Teorema 2.1.1 ga asoslanib, quyidagi limitlarni hisoblash mumkin:

$$\lim_{j\to\infty}\max_{0\leq t\leq x\leq\pi}|K_j(x,t) - K(x,t)| = 0, \tag{2.3.16}$$

$$\lim_{j\to\infty}\|q_j - q\|_{L_2} = 0, \quad \lim_{j\to\infty}h_j = h. \tag{2.3.17}$$

(2.3.1), (2.3.14) va (2.3.16) tenliklardan

$$\lim_{j\to\infty}\max_{0\leq x\leq\pi}\max_{|\lambda|\leq r}|\varphi_j(x,\lambda) - \varphi(x,\lambda)| = 0,$$

bo'lishi kelib chiqadi. Ikkinchi tomondan (2.3.17) ga asoslanib, ushbu

$$\lim_{j\to\infty}\max_{0\leq x\leq\pi}\max_{|\lambda|\leq r}|\varphi_j(x,\lambda) - \tilde{\varphi}(x,\lambda)| = 0,$$

tenglikni topamiz. Bu esa

$$\tilde{\varphi}(x,\lambda) \equiv \varphi(x,\lambda),$$

ekanini ko'rsatadi.■

Izoh. $K(0,0) = -F(0,0)$ bo'lgani uchun

$$h = -\left\{\sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n} - \frac{2}{\pi}\right) + \frac{1}{\alpha_0} - \frac{1}{\pi}\right\}$$

o'rinli bo'ladi. $\{\lambda_n\}$ sonlarning asimptotikasidan

$$H = \omega - h - \frac{1}{2}\int_0^{\pi} q(t)dt$$

kelib chiqadi.

4-§. Parseval tengligini keltirib chiqarish

Yuqorida biz $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari (2.2.3) shartlarni qanoatlantirgan holda ushbu

$$-y'' + q(x)y = \lambda y, \quad (0 \le x \le \pi), \tag{2.4.1}$$

$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0,$$

chegaraviy masalani qurishga muvaffaq bo'ldik. Gelfand-Levitan integral tenglamasini yechib, $K(x,t)$ funksiyani topamiz va ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt, \tag{2.4.2}$$

funksiyani aniqlaymiz. (2.4.2) tenglikni $\cos\sqrt{\lambda}x$ ga nisbatan tenglama deb qarab, uni yechsak quyidagi tenglikni olamiz:

$$\cos\sqrt{\lambda}x = \varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)dt. \tag{2.4.3}$$

Lemma 2.4.1. *Quyidagi ayniyat o'rinli:*

$$H(x,t) = F(x,t) + \int_0^t K(t,u)F(x,u)du, \quad 0 \le t \le x. \quad (2.4.4)$$

Isbot: I. Dastlab $a(x) \in W_2^2(0,2\pi)$ bo'lgan holni ko'rib chiqamiz. (2.4.3) tenglikni ikki marta differensiallash natijasida

$$-\sqrt{\lambda}\sin\sqrt{\lambda}x = \varphi'(x,\lambda) + H(x,x)\varphi(x,\lambda) + \int_0^x H'_x(x,t)\varphi(t,\lambda)dt, \quad (2.4.5)$$

$$-\lambda\cos\sqrt{\lambda}x = \varphi''(x,\lambda) + H(x,x)\varphi'(x,\lambda) +$$
$$+\left(\frac{dH(x,x)}{dx} + \frac{\partial H(x,t)}{\partial x}\bigg|_{t=x}\right)\varphi(x,\lambda) + \int_0^x H''_{xx}(x,t)\varphi(t,\lambda)dt, \quad (2.4.6)$$

tenglikni topamiz. Ikkinchi tomondan (2.4.3) tenglikning ikkala tarafini $(-\lambda)$ ga ko'paytirib, so'ngra (2.3.2) differensial tenglamani inobatga olsak quyidagi

$$-\lambda\cos\sqrt{\lambda}x = \varphi''(x,\lambda) - q(x)\varphi(x,\lambda) +$$
$$+\int_0^x H(x,t)\{\varphi''(t,\lambda) - q(t)\varphi(t,\lambda)\}dt,$$

hosil bo'ladi. Oxirgi integralni ikki marta bo'laklab integrallab, (2.3.3) boshlang'ich shartlarni e'tiborga olsak, ushbu

$$-\lambda\cos\sqrt{\lambda}x = \varphi''(x,\lambda) + H(x,x)\varphi'(x,\lambda) -$$
$$-\left(\frac{\partial H(x,t)}{\partial t}\bigg|_{t=x} + q(x)\right)\varphi(x,\lambda) +$$
$$+\left(\frac{\partial H(x,t)}{\partial t}\bigg|_{t=0} - hH(x,0)\right) + \int_0^x (H_{tt}(x,t) - q(t)H(x,t))\varphi(t,\lambda)dt,$$

tenglikka ega bo'lamiz. Oxirgi tenglikni (2.4.6) bilan solishtirib, ushbu

$$\frac{dH(x,x)}{dx} = \left(\frac{\partial H(x,t)}{\partial x} + \frac{\partial H(x,t)}{\partial t}\right)\Bigg|_{t=x},$$

ayniyatdan foydalansak, quyidagi

$$b_0(x) + b_1(x)\varphi(x,\lambda) + \int_0^x b(x,t)\varphi(t,\lambda)dt = 0, \qquad (2.4.7)$$

kelib chiqadi. Bu yerda

$$b_0(x) = -\left(\frac{\partial H(x,t)}{\partial t}\Bigg|_{t=0} - hH(x,0)\right), \quad b_1(x) = 2\frac{dH(x,x)}{dx} + q(x),$$

$$b(x,t) = H''_{xx}(x,t) - H''_{tt}(x,t) + q(t)H(x,t). \qquad (2.4.8)$$

(2.4.7) tenglikdagi $\varphi(x,\lambda)$ o'rniga uning (2.4.2) dagi ifodasini qo'yib,

$$b_0(x) + b_1(x)\cos\sqrt{\lambda}x + \int_0^x B(x,t)\cos\sqrt{\lambda}t dt = 0, \qquad (2.4.9)$$

ayniyatni topamiz. Bu yerda

$$B(x,t) = b(x,t) + b_1(x)K(x,t) + \int_t^x b(x,s)K(s,t)ds. \quad (2.4.10)$$

Agar (2.4.9) ayniyatda $\sqrt{\lambda} = \left(n + \frac{1}{2}\right)\frac{\pi}{x}$ deb olsak, quyidagi

$$b_0(x) + \int_0^x B(x,t)\cos\left(n + \frac{1}{2}\right)\frac{\pi t}{x}dt = 0,$$

tenglik hosil bo'ladi. Oxirgi tenglikda $n \to \infty$ bo'lganda limitga o'tib Riman-Lebeg teoremasidan foydalansak, $b_0(x) = 0$ kelib chiqadi. Agar (2.4.9) ayniyatda $\sqrt{\lambda} = \frac{2n\pi}{x}$ deb olsak,

$$b_1(x) + \int_0^x B(x,t)\cos\frac{2n\pi t}{x}dt = 0,$$

hosil bo'ladi. Bu yerda yana Riman-Lebeg teoremasidan foydalanib, $n \to \infty$ bo'lganda limitga o'tsak, $b_1(x) = 0$ kelib chiqadi. Natijada, (2.4.9) tenglik

$$\int_0^x B(x,t)\cos\sqrt{\lambda}t dt = 0,$$

ko'rinishni oladi. Oxirgi tenglikdan $B(x,t) = 0$ ekanligi kelib chiqadi. Nihoyat, (2.4.10) tenglik ushbu

$$b(x,t) + \int_t^x b(x,s)K(s,t)ds = 0,$$

ko'rinishni oladi. Bundan esa, $b(x,t) = 0$ ekanligi kelib chiqadi. (2.4.5) tenglikda $x = 0$ desak,

$$H(0,0) = -h, \tag{2.4.11}$$

munosabatni topamiz. $b_0(x) = b_1(x) = b(x,t) = 0$ bo'lgani uchun (2.4.8) va (2.4.11) tengliklardan $H(x,t)$ funksiya quyidagi masalaning yechimi ekani kelib chiqadi:

$$H''_{xx}(x,t) - H''_{tt}(x,t) + q(t)H(x,t) = 0, \quad 0 \le t \le x,$$

$$H(x,x) = -h - \frac{1}{2}\int_0^x q(t)dt, \quad \left.\frac{\partial H(x,t)}{\partial t}\right|_{t=0} - hH(x,0) = 0. \tag{2.4.12}$$

Aksincha, agar $H(x,t)$ funksiya (2.4.12) masalaning yechimi bo'lsa, u holda (2.4.3) o'rinli bo'ladi. Haqiqatan ham, ushbu

$$\gamma(x,\lambda) = \varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)dt,$$

belgilashni kiritib yuqoridagi mulohazalarni takrorlasak, quyidagi

$$\gamma''(x,\lambda) + \lambda\gamma(x,\lambda) = \left(2\frac{dH(x,x)}{dx} + q(x)\right)\varphi(x,\lambda) -$$

$$-\left(\frac{\partial H(x,t)}{\partial t}\bigg|_{t=0} - hH(x,0)\right)+$$

$$+\int_0^x (H''_{xx}(x,t) - H''_{tt}(x,t) + q(t)H(x,t))\varphi(t,\lambda)dt,$$

kelib chiqadi. Oxirgi tenglikda (2.4.12) dan foydalansak, $\gamma(x,\lambda)$ funksiya quyidagi tenglamani qanoatlantirishiga ishonch hosil qilamiz:

$$\gamma''(x,\lambda) + \lambda\gamma(x,\lambda) = 0.$$

Bu yerda

$$\gamma(0,\lambda) = 1, \quad \gamma'(0,\lambda) = 0,$$

bo'lgani uchun $\gamma(x,\lambda) = \cos\sqrt{\lambda}x$ bo'ladi.

Endi ushbu

$$\tilde{H}(x,t) = F(x,t) + \int_0^t K(t,u)F(x,u)du, \qquad (2.4.13)$$

belgilashni kiritib, $\tilde{H}(x,t)$ funksiya (2.4.12) masalani qanoatlantirishini ko'rsatamiz.

1) (2.4.13) tenglikni t o'zgaruvchi bo'yicha differensiallab, $t = 0$ deb olamiz, u holda

$$\frac{\partial \tilde{H}(x,t)}{\partial t}\bigg|_{t=0} = K(0,0)F(x,0) = hF(x,0),$$

bo'ladi. Bu yerda $\tilde{H}(x,0) = F(x,0)$ ekanini hisobga olsak, ushbu

$$\frac{\partial \tilde{H}(x,t)}{\partial t}\bigg|_{t=0} - h\tilde{H}(x,0) = 0,$$

tenglik hosil bo'ladi.

2) Gelfand-Levitan integral tenglamasi va (2.4.13) dan

$$\tilde{H}(x,x) = F(x,x) + \int_0^x K(x,u)F(x,u)du = -K(x,x)$$

tenglikni topamiz. Bu yerda (2.2.5) dan foydalansak,

$$\tilde{H}(x,x) = -h - \frac{1}{2}\int_0^x q(t)dt$$

hosil bo'ladi.

3) Yana (2.4.13) dan foydalanib, quyidagi hosilalarni hisoblaymiz:

$$\tilde{H}''_{tt}(x,t) = F''_{tt}(x,t) + \frac{dK(t,t)}{dt}F(x,t) + K(t,t)F'_t(x,t) +$$

$$+\left.\frac{\partial K(t,u)}{\partial t}\right|_{u=t} \cdot F(x,t) + \int_0^t K''_{tt}(t,u)F(x,u)du,$$

$$\tilde{H}''_{xx}(x,t) = F''_{xx}(x,t) + \int_0^t K(t,u)F''_{xx}(x,u)du =$$

$$= F''_{xx}(x,t) + \int_0^t K(t,u)F''_{uu}(x,u)du =$$

$$= F''_{xx}(x,t) + K(t,t)F'_t(x,t) + \left.\frac{\partial K(t,u)}{\partial t}\right|_{u=t} \cdot F(x,t) +$$

$$+\int_0^t K''_{uu}(t,u)F(x,u)du.$$

Yuqoridagi formulalardan foydalanib

$$\tilde{H}''_{xx}(x,t) - \tilde{H}''_{tt}(x,t) + q(t)\tilde{H}(x,t) = \left(q(t) - 2\frac{dK(t,t)}{dt}\right)F(x,t) -$$

$$-\int_0^t \left(K''_{tt}(t,u) - K''_{uu}(t,u) - q(t)K(t,u)\right)dt\,,$$

ayniyatni topamiz. (2.2.6) va (2.3.11) lardan

$$\tilde{H}''_{xx}(x,t) - \tilde{H}''_{tt}(x,t) + q(t)\tilde{H}(x,t) = 0,$$

kelib chiqadi. $\tilde{H}(x,t)$ funksiya (2.4.12) ni qanoatlantirganda ushbu

$$\cos\sqrt{\lambda}x = \varphi(x,\lambda) + \int_0^x \tilde{H}(x,t)\varphi(t,\lambda)dt,$$

tasvir o'rinli bo'lishini yuqorida ko'rsatgan edik. Bu tasvirni (2.4.3) bilan tenglashtirib,

$$\int_0^x \left(\tilde{H}(x,t) - H(x,t)\right)\varphi(t,\lambda)dt = 0,$$

ayniyatni hosil qilamiz. Bundan esa

$$\tilde{H}(x,t) = H(x,t),$$

kelib chiqadi.

II. Endi umumiy holni ko'rib chiqamiz, ya'ni $a(x) \in W_2^1(0,2\pi)$ bo'lsin. Bu holda $\{\lambda_{n,(j)}\}_{n=0}^{\infty}$ va $\{\alpha_{n,(j)}\}_{n=0}^{\infty}$, $j \geq 1$ haqiqiy sonlar ketma-ketliklarini shunday tanlaymizki, natijada

$$a_j(x) \in W_2^2(0,2\pi), \quad j \geq 1,$$

bo'lib,

$$\lim_{j\to\infty} \|a_j(x) - a(x)\|_{W_2^1} = 0,$$

bo'lsin. U holda ushbu

$$\lim_{j\to\infty} \max_{0\leq t\leq x\leq\pi} |F_j(x,t) - F(x,t)| = 0,$$

$$\lim_{j\to\infty} \max_{0\leq t\leq x\leq\pi} |K_j(x,t) - K(x,t)| = 0,$$

$$\lim_{j\to\infty} \max_{0\leq t\leq x\leq\pi} |H_j(x,t) - H(x,t)| = 0,$$

munosabatlarning bajaralishini ko'rsatish mumkin. Yuqorida isbotlangan

$$H_j(x,t) = F_j(x,t) + \int_0^t K_j(t,u)F_j(x,u)du,$$

tasvirda $j \to \infty$ da limitga o'tib (2.4.4) formulani hosil qilamiz.■

Teorema 2.4.1 *(Parseval tengligi). Ixtiyoriy $g(x) \in L^2(0,\pi)$ funksiya uchun*

$$\int_0^\pi g^2(x)dx = \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left(\int_0^\pi g(t)\varphi(t,\lambda_n)dt \right)^2, \qquad (2.4.14)$$

Parseval tengligi o'rinli bo'ladi.

Isbot. Dastlab ushbu

$$Q(\lambda) = \int_0^\pi g(t)\varphi(t,\lambda)dt$$

belgilashni kiritamiz. (2.4.2) almashtirish operatoridan foydalanib $Q(\lambda)$ ni quyidagi ko'rinishga keltiramiz:

$$Q(\lambda) = \int_0^\pi h(t)\cos\sqrt{\lambda}t dt.$$

Bu yerda

$$h(t) = g(t) + \int_t^\pi K(s,t)g(s)ds. \qquad (2.4.15)$$

Oxirgi tenglikdan $g(t)$ funksiyani topamiz:

$$g(t) = h(t) + \int_t^\pi H(s,t)h(s)ds. \qquad (2.4.16)$$

(2.4.15) dan foydalanib, quyidagi hisoblashlarni bajaramiz:

$$\int_0^\pi h(t)F(x,t)dt = \int_0^\pi \left(g(t) + \int_t^\pi K(u,t)g(u)du \right) F(x,t)dt =$$

$$= \int_0^\pi g(t) \left(F(x,t) + \int_0^t K(t,u)F(x,u)du \right) dt =$$

$$= \int_0^x g(t)\left(F(x,t)+\int_0^t K(t,u)F(x,u)du\right)dt+$$

$$+\int_x^\pi g(t)\left(F(x,t)+\int_0^t K(t,u)F(x,u)du\right)dt.$$

Oxirgi tenglikda Gelfand-Levitan integral tenglamasini va (2.4.4) ayniyatni e'tiborga olsak, quyidagi

$$\int_0^\pi h(t)F(x,t)dt = \int_0^x g(t)H(x,t)dt - \int_x^\pi g(t)K(t,x)dt, \quad (2.4.17)$$

kelib chiqadi. $F(x,t)$ funksiyaning (2.2.7) koʻrinishidan va Parseval tengligidan foydalanib

$$\int_0^\pi h^2(t)dt + \int_0^\pi\int_0^\pi h(x)h(t)F(x,t)dxdt = \int_0^\pi h^2(t)dt+$$

$$+\sum_{n=0}^\infty\left[\frac{1}{\alpha_n}\left(\int_0^\pi h(t)\cos\sqrt{\lambda_n}\,tdt\right)^2 - \frac{1}{\alpha_n^0}\left(\int_0^\pi h(t)\cos ntdt\right)^2\right] =$$

$$= \sum_{n=0}^\infty \frac{Q^2(n^2)}{\alpha_n^0} + \sum_{n=0}^\infty\left(\frac{Q^2(\lambda_n)}{\alpha_n} - \frac{Q^2(n^2)}{\alpha_n^0}\right) = \sum_{n=0}^\infty \frac{Q^2(\lambda_n)}{\alpha_n},$$

tenglikni topamiz. Yuqoridagi tenglikda (2.4.17) dan foydalansak,

$$\sum_{n=0}^\infty \frac{Q^2(\lambda_n)}{\alpha_n} = \int_0^\pi h^2(t)dt + \int_0^\pi h(x)\left(\int_0^x g(t)H(x,t)dt\right)dx-$$

$$-\int_0^\pi h(x)\left(\int_x^\pi g(t)K(t,x)dt\right)dx =$$

$$= \int_0^\pi h^2(t)dt + \int_0^\pi g(t)\left(\int_t^\pi H(x,t)h(x)dx\right)dt-$$

$$-\int_0^\pi h(x)\left(\int_x^\pi g(t)K(t,x)dt\right)dx$$

kelib chiqadi. Bu tenglikda (2.4.15) va (2.4.16) dan foydalansak, ushbu

$$\sum_{n=0}^{\infty}\frac{Q^2(\lambda_n)}{\alpha_n}=\int_0^\pi h^2(t)dt+\int_0^\pi g(t)(g(t)-h(t))dt-$$

$$\int_0^\pi h(x)(h(x)-g(x))dx=\int_0^\pi g^2(t)dt$$

hosil bo'ladi.■

Natija. Ixtiyoriy $f(x)$, $g(x)\in L_2(0,\pi)$ funksiyalar uchun umumlashgan Parseval tengligi

$$\int_0^\pi f(x)g(x)dx=\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\left(\int_0^\pi f(t)\varphi(t,\lambda_n)dt\right)\left(\int_0^\pi g(t)\varphi(t,\lambda_n)dt\right), \tag{2.4.18}$$

o'rinli bo'ladi.

Endi Gelfand-Levitan algoritmi natijasida topilgan

$$\varphi(x,\lambda_n)=\cos\sqrt{\lambda_n}x+\int_0^x K(x,t)\cos\sqrt{\lambda_n}tdt \tag{2.4.19}$$

funksiyalar sistemasining o'zaro ortogonalligini ko'rsatamiz.

Lemma 2.4.2. *Ushbu*

$$\int_0^\pi \varphi(t,\lambda_n)\varphi(t,\lambda_m)dt=\begin{cases}0, & n\neq m,\\ \alpha_n, & n=m,\end{cases} \tag{2.4.20}$$

munosabat o'rinli.

Isbot. I. Dastlab $f(x)\in W_2^2(0,\pi)$ bo'lgan holda quyidagi

$$\bar{f}(x)=\sum_{n=0}^{\infty}c_n\varphi(x,\lambda_n), \tag{2.4.21}$$

qatorni qaraylik. Bu yerda

$$c_n = \frac{1}{\alpha_n}\int_0^\pi f(x)\varphi(x,\lambda_n)dx. \tag{2.4.22}$$

Teorema 2.3.1 va bo'laklab integrallash qoidasidan foydalanib,

$$c_n = \frac{1}{\alpha_n\lambda_n}\int_0^\pi f(x)(-\varphi''(x,\lambda_n) + q(x)\varphi(x,\lambda_n))dx =$$

$$= \frac{1}{\alpha_n\lambda_n}[hf(0) - f'(0) + \varphi(\pi,\lambda_n)f'(\pi) - \varphi'(\pi,\lambda_n)f(\pi) +$$

$$+\int_0^\pi \varphi(x,\lambda_n)(-f''(x) + q(x)f(x))dx],$$

tenglikni topamiz. Bu yerda $\varphi(x,\lambda)$ yechim uchun olingan ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} +$$

$$+\int_0^x q(t)\frac{\sin\sqrt{\lambda}(x-2t)}{2\sqrt{\lambda}}dt + \underline{\underline{O}}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{\lambda}\right),$$

asimptotikadan hamda $\{\lambda_n\}_{n=0}^\infty$ va $\{\alpha_n\}_{n=0}^\infty$ spektral xarakteristikalar uchun berilgan (2.2.3) shartlardan foydalanib, $n \to \infty$ da c_n, $\varphi(x,\lambda_n)$ lar uchun ushbu

$$c_n = \underline{\underline{O}}\left(\frac{1}{n^2}\right), \quad \varphi(x,\lambda_n) = \underline{\underline{O}}(1), \quad x \in [0,\pi],$$

asimptotikalarni topish mumkin. Oxirgi asimptotikalardan (2.4.21) qatorning absolyut va tekis yaqinlashuvchiligi kelib chiqadi. Umumlashgan Parseval tengligidan va (2.4.22) formuladan foydalanib, quyidagini

$$\int_0^\pi f(x)g(x)dx = \sum_{n=0}^\infty c_n\int_0^\pi g(t)\varphi(t,\lambda_n)dt =$$

$$= \int_0^\pi g(t)\sum_{n=0}^\infty c_n\varphi(t,\lambda_n)dt = \int_0^\pi g(t)\tilde{f}(t)dt,$$

olamiz. $g(x)$ funksiyaning ixtiyoriyligidan $\tilde{f}(x) = f(x)$, ya'ni

$$f(x) = \sum_{n=0}^\infty c_n\varphi(x,\lambda_n), \tag{2.4.23}$$

yoyilma kelib chiqadi.

II. $k \geq 0$ sonini tayinlab, $f(x) = \varphi(x,\lambda_k)$ deylik, u holda (2.4.23) ga ko‘ra

$$\varphi(x,\lambda_k) = \sum_{n=0}^\infty c_{n,k}\varphi(x,\lambda_n),$$

$$c_{n,k} = \frac{1}{\alpha_n}\int_0^\pi \varphi(t,\lambda_k)\varphi(t,\lambda_n)dx,$$

bo‘ladi. $\{\cos\sqrt{\lambda_n}x\}_{n=0}^\infty$ funksiyalar $L^2(0,\pi)$ fazoda chiziqli erkli va minimal sistema bo‘lgani uchun (2.4.19) ga asosan $\{\varphi(x,\lambda_n)\}_{n=0}^\infty$ funksiyalar sistemasi ham minimal bo‘ladi. Shuning uchun

$$c_{n,k} = \delta_{n,k} = \begin{cases} 0, & n \neq k, \\ 1, & n = k, \end{cases}$$

bo‘ladi. ■

Navbatdagi ishimiz Shturm-Liuvill chegaraviy masalasining ikkinchi chegaraviy shartini keltirib chiqarishdan iborat.

Teorema 2.4.2. *Ixtiyoriy $n, m \geq 0$ butun sonlar uchun*

$$\frac{\varphi'(\pi,\lambda_n)}{\varphi(\pi,\lambda_n)} = \frac{\varphi'(\pi,\lambda_m)}{\varphi(\pi,\lambda_m)}, \tag{2.4.24}$$

tenglik bajariladi.

Isbot. Ushbu $\{\varphi(x,\lambda_n)\}_{n=0}^\infty$ funksiyalar sistemasining ortogonalligidan va Grin ayniyatidan foydalanib

$$\varphi'(\pi,\lambda_n)\varphi(\pi,\lambda_m) - \varphi'(\pi,\lambda_m)\varphi(\pi,\lambda_n) = 0, \quad (n \neq m), \tag{2.4.25}$$

tenglikni topamiz. Haqiqatan ham,

$$0 = (\lambda_n - \lambda_m)\int\limits_0^{\pi} \varphi(x,\lambda_n)\varphi(x,\lambda_m)dx = (\varphi(x,\lambda_n)\varphi'(x.\lambda_m)-$$

$$-\varphi'(x,\lambda_n)\varphi(x,\lambda_m))\,|_{x=0}^{x=\pi} = \varphi(\pi,\lambda_n)\varphi'(\pi,\lambda_m)-\varphi'(\pi,\lambda_n)\varphi(\pi,\lambda_m)-$$

$$-\varphi(0,\lambda_n)\varphi'(0,\lambda_m) + \varphi'(0,\lambda_n)\varphi(0,\lambda_m) =$$

$$= \varphi(\pi,\lambda_n)\varphi'(\pi,\lambda_m) - \varphi'(\pi,\lambda_n)\varphi(\pi,\lambda_m)-$$

$$-1\cdot h + h\cdot 1 = \varphi(\pi,\lambda_n)\varphi'(\pi,\lambda_m) - \varphi'(\pi,\lambda_n)\varphi(\pi,\lambda_m).$$

Agar biror n da $\varphi(\pi,\lambda_n) = 0$ bo'lsa, u holda (2.4.25) dan ixtiyoriy m uchun $\varphi(\pi,\lambda_m) = 0$ bo'ladi. Bu esa

$$\varphi(\pi,\lambda_m) = (-1)^m + O\left(\frac{1}{m}\right),$$

asimptotikaga ziddir. Shuning uchun ixtiyoriy $n \geq 0$ larda $\varphi(\pi,\lambda_n) \neq 0$ bo'lib, (2.4.25) dan

$$\frac{\varphi'(\pi,\lambda_n)}{\varphi(\pi,\lambda_n)} = \frac{\varphi'(\pi,\lambda_m)}{\varphi(\pi,\lambda_m)} = -\tilde{H} = \text{const},$$

kelib chiqadi. Bu holda

$$\varphi'(\pi,\lambda_n) + \tilde{H}\varphi(\pi,\lambda_n) = 0, \quad n \geq 0,$$

bo'ladi.

Teorema 2.3.1 va lemma 2.4.2 birgalikda $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari ushbu

$$-\varphi'' + q(x)\varphi = \lambda\varphi,$$

$$\varphi'(0,\lambda) - h\varphi(0,\lambda) = 0,$$

$$\varphi'(\pi,\lambda) + \tilde{H}\varphi(\pi,\lambda) = 0,$$

chegaraviy masalaning spektral xarakteristikalaridan iborat ekanligini ko'rsatadi. Bundan ushbu

$$\tilde{H} = H = \omega - h - \frac{1}{2}\int\limits_0^{\pi} q(t)dt,$$

tenglik kelib chiqadi. ■

Teskari masalani yechish jarayonida olingan natijalarni teorema shaklida bayon qilamiz.

Teorema 2.4.3. *$\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari (2.2.1)+ (2.2.2) ko'rinishdagi $q(x) \in L_2(0,\pi)$ koeffitsiyentli biror Shturm-Liuvill chegaraviy masalasining spektral xarakteristikalari bo'lishi uchun (2.2.3) shartlarning bajarilishi zarur va yetarli.*

Teorema 2.4.4. *$\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari (2.2.1)+ (2.2.2) ko'rinishdagi $q(x) \in W_2^N[0,\pi]$, $N \geq 1$ koeffitsiyentli biror Shturm-Liuvill chegaraviy masalasining spektral xarakteristikalari bo'lishi uchun ushbu*

$$\sqrt{\lambda_n} = n + \sum_{j=1}^{N+1} \frac{c_j}{n^j} + \frac{\chi_n}{n^{N+1}}, \quad c_{2p} = 0, \quad p \geq 0, \quad c_1 = \frac{\omega}{\pi}, \ \{\chi_n\} \in l_2,$$

$$\alpha_n = \frac{\pi}{2} + \sum_{j=1}^{N+1} \frac{a_j}{n^j} + \frac{\tilde{\chi}_n}{n^{N+1}}, \quad a_{2p+1} = 0, \quad p \geq 0, \ \{\tilde{\chi}_n\} \in l_2,$$

$$\lambda_n \neq \lambda_m, \quad n \neq m, \quad \alpha_n > 0, \quad n = 0, 1, 2, 3, \ldots,$$

shartlarning bajarilishi zarur va yetarlidir.

Shturm-Liuvill chegaraviy masalasi quyidagi algoritm bo'yicha quriladi:

1.Berilgan $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ sonlar ketma-ketliklari yordamida $F(x,t)$ funksiya (2.2.7) formuladan aniqlanadi.

2.Gelfand-Levitan integral tenglamasini yechib $K(x,t)$ funksiya topiladi.

3.Potentsial $q(x)$ va h, H sonlar quyidagi

$$q(x) = 2\frac{d}{dx}K(x,x), \quad h = K(0,0),$$

$$H = \omega - h - \frac{1}{2}\int_0^{\pi} q(t)dt$$

formulalar yordamida topiladi.

Misol 1. Faraz qilaylik berilgan $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ sonlar ketma-ketliklari

$$\lambda_n = n^2, \quad n \geq 0, \quad \alpha_n = \begin{cases} \frac{\pi}{2}, & n \geq 1, \\ \alpha_0, & n = 0, \alpha_0 \neq \pi \end{cases}$$

ko'rinishda bo'lsin. Bu yerda α_0 - berilgan musbat son.

Yuqoridagi algoritm yordamida, berilgan $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ ketma-ketliklarga mos keluvchi Shturm-Liuvill chegaraviy masalasini tuzamiz.

1. $F(x,t)$ funksiyani quyidagicha aniqlaymiz:

$$F(x,t) = \sum_{n=0}^{\infty}\left(\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{\alpha_n} - \frac{\cos nx\cos nt}{\alpha_n^0}\right) =$$

$$= \frac{\cos\sqrt{\lambda_0}x\cos\sqrt{\lambda_0}t}{\alpha_0} - \frac{1}{\alpha_0^0} = \frac{1}{\alpha_0} - \frac{1}{\pi} \equiv a.$$

2. Gelfand-Levitan inegral tenglamasini tuzib olamiz

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad (0 < t < x),$$

$$K(x,t) + a + a\int_0^x K(x,s)ds = 0.$$

Ushbu

$$\int_0^x K(x,s)ds = f(x),$$

belgilashni kiritib, quyidagi tenglamani hosil qilamiz:

$$K(x,t) = -a - af(x).$$

Buni belgilashga qo'yib, $f(x)$ ni topamiz:

$$f(x) = -\frac{ax}{1+ax}.$$

Bunga ko'ra

$$K(x,t) = -\frac{a}{1+ax}.$$

3. $q(x)$ potensial va h, H sonlarni hamda $\varphi(x,\lambda)$ yechimni aniqlaymiz:

$$q(x) = 2\frac{d}{dx}K(x,x) = \frac{2a^2}{(1+ax)^2},$$

$$h = K(0,0) = -a,$$

$$H = \omega - h - \frac{1}{2}\int_0^\pi q(t)dt = \frac{a}{1+a\pi} = \frac{a\alpha_0}{\pi},$$

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt = \cos\sqrt{\lambda}x - \frac{a}{1+ax}\cdot\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

Shunday qilib, ushbu

$$-y'' + \frac{2a^2}{(1+ax)^2}y = \lambda y,$$

$$y'(0) + ay(0) = 0,$$

$$y'(\pi) + \frac{a\alpha_0}{\pi}y(\pi) = 0,$$

Shturm-Liuvill chegaraviy masalasini topishga muvaffaq bo'ldik.

Mustaqil yechish uchun mashqlar

Gelfand-Levitan algoritmini qo'llab, quyida berigan $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ sonlar ketma-ketliklariga mos keluvchi Shturm-Liuvill chegaraviy masalasini tuzing.

1) $\lambda_n = n^2,\ n \geq 0;\ \alpha_n = \begin{cases} \frac{\pi}{2}, & n \geq 2, \\ \alpha_1, & n = 1, \\ \alpha_0, & n = 0. \end{cases}$

2) $\lambda_n = n^2,\ n \geq 0;\ \alpha_n = \begin{cases} \dfrac{\pi}{2}, & n \geq k, \\ \alpha_{k-1}, & n = k-1, \\ \alpha_{k-2}, & n = k-2, \\ \dots \\ \alpha_0, & n = 0. \end{cases}$

3) $\lambda_0 = a^2, (0 < a < 1),\ \lambda_n = n^2,\ n \geq 1;\ \alpha_n = \begin{cases} \dfrac{\pi}{2}, & n \geq 1, \\ \pi, & n = 0. \end{cases}$

4) $\lambda_0 = a^2, (0 < a < 1),\ \lambda_n = n^2,\ n \geq 1;\ \alpha_n = \begin{cases} \dfrac{\pi}{2}, & n \geq 1, \\ \alpha_0, & n = 0. \end{cases}$

5) $\lambda_0 = -a^2,\ \lambda_n = n^2,\ n \geq 1;\ \alpha_n = \begin{cases} \dfrac{\pi}{2}, & n \geq 1, \\ \pi, & n = 0. \end{cases}$

6) $\lambda_0 = -a^2,\ \lambda_n = n^2,\ n \geq 1;\ \alpha_n = \begin{cases} \dfrac{\pi}{2}, & n \geq 1, \\ \alpha_0, & n = 0. \end{cases}$

7) $\lambda_n = n^2,\ n \geq 0;\ \alpha_n^{-1} = \begin{cases} \dfrac{2}{\pi} + \dfrac{1}{2n^2}, & n \geq 1, \\ \dfrac{1}{\pi}, & n = 0. \end{cases}$

8) $\lambda_n = n^2;\ n \geq 0;\ \alpha_n^{-1} = \begin{cases} \dfrac{2}{\pi} + \dfrac{\omega}{n^2}, & n \geq 1, \\ \dfrac{1}{\pi}, & n = 0 \end{cases};\ \omega = \text{const}.$

5-§. Ambarsumyan sinfi haqida ma'lumot

Ta'rif 2.5.1. Ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.5.1}$$

Shturm-Liuvill chegaraviy masalasida $q(\pi - x) \equiv q(x) \in C[0,\ \pi]$ haqiqiy funksiya bo'lib, $H = h$ chekli haqiqiy son bo'lsa, bu masala V.A.Ambarsumyan sinfiga qarashli deyiladi.

$\varphi(x, \lambda)$ orqali Shturm-Liuvill tenglamasining $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = h$ boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaylik. (2.5.1) chegaraviy masalaning xos qiymatlarini $\{\lambda_n\}_{n=0}^{\infty}$ orqali belgilaymiz.

Teorema 2.5.1. *(2.5.1) Shturm-Liuvill chegaraviy masalasi V.A.Ambarsumyan sinfiga qarashli bo'lishi uchun, ushbu*

$$\varphi(\pi, \lambda_n) = (-1)^n,\ n \geq 0, \tag{2.5.2}$$

tengliklarning bajarilishi zarur va yetarli.

Isbot. (*Zaruriyligi*). (2.5.1) Shturm-Liuvill chegaraviy masalasi Ambarsumyan sinfiga qarashli bo'lsin, ya'ni $q(\pi - x) = q(x)$, $H = h$. Agar $\lambda = \lambda_n$ (2.5.1) chegaraviy masalaning xos qiymati bo'lsa, unga mos keluvchi xos funksiya $\varphi(x, \lambda_n)$ bo'ladi. Bu holda $y(x, \lambda_n) = \varphi(\pi - x, \lambda_n)$ ham shu $\lambda = \lambda_n$ xos qiymatga mos keluvchi xos funksiya bo'ladi. Buni ko'rsatish uchun (2.5.1) masalaga $\varphi(x, \lambda_n)$ funksiyani qo'yib, $\pi - x = t$ almashtirish bajarish kerak. Bu almashtirish natijasida tenglama o'zgarmaydi, ammo chegaraviy shartlarning o'rinlari almashadi xolos. (2.5.1) chegaraviy masalaning xos qiymatlari oddiy (karrasiz) bo'lgani uchun

$$\varphi(\pi - x, \lambda_n) = b_n \varphi(x, \lambda_n),\ n = 0, 1, 2, ... \tag{2.5.3}$$

bo'ladi. Bu yerda b_n – o'zgarmas sonlar.

Ossillyatsiya teoremasiga ko'ra, $\varphi(x, \lambda_n)$ xos funksiya $(0, \pi)$ oraliqda n ta ildizga ega. Bu ildizlarning hammasi karrasiz. (2.5.3) ayniyatga ko'ra $\varphi(x, \lambda_n)$ funksiyaning ildizlari $x = \frac{\pi}{2}$ nuqtaga nisbatan simmetrik joylashgan. Bunga ko'ra n toq bo'lsa, $x = \frac{\pi}{2}$ son $\varphi(x, \lambda_n)$ funksiyaning ildizi bo'ladi va n juft bo'lsa, $x = \frac{\pi}{2}$ son $\varphi(x, \lambda_n)$ funksiyaning ildizi bo'lmaydi.

Agar n juft bo'lsa, $\varphi\left(\frac{\pi}{2}, \lambda_n\right) \neq 0$ bo'lgani uchun, (2.5.3) ayniyatda $x = \frac{\pi}{2}$ desak,

$$\varphi\left(\frac{\pi}{2}, \lambda_n\right) = b_n \varphi\left(\frac{\pi}{2}, \lambda_n\right),$$

ya'ni $b_n = 1$ kelib chiqadi.

Agar n toq bo'lsa, $\varphi\left(\frac{\pi}{2}, \lambda_n\right) = 0$ bo'ladi, shuning uchun $\varphi'\left(\frac{\pi}{2}, \lambda_n\right) \neq 0$ o'rinli, aks holda differensial tenglamalar kursidagi mavjudlik va yagonalik teoremasiga ko'ra $\varphi(x, \lambda_n) \equiv 0$ bo'ladi. Ziddiyat, chunki $\varphi(0, \lambda_n) = 1$. (2.5.3) ayniyatdan hosila olib, so'ngra $x = \frac{\pi}{2}$ desak,

$$-\varphi'\left(\frac{\pi}{2}, \lambda_n\right) = b_n \varphi'\left(\frac{\pi}{2}, \lambda_n\right),$$

ya'ni $b_n = -1$ bo'ladi. Demak, $\varphi(\pi, \lambda_n) = (-1)^n$, $n = 0, 1, 2, \ldots$.

(*Yetarliligi*). Faraz qilaylik $\varphi(\pi, \lambda_n) = (-1)^n$, $n = 0, 1, 2, \ldots$ shart bajarilsin. U holda $\psi_n(x) = (-1)^n \varphi(\pi - x, \lambda_n)$ funksiya ushbu

$$\begin{cases} -y'' + q(\pi - x)y = \lambda y, \\ y'(0) - Hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0, \end{cases} \tag{2.5.4}$$

chegaraviy masalaning $\lambda = \lambda_n$ xos qiymatga mos keluvchi xos funksiyasi bo'ladi. Bundan tashqari ushbu

$$\int_0^\pi \psi_n^2(x)dx = \int_0^\pi \varphi^2(x, \lambda_n)dx$$

tenglik ham bajariladi. Shunday qilib (2.5.1) va (2.5.4) chegaraviy masalalarning xos qiymatlari va normallovchi o'zgarmaslari o'zaro teng ekan. V.A.Marchenko teoremasiga asosan $q(\pi - x) = q(x)$, $H = h$ kelib chiqadi.■

Teorema 2.5.2. *Agar (2.5.1) chegaraviy masala V.A.Ambarsumyan sinfiga qarashli bo'lsa, ushbu*

$$\alpha_n = (-1)^{n+1}\dot{\Delta}(\lambda_n), \quad n = 0,\ 1,\ 2,\ \ldots \tag{2.5.5}$$

tenglik o'rinli bo'ladi. Bu yerda

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda)\prod_{k=1}^{\infty}\frac{\lambda_k - \lambda}{k^2}.$$

Isbot. Ushbu

$$-\varphi''(x,\lambda) + q(x)\varphi(x,\lambda) = \lambda\varphi(x,\lambda), \tag{2.5.6}$$

ayniyatdan λ bo'yicha hosila olamiz

$$-\dot{\varphi}''(x,\lambda) + q(x)\dot{\varphi}(x,\lambda) = \varphi(x,\lambda) + \lambda\dot{\varphi}(x,\lambda). \tag{2.5.7}$$

(2.5.6) va (2.5.7) ayniyatlarni mos ravishda $\dot{\varphi}(x,\lambda)$ va $\varphi(x,\lambda)$ ga ko'paytirib, ikkinchisidan birinchisini ayiramiz:

$$\varphi^2(x,\lambda) = \varphi''(x,\lambda)\dot{\varphi}(x,\lambda) - \dot{\varphi}''(x,\lambda)\varphi(x,\lambda) =$$
$$= [\varphi'(x,\lambda)\dot{\varphi}(x,\lambda) - \dot{\varphi}'(x,\lambda)\varphi(x,\lambda)]'. \tag{2.5.8}$$

(2.5.8) tenglikni x bo'yicha $[0,\ \pi]$ oraliqda integrallasak,

$$\int_0^{\pi}\varphi^2(x,\lambda)dx = \varphi'(\pi,\lambda)\dot{\varphi}(\pi,\lambda) - \dot{\varphi}'(\pi,\lambda)\varphi(\pi,\lambda), \tag{2.5.9}$$

tenglik kelib chiqadi. Bu yerda $\dot{\varphi}(0,\lambda) = 0$, $\dot{\varphi}'(0,\lambda) = 0$ tengliklar ishlatildi.

(2.5.9) tenglikda $\lambda = \lambda_n$ deb, ushbu

$$\varphi'(\pi,\lambda_n) + h\varphi(\pi,\lambda_n) = 0,$$

tenglikni hisobga olsak, quyidagi

$$\alpha_n = \int_0^\pi \varphi^2(x,\lambda_n)dx = -h\varphi(\pi,\lambda_n)\dot\varphi(\pi,\lambda_n) - \dot\varphi'(\pi,\lambda_n)\varphi(\pi,\lambda_n) =$$

$$= -\varphi(\pi,\lambda_n)[\dot\varphi'(\pi,\lambda_n) + h\dot\varphi(\pi,\lambda_n)] = -\varphi(\pi,\lambda_n)\dot\Delta(\lambda_n), \quad (2.5.10)$$

formula hosil bo'ladi. Bu yerda $\Delta(\lambda) = \varphi'(\pi,\lambda) + h\varphi(\pi,\lambda)$ funksiya (2.5.1) chegaraviy masalaning xarakteristik funksiyasi. Oldingi paragraflarga ko'ra

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda)\prod_{k=1}^{\infty}\frac{\lambda_k - \lambda}{k^2}, \qquad (2.5.11)$$

yoyilma o'rinli.

Teorema 2.5.1 ga ko'ra $\varphi(\pi,\lambda_n) = (-1)^n$ bo'lgani uchun (2.5.10) tenglikdan (2.5.4) formula kelib chiqadi. ∎

Teskari masala. Faraz qilaylik bizga $\{\lambda_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketligi berilgan bo'lib, u ushbu

$$\sqrt{\lambda_n} = n + \frac{c}{n} + \frac{\gamma_n}{n}, \quad \{\gamma_n\} \in l_2 \qquad (2.5.12)$$

asimptotikani qanoatlantirsin. U holda (2.5.5) formula yordamida ushbu

$$\alpha_n = (-1)^{n+1}\dot\Delta(\lambda_n) \qquad (2.5.13)$$

haqiqiy sonlar ketma-ketligini tuzib olamiz. Bu ketma-ketlik uchun

$$\alpha_n = \frac{\pi}{2} + \frac{\beta_n}{n}, \quad \{\beta_n\} \in l_2, \qquad (2.5.14)$$

asimptotika o'rinli ekanini ko'rsatamiz. Buning uchun teorema 2.4.3 dan foydalanamiz. Spektri $\{\lambda_n\}_{n=0}^{\infty}$ bo'lgan $q_0(x) \in L^2(0,\pi)$, h_0, $H_0 \in R^1$ koeffitsiyentli Shturm-Liuvill chegaraviy masalalari cheksiz ko'p. Shulardan biri $\{\lambda_n, \alpha_n^0 = \frac{\pi}{2}\}_{n=0}^{\infty}$ spektral xarakteristikalarga mos keladi:

$$-y'' + q_0(x)y = \lambda y, \qquad (2.5.15)$$

$$\begin{cases} y'(0) - h_0 y(0) = 0, \\ y'(\pi) + H_0 y(\pi) = 0. \end{cases} \tag{2.5.16}$$

Bu chegaraviy masalaning xarakteristik funksiyasi uchun quyidagi tenglik o'rinli:

$$\Delta_0(\lambda) \equiv \varphi_0'(\pi, \lambda) + H_0 \varphi_0(\pi, \lambda) = \pi(\lambda_0 - \lambda) \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{k^2} = \Delta(\lambda).$$

Bu yerda $\varphi_0(x, \lambda)$ orqali (2.5.15) tenglamaning $\varphi_0(0, \lambda) = 1$, $\varphi_0'(0, \lambda) = h_0$ boshlang'ich shartlarni qanoatlantiruvchi yechimi belgilangan. (2.5.15)+(2.5.16) chegaraviy masalaning $\alpha_n^0 = \frac{\pi}{2}$, $n \geq 0$ normallovchi o'zgarmaslari uchun (2.5.10) formula o'rinli, ya'ni

$$\frac{\pi}{2} = \alpha_n^0 \equiv \int_0^{\pi} \varphi_0^2(x, \lambda_n) dx = -\varphi_0(\pi, \lambda_n) \dot{\Delta}(\lambda_n). \tag{2.5.17}$$

Bu tenglikdagi $\varphi_0(\pi, \lambda_n)$ ketma-ketlikning $n \to \infty$ dagi asimptotikasini topish uchun ushbu

$$\varphi_0(x, \lambda) = \cos \sqrt{\lambda} x + \left(h_0 + \frac{1}{2} \int_0^x q_0(t) dt \right) \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} -$$

$$- \frac{1}{2\sqrt{\lambda}} \int_0^x q_0(t) \sin \sqrt{\lambda}(2t - x) dt + \underline{O}\left(\frac{1}{\lambda}\right)$$

asimptotik formuladan foydalanamiz. Bunda $x = \pi$ va $\sqrt{\lambda} = \sqrt{\lambda_n} = n + \delta_n$, $\delta_n = \dfrac{c}{n} + \dfrac{\gamma_n}{n}$ deb olsak, u holda quyidagi

$$\varphi_0(\pi, \lambda_n) = \cos(n + \delta_n)\pi + \left(h_0 + \frac{1}{2} \int_0^{\pi} q_0(t) dt \right) \frac{\sin(n + \delta_n)\pi}{n + \delta_n} -$$

$$- \frac{1}{2(n + \delta_n)} \int_0^{\pi} q_0(t) \sin((n + \delta_n)(2t - \pi)) dt + \underline{O}\left(\frac{1}{(n + \delta_n)^2} \right) =$$

$$= (-1)^n + \frac{1}{n}\left\{ -\frac{1}{2}\int\limits_0^\pi q_0(t)\sin n(2t-\pi)dt + \underline{\underline{O}}\left(\frac{1}{n}\right)\right\} = (-1)^n + \frac{c_n}{n} \tag{2.5.18}$$

asimptotikaga ega bo‘lamiz. Bu yerda

$$c_n = -\frac{1}{2}\int\limits_0^\pi q_0(t)\sin(2t-\pi)ndt + \underline{\underline{O}}\left(\frac{1}{n}\right), \quad \{c_n\} \in l_2.$$

$\varphi_0(\pi, \lambda_n)$ ketma-ketlikning (2.5.18) asimptotikasini (2.5.17) ga qo‘yib, ushbu

$$\frac{\pi}{2} = -[(-1)^n + \frac{c_n}{n}] \cdot \dot{\Delta}(\lambda_n)$$

tenglamani hosil qilamiz. Bundan

$$\dot{\Delta}(\lambda_n) = \frac{\frac{\pi}{2}}{(-1)^{n+1} - \frac{c_n}{n}} = (-1)^{n+1} \cdot \frac{\pi}{2} + \frac{\tilde{c}_n}{n}, \quad \{\tilde{c}_n\} \in l_2 \tag{2.5.19}$$

tenglikni topamiz. Endi topilgan (2.5.19) asimptotikani (2.5.13) tenglikka qo‘yib ushbu

$$\alpha_n = (-1)^{n+1}[(-1)^{n+1}\frac{\pi}{2} + \frac{\tilde{c}_n}{n}] = \frac{\pi}{2} + \frac{\beta_n}{n}, \quad \{\beta_n\} \in l_2$$

asimptotikani hosil qilamiz. Shunday qilib $\{\lambda_n\}_{n=0}^\infty$ va $\{\alpha_n\}_{n=0}^\infty$ sonlar ketma-ketliklari teorema 2.4.3 ning shartlarini qanoatlantirishini ko‘rsatdik. Bunga ko‘ra Gelfand-Levitan algoritmi yordamida ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.5.20}$$

Shturm-Liuvill chegaraviy masalasini tuzamiz. Bu yerda $q(x) \in L^2(0,\pi)$, h, $H \in R^1$.

Endi bu chegaraviy masalaning Ambarsumyan sinfiga tegishli ekanini ko‘rsatamiz. Buning uchun $\varphi(\pi, \lambda_n) = (-1)^n$ ekanini ko‘rsatish yetarli. Shu maqsadda $\alpha_n = -\varphi(\pi, \lambda_n)\dot{\Delta}(\lambda_n)$ va

$\alpha_n = (-1)^{n+1}\dot{\Delta}(\lambda_n)$ ifodalarni solishtirib

$$\varphi(\pi, \lambda_n) = (-1)^n, \quad n = 0, 1, 2, ...$$

tengliklarni topamiz. Teorema 2.5.1 ga asosan (2.5.20) chegaraviy masala Ambarsumyan sinfiga tegishli bo'ladi, ya'ni $q(\pi - x) = q(x)$, $H = h$.

Shunday qilib, klassik asimptotikani qanoatlantiruvchi bitta $\{\lambda_n\}_{n=0}^{\infty}$ spektr yordamida Ambarsumyan sinfiga tegishli yagona Shturm-Liuvill chegaraviy masalasini topish mumkin.

6-§. Chegaraviy shartlarda $y(\pi) = 0$ qatnashgan Shturm-Liuvill chegaraviy masalasi uchun teskari masala

Quyidagi

$$-y'' + q(x)y = \lambda y, \quad (0 \le x \le \pi), \tag{2.6.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{2.6.2}$$

Shturm-Liuvill chegaraviy masalasini qaraylik. Bu yerda $q(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiya va h chekli haqiqiy son. Birinchi bobning oltinchi paragrafida (2.6.1)-(2.6.2) chegaraviy masalaning μ_n xos qiymatlari va $\alpha_n^{(1)}$ normallovchi o'zgarmaslari uchun ushbu

$$\sqrt{\mu_n} = n + \frac{1}{2} + \frac{a_1}{\left(n + \frac{1}{2}\right)\pi} + \frac{\chi_n}{n}, \quad \mu_n \ne \mu_m, \quad n \ne m, \quad \{\chi_n\} \in l_2, \tag{2.6.3}$$

$$\alpha_n^{(1)} = \frac{\pi}{2} + \frac{\chi_n^{(1)}}{n}, \quad \alpha_n^{(1)} > 0, \quad n \ge 0, \quad \left\{\chi_n^{(1)}\right\} \in l_2, \tag{2.6.4}$$

asimptotikalar olingan edi. Bu yerda

$$a_1 = h + \frac{1}{2}\int_0^{\pi} q(t)dt. \tag{2.6.5}$$

$\varphi(x,\lambda)$ orqali (2.6.1) tenglamaning

$$\varphi(0,\lambda)=1, \quad \varphi'(0,\lambda)=h,$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaylik. U holda (2.6.1)-(2.6.2) chegaraviy masalaning xarakteristik funksiyasi $d(\lambda)=\varphi(\pi,\lambda)$ ko'rinishda bo'lib, uning uchun ushbu

$$d(\lambda)=\prod_{n=0}^{\infty}\frac{\mu_n-\lambda}{\left(n+\frac{1}{2}\right)^2}, \tag{2.6.6}$$

formula o'rinli.

Birinchi bobning sakkizinchi paragrafida ushbu

$$-y''+q(x)y=\lambda y, \tag{2.6.7}$$

$$y'(0)-hy(0)=0, \quad y'(\pi)+Hy(\pi)=0, \tag{2.6.8}$$

chegaraviy masalaning $\Delta(\lambda)=\varphi'(\pi,\lambda)+H\varphi(\pi,\lambda)$ xarakteristik funksiyasi uchun

$$\Delta(\lambda)=\pi(\lambda_0-\lambda)\prod_{k=1}^{\infty}\frac{\lambda_k-\lambda}{k^2}, \tag{2.6.9}$$

formula o'rinli bo'lishi ko'rsatilgan edi. Bu yerda $\{\lambda_n\}_{n=0}^{\infty}$ orqali (2.6.7)-(2.6.8) chegaraviy masalaning xos qiymatlari belgilangan. Ular uchun ushbu

$$\sqrt{\lambda_n}=n+\frac{\omega}{n\pi}+\frac{\gamma_n}{n}, \quad \lambda_n\neq\lambda_m, \quad n\neq m, \quad \{\gamma_n\}\in l_2, \tag{2.6.10}$$

$$\omega=h+H+\frac{1}{2}\int_0^{\pi}q(t)dt.$$

asimptotika o'rinli.

Lemma 2.6.1. *(2.6.1)-(2.6.2) va (2.6.7)-(2.6.8) chegaraviy masalalarning xos qiymatlari almashinib keladi, ya'ni*

$$\lambda_n<\mu_n<\lambda_{n+1}, \quad n\geq 0, \tag{2.6.11}$$

tengsizlik o'rinli bo'ladi.

Isbot. Quyidagi

$$\frac{d}{dx}W\{\varphi(x,\lambda),\varphi(x,\mu)\} = (\lambda-\mu)\varphi(x,\lambda)\varphi(x,\mu),$$

ayniyatni $[0,\pi]$ oraliqda integrallaymiz:

$$(\lambda-\mu)\int_0^\pi \varphi(x,\lambda)\varphi(x,\mu)dx = W\{\varphi(x,\lambda),\varphi(x,\mu)\}|_{x=0}^{x=\pi} =$$

$$= \varphi(\pi,\lambda)\varphi'(\pi,\mu) - \varphi'(\pi,\lambda)\varphi(\pi,\mu) -$$

$$-\varphi(0,\lambda)\varphi'(0,\mu) + \varphi'(0,\lambda)\varphi(0,\mu) = d(\lambda)\varphi'(\pi,\mu) - \varphi'(\pi,\lambda)d(\mu) =$$

$$= d(\lambda)[\Delta(\mu) - Hd(\mu)] - [\Delta(\lambda) - Hd(\lambda)]d(\mu) =$$

$$= d(\lambda)\Delta(\mu) - \Delta(\lambda)d(\mu).$$

Oxirgi tenglikda $\mu \to \lambda$ limitga o'tsak,

$$\int_0^\pi \varphi^2(x,\lambda)dx = \dot{d}(\lambda)\Delta(\lambda) - \dot{\Delta}(\lambda)d(\lambda), \qquad (2.6.12)$$

kelib chiqadi. Bunda $\dot{F}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$, $\dot{d}(\lambda) = \frac{d}{d\lambda}d(\lambda)$. Xususan, (2.6.12) tenglikda $\lambda = \lambda_n$ deb olsak,

$$\alpha_n = \int_0^\pi \varphi^2(x,\lambda_n)dx = -\dot{\Delta}(\lambda_n)d(\lambda_n), \qquad (2.6.13)$$

hosil bo'ladi. (2.6.12) tenglikni quyidagi ko'rinshda yozish mumkin:

$$\frac{1}{d^2(\lambda)}\int_0^\pi \varphi^2(x,\lambda)dx =$$

$$= \frac{\dot{d}(\lambda)\Delta(\lambda) - \dot{\Delta}(\lambda)d(\lambda)}{d^2(\lambda)} = -\frac{d}{d\lambda}\left(\frac{\Delta(\lambda)}{d(\lambda)}\right), \quad d(\lambda) \neq 0.$$

Bu tenglikdan $\frac{\Delta(\lambda)}{d(\lambda)}$ funksiyaning $(-\infty, \mu_0)$, (μ_n, μ_{n+1}), $n \geq 0$ oraliqlarda monoton kamayuvchiligi va

$$\lim_{\lambda \to \mu_n \pm} \frac{\Delta(\lambda)}{d(\lambda)} = \mp\infty,$$

kelib chiqadi. Bundan va xos qiymatlarning (2.6.3), (2.6.10) asimptotikalaridan (2.6.11) tengsizlik kelib chiqadi. ■

$\varphi(x, \lambda)$ yechimning asimptotikasidan foydalanib quyidagi tasdiqni isbotlash mumkin.

Lemma 2.6.2. *(2.6.1)+(2.6.2) va (2.6.7)+(2.6.8) chegaraviy masalalarning xarakteristik funksiyalari uchun*

$$\dot{\Delta}(\lambda_n) = (-1)^{n+1}\frac{\pi}{2} + \frac{\chi_n}{n}, \quad \{\chi_n\} \in l_2,$$

$$d(\lambda_n) = (-1)^n + \frac{\tilde{\chi}_n'}{n}, \quad \{\tilde{\chi}_n'\} \in l_2, \tag{2.6.14}$$

asimptotikalar o'rinli. Bunda

$$\operatorname{sign} \dot{\Delta}(\lambda_n) = (-1)^{n+1}, \quad n \geq 0,$$

$$\operatorname{sign} d(\lambda_n) = (-1)^n, \quad n \geq 0.$$

Yuqoridagi paragraflarda qo'llanilgan usullar yordamida (2.6.1)-(2.6.2) ko'rinishdagi Shturm-Liuvill chegaraviy masalasi uchun ham teskari masalani yechish mumkin. Bu holda Gelfand-Levitan integral tenglamasining yadrosi ushbu

$$F(x,t) = \sum_{n=0}^{\infty} \left\{ \frac{\cos\sqrt{\mu_n}\, x \cos\sqrt{\mu_n}\, t}{\alpha_n^{(1)}} - \frac{2}{\pi}\cos(n+\frac{1}{2})\, x \cos(n+\frac{1}{2})t \right\}$$

ko'rinishda bo'ladi.

Teorema 2.6.1. $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n^{(1)}\}_{n=0}^{\infty}$ *haqiqiy sonlar ketma-ketliklari (2.6.1)+(2.6.2) ko'rinishdagi* $q(x) \in L^2(0, \pi)$ *koeffitsiyentli biror Shturm-Liuvill chegaraviy masalasining spektral xarakteristikalari bo'lishi uchun (2.6.3)-(2.6.4) shartlarning bajarilishi zarur va yetarlidir.*

Misol. Quyida

1) $\mu_n = \left(n + \frac{1}{2}\right)^2, \quad n \geq 0; \; \alpha_n^{(1)} = \begin{cases} \frac{\pi}{2}, & n \geq 1, \\ \alpha_0, & n = 0, \end{cases}$

2) $\mu_0 = 0, \; \mu_n = \left(n + \frac{1}{2}\right)^2, \; n \geq 1, \;\; \alpha_n^{(1)} = \frac{\pi}{2}, \; n \geq 0$

sonlar ketma-ketliklari berilgan. Bu yerda α_0 – ixtiyoriy musbat son. Gelfand-Levitan algoritmi yordamida berilgan $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n^{(1)}\}_{n=0}^{\infty}$ sonlar ketma-ketliklariga mos keluvchi (2.6.1)+(2.6.2) ko'rinishdagi Shturm-Liuvill chegaraviy masalasini tuzing.

7-§. Differensiallanuvchi va absolyut uzluksiz funksiyalar sinfida teskari masala

Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq \pi, \tag{2.7.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(0) + Hy(0) = 0, \end{cases} \tag{2.7.2}$$

chegaraviy masalani qaraylik. Bu yerda $q(x)$ potensial m-marta differensiallanuvchi haqiqiy funksiya bo'lib, $q^{(m)}(x) \in L^1(0,\pi)$ va h, H – chekli haqiqiy sonlar. (2.7.1)+(2.7.2) chegaraviy masalaning xos qiymatlarini $\{\lambda_n\}_{n=0}^{\infty}$ va normallovchi o'zgarmaslarini $\{\alpha_n\}_{n=0}^{\infty}$ orqali belgilaylik. U holda $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ ketma-ketliklar uchun quyidagi asimptotikalar o'rinli:

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + ... + \frac{a_l}{n^{2l+1}} + \frac{\gamma_n}{n^{2l+1}}, \tag{2.7.3}$$

$$\alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + ... + \frac{b_l}{n^{2l+1}} + \frac{\tau_n}{n^{2l+1}}. \tag{2.7.4}$$

Bu yerda

$$a_0 = \frac{h+H}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(t)dt, \tag{2.7.5}$$

$a_0, ..., a_l$ va $b_0, ..., b_l$ o'zgarmas sonlar, $l = \left[\frac{m}{2}\right]$. Agar m juft son bo'lsa, u holda

$$\gamma_n = \overline{\overline{o}}(1) \text{ va } \tau_n = \overline{\overline{o}}\left(\frac{1}{n}\right),$$

agar m toq son bo'lsa, u holda

$$\gamma_n = \overline{\overline{o}}\left(\frac{1}{n}\right) \text{ va } \tau_n = \overline{\overline{o}}(1)$$

bo'ladi. Bu asimptotik formulalarni $m = 0$, ya'ni $q(x) \in L^1(0, \pi)$ hol uchun isbotlaymiz. Umumiy hol ham shu tarzda amalga oshiriladi.

$\varphi(x, \lambda)$ orqali (2.7.1) tenglamaning

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \tag{2.7.6}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. U holda (2.7.1)+ (2.7.2) chegaraviy masalaning xos qiymatlari, ushbu

$$\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda) = 0, \tag{2.7.7}$$

xarakteristik tenglamaning ildizlaridan iborat bo'ladi.

Yuqoridagi paragraflarda $\varphi(x, \lambda)$ uchun

$$\varphi(x, \lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x, t)\cos\sqrt{\lambda}t dt, \tag{2.7.8}$$

$$\cos\sqrt{\lambda}x = \varphi(x, \lambda) + \int_0^x H(x, t)\varphi(t, \lambda)dt, \tag{2.7.9}$$

tasvirlar olingan edi. Bu yerda $K(x, t)$ va $H(x, t)$ – $(m + 1)$-tartibdagi hosilalari bilan jamlanuvchi funksiyalar bo'lib,

$$K(x, x) = h + \frac{1}{2}\int_0^x q(t)dt, \quad \left.\frac{\partial K(x, t)}{\partial t}\right|_{t=0} = 0, \tag{2.7.10}$$

$$H(x,x) = -h - \frac{1}{2}\int_0^x q(t)dt, \quad \left(\frac{\partial H(x,t)}{\partial t} - hH(x,t)\right)\Bigg|_{t=0} = 0, \tag{2.7.11}$$

$$K(x,-t) = 0, \ H(x,-t) = 0, \ t > 0, \tag{2.7.12}$$

$$K(x,t) = 0, \ H(x,t) = 0, \ t > x, \tag{2.7.13}$$

shartlarni qanoatlantiradi. Agar $m \geq 1$ bo'lsa, u holda

$$\frac{\partial^2 K(x,t)}{\partial x^2} - \frac{\partial^2 K(x,t)}{\partial t^2} = q(x)K(x,t), \tag{2.7.14}$$

$$\frac{\partial^2 H(x,t)}{\partial x^2} - \frac{\partial^2 H(x,t)}{\partial t^2} = -q(t)H(x,t), \tag{2.7.15}$$

o'rinli bo'ladi.

Agar $m = 0$ bo'lsa, u holda $K(x,t)$ funksiyaning birinchi hosilalari $L^1(0,\pi)$ fazoga tegishli bo'ladi. Shuning uchun (2.7.7) xarakteristik tenglama quyidagi ko'rinishni oladi:

$$-\sqrt{\lambda}\sin\sqrt{\lambda}\pi + [K(\pi,\pi) + H]\cos\sqrt{\lambda}\pi +$$

$$+\int_0^\pi [HK(\pi,t) + \frac{\partial}{\partial x}K(x,t)|_{x=\pi}]\cos\sqrt{\lambda}t dt = 0. \tag{2.7.16}$$

Bunga asosan, $n \to \infty$ da $\lambda_n \to +\infty$ bo'lishi ravshan. Bundan tashqari yetarli katta n larda (birinchi yaqinlashishda)

$$\sin\sqrt{\lambda_n}\pi + \underline{\underline{O}}\left(\frac{1}{\sqrt{\lambda_n}}\right) = 0,$$

bo'ladi. Rushye teoremasidan foydalanib,

$$\sqrt{\lambda_n}\pi = n\pi + \underline{\underline{O}}\left(\frac{1}{n}\right),$$

ya'ni

$$\sqrt{\lambda_n} = n + \underline{\underline{O}}\left(\frac{1}{n}\right),$$

asimptotik formulani keltirib chiqaramiz. Endi

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{\gamma_n}{n}, \tag{2.7.17}$$

deb olamiz. Bunda $\gamma_n \to 0$ $(n \to \infty)$. (2.7.16) tenglamaga asosan:

$$(-1)^n\pi\left[\frac{a_0}{n}+\frac{\gamma_n}{n}+\underline{\underline{O}}\left(\frac{1}{n^3}\right)\right]-\frac{(-1)^n}{n}[K(\pi,\pi)+H]\cdot\left[1+\underline{\underline{O}}\left(\frac{1}{n^2}\right)\right]-$$

$$-\frac{1}{\sqrt{\lambda_n}}\int_0^\pi\left[HK(x,t)+\frac{\partial}{\partial x}K(x,t)\bigg|_{x=\pi}\right]\cos\sqrt{\lambda_n}t\,dt=0. \tag{2.7.18}$$

Bu yerda

$$\int_0^\pi\left[HK(x,t)+\frac{\partial}{\partial x}K(x,t)\bigg|_{x=\pi}\right]\cos\sqrt{\lambda_n}t\,dt\to 0,\quad n\to\infty, \tag{2.7.19}$$

bo'lgani uchun (2.7.18) tenglikdan

$$a_0=\frac{K(\pi,\pi)+H}{\pi} \tag{2.7.20}$$

kelib chiqadi. Agar ushbu

$$K(\pi,\pi)=h+\frac{1}{2}\int_0^\pi q(t)dt,$$

tenglikdan foydalansak,

$$a_0=\frac{h+H}{\pi}+\frac{1}{2\pi}\int_0^\pi q(t)dt,$$

koeffitsiyent topiladi.

Shunday qilib, $m=0$, ya'ni $q(x)\in L^1(0,\pi)$ holda (2.7.3) asimptotika isbotlandi. Endi α_n lar uchun (2.7.4) asimptotikani keltirib chiqaramiz. Buning uchun (2.7.8) tasvirdan foydalanib

$$\varphi(x,\lambda_n)=\cos nx-a_0\frac{\sin nx}{n}+K(x,x)\frac{\sin nx}{n}+\bar{\bar{o}}\left(\frac{1}{n}\right),\ n\to\infty$$

asimptotik formulani topamiz. Oxirgi tenglikdan ko'rinadiki yetarli katta n larda

$$\int_0^\pi \varphi^2(x,\lambda_n)dx = \frac{\pi}{2} + \bar{\bar{o}}\left(\frac{1}{n}\right),$$

bajariladi. Bu esa $q(x) \in L^1(0,\pi)$ bo'lgan holda (2.7.4) asimptotikaning o'rinli ekanini bildiradi.

Berilgan $\{\lambda_n\}_{n=0}^\infty$ va $\{\alpha_n\}_{n=0}^\infty$ haqiqiy sonlar ketma-ketliklari qanday shartlarni qanoatlantirganda (2.7.1)+(2.7.2) ko'rinishdagi biror chegaraviy masalaning spektral xarakteristikalari bo'lishi mumkin degan savolga quyidagi teorema javob beradi.

Teorema 2.7.1 (*I.M.Gasimov, B.M.Levitan*). $\{\lambda_n\}_{n=0}^\infty$ *va* $\{\alpha_n\}_{n=0}^\infty$ *haqiqiy sonlar ketma-ketliklari (2.7.1) +(2.7.2) ko'rinishdagi,* $q(x)$ $(q^{(m)}(x) \in L^1(0,\pi))$ *koeffitsiyentli Shturm-Liuvill chegaraviy masalasining spektral xarakteristikalari bo'lishi uchun quyidagi shartlarning bajarilishi zarur va yetarlidir:*

1.
$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \bar{\bar{o}}\left(\frac{1}{n}\right), \quad \lambda_n \neq \lambda_k,\ n \neq k; \tag{2.7.21}$$

$$\alpha_n = \frac{\pi}{2} + \bar{\bar{o}}\left(\frac{1}{n}\right), \quad \alpha_n > 0, \quad n = 0,1,2,..., \tag{2.7.22}$$

2. *Ushbu*

$$F(x,t) = \frac{1}{\alpha_0}\cos\sqrt{\lambda_0}x\cos\sqrt{\lambda_0}t - \frac{1}{\pi} +$$

$$+\sum_{n=1}^\infty \left[\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{\alpha_n} - \frac{2}{\pi}\cos nx\cos nt\right], \tag{2.7.23}$$

funksiyaning $(m+1)$ *- tartibli hosilalari* $(0 \le x,t \le \pi)$ *sohada jamlanuvchi.*

Isbot (*Zaruriyligi*). Faraz qilaylik $\{\lambda_n\}_{n=0}^\infty$ va $\{\alpha_n\}_{n=0}^\infty$ haqiqiy sonlar ketma-ketliklari (2.7.1)+ (2.7.2) ko'rinishdagi $q^{(m)}(x) \in L^1(0,\pi)$ koeffitsiyentli Shturm-Liuvill chegaraviy masalasining spektral xarakteristikalaridan iborat bo'lsin. U holda yuqorida keltirilgan usul yordamida (2.7.21), (2.7.22) asimptotikalar o'rinli bo'lishini ko'rsatish mumkin.

Endi (2.7.23) tenglik yordamida aniqlangan $F(x,t)$ funksiyaning $(m+1)$ - tartibli hosilalarining jamlanuvchi funksiya ekanini ko'rsatish lozim. Buning uchun $t < x$ deb olamiz. (2.7.21)-(2.7.22) asimptotikalardan (2.7.23) qatorning yaqinlashuvchiligi (hech bo'lmaganda $L^2(0,\pi)$ fazoning normasi bo'yicha yaqinlashuvchiligi) kelib chiqadi. Shuning uchun quyidagi

$$\int_0^x\int_0^t F(u,v)dudv = \frac{1}{\alpha_0}\int_0^x\int_0^t \cos\sqrt{\lambda_0}u\cos\sqrt{\lambda_0}vdudv-$$

$$-\frac{xt}{\pi}+\sum_{n=1}^{\infty}\int_0^x\int_0^t\left\{\frac{\cos\sqrt{\lambda_n}u\cos\sqrt{\lambda_n}v}{\alpha_n}-\frac{2}{\pi}\cos nu\cos nv\right\}dudv,$$

tenglikda $\cos\sqrt{\lambda_n}x$ funksiyani $\varphi(x,\lambda_n)$ orqali ifodalaymiz, ya'ni

$$\cos\sqrt{\lambda_n}x = \varphi(x,\lambda_n)+\int_0^x H(x,t)\varphi(t,\lambda_n)dt,$$

tasvirdan foydalanamiz. Bunda $H(x,t)$ funksiyaning $(m+1)$-tartibli hosilalari jamlanuvchi, chunki $q^{(m)}(x)\in L^1(0,\pi)$. Bu almashtirish natijasida quyidagi tenglikka ega bo'lamiz:

$$\int_0^x\int_0^t F(u,v)dudv = \sum_{n=1}^{\infty}\frac{1}{\alpha_n}\int_0^x\int_0^t \varphi(u,\lambda_n)\varphi(v,\lambda_n)dudv+$$

$$+\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\int_0^t\varphi(v,\lambda_n)dv\int_0^x du\int_0^u H(u,s)\varphi(s,\lambda_n)ds+$$

$$+\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\int_0^x\varphi(u,\lambda_n)du\int_0^t dv\int_0^v H(v,s)\varphi(s,\lambda_n)ds+$$

$$+\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\int_0^x du\int_0^t dv\left\{\int_0^u\int_0^v H(u,t_1)H(v,t_2)\varphi(t_1,\lambda_n)\varphi(t_2,\lambda_n)dt_1dt_2\right\}-$$

$$-\frac{xt}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx\sin nt}{n^2}.$$

Oxirgi ifodada Parseval tengligidan foydalansak, ushbu

$$\int_0^x\int_0^t F(u,v)dudv=\int_0^t du\int_u^t H(s,u)ds+\int_0^t du\int_u^x H(s,u)ds+$$

$$+\int_0^t ds\int_0^x H(u,s)du\int_s^t H(v,s)dv,$$

tenglikka ega bo'lamiz.

Bu tenglikning ikkala tomoniga $\dfrac{\partial^2}{\partial x\partial t}$ operatorni qo'llaymiz. Natijada ushbu

$$F(x,t)=H(x,t)+H(t,x)+\int_0^t H(x,s)H(t,s)ds,$$

tenglik hosil bo'ladi. $t<x$ bo'lganda $H(t,x)=0$ bo'lgani uchun

$$F(x,t)=H(x,t)+\int_0^t H(x,s)H(t,s)ds, \qquad (2.7.24)$$

bo'ladi. (2.7.24) tenglikdan $F(x,t)$ funksiyaning $(m+1)$ - tartibdagi hosilalari jamlanuvchi ekani kelib chiqadi, chunki $H(x,t)$ funksiyaning $(m+1)$ -tartibli hosilalari jamlanuvchi.

Teorema isbotining yetarlilik qismi xuddi 2-4-paragraflardagidek ko'rsatiladi. ∎

Teorema 2.7.2 $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ *haqiqiy sonlar ketma-*

ketliklari ushbu

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + \underline{\underline{O}}\left(\frac{1}{n^4}\right), \lambda_n \neq \lambda_k, n \neq k,$$
$$\alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + \underline{\underline{O}}\left(\frac{1}{n^3}\right), \tag{2.7.25}$$
$$\alpha_n > 0, n = 0, 1, 2, ...; a_0, a_1, b_0 = \text{const},$$

shartlarni qanoatlantirsa, u holda shunday absolyut uzluksiz $q(x)$ funksiya va h, H haqiqiy sonlar topilib, $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ sonlar ketma-ketliklari (2.7.1)-(2.7.2) ko'rinishidagi chegaraviy masalaning spektral xarakteristikalaridan iborat bo'ladi.

Isbot. Teorema 2.7.1 ga asosan ushbu

$$F(x,t) = \frac{1}{\alpha_0}\cos\sqrt{\lambda_0}x\cos\sqrt{\lambda_0}t -$$
$$-\frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{\cos\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}t}{\alpha_n} - \frac{2}{\pi}\cos nx\cos nt\right),$$

funksiyaning birinchi tartibli hosilalari ($0 \leq x \leq \pi,\ 0 \leq t \leq \pi$) to'plamda absolyut uzluksiz ekanligini ko'rsatish kifoya. Buning uchun quyidagi

$$a(x) = \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n}\cos\sqrt{\lambda_n}x - \frac{2}{\pi}\cos nx\right), \tag{2.7.26}$$

funksiyaning birinchi tartibli hosilasi $0 \leq x \leq 2\pi$ oraliqda absolyut uzluksiz ekanligini ko'rsatish yetarli. Agar $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ ketma-ketliklarning asimptotikalarini (2.7.26) ga qo'ysak, u holda birinchi tomondan, ikki marta differensiallash mumkin bo'lgan qatorlar ajralib chiqadi, bu qatorlarning ikkinchi tartibli hosilalari $L^1(0,\pi)$ fazoda yaqinlashadi. Ikkinchi tomondan, ushbu

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

ko'rinishidagi qator ajraladi. Bu qatorning yig'indisi $\frac{\pi - x}{2}$, $x \in (0, 2\pi)$ ga tengligi matematik analiz kursida ko'rsatilgan. Xususan, bu funksiyaning birinchi hosilasi absolyut uzluksiz funksiya bo'ladi. ■

Teorema 2.7.3. $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ *haqiqiy sonlar ketma-ketliklari (2.7.1)-(2.7.2) ko'rinishdagi* $q(x)$ $(q''(x) \in L^1(0,\pi))$ *koeffitsiyentli Shturm-Liuvill masalasining spektral xarakteristikalari bo'lishi uchun ushbu*

$$\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + \frac{\gamma_n}{n^3}, \quad \{\gamma_n\} \in l_2,$$

$$\alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + \frac{\tau_n}{n^3}, \quad \{\tau_n\} \in l_2,$$

$$\lambda_n \neq \lambda_m, \quad n \neq m, \quad \alpha_n > 0, \quad n = 0, 1, 2, ...,$$

asimptotikalarni qanoatlantirishi zarur va yetarlidir.

Teorema 2.7.4.$\{\lambda_n\}_{n=0}^{\infty}$ *va* $\{\alpha_n\}_{n=0}^{\infty}$ *haqiqiy sonlar ketma-ketliklari (2.7.1)-(2.7.2) ko'rinishdagi* $q(x) \in C^{\infty}[0,\pi]$ *koeffitsiyentli Shturm-Liuvill masalasining spektral xarakteristikalari bo'lishi uchun ushbu*

$$\begin{aligned} &\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + ..., \quad \lambda_n \neq \lambda_m, \quad n \neq m, \\ &\alpha_n = \frac{\pi}{2} + \frac{b_0}{n^2} + \frac{b_1}{n^4} + ..., \quad \alpha_n > 0, \quad n = 0, 1, 2, ..., \end{aligned} \tag{2.7.27}$$

asimptotikalarni qanoatlantirishi zarur va yetarlidir.

Isbot (*Zaruriyligi*). Faraz qilaylik, $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ sonlar ketma-ketliklari (2.7.1)-(2.7.2) ko'rinishidagi $q(x) \in C^{\infty}[0,\pi]$ koeffitsiyentli Shturm-Liuvill masalasining spektral xarakteristikalari bo'lsin. U holda λ_n va α_n uchun (2.7.27) asimptotikalarning o'rinli ekanligi I bobda ko'rsatilgan edi.

(*Yetarliligi*). (2.7.27) asimptotikalarni qanoatlantiruvchi λ_n va α_n sonlar uchun (2.7.1)-(2.7.2) ko'rinishidagi $q(x) \in C^{\infty}[0,\pi]$ koeffitsiyentli chegaraviy masalaning mavjudligini ko'rsatamiz. Berilgan $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ sonlar ketma-ketliklari yordamida tuzilgan ushbu

$$a(x) = \sum_{n=1}^{\infty} \left\{ \frac{1}{\alpha_n} \cos \sqrt{\lambda_n} x - \frac{2}{\pi} \cos nx \right\}, \tag{2.7.28}$$

funksiyaning cheksiz differensiallanuvchi ekanligini ko'rsatish kifoya.

N ixtiyoriy natural son bo'lsin. Agar (2.7.28) tenglikka (2.7.27) asimptotikalarni qo'ysak, u holda bir tomondan N marta hadlab differensiallash mumkin bo'lgan qator hosil bo'ladi, ikkinchi tomondan x o'zgaruvchining ko'phadlariga ko'paytirilgan ushbu

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{2k+1}}, \quad (2k+1 \leq N+2), \tag{2.7.29}$$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2k}}, \quad (2k < N+2), \tag{2.7.30}$$

ko'rinishidagi qatorlarga ajraladi. Bu qatorlarning yig'indlari $0 \leq x \leq \pi$ bo'lganda cheksiz differensiallanuvchiligini ko'rsatsak teoremaning yetarlilik qismi isbotlangan bo'ladi. Shu maqsadda (2.7.29) qatorni $2k$ marta, (2.7.30) qatorni $(2k-1)$ marta differensiallab, $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n}$ ko'rinishdagi qatorni hosil qilamiz. Bu qatorning yig'indisi $\frac{\pi - x}{2}$, $x \in (0, 2\pi)$ ga tengligi matematik analiz kursidan ma'lum. Ushbu $f(x) = \frac{\pi - x}{2}$, $x \in [0, 2\pi]$ funksiyani cheksiz marta differensiallash mumkin. ∎

8-§. Shturm-Liuvill chegaraviy masalasini ikki spektr yordamida tuzish algoritmi

1946 yili G.Borg, Shturm-Liuvill chegaraviy masalasi uchun teskari spektral masalani o'zgacha qo'yilishini tavsiya qildi. Jumladan, Shturm-Liuvill differensial operatorini faqat bitta chegaraviy shart bilan farq qiluvchi ikkita Shturm-Liuvill chegaraviy masalasining spektrlari yordamida tuzishning yagonaligini ko'rsatib berdi.

1964 yilda I.M.Gasimov va B.M.Levitan ikki spektr yordamida Shturm-Liuvill chegaraviy masalasini tuzish algoritmini ishlab chiqdilar.

Bu paragrafda G.Borg yagonalik teoremasini va ikki spektr yordamida teskari masalani yechishning I.M.Gasimov va B.M.Levitan usulini keltiramiz.

Shu maqsadda quyidagi chegaraviy masalalarni qaraylik:

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.8.1}$$

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - h_1 y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.8.2}$$

$$\begin{cases} -y'' + \tilde{q}(x)y = \lambda y, \\ y'(0) - \tilde{h}y(0) = 0, \\ y'(\pi) + \tilde{H}y(\pi) = 0, \end{cases} \tag{2.8.3}$$

$$\begin{cases} -y'' + \tilde{q}(x)y = \lambda y, \\ y'(0) - \tilde{h}_1 y(0) = 0, \\ y'(\pi) + \tilde{H}y(\pi) = 0. \end{cases} \tag{2.8.4}$$

Bu yerda $q(x)$, $\tilde{q}(x) \in C[0, \pi]$ haqiqiy uzluksiz funksiyalar bo'lib, h, h_1, $\tilde{h}$, $\tilde{h}_1$, H va $\tilde{H}$ chekli haqiqiy sonlar.

(2.8.1), (2.8.2), (2.8.3) va (2.8.4) chegaraviy masalalarning xos qiymatlarini mos ravishda $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$, $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ va $\{\tilde{\mu}_n\}_{n=0}^{\infty}$ orqali belgilaylik.

$\varphi(x,\lambda)$ va $\tilde{\varphi}(x,\lambda)$ funksiyalar (2.8.1) va (2.8.3) masalalardagi differensial tenglamalarning mos ravishda ushbu

$$\varphi(0,\lambda)=1, \quad \varphi'(0,\lambda)=h, \quad \tilde{\varphi}(0,\lambda)=1, \quad \tilde{\varphi}'(0,\lambda)=\tilde{h},$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlari bo'lsin. Bu chegaraviy masalalarning normallovchi o'zgarmaslarini mos ravishda

$$\alpha_n=\int_0^{\pi}\varphi^2(x,\lambda_n)dx, \quad \tilde{\alpha}_n=\int_0^{\pi}\tilde{\varphi}^2(x,\lambda_n)dx, \quad n=0,1,2,\ldots$$

orqali belgilaymiz.

Teorema 2.8.1. (*G.Borg*) *Agar* $\lambda_n=\tilde{\lambda}_n$, $\mu_n=\tilde{\mu}_n$, $n=0,1,2,\ldots$ *bo'lsa, u holda*

$$q(x)=\tilde{q}(x), \quad h=\tilde{h}, \quad h_1=\tilde{h}_1, \quad H=\tilde{H},$$

bo'ladi.

Isbot. Birinchi bobdagi (1.16.25) formuladan foydalanib, berilgan $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ spektrlar yordamida $\{\alpha_n\}_{n=0}^{\infty}$ normallovchi o'zgarmaslar ketma-ketligini aniqlaymiz:

$$\frac{1}{\pi}(h_1-h)=\lim_{n\to\infty}n(\sqrt{\mu_n}-\sqrt{\lambda_n}), \tag{2.8.5}$$

$$\alpha_n=\frac{h_1-h}{\mu_n-\lambda_n}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\lambda_k-\lambda_n}{\mu_k-\lambda_n}, \quad n=0,1,2,\ldots. \tag{2.8.6}$$

Xuddi shuningdek, $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ va $\{\tilde{\mu}_n\}_{n=0}^{\infty}$ spektrlar yordamida $\{\tilde{\alpha}_n\}_{n=0}^{\infty}$ normallovchi o'zgarmaslar ketma-ketligini topamiz:

$$\frac{1}{\pi}(\tilde{h}_1-\tilde{h})=\lim_{n\to\infty}n(\sqrt{\tilde{\mu}_n}-\sqrt{\tilde{\lambda}_n}), \tag{2.8.7}$$

$$\tilde{\alpha}_n = \frac{\tilde{h}_1 - \tilde{h}}{\tilde{\mu}_n - \tilde{\lambda}_n} \prod_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{\tilde{\lambda}_k - \tilde{\lambda}_n}{\tilde{\mu}_k - \tilde{\lambda}_n}, \quad n = 0,\ 1,\ 2,\ \dots . \tag{2.8.8}$$

Teorema shartlaridan foydalanib, ushbu

$$\tilde{h}_1 - \tilde{h} = h_1 - h, \tag{2.8.9}$$

$$\tilde{\alpha}_n = \alpha_n, \quad n = 0,\ 1,\ 2,\ \dots, \tag{2.8.10}$$

tengliklarni hosil qilamiz.

Shunday qilib, teoremaning shartlari bajarilganda $\{\lambda_n,\ \alpha_n\}_{n=0}^{\infty}$ va $\{\tilde{\lambda}_n,\ \tilde{\alpha}_n\}_{n=0}^{\infty}$ spektral xarakteristikalar ustma-ust tushar ekan, ya'ni

$$\tilde{\lambda}_n = \lambda_n, \quad \tilde{\alpha}_n = \alpha_n, \quad n = 0,\ 1,\ 2,\ \dots$$

V.A.Marchenko yagonalik teoremasiga asosan $\tilde{q}(x) = q(x)$, $\tilde{h} = h$, $\tilde{H} = H$ tengliklar bajariladi. (2.8.9) tenglikda $\tilde{h} = h$ ekanligini e'tiborga olsak, $\tilde{h}_1 = h_1$ kelib chiqadi. ■

Izoh. Borg yagonalik teoremasi, chegaraviy shartlarida $y(\pi) = 0$ qatnashgan Shturm-Liuvill chegaraviy masalasi uchun ham o'rinli bo'lishini ko'rsatish mumkin.

Buning uchun quyidagi chegaraviy masalalarni qaraymiz:

$$-y'' + q(x)y = \lambda y, \tag{2.8.11}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.8.12}$$

$$-y'' + q(x)y = \lambda y, \tag{2.8.13}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{2.8.14}$$

$$-y'' + \tilde{q}(x)y = \lambda y, \tag{2.8.15}$$

$$\begin{cases} y'(0) - \tilde{h}y(0) = 0, \\ y'(\pi) + \tilde{H}y(\pi) = 0, \end{cases} \tag{2.8.16}$$

$$-y'' + \tilde{q}(x)y = \lambda y, \tag{2.8.17}$$

$$\begin{cases} y'(0) - \tilde{h}y(0) = 0, \\ y(\pi) = 0. \end{cases} \tag{2.8.18}$$

Bu yerda $q(x)$, $\tilde{q}(x) \in C[0,\pi]$ haqiqiy uzluksiz funksiyalar bo'lib, h, $\tilde{h}$, H va $\tilde{H}$ chekli haqiqiy sonlar. (2.8.11)-(2.8.12), (2.8.13)-(2.8.14), (2.8.15)-(2.8.16) va (2.8.17)-(2.8.18) chegaraviy masalalarning xos qiymatlarini mos ravishda $\{\lambda_n\}$, $\{\mu_n\}$, $\{\tilde{\lambda}_n\}$ va $\{\tilde{\mu}_n\}$ orqali belgilaylik. (2.8.11)-(2.8.12) va (2.8.15)-(2.8.16) chegaraviy masalalarning normallovchi o'zgarmaslarini mos ravishda α_n va $\tilde{\alpha}_n$ orqali belgilaylik. U holda quyidagi tasdiq o'rinli.

Terema 2.8.2 (*G.Borg*). *Agar $\tilde{\lambda}_n = \lambda_n$, $\mu_n = \tilde{\mu}_n$, $n \geq 0$ bo'lsa, u holda*

$$q(x) = \tilde{q}(x), \quad h = \tilde{h}, \quad H = \tilde{H},$$

bo'ladi.

Isbot. $\Delta(\lambda)$ va $d(\lambda)$ orqali mos ravishda (2.8.11)-(2.8.12) va (2.8.13)-(2.8.14) chegaraviy masalalarning xarakteristik funksiyalarini belgilaylik. U holda

$$\alpha_n = -\dot{\Delta}(\lambda_n)d(\lambda_n), \tag{2.8.19}$$

formula o'rinli bo'lishini oltinchi paragrafda ko'rsatgan edik.

Agar (2.8.15)-(2.8.16) va (2.8.17)-(2.8.18) chegaraviy masalalarning xarakteristik funksiyalarini mos ravishda $\tilde{\Delta}(\lambda)$ va $\tilde{d}(\lambda)$ orqali belgilab olsak, u holda

$$\tilde{\alpha}_n = -\dot{\tilde{\Delta}}(\tilde{\lambda}_n)\tilde{d}(\tilde{\lambda}_n), \quad n \geq 0,$$

o'rinli bo'ladi. Teorema shartlaridan foydalanib ushbu

$$\alpha_n = \tilde{\alpha}_n, \quad n \geq 0,$$

tenglikni hosil qilamiz. Bundan ko'rinadiki $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ va $\left\{\tilde{\lambda}_n, \tilde{\alpha}_n\right\}_{n=0}^{\infty}$ spektral xarakteristikalar ustma-ust tushadi. V.A.Marchenkoning yagonalik teoremasidan

$$q(x) = \tilde{q}(x), \quad h = \tilde{h}, \quad H = \tilde{H},$$

kelib chiqadi.■

Endi ikki spektr yordamida teskari masalani yechishning I.M.Gasimov va B.M.Levitan usulini keltiramiz.

Teorema 2.8.3. $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\mu_n\}_{n=0}^{\infty}$ *haqiqiy sonlar ketma-ketliklari (2.8.11)-(2.8.12) va (2.8.13)-(2.8.14) ko'rinishdagi* $q(x) \in L^2(0,\pi)$ *koeffitsiyentli Shturm-Liuvill chegaraviy masalalarining spektrlari bo'lishi uchun ushbu*

1) $\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 < ... < \lambda_n < \mu_n < \lambda_{n+1} < ...$,

2) $\sqrt{\lambda_n} = n + \frac{\omega}{n\pi} + \frac{\chi_n}{n}$, $\{\chi_n\} \in l_2$, $\sqrt{\mu_n} = n + \frac{1}{2} + \frac{\omega_1}{\left(n+\frac{1}{2}\right)\pi} + \frac{\chi_n^{(1)}}{n}$, $\omega \neq \omega_1$, $\{\chi_n^{(1)}\} \in l_2$,

shartlarning bajarilishi zarur va yetarli.

Isbot. Teorema shartlarining zarurligi yuqoridagi paragraflarda ko'rsatilgan edi. Bu yerda biz teorema shartlarining yetarliligini ko'rsatamiz.

Faraz qilaylik, $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\mu_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari teorema shartlarini qanoatlantirsin. Ushbu

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda)\prod_{n=1}^{\infty}\frac{\lambda_n - \lambda}{n^2}, \quad d(\lambda) = \prod_{n=0}^{\infty}\frac{\mu_n - \lambda}{\left(n+\frac{1}{2}\right)^2},$$

funksiyalarni tuzib olamiz. Teoremaning ikkinchi shartidagi asimptotikalarga asosan yuqoridagi cheksiz ko'paytmalar yaqinlashuvchi bo'ladi. (2.8.19) formula yordamida ushbu

$$\alpha_n = -\dot{\Delta}(\lambda_n)d(\lambda_n),$$

ketma-ketlikni tuzib olamiz. Endi $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\mu_n\}_{n=0}^{\infty}$ sonlar ketma-ketliklariga Gelfand-Levitan usulini qo'llab $q(x)$ funksiya va h, H sonlarni topamiz.

Teorema 2.4.3 dan foydalanish maqsadida $\{\alpha_n\}$ sonlar ketma-ketligi ushbu

$$\alpha_n = \frac{\pi}{2} + \frac{\tilde{\chi}_n}{n}, \quad \alpha_n > 0, \quad n \geq 0, \quad \{\tilde{\chi}_n\} \in l_2,$$

asimptotikani qanoatlantirishini ko'rsatamiz. λ_n, μ_n larning asimptotikalarini bilgan holda α_n larning asimptotikasini aniqlash qiyin masala. $\{\alpha_n\}$ ketma-ketlikning asimptotikasini topish uchun teorema 2.4.3 dan foydalanamiz. Spektri $\{\lambda_n\}_{n=0}^{\infty}$ bo'lgan, (2.8.11)-(2.8.12) ko'rinishdagi, $q_0(x) \in L^2(0,\pi)$, h_0, $H_0 \in R^1$ koeffitsiyentli Shturm-Liuvill chegaraviy masalasi cheksiz ko'p. Shulardan biri $\left\{\lambda_n, \alpha_n^0 = \frac{\pi}{2}\right\}$ spektral xarakteristikaga mos keladi:

$$\begin{cases} -y'' + q_0(x)y = \lambda y, \\ y'(0) - h_0 y(0) = 0, \quad y'(\pi) + H_0 y(\pi) = 0. \end{cases}$$

Bu chegaraviy masalaning xarakteristik funksiyasi uchun quyidagi tenglik o'rinli:

$$\Delta_0(\lambda) = \pi(\lambda_0 - \lambda)\prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2} = \Delta(\lambda).$$

Xuddi beshinchi paragrafdagidek, ushbu

$$\dot{\Delta}(\lambda_n) = (-1)^{n+1}\frac{\pi}{2} + \frac{\chi_n}{n}, \quad \{\chi_n\} \in l_2,$$

asimptotik formulani keltirib chiqarish mumkin. Xuddi shuningdek, spektri $\{\mu_n\}_{n=0}^{\infty}$ bo'lgan, (2.8.13)-(2.8.14) ko'rinishdagi, $q_1(x) \in L^2(0,\pi)$ koeffitsiyentli cheksiz ko'p Shturm-Liuvill chegaraviy masalasi mavjud. Shulardan bittasi $\left\{\mu_n, \alpha_n^{(1)} = \frac{\pi}{2}\right\}$ spektral xarakteristikalarga mos keladi:

$$\begin{cases} -y'' + q_1(x)y = \lambda y, \\ y'(0) - h_1 y(0) = 0, \quad y(\pi) = 0. \end{cases}$$

Bu chegaraviy masalaning xarakteristik funksiyasi uchun quyidagi tenglik o'rinli:

$$d_1(\lambda) = \prod_{n=0}^{\infty} \frac{\mu_n - \lambda}{\left(n + \frac{1}{2}\right)^2} = d(\lambda).$$

Bundan tashqari ushbu

$$d(\lambda_n) = (-1)^n + \frac{\chi_n^{(1)}}{n}, \quad \left\{\chi_n^{(1)}\right\} \in l_2,$$

asimptotik formula bajariladi. Bu yerda

$$\operatorname{sign} \dot{\Delta}(\lambda_n) = (-1)^{n+1}, \quad n \geq 0, \quad \operatorname{sign} d(\lambda_n) = (-1)^n, \quad n \geq 0.$$

Bu topilganlarga asoslanib, α_n uchun ushbu

$$\alpha_n = -\dot{\Delta}(\lambda_n)d(\lambda_n) = -\left[(-1)^{n+1}\frac{\pi}{2} + \frac{\chi_n}{n}\right] \cdot \left[(-1)^n + \frac{\chi_n^{(1)}}{n}\right] =$$

$$= \frac{\pi}{2} + \frac{\chi_n^{(2)}}{n}, \quad \left\{\chi_n^{(2)}\right\} \in l_2,$$

asimptotikani topamiz. Endi teorema 2.4.3 dan foydalanib, $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ spektral xarakteristikaga ega bo'lgan yagona

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases}$$

Shturm-Liuvill chegaraviy masalasini tuzamiz.

Endi quyidagi

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - hy(0) = 0, \\ y(\pi) = 0, \end{cases} \tag{2.8.20}$$

chegaraviy masalani qaraymiz. Bu chegaraviy masalaning spektrini $\{\tilde{\mu}_n\}$ orqali belgilaylik. U holda $\tilde{\mu}_n = \mu_n$, $n \geq 0$ ekanini ko'rsatsak teorema isbotlangan bo'ladi. (2.8.20) chegaraviy masalaning xarakteristik funksiyasi $\tilde{d}(\lambda)$ bo'lsin. U holda

$$\alpha_n = -\dot{\Delta}(\lambda_n)\tilde{d}(\lambda_n), \tag{2.8.21}$$

tenglik bajariladi. Bu yerda

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda)\prod_{n=1}^{\infty}\frac{\lambda_n - \lambda}{n^2}, \quad \tilde{d}(\lambda) = \prod_{n=0}^{\infty}\frac{\tilde{\mu}_n - \lambda}{\left(n + \frac{1}{2}\right)^2}.$$

Ikkinchi tomondan (2.8.20) chegaraviy masalaning tuzilishiga ko'ra

$$\alpha_n = -\dot{F}(\lambda_n) d(\lambda_n), \tag{2.8.22}$$

bo'ladi. (2.8.21) va (2.8.22) tengliklardan $d(\lambda_n) = \tilde{d}(\lambda_n)$, $n \geqslant 0$ kelib chiqadi.

Quyidagi

$$z(\lambda) = \frac{d(\lambda) - \tilde{d}(\lambda)}{\Delta(\lambda)},$$

butun funksiyani tuzib olamiz. Bu yerdagi $d(\lambda)$ va $\tilde{d}(\lambda)$ xarakteristik funksiyalar uchun ushbu

$$|d(\lambda)| \leq Ce^{|\tau|\pi}, \quad \left|\tilde{d}(\lambda)\right| \leq Ce^{|\tau|\pi}, \quad \tau = \operatorname{Im}\sqrt{\lambda}$$

asimptotikalarning o'rinli ekanini ko'rsatish mumkin. Buning uchun $\varphi(x, \lambda)$ yechimning quyidagi

$$\varphi(x, \lambda) = \underline{\underline{O}}\left(e^{|\tau|\pi}\right)$$

asimptotikasidan foydalanish kifoya. Endi $\Delta(\lambda)$ xarakteristik funksiya uchun topilgan

$$|\Delta(\lambda)| \geq C_\delta \left|\sqrt{\lambda}\right| e^{|\tau|\pi}, \quad \sqrt{\lambda} = \rho \in G_\delta, \ |\rho| \geq R^*$$

baholashdan foydalanib,

$$|z(\lambda)| \leqslant C|\sqrt{\lambda}|^{-1}, \quad \lambda \in G_\delta, \quad |\rho| \geq R^*$$

tengsizlikni topamiz. Analitik funksiya moduli maksimum printsipiga va Liuvill teoremasiga asosan $z(\lambda) \equiv 0$, ya'ni $\tilde{d}(\lambda) = d(\lambda)$ bo'ladi. Bundan esa $\mu_n = \tilde{\mu}_n$, $n \geq 0$ kelib chiqadi. ■

Teorema 2.8.4. $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\mu_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari mos ravishda ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, & (0 \leq x \leq \pi), \\ y'(0) - h_1 y(0) = 0, & h_1 \in R^1, \\ y'(\pi) + Hy(\pi) = 0, & H \in R^1, \end{cases} \tag{2.8.23}$$

$$\begin{cases} -y'' + q(x)y = \lambda y, & (0 \le x \le \pi), \\ y'(0) - h_2 y(0) = 0, & h_2 \in R^1, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.8.24}$$

ko'rinishdagi $q(x) \in L^2(0, \pi)$ koeffitsiyentli Shturm-Liuvill chegaraviy masalalarning spektral xarakteristikalari bo'lishi uchun quyidagi

1) $\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \ldots < \lambda_n < \mu_n < \lambda_{n+1} < \ldots$

2) $\sqrt{\lambda_n} = n + \frac{c_1}{n} + \frac{\gamma_n}{n}$, $\{\gamma_n\} \in l^2$, $\sqrt{\mu_n} = n + \frac{c_2}{n} + \frac{\tilde{\gamma}_n}{n}$, $\{\tilde{\gamma}_n\} \in l^2$,

shartlarning bajarilishi zarur va yetarli. Bu yerda

$$\operatorname{Im}\{q(x)\} = 0; \quad h_1 \ne h_2, \quad c_1 \ne c_2, \quad h_2 - h_1 = \pi(c_2 - c_1). \tag{2.8.25}$$

Isbot. Zaruriylik qismi yuqoridagi paragraflarda ko'rsatilgan edi. Quyida biz teorema isbotining yetarlilik qismini ko'rsatamiz.

Faraz qilaylik $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\mu_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari teorema shartlarini qanoatlantirsin. Quyidagi funksiyalarni tuzib olamiz:

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda)\prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2}, \quad \Delta_1(\lambda) = \pi(\mu_0 - \lambda)\prod_{n=1}^{\infty} \frac{\mu_n - \lambda}{n^2}.$$

Teorema ikkinchi shartidagi asimptotikalarga asosan bu cheksiz ko'paytmalar yaqinlashuvchi bo'ladi. Ushbu

$$\alpha_n = -(h_2 - h_1)\frac{\dot{\Delta}(\lambda_n)}{\Delta_1(\lambda_n)}, \tag{2.8.26}$$

formula yordamida $\{\alpha_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketligini tuzib olamiz. Endi $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\alpha_n\}_{n=0}^{\infty}$ sonlar ketma-ketliklariga Gelfand-Levitan algoritmini qo'llab $q(x)$ funksiyani va h, H sonlarni topamiz. Buning uchun avvalo $\{\alpha_n\}_{n=0}^{\infty}$ sonlar ketma-ketligi quyidagi

$$\alpha_n = \frac{\pi}{2} + \frac{\theta_n}{n}, \quad \{\theta_n\} \in l^2, \quad \alpha_n > 0, \quad n = 0, 1, 2, \ldots \tag{2.8.27}$$

shartlarni qanoatlantirishini ko'rsatishimiz lozim. Shu maqsadda teorema 2.3.4 dan foydalanamiz. Spektri $\{\lambda_n\}_{n=0}^{\infty}$ bo'lgan (2.8.23) ko'rinishdagi $q_0(x) \in L^2(0,\pi)$, h_0, $H_0 \in R^1$ koeffitsiyentli Shturm-Liuvill chegaraviy masalalari cheksiz ko'p. Shulardan bittasi $\left\{\lambda_n, \alpha_n^{(0)} = \frac{\pi}{2}\right\}_{n=0}^{\infty}$ spektral xarakteristikalarga mos keladi:

$$\begin{cases} -y'' + q_0(x)y = \lambda y, \quad 0 \le x \le \pi, \\ y'(0) - h_0 y(0) = 0, \\ y'(\pi) + H_0 y(\pi) = 0. \end{cases} \tag{2.8.28}$$

Bu chegaraviy masalaning xarakteristik funksiyasi

$$\Delta_0(\lambda) \equiv \varphi_0'(\pi,\lambda) + H_0\varphi_0(\pi,\lambda) = \pi(\lambda_0 - \lambda)\prod_{n=1}^{\infty}\frac{\lambda_n - \lambda}{n^2} = \Delta(\lambda)$$

ko'rinishda bo'ladi. Bu yerda $\varphi_0(x,\lambda)$ orqali (2.8.28) tenglamaning

$$\varphi_0(0,\lambda) = 1, \quad \varphi_0'(0,\lambda) = h_0$$

boshlang'ich shartlarni qanoatlantiruvchi yechimi belgilangan. (2.8.28) chegaraviy masalaning $\left\{\alpha_n^{(0)}\right\}_{n=0}^{\infty}$ normallovchi o'zgarmaslar ketma-ketligi uchun quyidagi

$$\alpha_n^{(0)} = \frac{\pi}{2} = \int_0^{\pi} \varphi_0^2(x,\lambda_n)dx = -\varphi_0(\pi,\lambda_n)\dot{\Delta}(\lambda_n) \tag{2.8.29}$$

formula o'rinli. Avvalo biz (2.8.29) tenglikdagi $\{\varphi_0(\pi,\lambda_n)\}_{n=0}^{\infty}$ ketma-ketlikning asimptotikasini topamiz. Buning uchun $\varphi_0(x,\lambda)$ yechimning ushbu

$$\varphi_0(x,\lambda) = \cos\sqrt{\lambda}x + \left(h_0 + \frac{1}{2}\int_0^x q_0(t)dt\right)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} -$$

$$-\frac{1}{2\sqrt{\lambda}}\int_0^x q_0(t)\sin\sqrt{\lambda}(2t-x)dt + O\left(\frac{1}{\lambda}\right), \quad \lambda \to +\infty$$

asimptotikasidan foydalanamiz. Bu yerda $x = \pi$, $\sqrt{\lambda} = \sqrt{\lambda_n} = n + \delta_n$, $\delta_n = \frac{c_1}{n} + \frac{\gamma_n}{n}$ deb, $\varphi_0(\pi, \lambda_n)$ uchun quyidagi

$$\varphi_0(\pi, \lambda_n) = (-1)^n + \frac{\theta_n}{n}, \quad \{\theta_n\} \in l^2,$$

$$\theta_n = \frac{(-1)^{n+1}}{2} \int_0^\pi q_0(t) \sin 2ntdt + \underline{\underline{Q}}\left(\frac{1}{n}\right)$$

asimptotikani topamiz. Bu ifodani (2.8.29) tenglikka qo'yib

$$\frac{\pi}{2} = -\left\{(-1)^n + \frac{\theta_n}{n}\right\}\dot{\Delta}(\lambda_n)$$

tenglamani hosil qilamiz. Bundan esa

$$\dot{\Delta}(\lambda_n) = (-1)^{n+1}\frac{\pi}{2} + \frac{\bar{\theta}_n}{n}, \quad \left\{\bar{\theta}_n\right\} \in l^2 \tag{2.8.30}$$

hosil bo'ladi.

Xuddi shuningdek, yana teorema 2.4.3 dan foydalanib, $\left\{\mu_n, \alpha_n^{(1)} = \frac{\pi}{2}\right\}_{n=0}^{\infty}$ spektral xarakteristikaga mos keluvchi (2.8.23) ko'rinishdagi $\tilde{q}(x) \in L^2(0, \pi)$, $\tilde{h}$, $\tilde{H} \in R^1$ koeffitsiyentli yagona Shturm-Liuvill chegaraviy masalani tuzib olamiz:

$$\begin{cases} -y'' + \tilde{q}(x)y = \lambda y, \quad (0 \le x \le \pi), \\ y'(0) - \tilde{h}y(0) = 0, \\ y'(\pi) + \tilde{H}y(\pi) = 0. \end{cases} \tag{2.8.31}$$

$\tilde{\varphi}(x, \lambda)$ orqali (2.8.31) chegaraviy masaladagi differensial tenglamaning $\tilde{\varphi}(0, \lambda) = 1$, $\tilde{\varphi}(0, \lambda) = \tilde{h}$ boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. U holda bu chegaraviy masalaning xarakteristik funksiyasi uchun

$$\tilde{\Delta}(\lambda) \equiv \tilde{\varphi}'(\pi, \lambda) + \tilde{H}\tilde{\varphi}(\pi, \lambda) = \pi(\mu_0 - \lambda)\prod_{n=1}^{\infty} \frac{\mu_n - \lambda}{n^2} = \Delta_1(\lambda),$$

tenglik o'rinli bo'ladi. Endi $\Delta_1(\lambda_n) = \tilde{\Delta}(\lambda_n)$ ketma-ketlikning asimptotikasini topish uchun almashtirish operatoridan, ya'ni $\tilde{\varphi}(x, \lambda)$ yechimning ushbu

$$\tilde{\varphi}(x, \lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt,$$

$$\tilde{\varphi}'(x, \lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x +$$

$$+K(x,x)\cos\sqrt{\lambda}x + \int_0^x K'_x(x,t)\cos\sqrt{\lambda}t dt,$$

integral tasviridan foydalanamiz. Bu yerda $x = \pi$, $\sqrt{\lambda} = \sqrt{\lambda_n} = n + \delta_n$, $\delta_n = \dfrac{c_1}{n} + \dfrac{\gamma_n}{n}$, $\{\gamma_n\} \in l_2$ deb

$$\Delta_1(\lambda_n) = -(n+\delta_n)\sin(n+\delta_n)\pi +$$

$$+\left(\tilde{h} + \tilde{H} + \frac{1}{2}\int_0^\pi \tilde{q}(t)dt\right)\cos(n+\delta_n)\pi +$$

$$+\int_0^\pi K'_x(\pi,t)\cos(n+\delta_n)t dt + \tilde{H}\int_0^\pi K(\pi,t)\cos(n+\delta_n)t dt$$

tasvirni hosil qilamiz. $\tilde{\varphi}(x, \lambda)$ va $\tilde{\varphi}'(x, \lambda)$ funksiyalarning integral tasvirida $x = \pi$, $\sqrt{\lambda} = \sqrt{\mu_n} = n + \tilde{\delta}_n$, $\tilde{\delta}_n = \dfrac{c_2}{n} + \dfrac{\tilde{\gamma}_n}{n}$, $\{\tilde{\gamma}_n\} \in l_2$ deb

$$0 = \Delta_1(\mu_n) = \tilde{\Delta}(\mu_n) = -(n+\tilde{\delta}_n)\sin(n+\tilde{\delta}_n)\pi + \Big(\tilde{h} + \tilde{H} +$$

$$+\frac{1}{2}\int_0^\pi \tilde{q}(t)dt\Big)\cos(n+\tilde{\delta}_n)\pi +$$

$$+\int_0^\pi K'_x(\pi,t)\cos(n+\tilde{\delta}_n)t dt + \tilde{H}\int_0^\pi K(\pi,t)\cos(n+\tilde{\delta}_n)t dt$$

tenglikni hosil qilamiz. Bu tengliklardan foydalanib

$$\Delta_1(\lambda_n) = \Delta_1(\lambda_n) - \Delta_1(\mu_n) = -(n+\delta_n)\sin(n+\delta_n)\pi +$$
$$+(n+\tilde{\delta}_n)\sin(n+\tilde{\delta}_n)\pi +$$

$$+\left(\tilde{h}+\tilde{H}+\frac{1}{2}\int_0^\pi \tilde{q}(t)dt\right)\left[\cos(n+\delta_n)\pi - \cos(n+\tilde{\delta}_n)\pi\right]+$$

$$+\int_0^\pi [K_x'(\pi,t)+\tilde{H}K(\pi,t)][\cos(n+\delta_n)t - \cos(n+\tilde{\delta}_n)t]dt$$

tasvirni topamiz. Bundan

$$\Delta_1(\lambda_n) = (-1)^n(c_2-c_1)\pi + \frac{\beta_n}{n}, \quad \{\beta_n\}\in l_2 \tag{2.8.32}$$

$$\beta_n = (c_2-c_1)\int_0^\pi [K_x'(\pi,t)+\tilde{H}K(\pi,t)]t\sin ntdt$$

kelib chiqadi. Topilgan (2.8.30) va (2.8.32) asimptotikalarni (2.8.26) tenglikka qo'yib, (2.8.25) dan foydalansak, (2.8.27) kelib chiqadi. (2.8.26) formulaga muvofiq barcha α_n, $n \geq 0$ sonlar bir xil ishorali, normallovchi o'zgarmaslarning asimptotikasiga binoan yetarli katta n larda $\alpha_n > 0$. Demak, $\alpha_n > 0$, $n \geq 0$.

Endi teorema 2.4.3 dan foydalanib, $\{\lambda_n, \alpha_n\}_{n=0}^\infty$ spektral xarakteristikaga ega bo'lgan yagona

$$\begin{cases} -y''+q(x)y=\lambda y, & q(x)\in L^2(0,\pi), \\ y'(0)-h_1y(0)=0, & h_1\in R^1, \\ y'(\pi)+Hy(\pi)=0, & H\in R^1, \end{cases} \tag{2.8.33}$$

Shturm-Liuvill chegaraviy masalasini topamiz. (2.8.25) formuladagi ushbu

$$h_2 = h_1 + (c_2-c_1)\pi$$

tenglikdan foydalanib h_2 sonini topib olamiz va quyidagi

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - h_2 y(0) = 0, \\ y'(\pi) + Hy(\pi) = 0, \end{cases} \tag{2.8.34}$$

chegaraviy masalani qaraymiz. Bu chegaraviy masalaning xos qiymatlarini $\{\tau_n\}_{n=0}^{\infty}$ orqali belgilaymiz. Agar $\tau_n = \mu_n$, $n = 0, 1, 2, \ldots$ ekanligini ko'rsatsak teorema isbotlangan bo'ladi. Buning uchun (2.8.33) chegaraviy masaladagi differensial tenglamaning

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h_1,$$

va

$$\psi(0, \lambda) = 1, \quad \psi'(0, \lambda) = h_2,$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini mos ravishda $\varphi(x, \lambda)$ va $\psi(x, \lambda)$ orqali belgilaymiz. U holda ushbu

$$\tilde{m}(\lambda) = -\frac{\psi'(\pi, \lambda) + H\psi(\pi, \lambda)}{\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda)},$$

funksiyaning qutb maxsus nuqtalari va nollari mos ravishda $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\tau_n\}_{n=0}^{\infty}$ sonlardan iborat bo'lib, ular almashinib keladi, ya'ni

$$\lambda_0 < \tau_0 < \ldots < \lambda_n < \tau_n < \lambda_{n+1} < \ldots .$$

$\varphi(x, \lambda)$ va $\psi(x, \lambda)$ yechimlarning $\lambda \to +\infty$ dagi asimptotikasidan foydalansak,

$$\tilde{m}(\lambda) \to -1, \quad \lambda \to +\infty$$

kelib chiqadi. Ikkinchi tomondan (2.8.26) formulaga asosan

$$\frac{1}{\alpha_n} = \frac{1}{h_2 - h_1} \operatorname*{res}_{\lambda=\lambda_n} \tilde{m}(\lambda).$$

Shuning uchun, Mittag-Leffler teoremasiga ko'ra

$$\frac{\tilde{m}(\lambda)}{h_2 - h_1} = -\frac{1}{h_2 - h_1} + \sum_{n=0}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \tag{2.8.35}$$

Endi quyidagi

$$m(\lambda) = -\frac{1}{h_2 - h_1}\prod_{k=0}^{\infty}\frac{\mu_k - \lambda}{\lambda_k - \lambda}$$

funksiyani qaraylik. Bu funksiyani ushbu

$$m(\lambda) = -\frac{1}{h_2 - h_1} + \sum_{n=0}^{\infty}\frac{1}{\alpha_n(\lambda - \lambda_n)} \qquad (2.8.36)$$

ko'rinishda yozish mumkin.

(2.8.35) va (2.8.36) tengliklardan

$$m(\lambda) = \frac{\tilde{m}(\lambda)}{h_2 - h_1}$$

hosil bo'ladi. Bundan esa, $\tau_n = \mu_n$, $n = 0, 1, 2, \ldots$ kelib chiqadi. ■

9-§. Xos qiymatlar asimptotikalari bo'yicha normallovchi o'zgarmaslar uchun asimptotik formula keltirib chiqarish

Ushbu

$$-y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi \qquad (2.9.1)$$

differensial tenglamani qaraymiz. Bu yerda $q(x)$ haqiqiy uzluksiz funksiya. Bu tenglamani quyidagi

$$y'(0) - h_1 y(0) = 0, \; y'(\pi) + Hy(\pi) = 0, \qquad (2.9.2)$$

yoki

$$y'(0) - h_2 y(0) = 0, \; y'(\pi) + Hy(\pi) = 0, \qquad (2.9.3)$$

chegaraviy shartlar bilan qarab, ikkita chegaraviy masala tuzib olamiz. Bu yerda h_1, h_2 va H lar haqiqiy sonlar bo'lib, $h_1 \neq h_2$.

(2.9.1)+(2.9.2) va (2.9.1)+(2.9.3) chegaraviy masalalarning xos qiymatlarini mos ravishda $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\mu_n\}_{n=0}^{\infty}$ orqali belgilaymiz.

Quyidagi

$$\alpha_n = \frac{h_2 - h_1}{\mu_n - \lambda_n} \cdot \prod_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{\lambda_k - \lambda_n}{\mu_k - \lambda_n}, \tag{2.9.4}$$

formula oldingi paragraflarda keltirilib chiqarilgan edi.

Bu paragrafda λ_n va μ_n ketma-ketliklarning asimptotikalari yordamida α_n uchun asimptotik formula keltirib chiqaramiz. Bunda (2.9.4) formuladan foydalanamiz.

Agar $q(x)$ yetarlicha silliq funksiya bo'lsa, u holda quyidagi

$$\left.\begin{aligned} \sqrt{\lambda_n} &= n + \frac{a_0}{n} + \frac{a_1}{n^3} + O\left(\frac{1}{n^4}\right), \\ \sqrt{\mu_n} &= n + \frac{a_0'}{n} + \frac{a_1'}{n^3} + O\left(\frac{1}{n^4}\right), \end{aligned}\right\} \tag{2.9.5}$$

asimptotikalar o'rinli bo'ladi. Bu yerda

$$a_0 = \frac{h_1 + H}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(s)ds, \quad a_0' = \frac{h_2 + H}{\pi} + \frac{1}{2\pi}\int_0^{\pi} q(s)ds,$$

bo'lib,

$$a_0' - a_0 = -\frac{1}{\pi}(h_1 - h_2) \neq 0. \tag{2.9.6}$$

(2.9.5) tengliklarni kvadratga ko'tarib ushbu

$$\left.\begin{aligned} \lambda_n &= n^2 + 2a_0 + \frac{c_0}{n^2} + O\left(\frac{1}{n^3}\right), \\ \mu_n &= n^2 + 2a_0' + \frac{c_0'}{n^2} + O\left(\frac{1}{n^3}\right), \end{aligned}\right\} \tag{2.9.7}$$

asimptotikalarni hosil qilamiz. Bunda $c_0 = a_0^2 + 2a_1$, $c_0' = {a_0'}^2 + 2a_1'$.

Avvalo (2.9.6) formulaga asosan $h_2 - h_1$ ayirmani hisoblaymiz. α_n sonlar ketma-ketligining asimptotikasini keltirib chiqarish oddiy emas, shuning uchun uni bir nechta etaplarga bo'lamiz.

a) $\Psi_n = \prod\limits_{\substack{k=0 \\ k\neq n}}^{\infty} \left(1 + \frac{\lambda_k - \mu_k}{\mu_k - \lambda_n}\right)$ cheksiz ko'paytmani qaraymiz. Bundan

$$\ln\Psi_n=\sum_{\substack{k=0\\k\neq n}}^{\infty}\ln\left(1+\frac{\lambda_k-\mu_k}{\mu_k-\lambda_n}\right)$$

bo'lishi kelib chiqadi. (2.9.7) asimptotik formulalardan foydalanib, yetarlicha katta $n\neq k$ larda

$$\left|\frac{\lambda_k-\mu_k}{\mu_k-\lambda_n}\right|<\frac{C}{n},$$

tengsizlikni hosil qilamiz. Bu yerda va keyinchalik C orqali o'zgarmas sonlarni belgilaymiz, umuman olganda, ular turlicha bo'lishi mumkin. Yig'indi belgisi ostidagi logarifmni darajali qatorga yoysak

$$\ln\Psi_n=\sum_{\substack{k=0\\k\neq n}}^{\infty}\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p}\left(\frac{\lambda_k-\mu_k}{\mu_k-\lambda_n}\right)^p,$$

tenglik kelib chiqadi.

Lemma 2.9.1. *Agar* $|\mu_k-\lambda_k|\leq a,\ k=0,1,2,\ldots$ *bo'lsa, u holda*

$$\sum_{\substack{k=0\\k\neq n}}^{\infty}\left|\frac{\lambda_k-\mu_k}{\mu_k-\lambda_n}\right|^p\leq\begin{cases}\dfrac{C\ln n}{n}, & p=1,\\ \dfrac{Ca^p}{n^p}, & p\geq 2,\end{cases}\tag{2.9.8}$$

bo'ladi.

Isbot. Lemma shartiga asosan

$$\sum_{\substack{k=0\\k\neq n}}^{\infty}\left|\frac{\lambda_k-\mu_k}{\mu_k-\lambda_n}\right|^p\leq a^p\sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{1}{|\mu_k-\lambda_n|^p},$$

bo'ladi. Bu tengsizlikning o'ng tomonidagi yig'indi yetarlicha katta n larda quyidagi

$$\int_0^{n-1}\frac{dx}{(\mu_n-x^2)^p}+\int_{n+1}^{\infty}\frac{dx}{(x^2-\mu_n)^p},$$

integrallar yig'indisi bilan bir xil tartibga ega bo'ladi. Bu integrallar quyidagicha baholanadi:

$$\int_0^{n-1} \frac{dx}{(\mu_n - x^2)^p} \le \frac{C}{n^p} \int_0^{n-1} \frac{dx}{(\sqrt{\mu_n} - x)^p} = \begin{cases} \underline{\underline{O}}\left(\frac{\ln n}{n}\right), & p = 1, \\ \underline{\underline{O}}\left(\frac{1}{n^p}\right), & p > 1, \end{cases}$$

$$\int_{n+1}^{\infty} \frac{dx}{(x^2 - \mu_n)^p} \le \frac{C}{n^p} \int_{n+1}^{\infty} \frac{dx}{(x - \sqrt{\mu_n})^p} = \underline{\underline{O}}\left(\frac{1}{n^p}\right), \quad p > 1,$$

$$\int_{n+1}^{\infty} \frac{dx}{x^2 - \mu_n} = \frac{1}{2\sqrt{\mu_n}} \ln \frac{x - \sqrt{\mu_n}}{x + \sqrt{\mu_n}}\Bigg|_{x=n+1}^{\infty} = \underline{\underline{O}}\left(\frac{\ln n}{n}\right).$$

Bu baholashlar yordamida lemma 2.9.1 isbotlanadi.

(2.9.8) baholashdan quyidagi

$$\left|\sum_{\substack{k=0 \\ k\neq n}}^{\infty} \sum_{p=3}^{\infty} \frac{(-1)^{p-1}}{p} \left(\frac{\lambda_k - \mu_k}{\mu_k - \lambda_n}\right)^p\right| \le \sum_{p=3}^{\infty} \frac{1}{p} \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left|\frac{\lambda_k - \mu_k}{\mu_k - \lambda_n}\right|^p \le C \sum_{p=3}^{\infty} \frac{a^p}{n^p} =$$

$$= C\frac{a^3}{n^3} \sum_{p=0}^{\infty} \frac{a^p}{n^p} = \underline{\underline{O}}\left(\frac{1}{n^3}\right),$$

kelib chiqadi. Shuning uchun

$$\ln \Psi_n = \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{\lambda_k - \mu_k}{\mu_k - \lambda_n} - \frac{1}{2} \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left(\frac{\lambda_k - \mu_k}{\mu_k - \lambda_n}\right)^2 + \underline{\underline{O}}\left(\frac{1}{n^3}\right). \tag{2.9.9}$$

b) Shunday qilib, bizga (2.9.9) tenglikning o'ng tomonidagi yig'indilarning asimptotikasini o'rganish kerak bo'ladi. Birinchi yig'indini quyidagicha yozib olamiz:

$$\sum_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{\lambda_k - \mu_k}{\mu_k - \lambda_n} = \frac{\lambda_0 - \mu_0}{\mu_0 - \lambda_n} + \sum_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{2(a_0 - a_0')}{\mu_k - \lambda_n} +$$

$$+\sum_{\substack{k=1\\k\neq n}}^{\infty}[(\lambda_k-\mu_k)-2(a_0-a_0')]\cdot\left[\frac{1}{\mu_k-\lambda_n}+\frac{1}{\lambda_n}\right]-$$

$$-\frac{1}{\lambda_n}\sum_{\substack{k=1\\n\neq k}}^{\infty}[(\lambda_k-\mu_k)-2(a_0-a_0')]=S_0+2(a_0-a_0')S_1+S_2+S_3.$$

(2.9.10)

Bu yerda tabiiy belgilashlar kiritilgan. (2.9.5) formulalardan

$$S_0=\frac{\lambda_0-\mu_0}{\mu_0-\lambda_n}=\frac{\mu_0-\lambda_0}{\lambda_n\left(1-\frac{\mu_0}{\lambda_n}\right)}=\frac{\mu_0-\lambda_0}{n^2}+\underline{\underline{O}}\left(\frac{1}{n^4}\right),$$

$$S_3=\frac{1}{\lambda_n}\sum_{k=1}^{\infty}[(\mu_k-\lambda_k)-2(a_0'-a_0)]-\frac{1}{\lambda_n}[(\mu_n-\lambda_n)-2(a_0'-a_0)]=$$

$$=\frac{A}{n^2}+\underline{\underline{O}}\left(\frac{1}{n^4}\right),$$

asimptotikalar kelib chiqadi. Bu yerda

$$A=\sum_{k=1}^{\infty}[(\mu_k-\lambda_k)-2(a_0'-a_0)].$$

c) Endi S_2 yig'indini qaraymiz:

$$S_2=\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{\mu_k[\lambda_k-\mu_k-2(a_0-a_0')]-(c_0-c_0')}{\lambda_n(\mu_k-\lambda_n)}+$$

$$+\frac{c_0-c_0'}{\lambda_n}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{\mu_k-\lambda_n}.$$

λ_k va μ_k sonlar uchun yozilgan (2.9.7)asimptotik formulalardan

$$\mu_k[\lambda_k-\mu_k-2(a_0-a_0')]-(c_0-c_0')=\underline{\underline{O}}(k^{-1}),$$

kelib chiqadi. Shuning uchun

$$\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{\mu_k[\lambda_k-\mu_k-2(a_0-a_0')]-(c_0-c_0')}{\lambda_n(\mu_k-\lambda_n)}\leq$$

$$\leq \frac{C}{n^2}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{k|\mu_k-\lambda_n|}=\underline{\underline{O}}\left(\frac{1}{n^3}\right),$$

bo'ladi. Demak,

$$S_2=\frac{c_0-c_0'}{\lambda_n}S_1+\underline{\underline{O}}\left(\frac{1}{n^3}\right).$$

S_0, S_2 va S_3 lar uchun topilgan asimptotik formulalarni (2.9.10) formulaga qo'ysak,

$$\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{\lambda_k-\mu_k}{\mu_k-\lambda_n}=\frac{A+\mu_0-\lambda_0}{n^2}+2(a_0-a_0')+\frac{c_0-c_0'}{\lambda_n}S_1+\underline{\underline{O}}\left(\frac{1}{n^3}\right) \tag{2.9.11}$$

kelib chiqadi. Endi S_1 yig'indining asimpotikasini o'rganamiz. Ravshanki,

$$S_1=\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{\mu_k-\lambda_n}=\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{k^2+2a_0'+\underline{\underline{O}}(k^{-2})-\lambda_n}=$$

$$=\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{(k^2-\lambda_n)\{1+\frac{2a_0'}{k^2-\lambda_n}+\underline{\underline{O}}\left(\frac{1}{k^2(k^2-\lambda_n)}\right)\}}=$$

$$=\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{k^2-\lambda_n}-2a_0'\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{(k^2-\lambda_n)^2}-\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{(k^2-\lambda_n)^2\underline{\underline{O}}(k^2)}+$$

$$+\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{k^2-\lambda_n}\cdot\frac{\left[\frac{2a_0'}{k^2-\lambda_n}+\underline{\underline{O}}\left(\frac{1}{k^2(k^2-\lambda_n)}\right)\right]^2}{1+\frac{2a_0'}{k^2-\lambda_n}+\underline{\underline{O}}\left(\frac{1}{k^2(k^2-\lambda_n)}\right)}.$$

Bu yerda biz $\frac{1}{1+x}=1-x+\frac{x^2}{1+x}$ tenglikdan foydalandik. Qiyudagi

$$\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{k^2(k^2-\lambda_n)^2}=\underline{\underline{O}}\left(\frac{1}{n^3}\right),$$

baholashdan, lemma 2.9.1 ga asosan

$$S_1 = \sum_{\substack{k=1\\k\neq n}}^{\infty} \frac{1}{k^2-\lambda_n} - 2a_0 \sum_{\substack{k=1\\k\neq n}}^{\infty} \frac{1}{(k^2-\lambda_n)^2} + \underline{\underline{O}}\left(\frac{1}{n^3}\right), \quad (2.9.12)$$

asimptotikani olamiz.

d) Endi (2.9.9) yoyilmadagi ikkinchi yig'indini qaraymiz. (2.9.7) asimptotik formulalardan foydalanib quyidagi

$$-\frac{1}{2}\sum_{\substack{k=0\\k\neq n}}^{\infty} \left(\frac{\lambda_k-\mu_k}{\mu_k-\lambda_n}\right)^2 = -2(a_0-a_0')^2 \sum_{\substack{k=1\\k\neq n}}^{\infty} \frac{1}{(k^2-\lambda_n)^2} + \underline{\underline{O}}\left(\frac{1}{n^3}\right), \quad (2.9.13)$$

baholashni olamiz. Shuning uchun (2.9.11), (2.9.12) va (2.9.13) asimptotikalardan ushbu

$$\ln \Psi_n = \frac{A+\mu_0-\lambda_0}{n^2} + \left[2(a_0-a_0') + \frac{c_0-c_0'}{n^2}\right] \cdot \sum_{\substack{k=1\\k\neq n}}^{\infty} \frac{1}{k^2-\lambda_n} -$$

$$-[4a_0'(a_0-a_0') + 2(a_0-a_0')^2] \cdot \sum_{\substack{k=1\\k\neq n}}^{\infty} \frac{1}{(k^2-\lambda_n)^2} + \underline{\underline{O}}\left(\frac{1}{n^3}\right), \quad (2.9.14)$$

asimptotika kelib chiqadi.

Endi

$$\sum_{\substack{k=1\\k\neq n}}^{\infty} \frac{1}{(k^2-\lambda_n)^p}, \quad p=1,2$$

qator yig'indisining asimptotikasini o'rganamiz. Shu maqsadda, quyidagi tenglikdan foydalanamiz:

$$\sum_{k=1}^{\infty} \frac{1}{k^2-\lambda} = -\left[\operatorname{ctg} \pi\sqrt{\lambda} - \frac{1}{\pi\sqrt{\lambda}}\right] \cdot \frac{\pi}{2\sqrt{\lambda}}. \quad (2.9.15)$$

$p=1$ holni ko'rib chiqamiz. $\sqrt{\lambda_n} = n + \varepsilon$ bo'lsin, bu yerda

$$\varepsilon = \frac{a_0}{n} + \frac{a_1}{n^3} + \underline{\underline{O}}\left(\frac{1}{n^4}\right).$$

(2.9.15) formuladan

$$\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{k^2-\lambda}=\frac{1}{2\lambda_n}-\left[\frac{\pi}{2\sqrt{\lambda_n}}\operatorname{ctg}\pi\sqrt{\lambda_n}+\frac{1}{n^2-\lambda_n}\right]=$$

$$=\frac{1}{2n^2}-\left[\frac{\pi}{2(n+\varepsilon)}\operatorname{ctg}\pi\varepsilon-\frac{1}{2n\varepsilon+\varepsilon^2}\right]+\underline{\underline{O}}\left(\frac{1}{n^4}\right)=$$

$$=\frac{1}{2n^2}-\frac{\pi\varepsilon(2n+\varepsilon)\cos\pi\varepsilon-2(n+\varepsilon)\sin\pi\varepsilon}{2\varepsilon(n+\varepsilon)(2n+\varepsilon)\sin\pi\varepsilon}+\underline{\underline{O}}\left(\frac{1}{n^4}\right)=$$

$$=\frac{3}{4n^2}+\frac{\pi^2 a_0}{6n^2}+\underline{\underline{O}}\left(\frac{1}{n^4}\right). \tag{2.9.16}$$

asimptotika kelib chiqadi. Quyidagi formula ham shu tarzda keltirib chiqariladi:

$$\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{(k^2-\lambda)^2}=\frac{1}{12}\frac{\pi}{n^2}+\underline{\underline{O}}\left(\frac{1}{n^4}\right). \tag{2.9.17}$$

(2.9.14), (2.9.16) va (2.9.17) formulalar va elementar almashtirishlar yordamida quyidagi

$$\ln\Psi_n=\frac{1}{n^2}\left[A+\mu_0-\lambda_0+\frac{\pi^2}{6}(a_0-a_0')^2+\frac{3}{2}(a_0-a_0')\right]+\underline{\underline{O}}\left(\frac{1}{n^3}\right),$$

asimptotika kelib chiqadi. Shunday qilib, biz ushbu

$$\Psi_n=1+\frac{1}{n^2}\left[A+\mu_0-\lambda_0+\frac{\pi^2}{6}(a_0-a_0')^2+\frac{3}{2}(a_0-a_0')\right]+\underline{\underline{O}}\left(\frac{1}{n^3}\right),$$

asimptotik formulaga ega bo‘ldik.

Endi (2.9.4) formuladagi $\dfrac{h_2-h_1}{\mu_n-\lambda_n}$ ko‘paytuvchini qaraymiz. (2.9.7) asimptotika va (2.9.6) dan quyidagiga ega bo‘lamiz:

$$\frac{h_2-h_1}{\mu_n-\lambda_n}=\frac{h_2-h_1}{2(a_0'-a_0)+\frac{c_0'-c_0}{n^2}+\underline{\underline{O}}\left(\frac{1}{n^3}\right)}=$$

$$=\frac{h_2-h_1}{2(a_0'-a_0)}\left[1-\frac{c_0'-c_0}{2(a_0'-a_0)n^2}\right]+\underline{\underline{O}}\left(\frac{1}{n^3}\right).$$

(2.9.5) va (2.9.6) formulalarga ko'ra

$$\frac{h_2 - h_1}{2(a_0' - a_0)} = \frac{\pi}{2}, \qquad \frac{c_0' - c_0}{2(a_0' - a_0)} = \frac{a_0' + a_0}{2} + \frac{a_1' - a_1}{a_0' - a_0}.$$

Nihoyat, α_n normallovchi o'zgarmaslar uchun quyidagi asimptotik formulani keltirib chiqaramiz:

$$\alpha_n = \frac{\pi}{2} + \frac{\pi}{2}\left[S + \frac{\pi^2}{6}(a_0 - a_0')^2 - a_0 - \frac{a_1 - a_1'}{a_0 - a_0'}\right] \cdot \frac{1}{n^2} + \underline{O}\left(\frac{1}{n^3}\right),$$

bu yerda

$$S = \mu_0 - \lambda_0 + \sum_{k=1}^{\infty} [(\mu_k - \lambda_k) + 2(a_0 - a_0')].$$

α_n normallovchi o'zgarmaslar ketma-ketligi uchun topilgan asimptotikadan foydalanib, quyidagi tasdiqni isbotlash mumkin ([82]).

Teorema 2.9.1. (*Yetarlilik shartlari*). Agar $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\mu_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari quyidagi

1. $\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \lambda_n < \mu_n < \lambda_{n+1} < \ldots$,

2. $\sqrt{\lambda_n} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + \underline{O}\left(\frac{1}{n^4}\right)$, $\sqrt{\mu_n} = n + \frac{a_0'}{n} + \frac{a_1'}{n^3} + \underline{O}\left(\frac{1}{n^4}\right)$, $a_0 \neq a_0'$,

shartlarni qanoatlantirsa, u holda shunday absolyut uzluksiz $q(x) \in AC[0, \pi]$ funksiya hamda h_1, h_2 va H haqiqiy sonlar topilib, $\{\lambda_n\}_{n=0}^{\infty}$ va $\{\mu_n\}_{n=0}^{\infty}$ haqiqiy sonlar ketma-ketliklari mos ravishda ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, \\ y'(0) - h_1 y(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \end{cases}$$

$$\begin{cases} -y'' + q(x)y = 0, \\ y'(0) - h_2 y(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \end{cases}$$

chegaraviy masalalarning xos qiymatlaridan iborat bo'ladi. Bu yerda

$$h_2 - h_1 = \pi(a_0' - a_0)$$

10-§. Shturm-Liuvill operatori xos qiymatlarining asimptotikasi va potensialning silliqligi orasidagi aniq bog'lanish

Teorema 2.10.1. *Agar haqiqiy $q(x)$ funksiya $W_2^n(0, \pi)$ fazoga tegishli bo'lsa, u holda ushbu*

$$y'' - q(x)y + \mu y = 0, \quad (0 \le x \le \pi), \tag{2.10.1}$$

Shturm-Liuvill tenglamasi $\lambda = \sqrt{\mu}$ haqiqiy bo'lganida quyidagi ko'rinishdagi

$$y(x, \lambda) = e^{i\lambda x}\left[1 + \frac{u_1(x)}{i\lambda} + ... + \frac{u_n(x)}{(i\lambda)^n} + \frac{u_{n+1}(x, \lambda)}{(i\lambda)^{n+1}}\right], \tag{2.10.2}$$

yechimga ega bo'ladi.

Bu yerda

$$u_1(x) = \frac{1}{2}\int_0^x q(t)dt,$$

$$u_k(x) = -\frac{1}{2}\int_0^x L[u_{k-1}(t)]dt, \quad k \ge 2; \; L \equiv \frac{d^2}{dx^2} - q(x), \tag{2.10.3}$$

bo'lib, ushbu

$$v_{n+1}(x, \lambda) = e^{i\lambda x}u_{n+1}(x, \lambda), \tag{2.10.4}$$

funksiya

$$L[v_{n+1}(x, \lambda)] + \lambda^2 v_{n+1}(x, \lambda) = -i\lambda e^{i\lambda x}L[u_n(x)],$$

differensial tenglamani va

$$v_{n+1}(0, \lambda) = 0, \quad v'_{n+1}(0, \lambda) = 0,$$

boshlang'ich shartlarni qanoatlantiradi. Bundan tashqari $\{y(x,\lambda),\ \bar{y}(x,\lambda)\}$ funksiyalar juftligi (2.10.1) differensial tenglama yechimlarining fundamental sistemasini tashkil qiladi.

Qisqalik uchun ushbu

$$P(x,\lambda) = 1 + \frac{u_1(x)}{i\lambda} + \frac{u_2(x)}{(i\lambda)^2} + \dots + \frac{u_n(x)}{(i\lambda)^n} + \frac{u_{n+1}(x,\lambda)}{(i\lambda)^{n+1}}; \qquad \sigma(x,\lambda) = \frac{P'(x,\lambda)}{P(x,\lambda)}, \tag{2.10.5}$$

belgilashlarni kiritib olamiz. U holda (2.10.2) formulani quyidagi tarzda yozish mumkin:

$$y(x,\lambda) = \exp\left\{ i\lambda x + \int_0^x \sigma(t,\lambda)dt \right\}. \tag{2.10.2'}$$

Lemma 2.10.1. *Agar $q(x) \in W_2^n(0,\pi)$ bo'lsa, u holda ushbu*

$$v_{n+1}(x,\lambda) = e^{i\lambda x}\left[u_{n+1}(x) + \varphi(x,\lambda) + \frac{1}{2i\lambda}\cdot \int_0^x q(t)u_{n+1}(t)dt \right] + $$

$$+\frac{\gamma_{n+1}(x,\lambda)}{2i\lambda}, \tag{2.10.6}$$

$$v'_{n+1}(x,\lambda) = i\lambda\, e^{i\lambda x}\left[u_{n+1}(x) - \varphi(x,\lambda) + \frac{1}{2i\lambda}\int_0^x q(t)u_{n+1}(t)dt \right] + $$

$$+\tilde{\gamma}_{n+1}(x,\lambda), \tag{2.10.7}$$

tasvirlar o'rinli. Bu yerda

$$\varphi(x,\lambda) = \frac{1}{2}\int_0^x e^{-2i\lambda(x-t)}\cdot L[u_n(t)]dt,$$

bo'lib, $\gamma_{n+1}(x,\lambda)$, $\tilde{\gamma}_{n+1}(x,\lambda)$ funksiyalar λ bo'yicha $L_2(-\infty;\infty)$ fazoga tegishli hamda x daraja bilan eksponensial turdagi funksiyalar.

Isbot. (2.10.1) tenglamaning $c(0,\lambda) = 1$, $c'(0,\lambda) = 0$, $s(0,\lambda) = 0$, $s'(0,\lambda) = 1$ boshlang'ich shartlarni qanoatlantiruvchi yechimlarini $c(x,\lambda)$, $s(x,\lambda)$ orqali belgilaymiz. U holda quyidagi

$$v_{n+1}(x,\lambda) = -i\lambda \int_0^x [s(x,\lambda)\,c(t,\lambda) - s(t,\lambda)\,c(x,\lambda)]\; e^{i\lambda t} L[u_n(t)]dt, \tag{2.10.7'}$$

tenglik o'rinli. Ushbu

$$\varphi(t,x,\lambda) = s(x,\lambda)\,c(t,\lambda) - s(t,\lambda)\,c(x,\lambda),$$

funksiya t o'zgaruvchi bo'yicha (2.10.1) differensial tenglamani hamda

$$\varphi(t,x,\lambda)|_{t=x} = 0, \quad \varphi'_t(t,x,\lambda)|_{t=x} = -1\,,$$

boshlang'ich shartlarni qanoatlantiradi. Bunga ko'ra u λ bo'yicha $(x-t)$ darajali eksponensial turdagi funksiya bo'ladi. Demak, (2.10.7') tenglikka asosan $v_{n+1}(x,\lambda)$ funksiya λ bo'yicha x darajali eksponensial turdagi funksiya bo'ladi.

$v_{n+1}(x,\lambda)$ funksiya quyidagi tenglamani qanoatlantirishi ravshan:

$$v_{n+1}(x,\lambda) = -i\int_0^x \sin\lambda(x-t)\cdot e^{i\lambda t}\cdot L[u_{n+1}(t)]dt +$$

$$+\int_0^x \frac{\sin\lambda(x-t)}{\lambda}\cdot q(t)\,v_{n+1}(t,\lambda)dt. \tag{2.10.8}$$

Bu tenglamani bir marotaba iteratsiya qilsak, ushbu

$$v_{n+1}(x,\lambda) = e^{i\lambda x}\cdot \left\{ -\frac{1}{2}\int_0^x L[u_n(t)]dt + \frac{1}{2}\int_0^x e^{-2i\lambda(x-t)} L[u_n(t)]dt + \right.$$

$$\left. + \frac{1}{2i\lambda}\int_0^x q(t)u_{n+1}(t)dt \right\} + \frac{\gamma_{n+1}(x,\lambda)}{2i\lambda},$$

tenglik kelib chiqadi. Bu tenglikka asosan $\gamma_{n+1}(x,\lambda)$ funksiya λ bo'yicha x darajali eksponensial turdagi funksiya bo'ladi.

$\gamma_{n+1}(x,\lambda) \in L^2(-\infty;\infty)$ bo'lishiga to'g'ridan-to'g'ri hisoblash yo'li bilan ishonch hosil qilishimiz mumkin.

(2.10.8) tenglikni x bo'yicha differensiallab, (2.10.7) formulani keltirib chiqaramiz. ■

Izoh 2.10.1. (2.10.3) rekurent formulalarga asosan

$$L[u_n(t)] = \frac{(-1)^{n-1}}{2^{n+1}} q^{(n)}(t) + g(t),$$

tenglik o'rinli bo'ladi. Bu yerda $g(t) \in W_2^1(0,\pi)$.

Bunga ko'ra ushbu

$$\varphi(x,\lambda) = \frac{1}{2}\int_0^x e^{-2i\lambda(x-t)} \cdot L[u_n(t)]dt =$$

$$= \frac{(-1)^{n-1}}{2^n}\int_0^x e^{-2i\lambda(x-t)} \cdot q^{(n)}(t)dt +$$

$$+\frac{1}{4i\lambda}g(x) - \frac{1}{4i\lambda}g(0)\, e^{-2i\lambda x} - \frac{1}{4i\lambda}\tilde{g}(x,\lambda), \qquad (2.10.9)$$

tasvir o'rinli bo'ladi. Bu yerda

$$\tilde{g}(x,\lambda) = \int_0^x e^{-2i\lambda(x-t)} g'(t)dt,$$

funksiya λ bo'yicha eksponensial turda bo'lib, uning darajasi $2x$ dan oshmaydi, hamda u $L^2(-\infty,\infty)$ fazoga tegishli bo'ladi.

(2.10.1) tenglama va ushbu

$$y'(0) - hy(0) = 0, \quad y'(\pi) - Hy(\pi) = 0, \qquad (2.10.10)$$

shartlar yordamida hosil qilingan Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu chegaraviy masalaning xos qiymatlarini

$$\mu_0, \mu_1, \mu_2,$$

orqali belgilaymiz. M.G. Gasimov, B.M. Levitanlarning [28] ishida xos qiymatlar uchun quyidagi asimptotik formula olingan:

$$\sqrt{\mu_k} = k + \frac{a_0}{k} + \ldots + \frac{a_l}{k^{2l+1}} + \frac{\gamma_k}{k^{2p+1}}, \quad l = \left[\frac{n}{2}\right], \quad \sum_{k=0}^{\infty} |\gamma_k|^2 < \infty.$$

Mazkur paragrafda bu asimptotik formulani yanada aniqlashtiramiz. Shu maqsadda, biz D.Sh.Lundinaning [90] maqolasida olingan natijani bayon qilamiz.

Teorema 2.10.2. *Agar $q(x)$ haqiqiy bo'lib, $W_2^n(0,\pi)$ fazoga tegishli bo'lsa, u holda (2.10.1)+(2.10.10) chegaraviy masalaning $\mu_k = \lambda_k^2$ xos qiymatlari $k \to \infty$ bo'lganida quyidagi asimptotik tenglikni qanoatlantiradi:*

$$\lambda_k = k + \sum_{1 \le 2j+1 \le n+3} \frac{b_{2j+1}}{k^{2j+1}} +$$

$$+\frac{1}{\pi} \operatorname{Im} \left\{ \frac{(-1)^{n-1}}{(2ik)^{n+1}} \cdot \int_0^{\pi} q^{(n)}(t) e^{2ikt} \left[1 + \frac{2h - \frac{2}{\pi} t (h + u_1(0) - H)}{ik} \right] dt \right\} +$$

$$+ \frac{\delta_k}{k^{n+2}} + \frac{\varepsilon_k(H,h)}{k^{n+3}}.$$

Bu yerda δ_k ketma-ketlik H, h larga bog'liq emas,

$$\sum_{k=0}^{\infty} |\delta_k|^2 < \infty, \qquad \sum_{k=0}^{\infty} |\varepsilon_k(H,h)|^2 < \infty.$$

Isbot. $y(x,\lambda)$ va $\overline{y}(x,\lambda)$ funksiyalar (2.10.1) tenglamaning yechimlar fundamental sistemasini hosil qilganligi sababli (2.10.1)+(2.10.10) masala noldan farqli yechimga ega bo'lishi uchun ushbu

$$\begin{cases} h[x_1 y(0,\lambda) + x_2 \overline{y}(0,\lambda)] - [x_1 y'(0,\lambda) + x_2 \overline{y}'(0,\lambda)] = 0, \\ H[x_1 y(\pi,\lambda) + x_2 \overline{y}(\pi,\lambda)] - [x_1 y'(\pi,\lambda) + x_2 \overline{y}'(\pi,\lambda)] = 0, \end{cases}$$

sistema noldan farqli x_1, x_2 yechimga ega bo'lishi zarur va yetarlidir. Bu sistemaning determinantini nolga tenglab, quyidagi xarakteristik tenglamani hosil qilamiz:

$$\begin{vmatrix} h\,y(0,\lambda) - y'(0,\lambda) & h\,\overline{y}(0,\lambda) - \overline{y}'(0,\lambda) \\ Hy(\pi,\lambda) - y'(\pi,\lambda) & H\overline{y}(\pi,\lambda) - \overline{y}'(\pi,\lambda) \end{vmatrix} = 0,$$

ya'ni

$$(h\,y(0,\lambda) - y'(0,\lambda)) \cdot (H\,\overline{y}(\pi,\lambda) - \overline{y}'(\pi,\lambda)) -$$
$$-(h\,\overline{y}(0,\lambda) - \overline{y}'(0,\lambda)) \cdot (Hy(\pi,\lambda) - y'(\pi,\lambda)) = 0.$$

Ushbu

$$y(0,\lambda) = 1, \quad y'(0,\lambda) = i\lambda + \sigma(0,\lambda),$$
$$y'(\pi,\lambda) = y(\pi,\lambda) \cdot (i\lambda + \sigma(\pi,\lambda)),$$

tengliklarni hisobga olib, xarakteristik tenglamani quyidagi tarzda yozamiz:

$$\frac{y(\pi,\lambda)}{\overline{y(\pi,\lambda)}} = \frac{\left[1 - \frac{h}{i\lambda} + \frac{\sigma(0,\lambda)}{i\lambda}\right] \cdot \left[1 + \frac{H}{i\lambda} - \frac{\bar{\sigma}(\pi,\lambda)}{i\lambda}\right]}{\left[1 + \frac{h}{i\lambda} - \frac{\bar{\sigma}(0,\lambda)}{i\lambda}\right] \cdot \left[1 - \frac{H}{i\lambda} + \frac{\sigma(\pi,\lambda)}{i\lambda}\right]}.$$

Bu tenglama ildizlarining kvadratlari (2.10.1)+(2.10.10) masalaning xos qiymatlari bilan ustma-ust tushadi.

Oxirgi tenglamani (2.10.2′) formula yordamida ushbu

$$\exp\left\{2i(\lambda\pi + \operatorname{Im}\int_0^\pi \sigma(t,\lambda)dt)\right\} =$$

$$= \exp\left\{2i\operatorname{Im}\left[\ln\left(1 - \frac{h}{i\lambda} + \frac{\sigma(0,\lambda)}{i\lambda}\right) - \ln\left(1 - \frac{H}{i\lambda} + \frac{\sigma(\pi,\lambda)}{i\lambda}\right)\right]\right\},$$

ko'rinishda yozish mumkin. Bundan (2.10.5) belgilashlarga muvofiq, $\lambda_k = \sqrt{\mu_k}$ sonlar quyidagi tenglamaning ildizlari ekani kelib chiqadi:

$$(\lambda - k)\pi = \operatorname{Im}\left\{\ln\left(1 - \frac{h}{i\lambda} + \frac{\sigma(0,\lambda)}{i\lambda}\right) -\right.$$

$$- \ln\left[\left(1 - \frac{H}{i\lambda}\right)\left(1 + \frac{u_1(\pi)}{i\lambda} + ... + \frac{u_n(\pi)}{(i\lambda)^n}\right) + \frac{u_1'(\pi)}{(i\lambda)^2} + ... + \frac{u_n'(\pi)}{(i\lambda)^{n+1}} +\right.$$

$$+\frac{u_{n+1}(\pi,\lambda)}{(i\lambda)^{n+1}}\cdot\left(1-\frac{H}{i\lambda}\right)+\frac{u'_{n+1}(\pi,\lambda)}{(i\lambda)^{n+2}}\Bigg]\Bigg\}, \qquad (2.9.11)$$

ya'ni

$$(\lambda-k)\pi=\operatorname{Im}\left\{\ln\left(1-\frac{h}{i\lambda}+\frac{\sigma(0,\lambda)}{i\lambda}\right)-\right.$$

$$-\ln\left[1+\frac{u_1(\pi)-H}{i\lambda}+\ldots+\frac{u'_n(\pi)-Hu_n(\pi)}{(i\lambda)^{n+1}}+\right.$$

$$\left.\left.+\frac{1}{(i\lambda)^{n+2}}\cdot\left[i\lambda\, u_{n+1}(\pi,\lambda)+u'_{n+1}(\pi,\lambda)-H\, u_{n+1}(\pi,\lambda)\right]\right]\right\}. \qquad (2.10.12)$$

Ushbu

$$i\lambda\cdot u_{n+1}(\lambda,\pi)+u'_{n+1}(\lambda,\pi)=e^{-i\lambda\pi}\cdot v'_{n+1}(\pi,\lambda),$$

tenglikni hisobga olsak, hamda (2.10.6), (2.10.7), (2.10.9) formulalardan foydalansak, quyidagi kelib chiqadi:

$$\frac{1}{(i\lambda)^{n+2}}\cdot\left[i\lambda\cdot u_{n+1}(\pi,\lambda)+u'_{n+1}(\pi,\lambda)-H\, u_{n+1}(\pi,\lambda)\right]=$$

$$=\frac{u_{n+1}(\pi)}{(i\lambda)^{n+1}}+\frac{1}{(i\lambda)^{n+2}}\cdot\left(\frac{1}{2}\int_0^\pi q(t)\cdot u_{n+1}(t)\,dt-Hu_{n+1}(\pi)\right)+$$

$$+\frac{H}{2(i\lambda)^{n+3}}\int_0^\pi q(t)\,u_{n+1}(t)\,dt+$$

$$+\frac{1}{4(i\lambda)^{n+2}}\left(e^{-i\lambda\pi}g(0)-g(\pi)\right)\left(1+\frac{H}{i\lambda}\right)+$$

$$+\frac{(-1)^n}{(2i\lambda)^{n+1}}\int_0^\pi e^{-2i\lambda\cdot(\pi-t)}\cdot q^{(n)}(t)dt\cdot\left(1+\frac{H}{i\lambda}\right)+\frac{\hat{\delta}(\lambda)}{(i\lambda)^{n+2}}+\frac{\sigma(H,h,\lambda)}{(i\lambda)^{n+3}}.$$

Bu yerda

$$\hat{\delta}(\lambda)=\frac{1}{4}\cdot\int_0^\pi e^{-2i\lambda(\pi-t)}\cdot g'(t)\,dt+e^{-i\lambda\pi}\cdot\tilde{\gamma}_{n+1}(\pi,\lambda),$$

$$\sigma(H,h,\lambda)=\frac{H}{4}\int_0^{\pi} e^{-2i\lambda(\pi-t)}\cdot g'(t)\,dt-He^{-i\lambda\pi}\cdot\gamma_{n+1}(\pi,\lambda),$$

funksiyalar eksponensial turda bo'lib, darajasi 2π dan oshmaydi, hamda ular $L^2(-\infty,\infty)$ fazoga tegishli. Shunday qilib, (2.10.12) xarakteristik tenglamani quyidagi tarzda yozish mumkin:

$$(\lambda-k)\pi=\operatorname{Im}\left\{\ln\left(1-\frac{h}{i\lambda}+\frac{\sigma(0,\lambda)}{i\lambda}\right)-\right.$$

$$-\ln\left(1+\frac{u_1(\pi)-H}{i\lambda}+\ldots+\frac{\frac{H}{2}\int_0^{\pi} q(t)u_{n+1}(t)\,dt}{(i\lambda)^{n+3}}\right)-$$

$$\left.-\ln\left(1+\frac{\frac{(-1)^n}{(2i\lambda)^{n+1}}\int_0^{\pi} e^{-2i\lambda(\pi-t)}\cdot q^{(n)}(t)dt\cdot\left(1+\frac{H}{i\lambda}\right)+\ldots+\frac{\hat{\delta}(\lambda)}{(i\lambda)^{n+2}}+\frac{\sigma(H,h,\lambda)}{(i\lambda)^{n+3}}}{1+\frac{u_1(\pi)-H}{i\lambda}+\ldots}\right)\right\}.$$

Bundan $\lambda\to\infty$ da quyidagi tenglik kelib chiqadi:

$$(\lambda-k)\pi+\left\{\frac{H-h-u_1(\pi)}{\lambda}+\ldots+\frac{C_{2p+1}}{\lambda^{2p+1}}+\ldots\right\}=$$

$$=-\operatorname{Im}\left[\frac{(-1)^n}{(2i\lambda)^{n+1}}\cdot\int_0^{\pi} e^{-2i\lambda(\pi-t)}\cdot q^{(n)}(t)\,dt\cdot\left(1+\frac{2H-u_1(\pi)}{i\lambda}\right)+\right.$$

$$\left.+\frac{e^{-i\lambda\pi}g(0)}{4\cdot(i\lambda)^{n+2}}\cdot\left(1+\frac{H}{i\lambda}\right)+\frac{\hat{\delta}(\lambda)}{(i\lambda)^{n+2}}+\frac{\sigma(H,h,\lambda)}{(i\lambda)^{n+3}}+\ldots\right].$$

$\mu_k=\lambda_k^2$ (2.10.1)+(2.10.10) masalaning xos qiymati bo'lsin. Bizga $k\to\infty$ da ushbu

$$\lambda_k=k+\theta_k,\qquad \theta_k=\underline{\underline{O}}\left(\frac{1}{k}\right),\qquad \lambda_k^{-p}=k^{-p}\cdot\left(1+\underline{\underline{O}}\left(\frac{1}{k^2}\right)\right),$$

asimptotikalarning o'rinli ekanligi ma'lum.

Oldingi tenglikda $\lambda=\lambda_k$ desak, quyidagi kelib chiqadi:

$$\theta_k\pi+\left\{\frac{H-h-u_1(\pi)}{k+\theta_k}+\ldots+\frac{C_{2p+1}}{(k+\theta_k)^{2p+1}}+\ldots\right\}=$$

$$= -\operatorname{Im}\left[\frac{(-1)^n}{(2ik)^{n+1}}\int_0^\pi e^{-2i(k+\theta_k)(\pi-t)}q^{(n)}(t)\,dt\left(1+\frac{2H-u_1(\pi)}{ik}\right)+\right.$$

$$\left.+\frac{\hat{\delta}(\lambda_k)}{(i\lambda)^{n+2}}+\frac{e^{-i(k+\theta_k)\pi}g(0)}{4(ik)^{n+2}}\left(1+\frac{H}{ik}\right)+\frac{\sigma(H,h,\lambda_k)}{(ik)^{n+3}}\right]+\frac{\alpha_k}{k^{n+3}},$$

Bu yerda

$$\sum_{k=0}^{\infty}|\alpha_k|^2<\infty.$$

Agar biror $f(\lambda)$ funksiya eksponensial turda bo‘lib, $L^2(-\infty,\infty)$ fazoga tegishli bo‘lsa, u holda quyidagi fikrni isbotlash qiyin emas:

$$f(k+\theta_k)=f(k)+\theta_k f'(k)+\frac{f_1(k)}{k},$$

$$\sum_{k=0}^{\infty}|f(k)|^2<\infty,\quad \sum_{k=0}^{\infty}|f'(k)|^2<\infty,\quad \sum_{k=0}^{\infty}|f_1(k)|^2<\infty.$$

Bu faktdan foydalanib, yuqoridagi tenglikni quyidagicha yozish mumkin:

$$\theta_k\pi+\left\{\frac{H-h-u_1(\pi)}{k+\theta_k}+\ldots+\frac{C_{2p+1}}{(k+\theta_k)^{2p+1}}+\ldots\right\}=$$

$$= -\operatorname{Im}\left[\frac{(-1)^n}{(2ik)^{n+1}}\int_0^\pi e^{2ikt}q^{(n)}(t)\left(1-2i\theta_k(\pi-t)\right)dt\times\right.$$

$$\left.\times\left(1+\frac{2H-u_1(\pi)}{ik}\right)+\frac{\hat{\delta}(k)}{(ik)^{n+2}}+\frac{e^{-ik\pi}(1-i\theta_k\pi)}{4(ik)^{n+2}}g(0)\left(1+\frac{H}{ik}\right)\right]+$$

$$+\frac{\varepsilon_k(H,h)}{k^{n+3}}.$$

Bu yerda $\hat{\delta}(k)$ miqdor h, H larga bog‘liq emas, hamda

$$\sum_{k=0}^{\infty}\left|\hat{\delta}(k)\right|^2<\infty,\quad \sum_{k=0}^{\infty}|\varepsilon_k(H,h)|^2<\infty,$$

shartlarni qanoatlantiradi. Bu yerdan, xususan hatto $n = 0$ bo'lganida ham

$$\theta_k \pi = \frac{h - H + u_1(\pi)}{\pi k} + \frac{\Delta_k}{k}, \qquad \sum_{k=0}^{\infty} |\Delta_k|^2 < \infty$$

bo'lishi kelib chiqadi. Demak, quyidagi tenglik o'rinli

$$\theta_k \pi + \left\{ \frac{H - h - u_1(\pi)}{k + \theta_k} + \dots + \frac{\tilde{C}_{2p+1}}{(k + \theta_k)^{2p+1}} + \dots \right\} =$$

$$= \operatorname{Im} \frac{(-1)^{n+1}}{(2ik)^{n+1}} \cdot \int_0^{\pi} e^{2ikt} \cdot q^{(n)}(t) \cdot \left[1 + \frac{2h - \frac{2}{\pi} t\,(h + u_1(\pi) - H)}{ik} \right] dt +$$

$$+ \frac{\delta_k}{k^{n+2}} + \frac{\varepsilon_k(H, h)}{k^{n+3}}. \qquad (2.10.13)$$

Faraz qilaylik, $\overline{\theta}_k$ lar ushbu

$$\overline{\theta}_k \pi + \left\{ \frac{H - h - u_1(\pi)}{k + \overline{\theta}_k} + \dots + \frac{\tilde{C}_{2p+1}}{(k + \overline{\theta}_k)^{2p+1}} + \dots \right\} = 0, \qquad (2.10.14)$$

tenglamadan aniqlanadigan bo'lsin. U holda

$$\overline{\theta}_k = \frac{h - H + u_1(\pi)}{\pi k} + \sum_{3 \le 2j+1 \le \infty} \frac{b_{2j+1}}{k^{2j+1}}. \qquad (2.10.15)$$

(2.10.13) tenglikdan (2.10.14) tenglikni ayirsak,

$$(\theta_k - \overline{\theta}_k)\pi \cdot \left(1 + \underline{\underline{O}}\left(\frac{1}{k^2} \right) \right) = \operatorname{Im} \frac{(-1)^{n+1}}{(2ik)^{n+1}} \cdot \int_0^{\pi} e^{2ikt} \cdot q^{(n)}(t) \times$$

$$\times \left[1 + \frac{2h - \frac{2}{\pi} t\,(h + u_1(\pi) - H)}{ik} \right] dt + \frac{\delta_k}{k^{n+2}} + \frac{\varepsilon_k(H, h)}{k^{n+3}}$$

kelib chiqadi. Bu yerda (2.10.15) tenglikni ishlatsak, (2.10.11) formula kelib chiqadi.■

Izoh 2.10.2. Yuqorida isbot qilingan teoremadan ushbu

$$\lambda_k = k + \sum_{1 \le 2j+1 \le n+2} \frac{b_{2j+1}}{k^{2j+1}} +$$

$$+\frac{1}{\pi}\operatorname{Im}\left\{\frac{(-1)^{n-1}}{(2ik)^{n+1}}\cdot\int\limits_0^\pi e^{2ikt}q^{(n)}(t)dt\right\}+\frac{\alpha_k}{k^{n+2}},\ \{\alpha_k\}\in l_2 \tag{2.10.16}$$

formula kelib chiqadi.

Izox 2.10.3. (2.10.1)+(2.10.10) masala bilan birgalikda (2.10.1) tenglama va ushbu

$$y'(0)-hy(0)=0,\quad y'(\pi)-H'y(\pi)=0, \tag{2.10.17}$$

chegaraviy shartlar bilan berilgan chegaraviy masalani ko'rib chiqamiz.

(2.10.1) + (2.10.17) masalaning $v_k=\xi_k^2$ xos qiymatlari uchun ham (2.10.11) tenglik o'rinli, ammo unda H o'rnida H' ni olish kerak xolos. Agar (2.10.11) formuladan, H ning o'rnida H' ni qo'yish yordamida xosil bo'lgan formulani ayirsak, u holda

$$\lambda_k-\xi_k=\sum_{1\le 2j+1\le n+3}\frac{C_{2j+1}}{k^{2j+1}}+$$

$$+\frac{1}{\pi}\cdot\operatorname{Im}\left\{\frac{(-1)^{n-1}}{(2ik)^{n+2}}\cdot\frac{4(H-H')}{\pi}\cdot\int\limits_0^\pi e^{2ikt}\cdot q^{(n)}(t)dt\right\}+\frac{\beta_k}{k^{n+3}}, \tag{2.10.18}$$

$$\sum_{k=0}^{\infty}|\beta_k|^2<\infty.$$

Teorema 2.10.3. *Haqiqiy $q(x)$ funksiya $W_2^n(0,\pi)$ fazoga tegishli bo'lishi uchun (2.10.1)+(2.10.10) va (2.10.1)+(2.10.17) masalalarning $\mu_k=\lambda_k^2$, $v_k=\xi_k^2$ xos qiymatlari uchun quyidagi asimptotikalar bajarilishi zarur va yetarlidir:*

$$\lambda_k=k+\sum_{1\le 2j+1\le n+2}\frac{\tilde{b}_{2j+1}}{k^{2j+1}}+\frac{\delta_k}{k^{n+1}},$$

$$\lambda_k-\xi_k=\sum_{1\le 2j+1\le n+3}\frac{\tilde{c}_{2j+1}}{k^{2j+1}}+\frac{\theta_k}{k^{n+2}}. \tag{2.10.19}$$

Bu yerda

$$\sum_{k=0}^{\infty} |\delta_k|^2 < \infty, \quad \sum_{k=0}^{\infty} |\theta_k|^2 < \infty.$$

Isbot. Teoremaning zaruriylik qismi (2.10.16) va (2.10.18) tengliklar yordamida oson isbotlanadi.

Yetarlilik qismini isbot qilamiz. $q(x) \in W_2^m(0,\pi)$ bo'lib, $q(x) \notin W_2^{m+1}(0,\pi)$ deb hisoblaymiz. Bu yerda $m < n$. Aniqlik uchun $m = 2l$ deb hisoblaymiz. U holda (2.10.16) va (2.10.18) ga asosan quyidagilarga egamiz:

$$\lambda_k = k + \sum_{1\le 2j+1\le m+2} \frac{b_{2j+1}}{k^{2j+1}} + \frac{1}{2}\cdot\frac{(-1)^l}{(2k)^{m+1}}\cdot C_{2k}^m + \frac{\alpha_k}{k^{m+2}},$$

$$\lambda_k - \xi_k = \sum_{1\le 2j+1\le m+3} \frac{c_{2j+1}}{k^{2j+1}} + \frac{2}{\pi}\cdot\frac{(H-H')}{(2k)^{m+2}}\cdot(-1)^l\cdot \hat{S}_{2k}^m + \frac{\beta_k}{k^{m+3}}.$$

Bu yerda

$$C_{2k}^m = \frac{2}{\pi}\int_0^\pi q^{(m)}(t)\,\cos 2kt dt,$$

$$\hat{S}_{2k}^m = \frac{2}{\pi}\int_0^\pi q^{(m)}(t)\,\cdot t\cdot \sin 2kt dt.$$

Bu formulalarni (2.10.19) shartlar bilan taqqoslab, quyidagilarni olamiz:

1) $\tilde{b}_{2j+1} = b_{2j+1}, 1 \le 2j+1 \le m+1, \quad \frac{1}{2}\cdot(-1)^l\cdot C_{2k}^m = \frac{\eta_k}{k},$ (2.10.20)

$$\sum_{k=1}^{\infty} |\eta_k|^2 < \infty, \tag{2.10.21}$$

2) $\tilde{C}_{2j+1} = C_{2j+1}, \ 1 \le 2j+1 \le m+1,$

$$\frac{2(H-H')}{\pi}\cdot(-1)^l\cdot \hat{S}_{2k}^m = \frac{\tilde{C}_{2l+3} - C_{2l+3}}{k} + \frac{\zeta_k}{k},$$

$$\sum_{k=1}^{\infty}|\zeta_k|^2<\infty. \tag{2.10.22}$$

Faraz qilaylik

$$q^{(m)}(x)=q_1^{(m)}(x)+q_2^{(m)}(x), \tag{2.10.23}$$

ko'rinishda bo'lsin. Bu yerda

$$q_1^{(m)}(\pi-x)=-\,q_1^{(m)}(x),\quad q_2^{(m)}(\pi-x)=q_2^{(m)}(x).$$

U holda (2.10.20) ga ko'ra

$$\frac{1}{2}\cdot(-1)^l\cdot q_2^{(m)}(x)=\sum_{k=1}^{\infty}\frac{\eta_k}{k}\cos 2kx,$$

bo'ladi. Bu yerda (2.10.21) shartni hisobga olsak, $q_2^{(m)}(x)\in W_2^1(0,\pi)$ kelib chiqadi.

Ikkinchi tomondan $x\cdot q^{(m)}(x)$ funksiyaning $x=\frac{\pi}{2}$ ga nisbatan toq qismi bo'lgan ushbu

$$q_2^{(m)}(x)\left(x-\frac{\pi}{2}\right)+\frac{\pi}{2}q_1^{(m)}(x),$$

funksiya uchun (2.10.22) ga asosan quyidagi tenglik o'rinli bo'ladi:

$$\frac{2\,(-1)^l\,(H-H')}{\pi}\left[q_2^{(m)}(x)\left(x-\frac{\pi}{2}\right)+\frac{\pi}{2}q_1^{(m)}(x)\right]=$$

$$=\sum_{k=1}^{\infty}\frac{\bar{C}_{2l+3}-C_{2l+3}}{k}\sin 2kx+\sum_{k=1}^{\infty}\frac{\zeta_k}{k}\sin 2kx.$$

Ushbu

$$\sum_{k=1}^{\infty}\frac{\bar{C}_{2l+3}-C_{2l+3}}{k}\sin 2kx=\left(\bar{C}_{2l+3}-C_{2l+3}\right)\left(\frac{\pi}{2}-x\right),$$

tenglikni, hamda (2.10.23) shartni hisobga olsak,

$$q_2^{(m)}(x)\left(x-\frac{\pi}{2}\right)+\frac{\pi}{2}q_1^{(m)}(x)\in W_2^1(0,\pi),$$

ekani kelib chiqadi. Bundan $q_1^{(m)}(x)$ funksiyaning $W_2^1(0,\pi)$ fazoga qarashli bo'lishi kelib chiqadi, ya'ni $q(x)\in W_2^{m+1}(0,\pi)$. Bu esa yuqorida qilgan farazimizga zid keladi. Demak, $q(x) \in W_2^n(0,\pi)$ ekan. m toq son bo'lgan hol ham xuddi shunday isbotlanadi. ∎

M.G. Gasimov va B.M. Levitanlarning [28] ishlarida $\{\mu_k\}$, $\{\nu_k\}$ ketma-ketliklar $q(x) \in L^2(0,\pi)$ haqiqiy potensialli (2.10.1) tenglama va (2.10.10), (2.10.17) chegaraviy shartlar yordamida hosil qilingan chegaraviy masalalarning xos qiymatlari bo'lishi uchun bu ketma-ketliklar almashinib kelishlari, hamda quyidagi

$$\sqrt{\mu_k} = k + \frac{a}{k} + \frac{\varepsilon_k}{k}, \quad \sqrt{\mu_k} - \sqrt{v_k} = \frac{c}{k} + \frac{\theta_k}{k^2},$$

$$\sum^{\infty} |\varepsilon_k|^2 < \infty, \quad \sum^{\infty} |\theta_k|^2 < \infty$$

asimptotikalarni qanoatlantirishlari zarur va yetarli ekanligi isbot qilingan.

Bu fikr teorema 2.10.3 dan, hamda quyidagi natijadan kelib chiqadi.

Natija 2.10.1. $\{\mu_k\}$, $\{v_k\}$ ketma-ketliklar $q(x) \in W_2^n(0,\pi)$ haqiqiy potensialli (2.10.1) tenglama va (2.10.10), (2.10.17) chegaraviy shartlar yordamida hosil qilingan chegaraviy masalalarning xos qiymatlari bo'lishi uchun bu ketma-ketliklar almashinib kelishlari hamda (2.10.19) asimptotik tengliklarni qanoatlantirishlari zarur va yetarlidir.

11-§. Teskari masala yechimining turg'unligi

Mazkur paragrafda Shturm-Liuvill chegaraviy masalasi uchun qo'yilgan teskari spektral masala yechimining turg'unligi ko'rsatiladi. Bu masala ilk bor xususiy holda X.Xoxshtadt tomonidan o'rganilgan. So'ngra bu masala V.A.Yurko tomonidan umumlashtirilgan.

Ushbu

$$L_j y \equiv -y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{2.11.1}$$

$$y'(0) - hy(0) = 0, \quad y^{(j-1)}(\pi) = 0, \quad j = 1,2, \tag{2.11.2}$$

Shturm-Liuvill chegaraviy masalalarini qaraymiz. Bu yerda $q(x) \in L^2(0,\pi)$ haqiqiy funksiya va h haqiqiy son. $\lambda_{n,j}$ $(n \geq 0,\ j = 1,2)$ orqali L_j $(j = 1,2)$ operatorning xos qiymatlarini, hamda $\varphi(x,\lambda)$ orqali (2.11.1) tenglamaning $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$ boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz. U holda $\lambda_{n,j}$ xos qiymatlar $\Delta_j(\lambda) = \varphi^{(j-1)}(\pi,\lambda)$ xarakteristik funksiyaning nollari bilan ustma-ust tushadi va ular ushbu

$$\sqrt{\lambda_{n,1}} = n + \frac{1}{2} + \frac{a}{\left(n+\frac{1}{2}\right)} + \frac{\chi_{n,1}}{n}, \quad \sqrt{\lambda_{n,2}} = n + \frac{a}{n} + \frac{\chi_{n,2}}{n}, \tag{2.11.3}$$

asimptotikalarni qanoatlantirishi yuqoridagi paragraflarda ko'rsatilgan edi. Bu yerda

$$a = \frac{1}{\pi}\left(h + \frac{1}{2}\int_0^\pi q(t)dt\right), \quad \{\chi_{n,j}\} \in l_2, \quad j = 1,2.$$

Hisoblashlarni soddalashtirish maqsadida, $a = 0$ deb hisoblaymiz. U holda (2.11.3) asimptotikalardan

$$\sum_{n=0}^{\infty}\left(\left|\sqrt{\lambda_{n,1}} - \left(n+\frac{1}{2}\right)\right| + \left|\sqrt{\lambda_{n,2}} - n\right|\right) < \infty, \tag{2.11.4}$$

sonli qatorning yaqinlashuvchiligi kelib chiqadi.

Endi $\tilde{L}_j$ operatorni shunday tanlaymizki, natijada ushbu

$$\Lambda = \sum_{n=0}^{\infty}\left(\left|\lambda_{n,1} - \tilde{\lambda}_{n,1}\right| + \left|\lambda_{n,2} - \tilde{\lambda}_{n,2}\right|\right) < \infty, \tag{2.11.5}$$

qator yaqinlashuvchi bo'lsin. Λ soniga $\lambda_{n,j}$ va $\tilde{\lambda}_{n,j}$ spektrlarning yaqinligini xarakterlovchi miqdor deb ataladi. Bu yerda $\tilde{\lambda}_{n,j}$ sonlar ketma-ketligi $\tilde{L}_j$ operatorning xos qiymatlaridan iborat bo'lib, ular ushbu

$$\sqrt{\tilde{\lambda}_{n,1}} = n + \frac{1}{2} + \frac{\tilde{a}}{\left(n+\frac{1}{2}\right)} + \frac{\tilde{\chi}_{n,1}}{n}, \quad \sqrt{\tilde{\lambda}_{n,2}} = n + \frac{\tilde{a}}{n} + \frac{\tilde{\chi}_{n,2}}{n}, \quad \{\tilde{\chi}_{n,j}\} \in l_2$$

asimptotikalarni qanoatlantiradi. Bu yerda

$$\tilde{a} = \frac{1}{\pi}\left(\tilde{h} + \frac{1}{2}\int_0^{\pi} \tilde{q}(t)dt\right).$$

(2.11.5) shartdan $\tilde{a} = a$ kelib chiqadi.

L_2 operatorning normallovchi o'zgarmaslar ketma-ketligini

$$\alpha_n = \int_0^{\pi} \varphi^2(x, \lambda_{n,2})dx, \quad n \geq 0$$

orqali belgilab olamiz.

$\tilde{L}_2$ operatorning normallovchi o'zgarmaslar ketma-ketligini esa

$$\tilde{\alpha}_n = \int_0^{\pi} \tilde{\varphi}^2(x, \tilde{\lambda}_{n,2})dx, \quad n \geq 0$$

orqali belgilaymiz. Bu yerda $\tilde{\varphi}(x, \lambda)$ funksiya quyidagi

$$-y'' + \tilde{q}(x)y = \lambda y, \;\; y(0) = 1, \;\; y'(0) = \tilde{h},$$

Koshi masalasining yechimi. U holda $\tilde{L}_j$ operatorning $\tilde{\lambda}_{nj}$ xos qiymatlari
$\tilde{\Delta}_j(\lambda) = \tilde{\varphi}^{(j-1)}(\pi, \lambda)$ xarakteristik funksiyaning nolaridan iborat bo'ladi.

Avvalo quyidagi yordamchi lemmalarni isbotlaymiz.

Lemma 2.11.1. *Shunday δ_1 soni topilib, $\Lambda < \delta_1$ bo'lsa, u holda*

$$\sum_{n=0}^{\infty} |\alpha_n - \tilde{\alpha}_n| < c\Lambda, \tag{2.11.6}$$

baholash o'rinli bo'ladi.

Isbot. L_2 operatorning α_n normallovchi o'zgarmaslari uchun ushbu

$$\alpha_n = -\dot{\Delta}_2(\lambda_{n,2})\Delta_1(\lambda_{n,2}), \;\; n \geq 0, \;\; \dot{\Delta}_2(\lambda) = \frac{d}{d\lambda}\Delta_2(\lambda), \tag{2.11.7}$$

tengliklarning o'rinli ekanligi oldingigi paragraflarda ko'rsatilgan edi. Bu yerda $\Delta_j(\lambda)$ yarim tartibli λ ning butun funksiyasi bo'lgani uchun Adamar teoremasiga ko'ra uni

$$\Delta_j(\lambda) = B_j \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{k,j}}\right), \quad j = 1, 2, \tag{2.11.8}$$

ko'rinishda yozish mumkin. Hisoblashlarni soddalashtirish maqsadida, $\lambda = 0$ soni L_j, $j = 1, 2$; operatorlarning xos qiymati emas deb hisoblaymiz. Bundan foydalanib, quyidagi

$$\frac{\tilde{\Delta}_j(\lambda)}{\Delta_j(\lambda)} = \frac{\tilde{B}_j}{B_j} \prod_{k=0}^{\infty} \frac{\lambda_{k,j}}{\tilde{\lambda}_{k,j}} \prod_{k=0}^{\infty} \left(1 + \frac{\tilde{\lambda}_{k,j} - \lambda_{k,j}}{\lambda_{k,j} - \lambda}\right),$$

ifodani hosil qilamiz.

Tenglama yechimlarining va xos qiymatlarining asimptotikalaridan foydalanib quyidagi limitlarni hisoblash mumkin:

$$\lim_{\lambda \to -\infty} \frac{\tilde{\Delta}_j(\lambda)}{\Delta_j(\lambda)} = 1, \quad \lim_{\lambda \to -\infty} \left(1 + \frac{\tilde{\lambda}_{k,j} - \lambda_{k,j}}{\lambda_{k,j} - \lambda}\right) = 1.$$

Demak,

$$\frac{\tilde{B}_j}{B_j} \prod_{k=0}^{\infty} \frac{\lambda_{k,j}}{\tilde{\lambda}_{k,j}} = 1. \tag{2.11.9}$$

(2.11.8) dan foydalanib

$$\dot{\Delta}_2(\lambda_{n,2}) = -\frac{B_2}{\lambda_{n,2}} \prod_{\substack{k=0 \\ k \neq n}}^{\infty} \left(1 - \frac{\lambda_{n,2}}{\lambda_{k,2}}\right),$$

tenglikni topamiz. Oxirgi tenglikda (2.11.9) inobatga olinsa, undan ushbu

$$\frac{\dot{\tilde{\Delta}}_2(\tilde{\lambda}_{n,2})}{\dot{\Delta}_2(\lambda_{n,2})} = \prod_{\substack{k=0 \\ k \neq n}}^{\infty} \left(\frac{\tilde{\lambda}_{k,2} - \tilde{\lambda}_{n,2}}{\lambda_{k,2} - \lambda_{n,2}}\right),$$

tenglik kelib chiqadi. Endi (2.11.7)-(2.11.9) formulalardan foydalanib,

$$\frac{\tilde{\alpha}_n}{\alpha_n}=\prod_{k=0}^{\infty}\left(\frac{\tilde{\lambda}_{k,1}-\tilde{\lambda}_{n,2}}{\lambda_{k,1}-\lambda_{n,2}}\right)\cdot\prod_{\substack{k=0\\k\neq n}}^{\infty}\left(\frac{\tilde{\lambda}_{k,2}-\tilde{\lambda}_{n,2}}{\lambda_{k,2}-\lambda_{n,2}}\right),$$

yoki

$$\frac{\tilde{\alpha}_n}{\alpha_n}=\prod_{k=0}^{\infty}(1-\theta_{k,n}^{(1)})\cdot\prod_{\substack{k=0\\k\neq n}}^{\infty}(1-\theta_{k,n}^{(2)}),$$

tenglikka ega bo'lamiz. Bu yerda

$$\theta_{k,n}^{(i)}=\frac{\lambda_{k,i}-\tilde{\lambda}_{k,i}}{\lambda_{k,i}-\lambda_{n,2}}+\frac{\tilde{\lambda}_{n,2}-\lambda_{n,2}}{\lambda_{k,i}-\lambda_{n,2}}.$$

Ushbu

$$\theta_n=\sum_{k=0}^{\infty}\left|\theta_{k,n}^{(1)}\right|+\sum_{\substack{k=0\\k\neq n}}^{\infty}\left|\theta_{k,n}^{(2)}\right|,$$

belgilashni kiritaylik. U holda quyidagi

$$\sum_{n=0}^{\infty}\theta_n\leq\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\left|\lambda_{k,1}-\tilde{\lambda}_{k,1}\right|}{|\lambda_{k,1}-\lambda_{n,2}|}+\frac{\left|\tilde{\lambda}_{n,2}-\lambda_{n,2}\right|}{|\lambda_{k,1}-\lambda_{n,2}|}\right)+$$

$$+\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k\neq n}}^{\infty}\left(\frac{\left|\lambda_{k,2}-\tilde{\lambda}_{k,2}\right|}{|\lambda_{k,2}-\lambda_{n,2}|}+\frac{\left|\tilde{\lambda}_{n,2}-\lambda_{n,2}\right|}{|\lambda_{k,2}-\lambda_{n,2}|}\right)\leq$$

$$\leq\sum_{n=0}^{\infty}\left|\lambda_{n,2}-\tilde{\lambda}_{n,2}\right|\left(2\sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{1}{|\lambda_{k,2}-\lambda_{n,2}|}+\sum_{k=0}^{\infty}\frac{1}{|\lambda_{k,1}-\lambda_{n,2}|}\right)+$$

$$+\sum_{n=0}^{\infty}\left|\lambda_{n,1}-\tilde{\lambda}_{n,1}\right|\sum_{k=0}^{\infty}\frac{1}{|\lambda_{n,1}-\lambda_{k,2}|},\qquad(2.11.10)$$

tengsizlikka ega bo'lamiz.

Quyidagi

$$\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{|k^2-n^2|}=\sum_{k=1}^{n-1}\frac{1}{n^2-k^2}+\sum_{k=n+1}^{\infty}\frac{1}{k^2-n^2}\leq$$

$$\leq \frac{1}{2n-1}+\frac{1}{2n+1}-\frac{1}{n^2}+\int\limits_0^{n-1}\frac{dx}{n^2-x^2}+\int\limits_{n+1}^{\infty}\frac{dx}{x^2-n^2}\leq$$

$$\leq \frac{1}{2n-1}+\frac{1}{2n+1}-\frac{1}{n^2}+\frac{1}{2n}\ln(2n-1)+\frac{1}{2n}\ln(2n+1),$$

tengsizlikdan foydalanib, ushbu

$$\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{1}{|k^2-n^2|}<1,\ n\geq 1, \qquad (2.11.11)$$

baholashni topamiz.

Xos qiymatlarning (2.11.3) asimptotikalaridan

$$\frac{1}{\lambda_{k,2}-\lambda_{n,2}}=\frac{1}{k^2-n^2}+\frac{\eta_{k,n}}{(k^2-n^2)(\lambda_{k,2}-\lambda_{n,2})},\quad |\eta_{k,n}|<C,$$

kelib chiqadi. Bunga muvofiq

$$\frac{1}{|\lambda_{k,2}-\lambda_{n,2}|}<\frac{C}{|k^2-n^2|},\quad k\neq n.$$

Oxirgi tengsizlikdan va (2.11.11) dan foydalanib, ushbu

$$\sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{1}{|\lambda_{k,2}-\lambda_{n,2}|}<C, \qquad (2.11.12)$$

tengsizlikka ega bo'lamiz.

Xuddi shuningdek, ushbu

$$\sum_{k=0}^{\infty}\left(\frac{1}{|\lambda_{k,1}-\lambda_{n,2}|}+\frac{1}{|\lambda_{n,1}-\lambda_{k,2}|}\right)<C, \qquad (2.11.13)$$

tengsizilik o'rinli ekanligini ham ko'rsatish mumkin. (2.11.10), (2.11.12) va (2.11.13) tengsizliklardan

$$\sum_{k=0}^{\infty}\theta_n<C\Lambda,$$

baholash kelib chiqadi.

Endi $\delta_1 > 0$ sonni shunday tanlaymizki, agar $\Lambda < \delta_1$ bo'lsa, u holda $\theta_n < \frac{1}{4}$. Ushbu $|\xi| < \frac{1}{2}$ shartni qanoatlantiruvchi barcha ξ larda quyidagi tengsizlik o'rinli:

$$|\ln(1-\xi)| \leq \sum_{k=1}^{\infty} \frac{|\xi|^k}{k} \leq \sum_{k=1}^{\infty} |\xi|^k = \frac{|\xi|}{1-|\xi|} \leq 2|\xi|.$$

Buni e'tiborga olsak, ushbu

$$\left|\ln \frac{\tilde{\alpha}_n}{\alpha_n}\right| \leq \sum_{k=0}^{\infty} \left|\ln(1-\theta_{k,n}^{(1)})\right| + \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \left|\ln(1-\theta_{k,n}^{(2)})\right| < 2\theta_n,$$

baholashga ega bo'lamiz. Logarifmik funksiyaning xossasidan foydalanib,

$$\left|\frac{\tilde{\alpha}_n}{\alpha_n} - 1\right| < 4\theta_n,$$

tengsizlikni topamiz. Bundan esa

$$|\tilde{\alpha}_n - \alpha_n| < C\theta_n,$$

kelib chiqadi. Bu yerda α_n ketma-ketlikning yaqinlashuvchiligi e'tiborga olindi. Yuqoridagi tengsizliklardan

$$\sum_{n=0}^{\infty} |\tilde{\alpha}_n - \alpha_n| < C\sum_{n=0}^{\infty} \theta_n < \tilde{C}\Lambda,$$

kelib chiqadi. ■

Shu bobning 2-paragrafida ushbu

$$\begin{gathered}
\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t\,dt, \\
K(x,x) = h + \frac{1}{2}\int_0^x q(t)dt, \\
\tilde{\varphi}(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x \tilde{K}(x,t)\cos\sqrt{\lambda}t\,dt, \\
\tilde{K}(x,x) = \tilde{h} + \frac{1}{2}\int_0^x \tilde{q}(t)dt
\end{gathered} \tag{2.11.14}$$

integral tasvirlardan foydalanib $K(x,t)$ va $\tilde{K}(x,t)$ yadrolar uchun

$$K(x,t)+F(x,t)+\int_0^x K(x,s)F(s,t)ds=0,\ \ 0<t<x,$$

$$\tilde{K}(x,t)+\tilde{F}(x,t)+\int_0^x \tilde{K}(x,s)\tilde{F}(s,t)ds=0,\ \ 0<t<x$$

Gelfand-Levitan integral tenglamalari keltirib chiqarilgan edi. Bu yerda

$$F(x,t)=\sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n}\cos\sqrt{\lambda_{n,2}}x\cos\sqrt{\lambda_{n,2}}t-\frac{1}{\alpha_n^0}\cos nx\cos nt\right),$$

$$\tilde{F}(x,t)=\sum_{n=0}^{\infty}\left(\frac{1}{\tilde{\alpha}_n}\cos\sqrt{\tilde{\lambda}_{n,2}}x\cos\sqrt{\tilde{\lambda}_{n,2}}t-\frac{1}{\alpha_n^0}\cos nx\cos nt\right),$$

$$\alpha_n^0=\frac{\pi}{2},\ (n\geq 1),\ \ \alpha_0^0=\pi.$$

Bu funksional qatorlar $0\leq x,t\leq\pi$ sohada absolyut va tekis yaqinlashishi ravshan, chunki

$$\sum_{n=0}^{\infty}\left(\left|\sqrt{\lambda_{n,2}}-n\right|+\left|\alpha_n-\frac{\pi}{2}\right|\right)<\infty,$$

$$\sum_{n=0}^{\infty}\left(\left|\sqrt{\tilde{\lambda}_{n,2}}-n\right|+\left|\tilde{\alpha}_n-\frac{\pi}{2}\right|\right)<\infty.$$

Lemma 2.11.2. *Shunday $\delta_2>0$ soni topilib, $\Lambda<\delta_2$ shart bajarilsa, u holda quyidagi*

$$\left|K(x,t)-\tilde{K}(x,t)\right|<C\Lambda,\ \ 0\leq x\leq t\leq\pi, \tag{2.11.15}$$

$$|\varphi'(\pi,\lambda_{n,1})\tilde{\varphi}(\pi,\lambda_{n,1})|<C\left|\lambda_{n,1}-\tilde{\lambda}_{n,1}\right|, \tag{2.11.16}$$

$$\left|\varphi'(\pi,\tilde{\lambda}_{n,2})\tilde{\varphi}(\pi,\tilde{\lambda}_{n,2})\right|<C\left|\lambda_{n,2}-\tilde{\lambda}_{n,2}\right|, \tag{2.11.17}$$

tengsizliklar o'rinli bo'ladi.

Isbot. Ushbu

$$\hat{F}(x,t)=\sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n}\cos\sqrt{\lambda_{n,2}}x\cos\sqrt{\lambda_{n,2}}t-\right.$$
$$\left.-\frac{1}{\tilde{\alpha}_n}\cos\sqrt{\tilde{\lambda}_{n,2}}x\cos\sqrt{\tilde{\lambda}_{n,2}}t\right),$$

yoki

$$\hat{F}(x,t)=\frac{1}{2}\sum_{n=0}^{\infty}\left(b_n(x+t)-b_n(x-t)\right),$$

funksiyani baholaymiz. Bu yerda

$$b_n(\xi)=\frac{1}{\alpha_n}\cos\sqrt{\lambda_{n,2}}\xi-\frac{1}{\tilde{\alpha}_n}\cos\sqrt{\tilde{\lambda}_{n,2}}\xi.$$

Faraz qilaylik, $\Lambda<\delta_1$ bo'lsin, u holda

$$|b_n(\xi)|\le\left|\left(\frac{1}{\alpha_n}-\frac{1}{\tilde{\alpha}_n}\right)\cos\sqrt{\tilde{\lambda}_{n,2}}\xi\right|+$$
$$+\frac{1}{|\alpha_n|}\left|\cos\sqrt{\tilde{\lambda}_{n,2}}\xi-\cos\sqrt{\lambda_{n,2}}\xi\right|\le$$
$$\le C\left(|\tilde{\alpha}-\tilde{\alpha}_n|+\left|\sqrt{\lambda_{n,2}}-\sqrt{\tilde{\lambda}_{n,2}}\right|\right),$$

baholash o'rinli bo'ladi. Bu yerda lemma 2.11.1 dan foydalansak,

$$\left|\hat{F}(x,t)\right|<C\Lambda,$$

ega bo'lamiz. Bundan va Gelfand-Levitan integral tenglamasining yagona yechimi mavjudligidan (2.11.15) kelib chiqadi.

(2.11.14) integral tasvirga asosan

$$\varphi'(\pi,\lambda_{n,1})=-\sqrt{\lambda_{n,1}}\sin\sqrt{\lambda_{n,1}}\pi+K(\pi,\pi)\cos\sqrt{\lambda_{n,1}}\pi+$$
$$+\int_0^{\pi}K'_x(\pi,t)\cos\sqrt{\lambda_{n,1}}t\,dt,$$

$$\hat{\varphi}(\pi,\lambda_{n,1})=\tilde{\varphi}(\pi,\lambda_{n,1})-\tilde{\varphi}(\pi,\tilde{\lambda}_{n,1})=\cos\sqrt{\lambda_{n,1}}\pi-\cos\sqrt{\tilde{\lambda}_{n,1}}\pi+$$
$$+\int_0^{\pi}\tilde{K}(\pi,t)(\cos\sqrt{\lambda_{n,1}}t-\cos\sqrt{\tilde{\lambda}_{n,1}}t)dt \qquad (2.11.18)$$

kelib chiqadi. $\varphi(x,\lambda)$ yechimning va $\sqrt{\lambda_{n,1}}$ larning asimptotikasidan foydalanib,

$$|\varphi'(\pi,\lambda_{n,1})| < C,$$

baholashni olish mumkin. Agar $\Lambda < \delta_2$ bo'lsa, u holda (2.11.15) baholash va (2.11.18) tasvirga asosan

$$|\tilde{\varphi}(\pi,\lambda_{n,1})| < C\left|\sqrt{\lambda_{n,1}} - \sqrt{\tilde{\lambda}_{n,1}}\right|,$$

tengsizlikni keltirib chiqarish mumkin. Bundan esa (2.11.16) kelib chiqadi. Xuddi shuningdek, (2.11.17) tengsizlikni ham isbotlash mumkin. ■

Lemma 2.11.3. *$g(x) \in C[0,\pi]$ uzluksiz funksiya va $\sum\limits_{n=0}^{\infty}|z_n - n| < \infty$ shartni qanoatlantiruvchi $\{z_n\}_{n=0}^{\infty}$ har xil sonlardan tuzilgan ketma-ketlik berilgan bo'lsin. Quyidagi*

$$\varepsilon_n = \int_0^\pi g(x)\cos z_n x dx,$$

sonlar uchun

$$\theta = \sum_{n=0}^{\infty}|\varepsilon_n| < \infty,$$

shart bajarilsa, u holda

$$|g(x)| < M\theta,$$

baholash o'rinli bo'ladi. Bu yerda M o'zgarmas son faqat $\{z_n\}_{n=0}^{\infty}$ to'plamga bog'liq.

Isbot. $\{\cos z_n x\}_{n=0}^{\infty}$ funksiyalar sistemasi $L^2[0,\pi]$ fazoda to'la bo'lgani uchun, ε_n koeffitsiyentlar $g(x)$ funksiyani yagona aniqlaydi. Quyidagi

$$\int_0^\pi g(x)\cos nx dx = \varepsilon_n + \int_0^\pi g(x)(\cos nx - \cos z_n x)dx,$$

tenglikdan

$$g(x) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\alpha_n^0} \cos nx + \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \cos nx \int_0^{\pi} g(t)(\cos nt - \cos z_n t) dt,$$

kelib chiqadi. Oxirgi tenglikdan $g(x)$ funksiya ushbu

$$g(x) = \varepsilon(x) + \int_0^{\pi} H(x,t) g(t) dt, \qquad (2.11.19)$$

integral tenglamaning yechimi ekanligi ko'rinadi. Bu yerdagi ushbu

$$\varepsilon(x) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\alpha_n^0} \cos nx, \qquad H(x,t) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \cos nx (\cos nt - \cos z_n t),$$

funksional qatorlar $0 \leq x, t \leq \pi$ sohada absolyut va tekis yaqinlashuvchi bo'lib,

$$|\varepsilon(x)| < \frac{2}{\pi}\theta, \qquad |H(x,t)| < C \sum_{n=0}^{\infty} |z_n - n|,$$

baholashlar o'rinli bo'ladi.

Endi ushbu

$$y(x) = \int_0^{\pi} H(x,t) y(t) dt, \quad y(x) \in C[0,\pi], \qquad (2.11.20)$$

bir jinsli integral tenglama faqat nol yechimga ega ekanligini ko'rsatamiz. Haqiqatan ham,

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \cos nx \int_0^{\pi} (\cos nt - \cos z_n t) y(t) dt,$$

tenglikdan ushbu

$$\int_0^{\pi} y(x) \cos nx dx = \int_0^{\pi} (\cos nt - \cos z_n t) y(t) dt,$$

yoki

$$\int_0^\pi y(x)\cos z_n x dx = 0,$$

hosil bo'ladi. Oxirgi tenglikdan esa $y(x) \equiv 0$ ekanligi kelib chiqadi. Shunday qilib, (2.11.19) integral tenglama yagona yechimga ega va undan

$$|g(x)| < M\theta,$$

baholash kelib chiqadi. ■

Izoh 2.11.1. Lemma 2.11.3 isbotidan ko'rinadiki, agar yetarlicha kichik $\delta > 0$ son uchun $\{\tilde{z}_n\}_{n=0}^{\infty}$ sonlar ketma-ketligini ushbu

$$\sum_{n=0}^{\infty} |z_n - \tilde{z}_n| < \delta$$

tengsizlikni qanoatlantiradigan qilib tanlasak, u holda M soni $\{\tilde{z}_n\}_{n=0}^{\infty}$ ketma-ketlikning tanlanishiga bog'liq bo'lmaydi.

Teorema 2.11.1. *Shunday $\delta > 0$ soni topiladiki, bunda agar $\Lambda < \delta$ shart bajarilsa, u holda*

$$\|q - \tilde{q}\|_{C[0,\pi]} = \max_{0\le x\le\pi} |q(x) - \tilde{q}(x)| < C\Lambda, \quad \left|h - \bar{h}\right| < C\Lambda, \tag{2.11.21}$$

baholashlar o'rinli bo'ladi.

Isbot. Quyidagi

$$-\varphi''(x,\lambda) + q(x)\varphi(x,\lambda) = \lambda\varphi(x,\lambda),$$
$$-\tilde{\varphi}''(x,\lambda) + \tilde{q}(x)\tilde{\varphi}(x,\lambda) = \lambda\tilde{\varphi}(x,\lambda),$$

tengliklardan birinchisini $\tilde{\varphi}(x,\lambda)$ ga, ikkinchisini esa $\varphi(x,\lambda)$ ga ko'paytirib, hosil bo'lgan ifodalarning ikkinchisidan birinchisini ayiramiz va uni 0 dan π gacha integrallab, ushbu

$$\int_0^\pi \hat{q}(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx = \varphi'(\pi,\lambda)\tilde{\varphi}(\pi,\lambda) - \varphi(\pi,\lambda)\tilde{\varphi}'(\pi,\lambda) + \bar{h} - h, \tag{2.11.22}$$

tenglikni hosil qilamiz. Bu yerda

$$\hat{q}(x) = q(x) - \tilde{q}(x).$$

Ushbu

$$\hat{h} + \frac{1}{2}\int_0^\pi \hat{q}(x)dx = 0, \quad \hat{h} = h - \tilde{h}, \tag{2.11.23}$$

tengliklarni e'tiborga olib, (2.11.22) ifodani quyidagi

$$\int_0^\pi \hat{q}(x)\left(\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) - \frac{1}{2}\right)dx =$$

$$= \varphi'(\pi,\lambda)\tilde{\varphi}(\pi,\lambda) - \varphi(\pi,\lambda)\tilde{\varphi}'(\pi,\lambda), \tag{2.11.24}$$

ko'rinishda yozish mumkin. $\varphi(x,\lambda)$ va $\tilde{\varphi}(x,\lambda)$ yechimlarning (2.11.14) integral tasvirlaridan foydalanib, $\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) - \frac{1}{2}$ ayirmani quyidagi tarzda yozib olamiz:

$$\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) - \frac{1}{2} =$$

$$= \frac{1}{2}\cos 2\sqrt{\lambda}x + \int_0^x [K(x,t) + \tilde{K}(x,t)]\cos\sqrt{\lambda}x\cos\sqrt{\lambda}t dt +$$

$$+ \int_0^x\int_0^x [K(x,t)\tilde{K}(x,s)]\cos\sqrt{\lambda}t\cos\sqrt{\lambda}s dt ds.$$

$K(x,t)$ va $\tilde{K}(x,t)$ funksiyalarni $K(x,-t) = K(x,t)$, $\tilde{K}(x,-t) = \tilde{K}(x,t)$ formulalar bo'yicha davom qildirsak, oxirgi tenglik

$$\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) - \frac{1}{2} =$$

$$= \frac{1}{2}\cos 2\sqrt{\lambda}x + \frac{1}{2}\int_{-x}^x [K(x,t) + \tilde{K}(x,t)]\cos\sqrt{\lambda}(x-t)dt +$$

$$+ \frac{1}{4}\int_{-x}^x\int_{-x}^x [K(x,t)\tilde{K}(x,s)]\cos\sqrt{\lambda}(t-s)dt ds,$$

ko'rinishni oladi. Bu yerdagi birinchi va ikkinchi integrallarda mos ravishda $\tau = \dfrac{x-t}{2}$ va $\tau = \dfrac{s-t}{2}$ almashtirish bajarib,

$$\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) - \frac{1}{2} = \frac{1}{2}\left\{ \cos 2\sqrt{\lambda}x + 2\int_0^x [K(x,x-2\tau)+ \right.$$

$$+\tilde{K}(x,x-2\tau)\,]\cos 2\sqrt{\lambda}\tau d\tau +$$

$$\left. + \int_{-x}^{x} \tilde{K}(x,s)\int_{\frac{s-x}{2}}^{\frac{s+x}{2}} K(x,s-2\tau)\cos 2\sqrt{\lambda}\tau d\tau ds \right\},$$

tenglikni hosil qilamiz.

$$\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) - \frac{1}{2} = \frac{1}{2}\left[\cos 2\sqrt{\lambda}x + \int_0^x V(x,\tau)\cos 2\sqrt{\lambda}\tau d\tau\right];$$

(2.11.25)

tasvir hosil bo'ladi. Bu yerda

$$V(x,\tau) = 2[K(x,x-2\tau) + \tilde{K}(x,x-2\tau)]+$$

$$+\int_{2\tau-x}^{x} \tilde{K}(x,s)K(x,s-2\tau)ds + \int_{-x}^{x-2\tau} \tilde{K}(x,s)K(x,s+2\tau)ds.$$

(2.11.26)

Bu ifodadan foydalanib va (2.11.15) tengsizlikni e'tiborga olib, $\Lambda < \delta_2$, $\delta_2 > 0$ bo'lganda

$$|V(x,t)| < C,$$

bo'lishini ko'rsatish mumkin. Endi (2.11.25) ni (2.11.24) tenglikka qo'yib, hosil bo'lgan ifodani $\lambda = \lambda_{n,1}$ va $\lambda = \tilde{\lambda}_{n,2}$ bo'lganda hisoblasak,

$$\int_0^\pi g(x)\cos 2\sqrt{\lambda_{n,1}}x dx = \varphi'(\pi,\lambda_{n,1})\tilde{\varphi}(\pi,\lambda_{n,1}),$$

$$\int_0^\pi g(x)\cos 2\sqrt{\tilde{\lambda}_{n,2}}x dx = \varphi'(\pi, \tilde{\lambda}_{n,2})\tilde{\varphi}(\pi, \tilde{\lambda}_{n,2}),$$

tengliklar hosil bo'ladi. Bu yerda

$$g(x) = 2\left(\hat{q}(x) + \int_x^\pi V(t,x)\hat{q}(t)dt\right). \tag{2.11.27}$$

Quyidagi

$$z_{2n+1} = 2\sqrt{\lambda_{n,1}}, \quad z_{2n} = 2\sqrt{\tilde{\lambda}_{n,2}},$$

$$\varepsilon_{2n+1} = \varphi'(\pi, \lambda_{n,1})\tilde{\varphi}(\pi, \lambda_{n,1}), \quad \varepsilon_{2n} = \varphi'(\pi, \tilde{\lambda}_{n,2})\tilde{\varphi}(\pi, \tilde{\lambda}_{n,2}),$$

belgilashlarni kiritib, lemma 2.11.3 dan foydalansak, $\Lambda < \delta_2$, $\delta_2 > 0$ bo'lganda

$$|g(x)| < C\Lambda,$$

baholash hosil bo'ladi. $\hat{q}(x)$ funksiya (2.11.27) Volterra integral tenglamasining yechimi bo'lgani uchun

$$|\hat{q}(x)| < C\Lambda,$$

tengsizlik o'rinli bo'ladi. Bunga va (2.11.23) tenglikka asosan $\left|\hat{h}\right| < C\Lambda$ kelib chiqadi. ■

III BOB. YARIM O'QDA BERILGAN SHTURM-LIUVILL CHEGARAVIY MASALASI

1-§. Parseval tengligi

Quyidagi

$$-y'' + q(x)y = \lambda y, \qquad 0 \le x < \infty, \tag{3.1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0 \tag{3.1.2}$$

Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \infty)$ haqiqiy funksiya, α berilgan haqiqiy son va λ kompleks parametr.

$\varphi(x, \lambda)$ orqali (3.1.1) tenglamaning

$$y(0) = \sin\alpha, \qquad y'(0) = -\cos\alpha, \tag{3.1.3}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

$f(x) \in L^2(0, \infty)$ ixtiyoriy haqiqiy funksiya va b - ixtiyoriy musbat son bo'lsin. Quyidagi regulyar chegaraviy masalaning

$$-y'' + q(x)y = \lambda y, \qquad (0 \le x \le b), \tag{3.1.4}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{3.1.5}$$

$$y(b)\cos\beta + y'(b)\sin\beta = 0, \tag{3.1.6}$$

xos qiymatlari va xos funksiyalarini mos ravishda $\lambda_{n,b}$ va $\varphi_{n,b}(x) = \varphi(x, \lambda_{n,b})$, $n = 0, 1, 2, \ldots$ orqali belgilaymiz. Normallovchi o'zgarmaslar ketma-ketligi quyidagi

$$\alpha_{n,b}^2 = \int_0^b \varphi_{n,b}^2(x)dx, n = 0, 1, 2, \ldots, \tag{3.1.7}$$

tengliklardan aniqlanadi.

Chekli oraliqdagi (3.1.4)-(3.1.6) chegaraviy masala uchun Parseval tengligi ushbu

$$\int_0^b f^2(x)dx = \sum_{n=0}^{\infty} \frac{1}{\alpha_{n,b}^2} \left\{ \int_0^b f(x)\varphi_{n,b}(x)dx \right\}^2 \tag{3.1.8}$$

ko'rinishda bo'ladi.

Yuqoridagi yig'indini Stiltes integrali orqali ifodalaymiz. Buning uchun quyidagi

$$\rho_b(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_{n,b}\le 0} \dfrac{1}{\alpha_{n,b}^2}, & \lambda \le 0, \\ \sum\limits_{0<\lambda_{n,b}\le\lambda} \dfrac{1}{\alpha_{n,b}^2}, & \lambda > 0 \end{cases} \tag{3.1.9}$$

funksiyani kiritamiz.

Stiltes integralining ta'rifiga ko'ra, (3.1.8) tenglik quyidagi ko'rinishni oladi

$$\int_0^b f^2(x)dx = \int_0^b F^2(\lambda)d\rho_b(\lambda), \tag{3.1.10}$$

bu yerda

$$F(\lambda) = \int_0^b f(x)\varphi(x,\lambda)dx\,, \tag{3.1.11}$$

belgilash kiritilgan.

(3.1.10) tenglikda cheksizlikka intiluvchi b_k ketma-ketlik bo'yicha limitga o'tish mumkin ekanligini ko'rsatamiz. Buning uchun avvalo quyidagi lemmalarni isbotlaymiz.

Lemma 3.1.1. *Ushbu*

$$P = \{(x,\lambda)|0 \le x \le a,\ -N \le \lambda \le N\}$$

soha berilgan bo'lsin. U holda ixtiyoriy $\varepsilon > 0$ son uchun shunday $h = h_N(\varepsilon) > 0$ son topilib, $0 \le x \le h$, $|\lambda| \le N$ shartlarni *qanoatlantiruvchi barcha (x, λ) nuqtalar uchun*

$$|\varphi(x,\lambda) - \sin\alpha| < \varepsilon, \qquad |\varphi'(x,\lambda) + \cos\alpha| < \varepsilon, \tag{3.1.12}$$

tengsizliklar bajariladi.

Isbot. $\varphi(x, \lambda)$ va $\varphi'(x, \lambda)$ funksiyalar P to‘g‘ri to‘rtburchakda tekis uzluksiz, ya’ni ixtiyoriy $\varepsilon > 0$ uchun shunday $h = h_N(\varepsilon) > 0$ topiladiki, barcha (x, λ), $(\tilde{x}, \tilde{\lambda}) \in P$, $|x - \tilde{x}| \le h$, $|\lambda - \tilde{\lambda}| \le h$ nuqtalar uchun

$$\left|\varphi(x,\lambda) - \varphi(\tilde{x},\tilde{\lambda})\right| < \varepsilon, \qquad \left|\varphi'(x,\lambda) - \varphi'(\tilde{x},\tilde{\lambda})\right| < \varepsilon,$$

tengsizliklar o‘rinli bo‘ladi. Bu tengsizliklarda $\tilde{x} = 0$, $\tilde{\lambda} = \lambda$ desak va boshlang‘ich shartlarni e’tiborga olsak lemma isbotlanadi.■

Natija 3.1.1. Agar $\sin\alpha \neq 0$ bo‘lsa, u holda shunday $h = h_N > 0$ son mavjud bo‘lib,

$$\left(\frac{1}{h}\int_0^h \varphi(x,\lambda)dx\right)^2 > \frac{1}{4}\sin^2\alpha, \tag{3.1.13}$$

tengsizlik bajariladi. Haqiqatan ham, $\sin\alpha > 0$ bo‘lgan holda lemma 3.1.1 da $\varepsilon = \frac{1}{2}\sin\alpha$ deylik. U holda

$$-\varepsilon < \varphi(x,\lambda) - \sin\alpha < \varepsilon,$$

$$-\frac{1}{2}\sin\alpha < \varphi(x,\lambda) - \sin\alpha < \frac{1}{2}\sin\alpha,$$

$$\varphi(x,\lambda) > \frac{1}{2}\sin\alpha,$$

$$\frac{1}{h}\int_0^h \varphi(x,\lambda)dx > \frac{1}{2}\sin\alpha > 0,$$

bo‘lgani uchun (3.1.13) tengsizlik o‘rinli bo‘ladi.

$\sin\alpha < 0$ bo‘lgan holda lemma 3.1.1 da $\varepsilon = -\frac{1}{2}\sin\alpha$ deylik. U holda

$$\frac{1}{h}\int_0^h \varphi(x;\lambda)dx < \frac{1}{2}\sin\alpha;$$

$$-\frac{1}{h}\int_0^h \varphi(x,\lambda)dx > -\frac{1}{2}\sin\alpha > 0,$$

bo'lgani uchun (3.1.13) tengsizlik bajariladi.

Natija 3.1.2. Agar $\sin\alpha = 0$ bo'lsa, u holda shunday $h = h_N > 0$ mavjud bo'lib,

$$\left(\frac{1}{h^2}\int_0^h \varphi(x,\lambda)dx\right)^2 > \frac{1}{16}, \tag{3.1.14}$$

tengsizlik o'rinli bo'ladi. Haqiqatan ham, lemma 3.1.1 da $\varepsilon = \frac{1}{2}$ desak, u holda

$$-\frac{1}{2} < \varphi'(x,\lambda) + 1 < \frac{1}{2}, \quad -\varphi'(x,\lambda) > \frac{1}{2},$$

$$\varphi'(x,\lambda)(x-h) > \frac{1}{2}(h-x),$$

$$\int_0^h \varphi'(x,\lambda)(x-h)dx > \frac{1}{2}\int_0^h (h-x)dx,$$

$$\int_0^h (x-h)d\varphi(x,\lambda) > \frac{1}{2}\left(hx - \frac{x^2}{2}\right)\Big|_0^h,$$

$$(x-h)\varphi(x,\lambda)|_0^h - \int_0^h \varphi(x,\lambda)dx > \frac{1}{4}h^2,$$

$$-\frac{1}{h^2}\int_0^h \varphi(x,\lambda)dx > \frac{1}{4},$$

bo'lgani uchun (3.1.14) tengsizlik bajariladi.■

Lemma 3.1.2. *Ixtiyoriy musbat N soni uchun b ga bog'liq bo'lmagan $A = A(N) > 0$ soni topilib,*

$$\mathop{V}_{-N}^{N}\{\rho_b(\lambda)\} = \sum_{-N<\lambda_{n,b}\leq N} \frac{1}{\alpha_{n,b}^2} = \rho_b(N) - \rho_b(-N) < A,$$

tengsizlik bajariladi, ya'ni $\rho_b(\lambda)$ funksiyalarning o'zgarishlari b ga nisbatan tekis chegaralangan bo'ladi.

Isbot. Ikkita holni ko'rib chiqamiz:

$\sin\alpha \neq 0$ bo'lgan holda (3.1.10) Parseval tengligini

$$f(x) = \begin{cases} \frac{1}{h}, & 0 \leq x \leq h, \\ 0, & x > h, \end{cases}$$

funksiyaga qo'llaymiz, bu yerda h soni natija 3.1.1 dan olindi:

$$\int_0^h f^2(x)dx = \int_0^h \frac{1}{h^2}dx = \frac{1}{h} = \int_{-\infty}^{\infty} \left\{ \frac{1}{h}\int_0^h \varphi(x,\lambda)dx \right\}^2 d\rho_b(\lambda) \geq$$

$$\geq \int_{-N}^{N} \left\{ \frac{1}{h}\int_0^h \varphi(x,\lambda)dx \right\}^2 d\rho_b(\lambda) \geq$$

$$\geq \frac{1}{4}\sin^2\alpha \int_{-N}^{N} d\rho_b(\lambda) = \frac{1}{4}\sin^2\alpha[\rho_b(N) - \rho_b(-N)],$$

ya'ni quyidagi

$$\rho_b(N) - \rho_b(-N) \leq \frac{4}{h\sin^2\alpha},$$

baholash o'rinli.

$\sin\alpha = 0$ bo'lgan holda (3.1.10) Parseval tengligini ushbu

$$f(x) = \begin{cases} \frac{1}{h^2}, & 0 \leq x \leq h, \\ 0, & x > h, \end{cases}$$

funksiyaga qo'llaymiz, bu yerda h soni natija 3.1.2 dan olingan:

$$\int_0^h f^2(x)dx = \int_0^h \frac{1}{h^4}dx = \frac{1}{h^3} = \int_{-\infty}^{\infty} \left\{ \frac{1}{h^2}\int_0^h \varphi(x,\lambda)dx \right\}^2 d\rho_b(\lambda) \geq$$

$$\geq \int_{-N}^{N} \left\{ \frac{1}{h^2} \int_0^h \varphi(x,\lambda)dx \right\}^2 d\rho_b(\lambda) \geq$$

$$\geq \frac{1}{16} \int_{-N}^{N} d\rho_b(\lambda) = \frac{1}{16}[\rho_b(N) - \rho_b(-N)] \ .$$

Bundan, esa

$$\rho_b(N) - \rho_b(-N) \leq \frac{16}{h^3},$$

kelib chiqadi. ■

Natija 3.1.3. $\{\rho_b(\lambda)\}$ funksiyalar to'plami $[-N, N]$ kesmada b ga nisbatan tekis chegaralangan to'plam. Haqiqatan ham,

$$\rho_b(\lambda) - \rho_b(-\lambda) \leq A(N), \quad \lambda \in [0, \ N],$$

bo'lgani uchun

$$0 \leq \rho_b(\lambda) \leq A(N) + \rho_b(-\lambda) \leq A(N),$$

tengsizlik o'rinli bo'ladi. $\lambda \in [-N, 0]$ bo'lsa, $-\rho(\lambda) \leq \rho(-\lambda) - \rho(\lambda) \leq A(N)$ bo'lgani uchun $-A(N) \leq \rho(\lambda) \leq 0$ bo'ladi.

Teorema 3.1.1. (*Xellining birinchi teoremasi.*) *Chegaralangan kesmada aniqlangan tekis chegaralangan funksiyalar sinfining variatsiyalari ham tekis chegaralangan bo'lsa, bu sinfdan yaqinlashuvchi funksiyalar ketma-ketligini tanlash mumkin.*

Natija 3.1.4. Lemma 3.1.2 va uning natijasiga ko'ra $\rho_b(\lambda)$, $\lambda \in [-N, \ N]$ funksiyalar sinfidan $\rho_{b_k}(\lambda) \to \rho(\lambda)$ yaqinlashuvchi ketma-ketlik tanlash mumkin.

Teorema 3.1.2. (*Xellining ikkinchi teoremasi.*) *Chegaralangan kesmada aniqlangan yaqinlashuvchi funksiyalar ketma-ketligining variatsiyalari ham tekis chegaralangan bo'lsa, u holda limitik funksiyaning variatsiyasi chegaralangan bo'ladi va har* *qanday $F(\lambda)$ uzluksiz funksiya uchun*

$$\lim_{k\to\infty}\int\limits_{-N}^{N} F(\lambda)d\rho_{b_k}(\lambda) = \int\limits_{-N}^{N} F(\lambda)d\rho(\lambda);$$

tenglik o'rinli bo'ladi.

Lemma 3.1.3. *Quyidagi*

$$f_n(x) \in C^2[0,\infty), \quad f_n(x) \equiv 0, \quad x > n, \ (n < b).$$

$$f_n(0)\cos\alpha + f_n'(0)\sin\alpha = 0,$$

shartlarni qanoatlantiruvchi har qanday $f_n(x)$ funksiya uchun

$$\left|\int\limits_0^n f_n^2(x)dx - \int\limits_{-N}^{N} F_n^2(\lambda)d\rho_b(\lambda)\right| \leq \frac{1}{N^2}\int\limits_0^n [f_n''(x) - q(x)f_n(x)]^2 dx, \tag{3.1.15}$$

tengsizlik bajariladi.

Isbot. (3.1.10) Parseval tengligiga binoan

$$\int\limits_0^n f_n^2(x)dx = \int\limits_{-\infty}^{\infty} F_n^2(\lambda)d\rho_b(\lambda), \tag{3.1.16}$$

bu yerda

$$F_n(\lambda) = \int\limits_0^n f_n(x)\varphi(x,\lambda)dx\ .$$

Bo'laklab integrallash natijasida

$$F_n(\lambda) = -\frac{1}{\lambda}\int\limits_0^b f_n(x)[\varphi''(x,\lambda) - q(x)\varphi(x,\lambda)]dx =$$

$$= -\frac{1}{\lambda}\int\limits_0^b \varphi(x,\lambda)[f_n''(x) - q(x)f_n(x)]dx, \tag{3.1.17}$$

tenglik hosil bo'ladi.

(3.1.16) va (3.1.17) tengliklarga ko'ra

$$\left|\int_0^n f_n^2(x)dx - \int_{-N}^{N} F_n^2(\lambda)d\rho_b(\lambda)\right| = \left|\int_{|\lambda|>N} F_n^2(\lambda)d\rho_b(\lambda)\right| =$$

$$= \left|\int_{|\lambda|>N} \frac{1}{\lambda^2}\left\{\int_0^b \varphi(x,\lambda)[f_n''(x) - q(x)f_n(x)]dx\right\}^2 d\rho_b(\lambda)\right| \leq$$

$$\leq \frac{1}{N^2}\int_{|\lambda|>N} \left\{\int_0^b \varphi(x,\lambda)[f_n''(x) - q(x)f_n(x)]dx\right\}^2 d\rho_b(\lambda) \leq$$

$$\leq \frac{1}{N^2}\int_{-\infty}^{\infty} \left\{\int_0^b \varphi(x,\lambda)[f_n''(x) - q(x)f_n(x)]dx\right\}^2 d\rho_b(\lambda) =$$

$$= \frac{1}{N^2}\int_0^n [f_n''(x) - q(x)f_n(x)]^2 dx \;.\blacksquare$$

Xellining ikkinchi teoremasiga asoslanib, (3.1.15) tengsizlikda $b_k \to \infty$ ketma-ketlik bo'yicha limitga o'tish mumkin:

$$\left|\int_0^n f_n^2(x)dx - \int_{-N}^{N} F_n^2(\lambda)d\rho(\lambda)\right| \leq \frac{1}{N^2}\int_0^n [f_n''(x) - q(x)f_n(x)]^2 dx. \tag{3.1.18}$$

(3.1.18) tengsizlikda $N \to \infty$ limitga o'tsak,

$$\int_0^n f_n^2(x)dx = \int_{-\infty}^{\infty} F_n^2(\lambda)d\rho(\lambda), \tag{3.1.19}$$

tenglik kelib chiqadi.

(3.1.19) tenglik ixtiyoriy $f(x) \in L^2(0,\infty)$ funksiya uchun o'rinli ekanligini ko'rsatamiz. Funksiyalar nazariyasi kursidan ma'lumki, ixtiyoriy $f(x) \in L^2(0,\infty)$ funksiya uchun quyidagi shartlarni qanoatlantiruvchi $f_n(x) \in C^2[0,\infty)$ ketma-ketlik topiladi:

1) $f_n(x) \equiv 0, \quad x > n,$

2) $f_n(0)\cos\alpha + f_n'(0)\sin\alpha = 0,$

3) $\lim\limits_{n\to\infty} \int\limits_0^\infty [f_n(x) - f(x)]^2 dx = 0.$

Ushbu $f_n(x) - f_m(x)$ funksiyalar uchun (3.1.19) tenglikni yozamiz:

$$\int\limits_{-\infty}^{\infty} [F_n(\lambda) - F_m(\lambda)]^2 d\rho(\lambda) = \int\limits_0^\infty [f_n(x) - f_m(x)]^2 dx\,. \qquad (3.1.20)$$

(3.1.20) tenglikdan

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int\limits_{-\infty}^{\infty} [F_n(\lambda) - F_m(\lambda)]^2 d\rho(\lambda) = 0, \qquad (3.1.21)$$

kelib chiqadi, ya'ni $F_n(\lambda)$ ketma-ketlik $L^2_{\rho(\lambda)}(-\infty,\infty)$ fazoda fundamental ekan. Ushbu fazo to'la bo'lganligi uchun $F_n(\lambda)$ ketma-ketlikning $F(\lambda)$ limiti mavjud bo'ladi. Quyidagi

$$\left| \sqrt{\int\limits_0^\infty f_n^2(x)dx} - \sqrt{\int\limits_0^\infty f^2(x)dx} \right| \le \sqrt{\int\limits_0^\infty [f_n(x) - f(x)]^2 dx},$$

$$\left| \sqrt{\int\limits_{-\infty}^{\infty} F_n^2(\lambda)d\rho(\lambda)} - \sqrt{\int\limits_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda)} \right| \le$$

$$\le \sqrt{\int\limits_{-\infty}^{\infty} [F_n(\lambda) - F(\lambda)]^2 d\rho(\lambda)},$$

tengsizliklarni ishlatib, (3.1.19) tenglikda $n \to \infty$ limitga o'tsak,

$$\int\limits_0^\infty f^2(x)dx = \int\limits_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda), \qquad (3.1.22)$$

tenglik kelib chiqadi.

$\rho_b(\lambda)$ funksiyalar monoton o'suvchi bo'lganligidan, $\rho(\lambda)$ funksiyaning monoton o'suvchiligi kelib chiqadi.

Shunday qilib, biz quyidagi teoremani isbot qildik.

Teorema (*Veyl*). *(3.1.1)+(3.1.2) chegaraviy masala uchun butun o'qda aniqlangan, monoton o'suvchi, chapdan uzluksiz, $\rho(-0) = 0$ shart bilan normallangan shunday $\rho(\lambda)$ funksiya mavjudki, $L^2(0,\infty)$ fazodan olingan ixtiyoriy $f(x)$ funksiya uchun*

$$\int_0^\infty f^2(x)dx = \int_{-\infty}^\infty F^2(\lambda)d\rho(\lambda),$$

tenglik bajariladi. Bu yerda $F(\lambda)$ funksiya

$$F_n(\lambda) = \int_0^n f(x)\varphi(x,\lambda)dx,$$

ketma-ketlikning $L^2_{\rho(\lambda)}(-\infty,\infty)$ fazodagi limitini bildiradi.

(3.1.22) tenglikka yarim o'qda berilgan Shturm-Liuvill chegaraviy masalasi uchun Parseval tengligi deyiladi, $\rho(\lambda)$ funksiyaga esa (3.1.1)+(3.1.2) chegaraviy masalaning spektral funksiyasi deyiladi, $F(\lambda)$ funksiyaga $f(x)$ funksiyaning $\varphi(x,\lambda)$ funksiyalar bo'yicha Furye almashtirishi deyiladi va

$$F(\lambda) = \underset{n\to\infty}{l\cdot i\cdot m}\int_0^n f(x)\varphi(x,\lambda)dx,$$

orqali belgilanadi. Parseval tengligini isbotlashning bu usuliga B.M.Levitan usuli deyiladi.

$\rho(\lambda)$ spektral funksiya biror λ_0 nuqtaning kichik atrofida o'zgarmas bo'lsa, λ_0 ga regulyar nuqta deyiladi, regulyar bo'lmagan nuqtalar to'plamiga spektr deyiladi va E harfi bilan belgilanadi. Ma'lumki, Shturm-Liuvill regulyar masalasining spektri xos qiymatlardan iborat. Biz qarayotgan holda esa, xos qiymatlar spektrning biror qismi bo'ladi xolos. Ular mavjud bo'lmasligi ham mumkin.

Izoh. Spektral funksiya umuman olganda yagona emas. (3.1.1)+(3.1.2) chegaraviy masalaning spektral funksiyasini topish masalasiga to'g'ri masala deyiladi.

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$$\begin{cases} -y'' = \lambda y, & 0 \leq x < \infty, \\ y'(0) = 0, \end{cases}$$

chegaraviy masalaning spektral funksiyasini topamiz.

Dastlab,

$$\begin{cases} -y'' = \lambda y, & 0 \leq x \leq b, \\ y'(0) = 0, \\ y'(b) = 0, \end{cases}$$

regulyar chegarvaviy masalaning xos qiymatlari va ortonormallangan xos funksiyalarini topamiz. Buning uchun berilgan differensial tenglamaning umumiy yechimini topamiz:

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}.$$

Ushbu

$$\varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = 0$$

boshlang'ich shartlarni qanoatlantiruvchi yechim esa

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x,$$

bo'ladi. Bu yechim

$$y'(0) = 0,$$

chegaraviy shartni qanoatlantiradi, ushbu

$$y'(b) = 0,$$

chegaraviy shartdan

$$-\sqrt{\lambda}\sin\sqrt{\lambda}b = 0,$$

xarakteristik tenglama kelib chiqadi. Xarakteristik tenglamadan xos qiymatlarni aniqlaymiz:

$$\lambda_n = \left(\frac{\pi n}{b}\right)^2, \; n = 0,\ 1,\ 2,\ \dots\ .$$

Endi esa normallovchi o'zgarmaslarni topamiz:

$$\alpha_0^2 = \int_0^b 1dx = b,$$

$$\alpha_n^2 = \int_0^b \cos^2\frac{\pi nx}{b}dx = \int_0^b\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2\pi nx}{b}\right)dx = \frac{b}{2},\ n = 1, 2, \dots .$$

Demak, ortonormallangan xos funksiyalar quyidagilardan iborat:

$$u_0(x) = \sqrt{\frac{1}{b}}, \quad u_n(x) = \sqrt{\frac{2}{b}}\cos\frac{\pi nx}{b}, \quad n = 1,\ 2,\ \dots\ .$$

Agar $s_{n,b} = \sqrt{\lambda_n}$ belgilash kiritsak, u holda $s_{n+1,b} - s_{n,b} = \frac{\pi}{b}$ bo'ladi.

$f(x) \in L^2(0,\infty)$ ixtiyoriy funksiya bo'lsin. U holda Parseval tengligi quyidagi ko'rinishda bo'ladi:

$$\int_0^b f^2(x)dx = \sum_{n=-\infty}^{\infty}\frac{1}{b}F_b^2(s_{n,b}^2) = \frac{1}{\pi}\sum_{-\infty<s_{n,b}<\infty} F_b^2(s_{n,b}^2)\cdot\Delta s_{n,b}.$$

Oxirgi tenglikda $b\to\infty$ da limitga o'tsak,

$$\int_0^\infty f^2(x)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} F^2(s^2)ds = \frac{2}{\pi}\int_0^\infty F^2(s^2)ds = \frac{2}{\pi}\int_0^\infty F^2(\lambda)d(\sqrt{\lambda}),$$

tenglik kelib chiqadi. Demak, berilgan masalaning spektral funksiyasi

$$\rho(\lambda) = \begin{cases} \frac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

bo'lib, spektri $E = [0, \infty)$ to'plamdan iborat bo'ladi.

$\lambda \in E = [0, \infty)$ bo'lganda berilgan tenglamaning kamida bitta noldan farqli, chegaralangan yechimi mavjud bo'lib, bu yechim $L^2(0, \infty)$ fazoga qarashli emas, ya'ni bu holda spektr uzluksiz, xos qiymat yoq.

2-§. Yoyilma haqidagi teorema

Ushbu paragrafda avvalo Parsevalning umumlashgan tengligi deb ataluvchi ayniyatni keltirib chiqaramiz, so'ngra bu tenglikdan foydalanib, yoyilma haqidagi teoremani isbotlaymiz.

Kvadrati bilan jamlanuvchi $f(x), g(x) \in L^2(0, \infty)$ haqiqiy funksiyalar berilgan bo'lib,

$$F(\lambda) = \int_0^\infty f(x)\varphi(x, \lambda)dx, \quad G(\lambda) = \int_0^\infty g(x)\varphi(x, \lambda)dx,$$

bo'lsin. U holda $f(x)+g(x)$ va $f(x)-g(x)$ funksiyalarning $\varphi(x, \lambda)$ bo'yicha Furye almashtirishlari mos ravishda $F(\lambda) + G(\lambda)$ va $F(\lambda) - G(\lambda)$ bo'ladi. Parseval tengligiga ko'ra

$$\int_0^\infty [f(x) + g(x)]^2 dx = \int_{-\infty}^\infty [F(\lambda) + G(\lambda)]^2 d\rho(\lambda),$$

$$\int_0^\infty [f(x) - g(x)]^2 dx = \int_{-\infty}^\infty [F(\lambda) - G(\lambda)]^2 d\rho(\lambda),$$

tengliklar o'rinli bo'ladi. Bu tengliklarning birinchisidan ikkinchisini ayirsak,

$$\int_0^\infty f(x)g(x)dx = \int_{-\infty}^\infty F(\lambda)G(\lambda)d\rho(\lambda), \qquad (3.2.1)$$

ayniyat hosil bo'ladi. (3.2.1) ayniyatga umumlashgan Parseval tengligi deyiladi. $f(x) \equiv g(x)$ bo'lgan holda (3.2.1) oddiy Parseval tengligiga aylanadi.

Teorema 3.2.1 (*Yoyilma haqida*). *Agar $f(x) \in L^2(0,\infty)$ funksiya uzluksiz bo'lib,*

$$\int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)d\rho(\lambda), \tag{3.2.2}$$

integral absolyut va har bir chekli oraliqda tekis yaqinlashuvchi bo'lsa, u holda $f(x)$ funksiya uchun

$$f(x) = \int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)d\rho(\lambda), \tag{3.2.3}$$

tasvir o'rinli bo'ladi.

Isbot. (3.2.1) tenglikdagi $g(x)$ funksiya $[0,\infty)$ da uzluksiz bo'lib, $[0,\ n]$ kesmadan tashqarida aynan nolga teng bo'lsin. U holda

$$\int_0^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} F(\lambda)\left\{\int_0^{n} g(x)\varphi(x,\lambda)dx\right\}d\rho(\lambda), \tag{3.2.4}$$

tenglikdagi integral absolyut yaqinlashuvchi bo'lgani uchun integrallash tartibini almashtirish mumkin:

$$\int_0^{\infty} f(x)g(x)dx = \int_0^{n}\left\{\int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)d\rho(\lambda)\right\}g(x)dx,$$

$$\int_0^{n}\left\{f(x) - \int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)d\rho(\lambda)\right\}g(x)dx = 0. \tag{3.2.5}$$

(3.2.2) integral tekis yaqinlashuvchi bo'lgani uchun u x ga nisbatan uzluksiz bo'ladi. (3.2.5) tenglikda qavs ichidagi ifoda uzluksiz bo'lgani va $g(x)$ funksiya ixtiyoriy ekanligidan

$$f(x) - \int_{-\infty}^{\infty} F(\lambda)\varphi(x,\lambda)d\rho(\lambda) = 0,$$

tenglik kelib chiqadi, ya'ni (3.2.3) tasvir o'rinli bo'ladi.■

Misol. Ushbu

$$\begin{cases} -y'' = \lambda y, & 0 \le x < \infty, \\ y'(0) = 0, \end{cases}$$

chegaraviy masala uchun yoyilma teoremasini yozamiz.

Bu masala uchun

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x, \quad \rho(\lambda) = \begin{cases} \frac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

bo'lishini yuqorida ko'rsatgan edik.

$f(x) \in L^2(0,\infty)$ ixtiyoriy uzluksiz finit funksiya bo'lsin. U holda

$$F(\lambda) = \int_{0}^{\infty} f(x)\cos\sqrt{\lambda}x dx, \tag{3.2.6}$$

bo'lib,

$$f(x) = \int_{0}^{\infty} F(\lambda)\cos\sqrt{\lambda}xd\left(\frac{2}{\pi}\sqrt{\lambda}\right)$$

bo'ladi. Oxirgi tenglikni quyidagi ko'rinishda ham yozish mumkin

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left\{\int_{0}^{\infty} f(t)\cos pt dt\right\}\cos px dp. \tag{3.2.7}$$

(3.2.7) tenglikdan

$$f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} H(p)\cos px dp,$$

kelib chiqadi. Bu yerda

$$H(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos pt dt,$$

Furyening kosinus almashtirishidir.

3-§. Veyl doirasi va nuqtasi[3]

Quyidagi

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad 0 \leq x < \infty, \tag{3.3.1}$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0, \tag{3.3.2}$$

chegaraviy masala yordamida hosil bo'lgan L operatorni ko'rib chiqamiz. Bu yerda $q(x) \in C[0, \infty)$ haqiqiy uzluksiz funksiya. Mazkur paragrafda L operatorning rezolventasini topishda kerak bo'ladigan, (3.3.1) tenglamaning $L^2(0, \infty)$ fazoga tegishli yechimlarini o'rganamiz. Buning uchun quyidagi ayniyatlardan foydalanamiz:

1) Agar $F(x)$ va $G(x)$ funksiyalar (3.3.1) tenglamaning $\lambda = \lambda_1$ va $\lambda = \lambda_2$ qiymatlarga mos keluvchi biror yechimlari bo'lsa, u holda

$$(\lambda_1 - \lambda_2) \int_0^b F(x)G(x)dx = \int_0^b [G \cdot (\lambda_1 F) - F \cdot (\lambda_2 G)]dx =$$

$$= \int_0^b [G(-F'' + qF) - F(-G'' + qG)]dx =$$

$$= \int_0^b (FG'' - GF'')dx = (FG' - GF')|_0^b = W_b\{F, G\} - W_0\{F, G\},$$

[3] Bu paragrafni yozishda akademik Sh.A.Alimovning "Математические принципы электроразведки" nomli ma'ruzalar matnidan foydalanildi.

ya'ni

$$(\lambda_1 - \lambda_2)\int\limits_0^b F(x)G(x)dx = W_b\{F,G\} - W_0\{F,G\}, \qquad (3.3.3)$$

tenglik o'rinli bo'ladi.

2) Agar (3.3.3) tenglikda $\lambda_1 = u+iv$, $\lambda_2 = u-iv$ bo'lib, $G = \bar{F}$ bo'lsa, u holda (3.3.3) tenglikni quyidagicha yozish mumkin

$$2iv\int\limits_0^b |F(x)|^2dx = W_b\{F,\bar{F}\} - W_0\{F,\bar{F}\}\ .$$

Bu tenglikdan

$$2v\int\limits_0^b |F(x)|^2dx = iW_0\{F,\bar{F}\} - iW_b\{F,\bar{F}\}, \qquad (3.3.4)$$

kelib chiqadi.

3) $\theta(x,\lambda)$ va $\varphi(x,\lambda)$ orqali (3.3.1) tenglamaning

$$\begin{cases} \theta(0,\lambda) = \cos\alpha \\ \theta'(0,\lambda) = \sin\alpha \end{cases} \quad \text{va} \quad \begin{cases} \varphi(0,\lambda) = \sin\alpha \\ \varphi'(0,\lambda) = -\cos\alpha \end{cases} \qquad (3.3.5)$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini belgilaylik va uning

$$y(b)\cos\beta + y'(b)\sin\beta = 0, \qquad (3.3.6)$$

shartni qanoatlantiruvchi quyidagi

$$\psi(x,\lambda) = \theta(x,\lambda) + l\varphi(x,\lambda), \qquad (3.3.7)$$

yechimlarini ko'rib chiqaylik. $\psi(x,\lambda)$ yechimni (3.3.6) shartga qo'yib l ni topsak,

$$l = -\frac{\theta(b,\lambda)\cos\beta + \theta'(b,\lambda)\sin\beta}{\varphi(b,\lambda)\cos\beta + \varphi'(b,\lambda)\sin\beta},$$

tenglik hosil bo'ladi. Bundan esa

$$l = -\frac{\theta(b,\lambda)\operatorname{ctg}\beta + \theta'(b,\lambda)}{\varphi(b,\lambda)\operatorname{ctg}\beta + \varphi'(b,\lambda)}, \qquad (3.3.8)$$

ekanligi kelib chiqadi. Bu yerda $x = \operatorname{ctg}\beta$, $\beta \in R^1$ belgilashni olib,

$$-\frac{\theta(b,\lambda)x + \theta'(b,\lambda)}{\varphi(b,\lambda)x + \varphi'(b,\lambda)}$$

haqiqiy argumentli ratsional funksiyani tuzib olamiz va uni $z = x + iy$, $i = \sqrt{-1}$ o'zgaruvchiga nisbatan kompleks tekislikga davom qildirib ushbu

$$l(z) \equiv -\frac{\theta(b,\lambda)z + \theta'(b,\lambda)}{\varphi(b,\lambda)z + \varphi'(b,\lambda)} \tag{3.3.9}$$

kasr-chiziqli akslantirishni hosil qilamiz. Bu kasr-chiziqli akslantirish haqiqiy o'qni aylana yoki to'g'ri chiziqqa o'tkazishi mumkin.

Avvalo, kompleks o'zgaruvchili funksiyalar nazariyasida uchraydigan quyidagi tasdiqni isbotsiz keltiramiz.

Lemma 3.3.1. *Agar* $\bar{Q}P - \bar{P}Q \neq 0$ bo'lsa, u holda ushbu

$$f(z) = \frac{Mz + N}{Pz + Q}, \quad MQ - PN \neq 0 \tag{3.3.10}$$

formula bilan berilgan $f : \mathbb{C} \to \mathbb{C}$ *kasr-chiziqli akslantirish, haqiqiy o'qni aylanaga akslantiradi. Bu aylananing radiusi va markazi mos ravishda*

$$R = \frac{|NP - MQ|}{|\bar{Q}P - \bar{P}Q|}, \tag{3.3.11}$$

$$\omega_0 = f(\bar{C}), \;\; C = -\frac{Q}{P} \tag{3.3.12}$$

formulalar yordamida topiladi.

Bu lemmani isbotlash uchun $|f(x) - \omega_0| = R$ ayniyat bajarilishini tekshirish yetarli.

Lemma 3.3.2. *Agar* $\operatorname{Im}\lambda \neq 0$ *bo'lsa, u holda (3.3.9) tenglik bilan aniqlangan* $l(z)$ *funksiya haqiqiy sonlar o'qini* C_b *aylanaga akslantiradi. Bu aylananing radiusi va markazi mos ravishda*

$$R_b = \left(2|\operatorname{Im}\lambda| \cdot \int_0^b |\varphi(x,\lambda)|^2 dx\right)^{-1}, \omega_0 = -\frac{W_b\{\theta,\bar{\varphi}\}}{W_b\{\varphi,\bar{\varphi}\}} \tag{3.3.13}$$

formulalar orqali aniqlanadi.

Isbot. Yuqoridagi $l(z)$ akslantirishga lemma 3.3.1 ni qo'llaymiz. Buning uchun avvalo $MQ-PN \neq 0$ va $\bar{Q}P-\bar{P}Q \neq 0$ shartlar bajarilishini ko'rsatamiz. Qaralayotgan holda

$$M = -\theta(b,\lambda), \quad N = -\theta'(b,\lambda), \; P = \varphi(b,\lambda), \; Q = \varphi'(b,\lambda).$$

Bu ifodalardan

$$MQ - PN = W_b\{\varphi,\theta\} = W_0\{\varphi,\theta\} = 1$$

va

$$\bar{Q}P - \bar{P}Q = W_b\{\varphi,\bar{\varphi}\}$$

kelib chiqadi.

(3.3.4) formulada $F(x) = \varphi(x,\lambda)$ deb olib, $W_0\{\varphi,\bar{\varphi}\} = 0$ tenglikdan foydalansak,

$$W_b\{\varphi,\bar{\varphi}\} = 2iv\int_0^b |\varphi(x,\lambda)|^2 dx \qquad (3.3.13')$$

formula hosil bo'ladi. Demak, $v = \operatorname{Im}\lambda \neq 0$ bo'lgani uchun $\bar{Q}P - \bar{P}Q \neq 0$ bo'ladi. Endi (3.3.11) formuladan foydalanib, quyidagi hisoblashlarni bajaramiz:

$$R_b \equiv R = \frac{|NP-MQ|}{|\bar{Q}P-\bar{P}Q|} = \frac{|\theta(b,\lambda)\varphi'(b,\lambda)-\theta'(b,\lambda)\varphi(b,\lambda)|}{|\varphi(b,\lambda)\bar{\varphi}'(b,\lambda)-\varphi'(b,\lambda)\bar{\varphi}(b,\lambda)|} =$$

$$= \frac{|W_b\{\theta,\varphi\}|}{|W_b\{\varphi,\bar{\varphi}\}|} = \frac{1}{|W_b\{\varphi,\bar{\varphi}\}|}.$$

Bu yerda (3.3.13′) va $W_b\{\theta,\varphi\} = -1$ tengliklardan foydalansak, (3.3.13) formulalardan birinchisi kelib chiqadi.

C_b aylananing markazini topish uchun (3.3.12) formuladan foydalanamiz:

$$\omega_0 = l\left(-\frac{\bar{\varphi}'(b,\lambda)}{\bar{\varphi}(b,\lambda)}\right) = -\frac{\theta(b,\lambda)\bar{\varphi}'(b,\lambda)-\theta'(b,\lambda)\bar{\varphi}(b,\lambda)}{\varphi(b,\lambda)\bar{\varphi}'(b,\lambda)-\varphi'(b,\lambda)\bar{\varphi}(b,\lambda)} =$$

$$= -\frac{W_b\{\theta, \bar{\varphi}\}}{W_b\{\varphi, \bar{\varphi}\}}. \blacksquare$$

Haqiqiy sonlar o'qi kompleks tekislikni yuqori va pastki yarim tekisliklarga ajratadi. Shu bois (3.3.9) tenglik bilan aniqlangan $l(z)$ akslantirishning uzluksizligidan yuqori yoki pastki yarim tekisliklarning bittasi D_b doira ichiga, ikkinchisi esa D_b doira tashqarisiga akslanishi kelib chiqadi. D_b doiraning markaziga $\bar{z}_0$ nuqta akslanadi. Bu yerda $z_0 = -\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}$. Bu kompleks sonning mavhum qismini hisoblaymiz:

$$\operatorname{Im} z_0 = \operatorname{Im}\left\{-\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}\right\} = \frac{i}{2}\left\{\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)} - \frac{\bar{\varphi}'(b,\lambda)}{\bar{\varphi}(b,\lambda)}\right\} =$$

$$= \frac{i}{2}\cdot\frac{\varphi'(b,\lambda)\bar{\varphi}(b,\lambda) - \bar{\varphi}'(b,\lambda)\varphi(b,\lambda)}{|\varphi(b,\lambda)|^2} = -\frac{i}{2}\cdot\frac{W_b\{\varphi,\bar{\varphi}\}}{|\varphi(b,\lambda)|^2}. \quad (3.3.15)$$

Bu tenglikni (3.3.13′) formuladan foydalanib, ushbu

$$\operatorname{Im} z_0 = \operatorname{Im}\left\{-\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}\right\} = \frac{v}{|\varphi(b,\lambda)|^2}\int_0^b |\varphi(x,\lambda)|^2 dx \neq 0 \quad (3.3.16)$$

ko'rinishda yozish mumkin. Shunday qilib, doiraning markaziga o'tadigan $\bar{z}_0$ nuqta uchun

$$\operatorname{Im} \bar{z}_0 = -\frac{v}{|\varphi(b,\lambda)|^2}\int_0^b |\varphi(x,\lambda)|^2 dx \neq 0 \quad (3.3.17)$$

o'rinli bo'ladi.

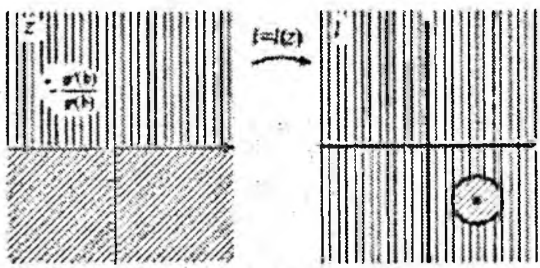
4) $v = \operatorname{Im}\lambda > 0$ bo'lsin. U holda (3.3.16) tenglikka ko'ra

$$\operatorname{Im} z_0 = \operatorname{Im}\left\{-\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}\right\} > 0 \quad (3.3.18)$$

bo'ladi. Bundan esa $z_0 = -\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}$ nuqtaning yuqori yarim tekislikda joylashganligi kelib chiqadi. Ushbu $z_0 = -\frac{\varphi'(b,\lambda)}{\varphi(b,\lambda)}$ nuqta $l(z)$ akslantirish natijasida ∞ nuqtaga o'tadi, ya'ni doira tashqarisiga o'tadi. Demak, $\operatorname{Im} z > 0$ yuqori yarim tekislik doira tashqarisiga o'tadi. Xuddi shuningdek, (3.3.17) tenglikdan

$$\operatorname{Im} \bar{z}_0 = \operatorname{Im}\left\{-\frac{\bar{\varphi}'(b,\lambda)}{\bar{\varphi}(b,\lambda)}\right\} < 0$$

kelib chiqadi. Bundan esa, $\operatorname{Im} z < 0$ quyi yarim tekislikning doira ichiga akslanishi ko'rinadi.



3-rasm.

5) $l(z)$ akslantirish natijasida hosil bo'lgan doirani D_b bilan belgilaylik. $v = \operatorname{Im}\lambda > 0$ bo'lgan holda $l \in D_b$ bo'lishi uchun $\operatorname{Im} z \leq 0$ bo'lishi zarur va yetarlidir, ya'ni $i(\bar{z} - z) \leq 0$. (3.3.9) formuladan

$$z = -\frac{\theta'(b,\lambda) + l\varphi'(b,\lambda)}{\theta(b,\lambda) + l\varphi(b,\lambda)}$$

ifodani topamiz va $\operatorname{Im} z$ ni hisoblaymiz:

$$2\operatorname{Im} z = i(z - \bar{z}) = -i\frac{\bar{\theta}'(b,\lambda) + \bar{l}\bar{\varphi}'(b,\lambda)}{\bar{\theta}(b,\lambda) + \bar{l}\bar{\varphi}(b,\lambda)} + i\frac{\theta'(b,\lambda) + l\varphi'(b,\lambda)}{\theta(b,\lambda) + l\varphi(b,\lambda)} =$$

$$= -i\frac{|l|^2 W_b\{\varphi,\bar{\varphi}\} + lW_b\{\varphi,\bar{\theta}\} + \bar{l}W_b\{\theta,\bar{\varphi}\} + W_b\{\theta,\bar{\theta}\}}{|\theta(b,\lambda) + l\varphi(b,\lambda)|^2} =$$

$$= -i\frac{W_b\{\theta + l\varphi, \overline{\theta + l\varphi}\}}{|\theta(b,\lambda) + l\varphi(b,\lambda)|^2}.$$

Demak, $v = \operatorname{Im}\lambda > 0$ bo'lganda $l \in D_b$ bo'lishi uchun

$$iW_b\{\theta + l\varphi, \overline{\theta + l\varphi}\} \geq 0$$

tengsizlikning bajarilishi zarur va yetarli. Xuddi shuningdek, $v = \operatorname{Im}\lambda < 0$ bo'lganda $l \in D_b$ bo'lishi uchun

$$iW_b\{\theta + l\varphi, \overline{\theta + l\varphi}\} \leq 0$$

tengsizlikning bajarilishi zarur va yetarli ekanligini ko'rsatish mumkin.

Bu ikki fikrni quyidagicha umumlashtirishimiz mumkin. Agar $\operatorname{Im}\lambda \neq 0$ bo'lsa, u holda $l \in D_b$ bo'lishi uchun ushbu

$$\frac{i}{\operatorname{Im}\lambda} W_b\{\theta + l\varphi, \overline{\theta + l\varphi}\} \geq 0 \tag{3.3.19}$$

tengsizlikning bajarilishi zarur va yetarlidir.

Endi (3.3.19) shartga teng kuchli bo'lgan quyidagi tasdiqni keltiramiz.

Lemma 3.3.3. *Agar* $v = \operatorname{Im}\lambda \neq 0$ *bo'lsa, u holda* l *nuqta* D_b *doiraga tegishli bo'lishi uchun*

$$\int_0^b |\theta(x,\lambda) + l\varphi(x,\lambda)|^2 dx \leq -\frac{\operatorname{Im} l}{v} \tag{3.3.20}$$

tengsizlikning bajarilishi zarur va yetarlidir.

Isbot. (3.3.4) tenglikda $F(x) = \theta(x,\lambda) + l\varphi(x,\lambda)$ deb olsak, u holda

$$2v\int_0^b |\theta(x,\lambda) + l\varphi(x,\lambda)|^2 dx =$$

$$= iW_0\{\theta + l\varphi, \overline{\theta + l\varphi}\} - iW_b\{\theta + l\varphi, \overline{\theta + l\varphi}\} \tag{3.3.21}$$

bo'ladi. Bu yerda W_0 ni hisoblaymiz:

$$W_0\{\theta + l\varphi, \overline{\theta + l\varphi}\} =$$

$$= W_0\{\theta, \bar\theta\} + lW_0\{\varphi, \bar\theta\} + \bar l W_0\{\theta, \bar\varphi\} + |l|^2 W_0\{\varphi, \bar\varphi\}.$$

(3.3.5) boshlang'ich shartlardan foydalanib,

$$W_0\{\theta, \bar\theta\} = W_0\{\varphi, \bar\varphi\} = 0,$$

$$W_0\{\theta, \bar{\varphi}\} = -1, \quad W_0\{\varphi, \bar{\theta}\} = 1$$

tengliklarni topamiz. Bulardan

$$W_0\{\theta + l\varphi, \overline{\theta + l\varphi}\} = l - \bar{l} = 2i \operatorname{Im} l$$

tenglik kelib chiqadi. Bundan foydalanib, (3.3.21) tenglikni quyidagicha yozish mumkin:

$$-2\frac{\operatorname{Im} l}{v} - 2\int_0^b |\theta(x,\lambda) + l\varphi(x,\lambda)|^2 dx = \frac{i}{v} W_b\{\theta + l\varphi,\ \overline{\theta + l\varphi}\}.$$

Bu yerda (3.3.19) tengsizlikni qo'llasak, (3.3.20) kelib chiqadi.■

Lemma 3.3.4. *Agar $b' > b$ bo'lsa, u holda $D_{b'}$ doira D_b doiraning ichida yotadi.*

Isbot. Faraz qilaylik, l nuqta $D_{b'}$ doirada yotsin. U holda lemma 3.3.3 ga asosan

$$\int_0^{b'} |\theta(x,\lambda) + l\varphi(x,\lambda)|^2 dx \le -\frac{\operatorname{Im} l}{v}$$

tengsizlik bajariladi. $b' > b$ bo'lgani uchun

$$\int_0^{b} |\theta(x,\lambda) + l\varphi(x,\lambda)|^2 dx < \int_0^{b'} |\theta(x,\lambda) + l\varphi(x,\lambda)|^2 dx \le -\frac{\operatorname{Im} l}{v}$$

bo'ladi. Bunga va lemma 3.3.3 ga asosan $l \in D_b$ kelib chiqadi. ■

Demak, $\{D_b\}$ doiralar ichma-ich joylashgan ekan. Bu holda ularning kesishmasi $D_\infty(\lambda)$ doiradan yoki $m(\lambda)$ nuqtadan iborat bo'ladi. Bu limitik $D_\infty(\lambda)$ doiraga Veyl doirasi, limitik $m(\lambda)$ nuqtaga esa Veyl nuqtasi deyiladi.

Teorema 3.3.1. *Agar $m(\lambda)$ - Veyl nuqtasi yoki $D_\infty(\lambda)$ Veyl doirasiga tegishli bo'lgan biror nuqta bo'lsa, u holda ixtiyoriy $\lambda \in C \backslash R$ kompleks son uchun (3.3.1) tenglamaning*

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$$

yechimi $L^2(0,\infty)$ fazoga tegishli bo'ladi va ushbu

$$\int_0^\infty |\psi(x,\lambda)|^2 dx \le -\frac{\operatorname{Im} m(\lambda)}{\operatorname{Im}\lambda} \tag{3.3.22}$$

tengsizlikni qanoatlantiradi.

Isbot. Bu $m(\lambda)$ nuqta doiralarning barchasiga tegishli bo'lgani uchun

$$\int_0^b |\theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)|^2 dx \le -\frac{\operatorname{Im} m(\lambda)}{v}, \quad v = \operatorname{Im}\lambda$$

tengsizliklar o'rinli bo'ladi. $m(\lambda)$ son b ga bog'liq emasligini hisobga olib, bu yerda $b \to \infty$ da limitga o'tsak, (3.3.22) hosil bo'ladi. ∎

Ta'rif 3.3.1. $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$ yechimga Veyl yechimi, $m(\lambda)$ funksiyaga esa Veyl-Titchmarsh funksiyasi deyiladi.

Teorema 3.3.2. $D_\infty(\lambda)$ - *Veyl doirasi bo'lsin, u holda* $\operatorname{Im}\lambda \ne 0$ *bo'lganda, (3.3.1) tenglamaning barcha yechimlari $L^2(0,\infty)$ fazoga tegishli bo'ladi va $D_\infty(\lambda)$ doiraning radiusi R_∞ uchun*

$$R_\infty = \left(2|\operatorname{Im}\lambda| \cdot \int_0^\infty |\varphi(x,\lambda)|^2 dx\right)^{-1} \tag{3.3.23}$$

tenglik bajariladi.

Isbot. (3.3.13) formulada $b \to \infty$ da limitga o'tib, (3.3.23) tenglikka ega bo'lamiz. Xususan, (3.3.23) formulaga asosan $\operatorname{Im}\lambda \ne 0$ bo'lganda $\varphi(x,\lambda) \in L^2(0,\infty)$ bo'ladi. Ushbu $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$ tenglikdan $\theta(x,\lambda) \in L^2(0,\infty)$ kelib chiqadi. (3.3.1) tenglamaning barcha yechimi $\theta(x,\lambda)$ va $\varphi(x,\lambda)$ yechimlar orqali chiziqli ifodalanganligi uchun, (3.3.1) tenglamaning barcha yechimlari $L^2(0,\infty)$ fazoga tegishli bo'ladi. ∎

4-§. Veyl doirasi va nuqtasi haqida teoremalar

Veyl doirasi yoki nuqtasi holi o'rinli bo'lishi λ ga bog'liqmi, yoki $q(x)$ funksiyaga bog'liqmi, degan tabiiy savol tug'iladi. Bu savolga quyidagi teoremalar javob beradi.

Teorema 3.4.1. *Agar biror λ_0 kompleks son uchun*

$$-y'' + q(x)y = \lambda_0 y,$$

tenglamaning barcha yechimlari $L^2(0,\infty)$ fazoga tegishli bo'lsa, ixtiyoriy λ kompleks son uchun

$$-y'' + q(x)y = \lambda y,$$

tenglamaning ham barcha yechimlari $L^2(0,\infty)$ fazoga tegishli bo'ladi.

Isbot. $\operatorname{Im}\lambda_0 \neq 0, \operatorname{Im}\lambda \neq 0$ bo'lib, $\theta(x)$ va $\varphi(x)$ funksiyalar

$$-y'' + q(x)y = \lambda_0 y,$$

tenglamaning

$$\theta(0) = 1, \theta'(0) = 0 \quad \text{va} \quad \varphi(0) = 0, \varphi'(0) = 1$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlari bo'lsin. $y(x)$ funksiya esa

$$-y'' + q(x)y = \lambda y, \tag{3.4.1}$$

tenglamaning ixtiyoriy yechimi bo'lsin. (3.4.1) tenglamani quyidagi ko'rinishda yozib olamiz

$$-y'' + q(x)y - \lambda_0 y = (\lambda - \lambda_0)y. \tag{3.4.2}$$

Ushbu

$$-y'' + q(x)y - \lambda_0 y = f(x),$$

tenglama uchun Koshi funksiyasi

$$K(x,t) = \theta(x)\varphi(t) - \varphi(x)\theta(t),$$

bo'ladi. Koshi funksiyasi yordamida (3.4.2) tenglamani quyidagi integral tenglamaga keltiramiz:

$$y(x) = c_1\theta(x) + c_2\varphi(x) + (\lambda - \lambda_0)\int_a^x [\theta(x)\varphi(t) - \varphi(x)\theta(t)]y(t)dt, \tag{3.4.3}$$

bu yerda a, c_1 va c_2 ixtiyoriy o'zgarmas sonlarni bildiradi.

$[a,\ x]$ tayinlangan kesma bo'lsin. Ma'lumki, ushbu

$$||y|| = \sqrt{\int_a^x |y(t)|^2 dt}, \tag{3.4.4}$$

funksional norma shartlarini qanoatlantiradi. M sonini quyidagicha aniqlaymiz:

$$M = \max\left\{\sqrt{\int_a^\infty |\theta(t)|^2 dt},\ \ \sqrt{\int_a^\infty |\varphi(t)|^2 dt}\right\}. \tag{3.4.5}$$

U holda Koshi-Bunyakovskiy tengsizligidan foydalanib, quyidagi baholashlarni bajaramiz:

$$\left|\int_a^x [\theta(x)\varphi(t) - \varphi(x)\theta(t)]y(t)dt\right| \leq$$

$$\leq |\theta(x)|\int_a^x |\varphi(t)|\cdot|y(t)|dt + |\varphi(x)|\int_a^x |\theta(t)|\cdot|y(t)|dt \leq$$

$$\leq\ |\theta(x)|\sqrt{\int_a^x |\varphi(t)|^2 dt}\cdot\sqrt{\int_a^x |y(t)|^2 dt} +$$

$$+\ |\varphi(x)|\sqrt{\int_a^x |\theta(t)|^2 dt}\cdot\sqrt{\int_a^x |y(t)|^2 dt} \leq$$

$$\leq M\left(|\theta(x)|+|\varphi(x)|\right)\sqrt{\int_a^x |y(t)|^2 dt}\,. \qquad (3.4.6)$$

Minkovskiy tengsizligidan esa ushbu

$$||y|| \leq |c_1|\cdot||\theta||+|c_2|\cdot||\varphi||+$$

$$+|\lambda-\lambda_0|\sqrt{\int_a^x \left|\int_a^t [\theta(t)\varphi(s)-\varphi(t)\theta(s)]y(s)ds\right|^2 dt} \leq$$

$$\leq \left(|c_1|+|c_2|\right)M+|\lambda-\lambda_0|\sqrt{\int_a^x M^2\left(|\theta(t)|+|\varphi(t)|\right)^2\int_a^t |y(s)|^2 ds dt} \leq$$

$$\leq \left(|c_1|+|c_2|\right)M+M|\lambda-\lambda_0|\sqrt{\int_a^x 2\left(|\theta(t)|^2+|\varphi(t)|^2\right)\int_a^t |y(s)|^2 ds dt} \leq$$

$$\leq \left(|c_1|+|c_2|\right)M+$$

$$+M|\lambda-\lambda_0|\sqrt{\int_a^x \left(\int_a^t |y(s)|^2 ds\right) d\left(\int_a^t 2\left(|\theta(s)|^2+|\varphi(s)|^2\right)ds\right)} =$$

$$= \left(|c_1|+|c_2|\right)M+M|\lambda-\lambda_0|\left\{\int_a^x |y(s)|^2 ds\int_a^x 2\left(|\theta(s)|^2+|\varphi(s)|^2\right)ds-\right.$$

$$\left.-\int_a^x |y(t)|^2\left[\int_a^t 2\left(|\theta(s)|^2+|\varphi(s)|^2\right)ds\right]dt\right\}^{\frac{1}{2}} \leq$$

$$\leq \left(|c_1|+|c_2|\right)M+M|\lambda-\lambda_0|\sqrt{\int_a^x |y(s)|^2 ds\cdot 4M^2} =$$

$$= \left(|c_1|+|c_2|\right)M+2M^2|\lambda-\lambda_0|||y|| \qquad (3.4.7)$$

kelib chiqadi. Oxirgi tengsizlikdan quyidagi baholash

$$\left(1-2M^2|\lambda-\lambda_0|\right)||y|| \leq \left(|c_1|+|c_2|\right)M, \tag{3.4.8}$$

hosil bo'ladi. a parametrni tanlash hisobiga, $M^2|\lambda-\lambda_0|<\frac{1}{4}$ tengsizlik bajariladigan qilish mumkin. U holda (3.4.8) tengsizlikdan

$$\sqrt{\int_a^x |y(t)|^2 dt} \leq 2M\left(|c_1|+|c_2|\right), \tag{3.4.9}$$

munosabat kelib chiqadi. (3.4.9) tengsizlikda x ni cheksizlikka intiltirsak,

$$\sqrt{\int_a^\infty |y(t)|^2 dt} \leq 2M\left(|c_1|+|c_2|\right), \tag{3.4.10}$$

o'rinli bo'lishi ko'rinadi. Demak, $y(x)\in L_2(0,\infty)$.■

Quyidagi teorema amaliy jihatdan muhim ahamiyatga ega.

Teorema 3.4.2. *Agar biror k musbat son uchun $q(x)\geq -kx^2$ tengsizlik bajarilsa, $Ly\equiv -y''+q(x)y$ Shturm-Liuvill operatori uchun Veyl nuqtasi holi o'rinlidir, ya'ni λ parametrning kompleks qiymatida $-y''+q(x)y=\lambda y$ tenglamaning $L^2(0,\infty)$ fazoga tegishli yechimlari o'zaro proporsionaldir.*

Isbot. Bundan oldingi teoremaga ko'ra Veyl nuqtasi yoki doirasi holi bo'lishi λ ga bog'lik emas. $\lambda=0$ nuqta spektrga tegishli emas deb hisoblaymiz. $\varphi(x)$ va $\theta(x)$ haqiqiy yechimlar chiziqli erkli bo'lib, $L_2(0,\infty)$ fazoga tegishli bo'lsin deb faraz qilaylik.

$\varphi''(x)=q(x)\varphi(x)$ ekanligidan

$$\int_a^x \frac{\varphi''(t)\varphi(t)}{t^2}dt=\int_a^x \frac{q(t)\varphi^2(t)}{t^2}dt \geq -k\int_a^x \varphi^2(t)dt, \tag{3.4.11}$$

tengsizlik kelib chiqadi. Bu tengsizlik chap tomonidagi integralni bo'laklab integrallaymiz:

$$\int_a^x \frac{\varphi''(t)\varphi(t)}{t^2}dt = \int_a^x \frac{\varphi(t)}{t^2}d\varphi'(t) =$$

$$= \frac{\varphi(t)\varphi'(t)}{t^2}\Big|_a^x - \int_a^x \frac{\varphi'(t)t^2 - 2t\varphi(t)}{t^4}\varphi'(t)dt =$$

$$= \frac{\varphi(x)\varphi'(x)}{x^2} - \frac{\varphi(a)\varphi'(a)}{a^2} - \int_a^x \frac{\varphi'^2(t)}{t^2}dt + 2\int_a^x \frac{\varphi(t)\varphi'(t)}{t^3}dt. \quad (3.4.12)$$

(3.4.11) tengsizlikdan

$$\begin{gathered}-\frac{\varphi(x)\varphi'(x)}{x^2} + \frac{\varphi(a)\varphi'(a)}{a^2} + \int_a^x \frac{\varphi'^2(t)}{t^2}dt - 2\int_a^x \frac{\varphi(t)\varphi'(t)}{t^3}dt \le \\ \le k\int_a^x \varphi^2(t)dt,\end{gathered} \quad (3.4.13)$$

ekanligi ko'rinadi. $\varphi(t) \in L^2(0,\infty)$ bo'lgani uchun (3.4.13) dan

$$-\frac{\varphi(x)\varphi'(x)}{x^2} + \int_a^x \frac{\varphi'^2(t)}{t^2}dt - 2\int_a^x \frac{\varphi(t)\varphi'(t)}{t^3}dt \le k_1, \quad (3.4.14)$$

kelib chiqadi. Koshi-Bunyakovskiy tengsizligiga asosan

$$\left|\int_a^x \frac{2\varphi'(t)\varphi(t)}{t^3}dt\right| \le \frac{2}{a^2}\int_a^x \frac{|\varphi'(t)|}{t}|\varphi(t)|dt \le$$

$$\le \frac{2}{a^2}\sqrt{\int_a^x \frac{\varphi'^2(t)}{t^2}dt} \cdot \sqrt{\int_a^x \varphi^2(t)dt} \le k_2\sqrt{\int_a^x \frac{\varphi'^2(t)}{t^2}dt}, \quad (3.4.15)$$

baholash o'rinli bo'ladi.

Ushbu

$$H(x) = \int_a^x \frac{\varphi'^2(t)}{t^2}dt, \quad (3.4.16)$$

yordamchi funksiyani kiritamiz. (3.4.14) va (3.4.15) tengsizliklardan

$$-\frac{\varphi(x)\varphi'(x)}{x^2} + H(x) - k_2\sqrt{H(x)} \le k_1, \qquad (3.4.17)$$

hosil bo'ladi.

Agar $x \to \infty$ bo'lganda $H(x) \to \infty$ bo'lsin desak, biror x_0 dan boshlab,

$$H(x) - k_2\sqrt{H(x)} - k_1 > \frac{1}{2}H(x),$$

tengsizlik bajariladi. Bundan esa (3.4.17) ga ko'ra

$$2\varphi'(x)\varphi(x) > x^2 H(x), \qquad x > x_0,$$

kelib chiqadi. Oxirgi tengsizlikni integrallaymiz:

$$\int_{x_0}^{x} 2\varphi'(t)\varphi(t)dt > \int_{x_0}^{x} t^2 H(t)dt,$$

$$\varphi^2(x) > \varphi^2(x_0) + \int_{x_0}^{x} t^2 H(t)dt.$$

Natijada, $\varphi^2(x) \to \infty$ ekanligi kelib chiqadi. Bu esa $\varphi(x) \in L^2(0,\infty)$ ekanligiga ziddir.

Demak, $H(x)$ funksiya chegaralangan bo'ladi, ya'ni

$$\int_{a}^{x} \frac{\varphi'^2(t)}{t^2}dt < \infty. \qquad (3.4.18)$$

Xuddi shunday qilib,

$$\int_{a}^{x} \frac{\theta'^2(t)}{t^2}dt < \infty, \qquad (3.4.19)$$

tengsizlik ham ko'rsatiladi. (3.4.18) va (3.4.19) dan ixtiyoriy $x > a$ da

$$\int_{a}^{x} \frac{1}{t}dt = \int_{a}^{x} \frac{\theta(t)\varphi'(t) - \varphi(t)\theta'(t)}{t}dt \le$$

$$\leq \int_a^x |\theta(t)|\frac{|\varphi'(t)|}{t}dt + \int_a^x |\varphi(t)|\frac{|\theta'(t)|}{t}dt \leq$$

$$\leq \sqrt{\int_a^x \theta^2(t)dt} \cdot \sqrt{\int_a^x \frac{\varphi'^2(t)}{t^2}dt} + \sqrt{\int_a^x \varphi^2(t)dt} \cdot \sqrt{\int_a^x \frac{\theta'^2(t)}{t^2}dt} < C,$$

ekanligi kelib chiqadi, ya'ni ziddiyat hosil bo'ladi. Demak, farazimiz noto'g'ri.■

Natija 3.4.1. $q(x)$ uzluksiz funksiya quyidan chegaralangan bo'lgan holda, Veylning nuqta holi o'rinli bo'ladi.

Natija 3.4.2. $q(x)$ uzluksiz funksiya davriy bo'lgan holda, Veylning nuqta holi o'rinli bo'ladi.

Endi β ning berilgan qiymatida

$$l = l(\lambda) = -\frac{\theta(b,\lambda)\cos\beta + \theta'(b,\lambda)\sin\beta}{\varphi(b,\lambda)\cos\beta + \varphi'(b,\lambda)\sin\beta},$$

funksiyani ko'rib chiqamiz. $l(\lambda)$ funksiya meromorf bo'lib, uning qutblari

$$\varphi(b,\lambda)\cos\beta + \varphi'(b,\lambda)\sin\beta = 0,$$

tenglamaning ildizlaridan iborat. Bu tenglamaning nollari

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x \leq b,$$

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(b)\cos\beta + y'(b)\sin\beta = 0, \end{cases}$$

chegaraviy masalaning xos qiymatlari bo'lgani uchun ular haqiqiy o'qda joylashgan va karrasizdir.

Ma'lumki, b ortgan sari D_b doiralar kichrayib boradi va ichma ich joylashadi. Shuning uchun $l(\lambda) = l(\lambda, b, \beta)$ funksiyalar sinfi yuqori yarim tekislikda yotgan chegaralangan sohada b ga nisbatan tekis chegaralangan bo'ladi. Pastki yarim tekislik uchun ham ushbu fikrlar o'rinli bo'ladi. Kompleks analizning klassik teoremasiga ko'ra bu sinfdan tekis yaqinlashuvchi qismiy ketma-ketlik tanlab olish mumkin, ya'ni, cheksizlikka intiluvchi shunday b_k va β_k ketma-ketliklar topilib, quyidagi limit mavjud bo'ladi:

$$\lim_{k\to\infty} l(\lambda, b_k, \beta_k) = m(\lambda).$$

Veyershtrass teoremasiga ko'ra $m(\lambda)$ funksiya $\operatorname{Im}\lambda \neq 0$ sohada analitik bo'ladi.

Teorema 3.4.3. $\psi_b(x,\lambda) = \theta(x,\lambda) + l\varphi(x,\lambda)$ *bo'lib,* $l = l(\lambda, b)$ *bo'lsin. U holda* $\operatorname{Im}\lambda \neq 0$ *uchun ushbu*

$$\psi_b(x,\lambda) \to \psi(x,\lambda), \quad (b \to \infty), \tag{3.3.19'}$$

$$\int_0^b |\psi_b(x,\lambda)|^2 dx \to \int_0^\infty |\psi(x,\lambda)|^2 dx, \quad (b \to \infty), \tag{3.3.19''}$$

munosabatlar o'rinli bo'ladi.

Isbot. Quyidagi ikkita holni ko'rib chiqamiz:

1) Veyl doirasi holida $l(\lambda) \to m(\lambda)$, $(b \to \infty)$ bo'lgani uchun x va λ ning tayinlangan qiymatida

$$\psi_b(x,\lambda) - \psi(x,\lambda) = [l(\lambda) - m(\lambda)]\varphi(x,\lambda) \to 0, \quad (b \to \infty),$$

bo'ladi va bundan

$$\int_0^b |\psi_b(x,\lambda) - \psi(x,\lambda)|^2 dx =$$

$$= |l(\lambda) - m(\lambda)|^2 \int_0^b |\varphi(x,\lambda)|^2 dx \to 0, \quad (b \to \infty),$$

ekanligi kelib chiqadi.

2) Veyl nuqtasi holida quyidagi tengsizliklar o'rinli:

$$\int_0^b |\psi_b(x,\lambda) - \psi(x,\lambda)|^2 dx = |l(\lambda) - m(\lambda)|^2 \int_0^b |\varphi(x,\lambda)|^2 dx \leq$$

$$\leq 4R_b^2\int_0^b|\varphi(x,\lambda)|^2dx \leq \frac{1}{v^2\left(\int\limits_0^b|\varphi(x,\lambda)|^2dx\right)^2}\cdot\int_0^b|\varphi(x,\lambda)|^2dx =$$

$$= \frac{1}{v^2\int\limits_0^b|\varphi(x,\lambda)|^2dx} \to 0, \quad (b\to\infty).$$

Bunga asosan ushbu

$$\left|\sqrt{\int_0^b|\psi_b(x,\lambda)|^2dx} - \sqrt{\int_0^\infty|\psi(x,\lambda)|^2dx}\right| \leq$$

$$\leq \left|\sqrt{\int_0^b|\psi_b(x,\lambda)|^2dx} - \sqrt{\int_0^b|\psi(x,\lambda)|^2dx}\right| +$$

$$+\left|\sqrt{\int_0^b|\psi(x,\lambda)|^2dx} - \sqrt{\int_0^\infty|\psi(x,\lambda)|^2dx}\right| \leq$$

$$\leq \sqrt{\int_0^b|\psi_b(x,\lambda)-\psi(x,\lambda)|^2dx} +$$

$$+\left|\sqrt{\int_0^b|\psi(x,\lambda)|^2dx} - \sqrt{\int_0^\infty|\psi(x,\lambda)|^2dx}\right|$$

baholashlardan (3.4.19″) munosabat kelib chiqadi. ■

Lemma 3.4.1. *Veyl-Titchmarsh funksiyasi uchun quyidagi munosabatlar o'rinli:*

1) $\overline{m(\lambda)} = m(\bar{\lambda})$;

2) λ_0 *son xos qiymat bo'lsa, u holda*

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\cdot m(\lambda)=c\neq 0$$

bo'ladi.

Bu lemmaning isbotlashni o'quvchiga vazifa sifatida qoldiramiz.

Teorema 3.4.4. *Haqiqiy bo'lmagan λ va $\tilde{\lambda}$ sonlar uchun quyidagi tengliklar o'rinli:*

$$a)\ \lim_{b\to\infty}W_b\Big\{\psi(x,\lambda),\ \psi(x,\tilde{\lambda})\Big\}=0; \qquad (3.4.20)$$

$$b)\ \int_0^\infty \psi(x,\lambda)\psi(x,\tilde{\lambda})dx=\frac{m(\lambda)-m(\tilde{\lambda})}{\tilde{\lambda}-\lambda}; \qquad (3.4.21)$$

$$c)\ \int_0^\infty |\psi(x,\lambda)|^2dx=-\frac{\operatorname{Im}\{m(\lambda)\}}{\operatorname{Im}\lambda}. \qquad (3.4.22)$$

Isbot. *a)* $\theta(x,\lambda)+l(\lambda)\varphi(x,\lambda)$ va $\theta(x,\tilde{\lambda})+l(\tilde{\lambda})\varphi(x,\tilde{\lambda})$ funksiyalar

$$y(b)\cos\beta+y'(b)\sin\beta=0,$$

chegaraviy shartni qanoatlantirganligi uchun

$$W_b\Big\{\theta(x.\lambda)+l(\lambda)\varphi(x,\lambda),\ \theta(x,\tilde{\lambda})+l(\tilde{\lambda})\varphi(x,\tilde{\lambda})\Big\}=0,$$

bo'ladi. Bu tenglikdan

$$W_b\{\psi(x,\lambda)+[l(\lambda)-m(\lambda)]\varphi(x,\lambda),$$
$$\psi(x,\tilde{\lambda})+[l(\tilde{\lambda})-m(\tilde{\lambda})]\varphi(x,\tilde{\lambda})\Big\}=0,$$

ya'ni

$$W_b\{\psi(x,\lambda),\ \psi(x,\tilde{\lambda})\}+[l(\lambda)-m(\lambda)]W_b\{\varphi(x,\lambda),\ \psi(x,\tilde{\lambda})\}+$$
$$+[l(\tilde{\lambda})-m(\tilde{\lambda})]W_b\{\psi(x,\lambda),\ \varphi(x,\tilde{\lambda})\}+$$
$$+[l(\lambda)-m(\lambda)][l(\tilde{\lambda})-m(\tilde{\lambda})]W_b\{\varphi(x,\lambda),\ \varphi(x,\tilde{\lambda})\}=0, \quad (3.4.23)$$

kelib chiqadi. Grin ayniyatiga asosan

$$W_b\{\varphi(x,\lambda),\ \psi(x,\tilde{\lambda})\} =$$

$$= (\lambda - \tilde{\lambda})\int_0^b \varphi(x,\lambda)\psi(x,\tilde{\lambda})dx + W_0\{\varphi(x,\lambda),\ \psi(x,\tilde{\lambda})\},$$

hosil bo'ladi. Koshi-Bunyakovskiy tengsizligiga ko'ra

$$W_b\{\varphi(x,\lambda),\ \psi(x,\tilde{\lambda})\} = \underline{\underline{O}}\left\{\sqrt{\int_0^b |\varphi(x,\lambda)|^2 dx}\right\} + \underline{\underline{O}}(1), \quad b \to \infty,$$

bajariladi. Bundan ushbu

$$\lim_{b\to\infty}[l(\lambda) - m(\lambda)]W_b\{\varphi(x,\lambda),\ \psi(x,\tilde{\lambda})\} = 0,$$

o'rinli bo'lishi ko'rinadi, bu tenglik Veyl nuqtasi holida

$$|l(\lambda) - m(\lambda)| \le 2R_b = \left\{|v|\int_0^b |\varphi(x,\lambda)|^2 dx\right\}^{-1},$$

tengsizlikdan kelib chiqadi, Veyl doirasi holida esa ushbu

$$\int_0^b |\varphi(x,\lambda)|^2 dx,$$

integral chegaralangan bo'lib, $l(\lambda)$ biror b_k ketma-ketlik bo'yicha $m(\lambda)$ ga intilishidan kelib chiqadi. (3.4.23) tenglikdagi qolgan integrallar ham xuddi shu kabi baholanadi va (3.4.20) tenglik hosil bo'ladi;

b) Grin ayniyatiga asosan ushbu

$$(\lambda - \tilde{\lambda})\int_0^b \psi(x,\lambda)\psi(x,\tilde{\lambda})dx =$$

$$= W_0\{\psi(x,\lambda),\ \psi(x,\tilde{\lambda})\} - W_b\{\psi(x,\lambda),\ \psi(x,\tilde{\lambda})\},$$

tenglik bajariladi. Boshlang'ich shartlar va (3.4.20) ga binoan (3.4.21) formula kelib chiqadi.

c) Agar (3.4.21) formulada $\bar{\lambda}$ sifatida $\bar{\lambda}$ ni olsak (3.4.22) kelib chiqadi.■

Teorema 3.4.5. λ_n, λ_k - *qaralayotgan masalaning xos qiymatlari bo'lib,* $r_n = \underset{\lambda=\lambda_n}{\text{res}}\, m(\lambda)$ - *Veyl-Titchmarsh funksiyasining* λ_n *xos qiymatdagi chegirmasi bo'lsin. U holda quyidagi tengliklar bajariladi:*

$$a) \qquad \int_0^\infty \varphi(x,\lambda_n)\psi(x,\lambda)dx = \frac{1}{\lambda-\lambda_n}, \qquad (3.4.24)$$

$$b) \qquad \int_0^\infty \varphi^2(x,\lambda_n)dx = \frac{1}{r_n}, \qquad (3.4.25)$$

$$c) \qquad \int_0^\infty \varphi(x,\lambda_n)\varphi(x,\lambda_k)dx = 0, (n \neq k). \qquad (3.4.26)$$

Isbot. (3.4.22)tenglikdan

$$\int_0^\infty |v\psi(x,\lambda_n+iv)|^2dx = |v\,\mathrm{Im}\,\{m(\lambda_n+iv)\}| \leq |vm(\lambda_n+iv)| < \infty,$$

kelib chiqadi. Bundan $v \to 0$ bo'lganda

$$\int_0^\infty \varphi^2(x,\lambda_n)dx < \infty,$$

hosil bo'ladi. (3.4.21) formulaga binoan

$$\int_0^\infty \frac{iv}{r_n}\psi(x,\lambda_n+iv)\psi(x,\lambda)dx = \frac{iv[m(\lambda)-m(\lambda_n+iv)]}{r_n}\cdot\frac{1}{\lambda_n+iv-\lambda},$$

o'rinli. Bu tenglikni ushbu

$$\int\limits_0^{\infty}\left[\frac{iv}{r_n}\theta(x,\lambda_n+iv)+\frac{ivm(\lambda_n+iv)}{r_n}\varphi(x,\lambda_n+iv)\right]\psi(x,\lambda)dx =$$

$$=\left[\frac{ivm(\lambda)}{r_n}-\frac{ivm(\lambda_n+iv)}{r_n}\right]\cdot\frac{1}{\lambda_n+iv-\lambda},$$

ko'rinishda yozib olamiz va v ni nolga intiltiramiz, natijada (3.4.24) kelib chiqadi. (3.4.24) tenglikda $\lambda=\lambda_n+iv$ deb, uni $\frac{iv}{r_n}$ ga ko'paytirib, v ni nolga intiltirsak, (3.4.25) hosil bo'ladi. Agar (3.4.24) formulada $\lambda=\lambda_k+iv$ deb, uni $\frac{iv}{r_k}$ ga ko'paytirib v ni nolga intiltirsak, (3.4.26) o'rinli bo'lishi ko'rinadi.■

Natija. (3.4.25) formulaga asosan Veyl-Titchmarsh funksiyasining xos qiymatdagi qoldiqlari musbat bo'ladi.

Teorema 3.4.6. Agar ushbu

$$\begin{cases} -y''+q(x)y=\lambda y, & 0\le x<\infty,\\ y(0)\cos\alpha+y'(0)\sin\alpha=0 \end{cases}$$

chegaraviy masala uchun Veylning nuqta holi o'rinli bo'lib, $h(x,\lambda), \operatorname{Im}\lambda\neq 0$ funksiya

$$-y''+q(x)y=\lambda y,\ \ 0\le x<\infty$$

tenglamaning $L^2(0,\infty)$ fazoga tegishli biror yechimi bo'lsa, u holda berilgan chegaraviy masalaning Veyl-Titchmarsh funksiyasi uchun quyidagi tenglik o'rinli:

$$m_\alpha(\lambda)=\frac{h(0,\lambda)\sin\alpha-h'(0,\lambda)\cos\alpha}{h'(0,\lambda)\sin\alpha+h(0,\lambda)\cos\alpha}.$$

Isbot. $\psi(x,\lambda)=\theta(x,\lambda)+m_\alpha(\lambda)\varphi(x,\lambda)\in L^2(0,\infty), \operatorname{Im}\lambda\neq 0$ Veyl yechimi bo'lsin, u holda $\psi(x,\lambda)=C\cdot h(x,\lambda)$ bo'ladi. Bunga ko'ra

$$\frac{\sin\alpha-m_\alpha(\lambda)\cos\alpha}{\cos\alpha+m_\alpha(\lambda)\sin\alpha}=\frac{h'(0,\lambda)}{h(0,\lambda)}.$$

Bu tenglikdan $m_\alpha(\lambda)$ funksiyaning ifodasini topamiz. ■

Teorema 3.4.7. Agar ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \leq x < \infty, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0 \end{cases}$$

chegaraviy masala uchun Veylning nuqta holi o'rinli bo'lib. bu masalaning Veyl-Titchmarsh funksiyasi $m_\alpha(\lambda)$ bo'lsa, u holda

$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \leq x < \infty, \\ y(0)\cos\beta + y'(0)\sin\beta = 0 \end{cases}$$

chegaraviy masalaning Veyl-Titchmarsh funksiyasi $m_\beta(\lambda)$ uchun quyidagi tenglik bajariladi:

$$m_\beta(\lambda) = \frac{m_\alpha(\lambda)\cos(\alpha-\beta) - \sin(\alpha-\beta)}{\cos(\alpha-\beta) + m_\alpha(\lambda)\sin(\alpha-\beta)}. \tag{3.4.27}$$

Isbot. $\theta(x,\lambda)$, $\varphi(x,\lambda)$, $\tilde{\theta}(x,\lambda)$, $\tilde{\varphi}(x,\lambda)$ orqali ushbu

$$-y'' + q(x)y = \lambda y$$

tenglamaning

$$\begin{cases} \theta(0,\lambda) = \cos\alpha, \\ \theta'(0,\lambda) = \sin\alpha, \end{cases} \quad \begin{cases} \varphi(0,\lambda) = \sin\alpha, \\ \varphi'(0,\lambda) = -\cos\alpha, \end{cases}$$

$$\begin{cases} \tilde{\theta}(0,\lambda) = \cos\beta, \\ \tilde{\theta}'(0,\lambda) = \sin\beta, \end{cases} \quad \begin{cases} \tilde{\varphi}(0,\lambda) = \sin\beta, \\ \tilde{\varphi}'(0,\lambda) = -\cos\beta \end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini belgilaymiz. Quyidagi

$$\psi(x,\lambda) = \theta(x,\lambda) + m_\alpha(\lambda)\varphi(x,\lambda) \in L^2(0,\infty), \mathrm{Im}\,\lambda \neq 0,$$

$$\tilde{\psi}(x,\lambda) = \tilde{\theta}(x,\lambda) + m_\beta(\lambda)\tilde{\varphi}(x,\lambda) \in L^2(0,\infty), \mathrm{Im}\,\lambda \neq 0,$$

munosabatlardan, Veylning nuqta holida $\tilde{\psi}(x,\lambda) = C \cdot \psi(x,\lambda)$ bo'lgani uchun

$$\begin{cases} \tilde{\theta}(0,\lambda) + m_\beta(\lambda)\tilde{\varphi}(0,\lambda) = C \cdot [\theta(0,\lambda) + m_\alpha(\lambda)\varphi(0,\lambda)], \\ \tilde{\theta}'(0,\lambda) + m_\beta(\lambda)\tilde{\varphi}'(0,\lambda) = C \cdot [\theta'(0,\lambda) + m_\alpha(\lambda)\varphi'(0,\lambda)], \end{cases}$$

ya'ni

$$\begin{cases} \cos\beta + m_\beta(\lambda)\sin\beta = C\cdot[\cos\alpha + m_\alpha(\lambda)\sin\alpha] \\ \sin\beta - m_\beta(\lambda)\cos\beta = C\cdot[\sin\alpha - m_\alpha(\lambda)\cos\alpha] \end{cases}$$

tengliklar bajariladi. Bunga asosan

$$\frac{\cos\beta + m_\beta(\lambda)\sin\beta}{\sin\beta - m_\beta(\lambda)\cos\beta} = \frac{\cos\alpha + m_\alpha(\lambda)\sin\alpha}{\sin\alpha - m_\alpha(\lambda)\cos\alpha}.$$

Bu tenglikdan (3.4.27) kelib chiqadi.■

5-§.Rezolventa uchun integral tasvir

Bu paragrafda Shturm-Liuvill operatorining rezolventasi o'rganiladi.

Lemma 3.5.1. *Agar* $f(x) \in L^2(0,\infty)$ *uzluksiz funksiya bo'lsa, u holda ushbu*

$$\Phi(x,\lambda) = \psi(x,\lambda)\int_0^x \varphi(t,\lambda)f(t)dt + \varphi(x,\lambda)\int_x^\infty \psi(t,\lambda)f(t)dt, \tag{3.5.1}$$

funksiya

$$\Phi'' + [\lambda - q(x)]\Phi = f(x), \tag{3.5.2}$$

tenglama va

$$\Phi(0,\lambda)\cos\alpha + \Phi'(0,\lambda)\sin\alpha = 0, \tag{3.5.3}$$

chegaraviy shartni qanoatlantiradi. Bu yerda $\operatorname{Im}\lambda \neq 0$.

Isbot. $\Phi(x,\lambda)$ funksiyaning hosilalarini hisoblaymiz:

$$\Phi'(x,\lambda) = \psi'(x,\lambda)\int_0^x \varphi(t,\lambda)f(t)dt + \varphi'(x,\lambda)\int_x^\infty \psi(t,\lambda)f(t)dt,$$

$$\Phi''(x,\lambda) = \psi''(x,\lambda)\int_0^x \varphi(t,\lambda)f(t)dt + \varphi''(x,\lambda)\int_x^\infty \psi(t,\lambda)f(t)dt+$$

$$+[\psi'(x.\lambda)\varphi(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda)]f(x)=$$

$$=[q(x)-\lambda]\psi(x,\lambda)\int_0^x\varphi(t,\lambda)f(t)dt+$$

$$+[q(x)-\lambda]\varphi(x,\lambda)\int_x^\infty\psi(t,\lambda)f(t)dt+$$

$$+W(\varphi,\psi)f(x)=[q(x)-\lambda]\Phi(x,\lambda)+W(\varphi,\psi)f(x). \qquad (3.5.4)$$

Boshlangich shartlarga ko'ra

$$W(\varphi,\psi)=\begin{vmatrix}\sin\alpha & \cos\alpha+m\sin\alpha\\ -\cos\alpha & \sin\alpha-m\cos\alpha\end{vmatrix}=$$

$$=\sin^2\alpha\ -m\sin\alpha\cos\alpha\ +\cos^2\alpha\ +m\sin\alpha\cos\alpha\ =1,$$

bo'ladi. Demak, (3.5.2) tenglik o'rinli ekan. Ushbu

$$\Phi(0,\lambda)=\varphi(0,\lambda)\int_0^\infty\psi(t,\lambda)f(t)dt,$$

$$\Phi'(0,\lambda)=\varphi'(0,\lambda)\int_0^\infty\psi(t,\lambda)f(t)dt,$$

tengliklardan (3.5.3) chegaraviy shartning bajarilishi kelib chiqadi.

Lemma 3.5.1 dagi $\Phi(x,\lambda)$ funksiyaga yarim o'qda berilgan Shturm-Liuvill operatorining rezolventasi deyiladi va $R_\lambda f(x)$ kabi belgilanadi.

Ushbu

$$-y''\ +\ q(x)y=\lambda y,\qquad 0\le x\le b, \qquad (3.5.5)$$

$$y(0)\cos\alpha+y'(0)\sin\alpha=0, \qquad (3.5.6)$$

$$y(b)\cos\beta+y'(b)\sin\beta=0, \qquad (3.5.7)$$

yordamchi chegaraviy masalaning xos qiymatlarini $\lambda_{n,b}$ va xos funksiyalarini $\varphi_{n,b}(x)=\varphi(x,\lambda_{n,b})$ orqali belgilaymiz.

$\psi_b(x,\lambda)$ oldingi mavzudagi yechim bo'lsin. Chekli oraliqda berilgan (3.5.5)+ (3.5.6)+(3.5.7) Shturm-Liuvill operatorining Grin funksiyasi va rezolventasi quyidagi tengliklar bilan aniqlangan edi:

$$G_b(x,t,z) = \begin{cases} \psi_b(x,z)\varphi(t,z), & t \le x, \\ \varphi(x,z)\psi_b(t,z), & t > x, \end{cases} \tag{3.5.8}$$

$$R_{z,b}f(x) = \int_0^b G_b(x,t,z)f(t)dt, \quad (z = u + iv). \tag{3.5.9}$$

Bundan tashqari $R_{z,b}f(x)$ rezolventa uchun quyidagicha tasvir olingan edi:

$$R_{z,b}f(x) = \sum_{n=0}^{\infty} \frac{a_{n,b}}{z - \lambda_{n,b}} \cdot u_{n,b}(x), \tag{3.5.10}$$

bu yerda

$$u_{n,b}(x) = \frac{1}{\alpha_{n,b}} \cdot \varphi_{n,b}(x), \quad a_{n,b} = \int_0^b f(t)u_{n,b}(t)dt,$$

bo'lib, $\alpha_{n,b}$ sonlar normallovchi o'zgarmaslarni bildiradi.

(3.5.10) tenglikni quyidagi

$$R_{z,b}f(x) = \sum_{k=0}^{\infty} \frac{\varphi_{k,b}(x)}{\alpha_{k,b}^2(z - \lambda_{k,b})} \cdot \int_0^b f(t)\varphi_{k,b}(t)dt, \tag{3.5.11}$$

ko'rinishda yozib olib, Stiltes integrali va $\rho_b(\lambda)$ funksiya ta'rifiga asosan ushbu

$$R_{z,b}f(x) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)\int_0^b f(t)\varphi(t,\lambda)dt}{z - \lambda} d\rho_b(\lambda), \tag{3.5.12}$$

tasvirni hosil qilamiz.

Lemma 3.5.2. *Haqiqiy bo'lmagan ixtiyoriy z son va tayinlangan x uchun quyidagi tengsizlik o'rinlidir*

$$\int_{-\infty}^{\infty} \left|\frac{\varphi(x,\lambda)}{z-\lambda}\right|^2 d\rho_b(\lambda) < K. \tag{3.5.13}$$

Isbot. (3.5.11) tenglikda $f(t) = \varphi_{n,b}(t)$ desak,

$$R_{z,b}\varphi_{n,b}(x) = \sum_{k=0}^{\infty} \frac{\varphi_{k,b}(x)}{\alpha_k^2(z-\lambda_{k,b})} \cdot \int_0^b \varphi_{n,b}(t)\varphi_{k,b}(t)dt,$$

hosil bo'ladi. Rezolventa xossasiga ko'ra

$$\int_0^b G_b(x,t,z)\varphi_{n,b}(t)dt = \frac{\varphi_{n,b}(x)}{z-\lambda_{n,b}},$$

$$\int_0^b G_b(x,t,z) \cdot \left\{\frac{1}{\alpha_{n,b}}\varphi_{n,b}(t)\right\}dt = \frac{\varphi_{n,b}(x)}{\alpha_{n,b}(z-\lambda_{n,b})}, \tag{3.5.14}$$

tengliklar o'rinli. (3.5.14) tenglik $G_b(x,t,z)$ funksiyaning (t bo'yicha) Furye koeffitsiyentlarini berayapti. Parseval tengligiga asosan

$$\int_0^b |G_b(x,t,z)|^2 dt = \sum_{n=0}^{\infty} \frac{|\varphi_{n,b}(x)|^2}{\alpha_{n,b}^2|z-\lambda_{n,b}|^2} = \int_{-\infty}^{\infty} \left|\frac{\varphi(x,\lambda)}{z-\lambda}\right|^2 d\rho_b(\lambda). \tag{3.5.15}$$

(3.5.15) tenglikdan ushbu

$$\int_{-\infty}^{\infty} \left|\frac{\varphi(x,\lambda)}{z-\lambda}\right|^2 d\rho_b(\lambda) \leq \int_0^{\infty} |G(x,t,z)|^2 dt,$$

tengsizlik kelib chiqadi.

Natija 3.5.1. Haqiqiy bo'lmagan ixtiyoriy z son va tayinlangan x uchun quyidagi tengsizlik o'rinli:

$$\int_{-\infty}^{\infty} \left|\frac{\varphi(x,\lambda)}{z-\lambda}\right|^2 d\rho(\lambda) \le K. \qquad (3.5.16)$$

Bu yerdagi K o'zgarmas son lemma 3.5.2 dagi o'zgarmas sonning aynan o'zidir.

Isbot. $a > 0$ ixtiyoriy son bo'lsin. Lemma 3.5.2 ga ko'ra

$$\int_{-a}^{a} \left|\frac{\varphi(x,\lambda)}{z-\lambda}\right|^2 d\rho_b(\lambda) < K,$$

tengsizlik bajariladi. Xellining ikkinchi teoremasiga asosan oxirgi tengsizlikda b_k ketma-ketlik bo'yicha limitga o'tish mumkin:

$$\int_{-a}^{a} \left|\frac{\varphi(x,\lambda)}{z-\lambda}\right|^2 d\rho(\lambda) \le K. \qquad (3.5.17)$$

Oxirgi tengsizlikda a ni cheksizlikka intiltirsak, (3.5.16) kelib chiqadi.■

Natija 3.5.2. Ixtiyoriy $a > 0$ son uchun quyidagi tengsizliklar o'rinli:

$$\int_{-\infty}^{-a} \frac{d\rho(\lambda)}{\lambda^2} < \infty, \qquad \int_{a}^{\infty} \frac{d\rho(\lambda)}{\lambda^2} < \infty. \qquad (3.5.18)$$

Isbot. 1) $\sin\alpha \neq 0$ bo'lsin. (3.5.16) tengsizlikda $x = 0$ va $z = iv$, $|v| \le a$ desak,

$$\int_{-\infty}^{\infty} \frac{\sin^2\alpha d\rho(\lambda)}{\lambda^2 + v^2} \le K,$$

hosil bo'ladi. Bundan esa

$$\int_{-\infty}^{-a} \frac{d\rho(\lambda)}{\lambda^2 + v^2} \le K_1, \qquad \int_{a}^{\infty} \frac{d\rho(\lambda)}{\lambda^2 + v^2} \le K_1, \qquad (3.5.19)$$

kelib chiqadi. Ushbu

$$\frac{1}{2\lambda^2} \leq \frac{1}{\lambda^2+a^2}, \quad (\ |\lambda| \geq a),$$

tengsizlik va (3.5.19) tengsizliklardan quyidagi baholashlarga ega bo'lamiz:

$$\int_{-\infty}^{-a} \frac{d\rho(\lambda)}{\lambda^2} \leq 2K_1 \qquad \int_{a}^{\infty} \frac{d\rho(\lambda)}{\lambda^2} \leq 2K_1 \quad .$$

2) $\sin\alpha = 0$ bo'lsin. (3.5.14) tenglikdan x bo'yicha hosila olsak,

$$\int_0^b G_b'(x,t,z)\left\{\frac{1}{\alpha_{n,b}}\varphi_{n,b}(t)\right\}dt = \frac{\varphi_{n,b}'(x)}{\alpha_{n,b}(z-\lambda_{n,b})}, \tag{3.5.20}$$

bo'ladi. Parseval tengligiga ko'ra ushbu

$$\int_0^b |G_b'(x,t,z)|^2 dt = \int_{-\infty}^{\infty} \left|\frac{\varphi'(x,\lambda)}{z-\lambda}\right|^2 d\rho_b(\lambda), \tag{3.5.21}$$

ayniyat o'rinli. Xuddi yuqoridagidek

$$\int_{-\infty}^{\infty} \left|\frac{\varphi'(x,\lambda)}{z-\lambda}\right|^2 d\rho_b(\lambda) \leq K, \tag{3.5.22}$$

bo'lishini keltirib chiqarish mumkin. (3.5.22) tengsizlikda $x = 0$ va $z = iv$, $|v| \leq a$ desak,

$$\int_{-\infty}^{\infty} \frac{\cos^2\alpha d\rho(\lambda)}{\lambda^2+v^2} \leq K,$$

hosil bo'ladi. Bundan, xuddi yuqoridagidek qilib, (3.5.18) tengsizliklar keltirilib chiqariladi. ■

Lemma 3.5.3. *Ixtiyoriy* $f(x) \in L^2(0,\infty)$ *haqiqiy funksiya va ixtiyoriy* $z = u + iv$ *haqiqiy bo'lmagan son uchun*

$$\int_0^{\infty} |R_z f(x)|^2 dx \leq \frac{1}{v^2}\int_0^{\infty} f^2(x)dx, \tag{3.5.23}$$

tengsizlik o'rinli.

Isbot. $b > 0$ ixtiyoriy son bo'lsin. (3.5.11) tasvir va Parseval tengligiga asosan

$$\int_0^b |R_{z,b}f(x)|^2 dx = \sum_{k=0}^{\infty} \frac{1}{\alpha_{k,b}^2 |z - \lambda_{k,b}|^2} \cdot \left\{ \int_0^b f(t)\varphi_{k,b}(t)dt \right\}^2 \le$$

$$\le \frac{1}{v^2} \sum_{k=0}^{\infty} \left\{ \int_0^b f(t)\frac{\varphi_{k,b}(t)}{\alpha_{k,b}} dt \right\}^2 = \frac{1}{v^2} \cdot \int_0^b f^2(t)dt, \qquad (3.5.24)$$

o'rinli. $b > a > 0$ ixtiyoriy sonlar bo'lsin. U holda

$$\int_0^a |R_{z,b}f(x)|^2 dx \le \int_0^b |R_{z,b}f(x)|^2 dx \le \frac{1}{v^2} \int_0^b f^2(t)dt,$$

bo'lgani uchun

$$\int_0^a |R_z f(x)|^2 dx \le \frac{1}{v^2} \int_0^{\infty} f^2(t)dt, \qquad (3.5.25)$$

bo'ladi. (3.5.25) tengsizlikdan (3.5.23) kelib chiqadi.■

Teorema 3.5.1. (*Rezolventa uchun integral tasvir.*) *Ixtiyoriy* $f(x) \in L^2(0, \infty)$ *funksiya va ixtiyoriy* z *haqiqiy bo'lmagan son uchun quyidagi tasvir o'rinli:*

$$R_z f(x) = \int_{-\infty}^{\infty} \frac{\varphi(x, \lambda)F(\lambda)}{z - \lambda} d\rho(\lambda). \qquad (3.5.26)$$

Bu yerda

$$F(\lambda) = l \cdot i \cdot \underset{n \to \infty}{m} \int_0^n f(x)\varphi(x, \lambda)dx.$$

Isbot. Dastlab $[0, n]$ kesmadan tashqarida aynan nol bo'ladigan va

$$f_n(0)\cos\alpha + f_n'(0)\sin\alpha = 0, \qquad (3.5.27)$$

chegaraviy shartni qanoatlantiradigan $f_n(x) \in C^2[0, \infty)$ funksiyalar uchun (3.5.26) tasvirni isbotlaymiz. $b > n$ va $a > 0$ ixtiyoriy son bo'lsin. U holda

$$F_n(\lambda) = \int_0^n f_n(x)\varphi(x,\lambda)dx, \tag{3.5.28}$$

belgilash kiritib, (3.5.12) tenglikni

$$R_{z,b}f_n(x) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)F_n(\lambda)}{z-\lambda}d\rho_b(\lambda), \tag{3.5.29}$$

ko'rinishda yozib olamiz. (3.5.29) tenglikdan ushbu

$$R_{z,b}f_n(x) - \int_{-a}^{a} \frac{\varphi(x,\lambda)F_n(\lambda)}{z-\lambda}d\rho_b(\lambda) =$$

$$= \int_{-\infty}^{-a} \frac{\varphi(x,\lambda)F_n(\lambda)}{z-\lambda}d\rho_b(\lambda) + \int_{a}^{\infty} \frac{\varphi(x,\lambda)F_n(\lambda)}{z-\lambda}d\rho_b(\lambda), \tag{3.5.30}$$

ayniyat kelib chiqadi. (3.5.30) tenglikning o'ng tomonidagi integrallarni baholaymiz:

$$\left| \int_{-\infty}^{-a} \frac{\varphi(x,\lambda)F_n(\lambda)}{z-\lambda}d\rho_b(\lambda) \right| =$$

$$= \left| \sum_{\lambda_{k,b}<a} \frac{\varphi_{k,b}(x)}{\alpha_{k,b}(z-\lambda_{k,b})} \left\{ \frac{1}{\alpha_{k,b}} \int_0^n f_n(t)\varphi_{k,b}(t)dt \right\} \right| \le$$

$$\le \sqrt{\sum_{\lambda_{k,b}<a} \frac{\varphi_{k,b}^2(x)}{\alpha_{k,b}^2|z-\lambda_{k,b}|^2}} \cdot \sqrt{\sum_{\lambda_{k,b}<a} \frac{1}{\alpha_{k,b}^2} \left\{ \int_0^n f_n(t)\varphi_{k,b}(t)dt \right\}^2} \le$$

$$\le \sqrt{K} \cdot \sqrt{\sum_{\lambda_{k,b}<a} \frac{1}{\alpha_{k,b}^2} \left\{ \frac{1}{\lambda_{k,b}} \int_0^n f_n(t)[\varphi''_{k,b}(t) - q(t)\varphi_{k,b}(t)]dt \right\}^2} =$$

$$= \sqrt{K} \cdot \sqrt{\sum_{\lambda_{k,b}<a} \frac{1}{\alpha_{k,b}^2} \frac{1}{\lambda_{k,b}^2} \left\{ \int_0^n [f_n''(t) - q(t) f_n(t)] \varphi_{k,b}(t) dt \right\}^2} \leq$$

$$\leq \sqrt{K} \cdot \frac{1}{a} \sqrt{\sum_{\lambda_{k,b}<a} \left\{ \int_0^n [f_n''(t) - q(t) f_n(t)] \left(\frac{\varphi_{k,b}(t)}{\alpha_{k,b}} \right) dt \right\}^2} \leq$$

$$\leq \sqrt{K} \cdot \frac{1}{a} \sqrt{\int_0^n [f_n''(t) - q(t) f_n(t)]^2 dt} = \frac{C}{a}. \qquad (3.5.31)$$

Xuddi shunday qilib,

$$\left| \int_a^{\infty} \frac{\varphi(x, \lambda) F_n(\lambda)}{z - \lambda} d\rho_b(\lambda) \right| \leq \frac{C}{a}; \qquad (3.5.32)$$

tengsizlik isbot qilinadi. (3.5.31) va (3.5.32) tengsizliklardan

$$\left| R_{z,b} f_n(x) - \int_{-a}^{a} \frac{\varphi(x, \lambda) F_n(\lambda)}{z - \lambda} d\rho_b(\lambda) \right| \leq \frac{2C}{a}, \qquad (3.5.33)$$

kelib chiqadi. Xellining ikkinchi teoremasiga ko'ra $b_k \to \infty$ bo'yicha limitga o'tsak,

$$\left| R_z f_n(x) - \int_{-a}^{a} \frac{\varphi(x, \lambda) F_n(\lambda)}{z - \lambda} d\rho(\lambda) \right| \leq \frac{2C}{a}, \qquad (3.5.34)$$

hosil bo'ladi. (3.5.34) tengsizlikdan ushbu

$$R_z f_n(x) = \int_{-\infty}^{\infty} \frac{\varphi(x, \lambda) F_n(\lambda)}{z - \lambda} d\rho(\lambda), \qquad (3.5.35)$$

tenglik kelib chiqadi.

Endi teoremani umumiy holda isbotlaymiz. Funksiyalar nazariyasi kursidan ma'lumki ixtiyoriy $f(x) \in L^2(0, \infty)$ funksiya uchun quyidagi shartlarni qanoatlantiruvchi $f_n(x) \in C^2[0,\infty)$ funksiyalar ketma-ketligi mavjud:

1) $f_n(0)\cos\alpha + f_n'(0)\sin\alpha = 0$,

2) $f_n(x) \equiv 0, \quad x \notin [0,n]$,

3) $\lim\limits_{n\to\infty} \int\limits_0^\infty [f_n(x) - f(x)]^2 dx = 0.$

Parseval tengligiga ko'ra $f_n(x)$ funksiyalarning $F_n(\lambda)$ Furye almashtirishlari $L^2_{\rho(\lambda)}(-\infty,\infty)$ fazoning normasi bo'yicha $f(x)$ funksiyaning $F(\lambda)$ Furye almashtirishiga yaqinlashadi. Lemma 3.5.2 ning birinchi natijasi va lemma 3.5.3 ga asosan (3.5.35) tenglikda limitga o'tish mumkin.■

Natija 3.5.3. (3.5.26) tenglikdan foydalanib, ushbu

$$\int\limits_0^\infty R_z f(x)\cdot g(x)dx = \int\limits_{-\infty}^\infty \frac{F(\lambda)G(\lambda)}{z-\lambda}d\rho(\lambda), \qquad (3.5.36)$$

formula o'rinli ekanligini ko'rsatish mumkin.■

Teorema 3.5.2. $f(x) \in L^2(0,\infty)$ *funksiya quyidagi shartlarni qanoatlantirsin:*

1. $f''(x) - q(x)f(x) \in L^2(0,\infty)$,

2. $f(0)\cos\alpha + f'(0)\sin\alpha = 0$,

3. $\lim\limits_{x\to\infty} W\{f(x), E_\lambda(x)\} = 0, E_\lambda(x) = \int\limits_{+0}^{\lambda} \varphi(x,\mu)d\rho(\mu), (\lambda \neq 0).$

U holda

$$f(x) = \int\limits_{-\infty}^\infty \varphi(x,\lambda)d\sigma(\lambda), \qquad (3.5.37)$$

tasvir o'rinlidir, bu yerda

$$\sigma(\lambda) = \int\limits_0^\infty f(t)E_\lambda(t)dt,$$

bo'lib, (3.5.37) integral absolyut yaqinlashuvchidir.

Isbot. Agar $f(x)$ funksiya chekli oraliqdan tashqarida nolga aylanadigan bo'lsa, quyidagi

$$\sigma(\lambda) = \int_0^\infty f(t) \left(\int_{+0}^{\lambda} \varphi(t, \mu) d\rho(\mu) \right) dt,$$

tenglikda integrallash tartibini o'zgartirsak,

$$\sigma(\lambda) = \int_0^\lambda \left(\int_0^\infty f(t)\varphi(t, \mu) dt \right) d\rho(\mu),$$

ya'ni

$$\sigma(\lambda) = \int_0^\lambda F(\mu) d\rho(\mu), \tag{3.5.38}$$

tasvir hosil bo'ladi. Bu yerda

$$F(\lambda) = \int_0^\infty f(t)\varphi(t, \lambda) dt,$$

belgilash kiritilgan. Bu tasvir $f(x)$ funksiya finit bo'lmagan holda ham o'rinli bo'lishini limitga o'tish usuli bilan ko'rsatish mumkin. (3.5.38) tenglik, rezolventa uchun olingan (3.5.26) tasvirni quyidagicha qilib yozishga imkon beradi

$$R_z f(x) = \int_{-\infty}^{\infty} \frac{\varphi(x, \lambda)}{z - \lambda} d\sigma(\lambda).$$

Oxirgi integral lemma 3.5.2 ga asosan absolyut yaqinlashadi. Grin ayniyati va teoremadagi uchinchi shartga muvofiq quyidagi tenglik bajariladi:

$$\tilde{\sigma}(\lambda) = \int_0^\infty [f''(t) - q(t) f(t)] E_\lambda(t) dt =$$

$$= \int_{0}^{\infty} f(t)\left\{\int_{+0}^{\lambda} \mu\varphi(t,\mu)d\rho(\mu)\right\}dt.$$

Agar $f(x)$ funksiya chekli oraliqdan tashqarida nolga aylanadigan bo'lsa,

$$\tilde{\sigma}(\lambda) = -\int_{+0}^{\lambda} \mu F(\mu)d\rho(\mu) = -\int_{+0}^{\lambda} \mu d\sigma(\mu), \qquad (3.5.39)$$

bo'ladi. Bu formula umumiy hol uchun ham to'g'ri bo'ladi, chunki

$$\int_{+0}^{\lambda} \mu\varphi(t,\mu)d\rho(\mu) \in L^2(0,\ \infty)\ .$$

Haqiqatan ham,

$$\int_{0}^{b}\left\{\int_{+0}^{\lambda} \mu\varphi(t,\mu)d\rho_b(\mu)\right\}^2 dt = \int_{0}^{b}\left\{\sum_{0\le\lambda_{n,b}\le\lambda}\frac{\lambda_{n,b}}{\alpha_n^2}\varphi_{n,b}(t)\right\}^2 dt =$$

$$= \sum_{0\le\lambda_{n,b}\le\lambda}\frac{\lambda_{n,b}^2}{\alpha_n^2} = \int_{+0}^{\lambda} \mu^2 d\rho_b(\mu).$$

Ushbu tenglikdan ixtiyoriy $0 < a < b$ sonlar uchun

$$\int_{0}^{a}\left\{\int_{+0}^{\lambda} \mu\varphi(t,\mu)d\rho_b(\mu)\right\}^2 dt \le \int_{+0}^{\lambda} \mu^2 d\rho_b(\mu)$$

tengsizlik o'rinli ekanligi kelib chiqadi. Bu tengsizlikda avvalo b ni, keyinchalik a ni cheksizlikka intiltirsak, isbotlanayotgan fikrning bajarilishi kelib chiqadi. (3.5.39) formuladan

$$d\tilde{\sigma}(\lambda) = -\lambda d\sigma(\lambda) \quad \text{va} \quad d\sigma(\lambda) = -\frac{d\tilde{\sigma}(\lambda)}{\lambda}, \qquad (3.5.40)$$

bog'lanishlar hosil qilinadi. Teoremadagi ikkinchi shartdan

$$\int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{z-\lambda}d\tilde{\sigma}(\lambda),$$

integralning absolyut yaqinlashuvchiligi ko'rinadi. Bundan (3.5.40) tengliklarga asosan

$$\int_{-\infty}^{\infty} \varphi(x,\lambda) d\sigma(\lambda),$$

integralning absolyut yaqinlashishi kelib chiqadi. Yoyilma haqidagi teoremadan foydalansak, isbot qilinishi zarur bo'lgan fikrning to'g'riligi namoyon bo'ladi.■

6-§. Veyl-Titchmarsh funksiyasi va spektral funksiya orasidagi bog'lanishlar

Teorema 3.6.1. *Haqiqiy bo'lmagan z ning har bir qiymatida ushbu*

$$m(z) = -\operatorname{ctg}\alpha + \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{z-\lambda}, \quad (\sin\alpha \neq 0) \tag{3.6.1}$$

formula o'rinli.

Isbot. Rezolventa uchun olingan integral tasvirda $f(x)$ ixtiyoriy bo'lganligidan

$$G(x,t;z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)\varphi(t,\lambda)}{z-\lambda} d\rho(\lambda),$$

tenglik $G(x,t;z) =$ Grin funksiyasining ta'rifidan

$$G(x,t;z) = \begin{cases} [\vartheta(x,z) + m(z)\varphi(x,z)]\varphi(t,z), & t \leq x, \\ [\theta(t,z) + m(z)\varphi(t,z)]\varphi(x,z), & t > x, \end{cases}$$

boshlang'ich shartlarga asosan

$$G(0,0;z) = [\cos\alpha + m(z)\sin\alpha]\sin\alpha$$

bajariladi. Demak,

$$\sin\alpha\cos\alpha + m(z)\sin^2\alpha = \int_{-\infty}^{\infty} \frac{\sin^2\alpha}{z-\lambda} d\rho(\lambda),$$

ya'ni

$$m(z) = -\operatorname{ctg}\alpha + \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{z-\lambda},$$

tenglik o'rinli.■

Izoh 3.6.1. Agar $\sin\alpha = 0$ bo'lsa, u holda

$$m(z) = \int_{-\infty}^{\infty} \left(\frac{1}{z-\lambda} + \frac{\lambda}{1+\lambda^2}\right) d\rho\lambda + a$$

formula o'rinli. Bu yerda a o'zgarmas son

$$m(z) = -i\sqrt{z} + \rho(-\infty) + \bar{\bar{o}}(1)$$

tenglikdan aniqlanadi.

Teorema 3.6.2. *Agar $(a,\ b)$ oraliqning chetki nuqtalari $\rho(\lambda)$ spektral funksiyaning uzluksizlik nuqtalaridan iborat bo'lsa, u holda*

$$\rho(b) - \rho(a) = -\frac{1}{\pi} \lim_{v\to+0} \int_a^b \operatorname{Im}\{m(u+iv)\} du, \qquad (3.6.2)$$

tenglik o'rinli bo'ladi.

Isbot. $z = u + iv,\ \ v > 0$ bo'lsin. Quyidagi

$$H(u,v) = \frac{1}{\pi} \cdot \frac{m(z) - m(\bar{z})}{2i}, \qquad (3.6.3)$$

yordamchi funksiya kiritamiz. Teorema 3.6.1 ga ko'ra

$$H(u,v) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v\,d\rho(\lambda)}{(\lambda-u)^2 + v^2}, \qquad (3.6.4)$$

bo'ladi. Oxirgi integralni bo'laklab integrallaymiz:

$$-H(u,v) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial\lambda}\left(\frac{v}{(\lambda-u)^2+v^2}\right)\rho(\lambda)d\lambda =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial u}\left(\frac{v}{(\lambda-u)^2+v^2}\right)\rho(\lambda)d\lambda.$$

Bu tenglikni $[a, b]$ kesmada u bo'yicha integrallasak,

$$-\int_a^b H(u,v)du = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v}{(\lambda-b)^2+v^2}\rho(\lambda)d\lambda -$$

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v}{(\lambda-a)^2+v^2}\rho(\lambda)d\lambda, \tag{3.6.5}$$

hosil bo'ladi. Oxirgi tenglikni quyidagi ko'rinishda yozib olamiz:

$$-\int_a^b H(u,v)du = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v}{\lambda^2+v^2}[\rho(\lambda+b)-\rho(b)]d\lambda -$$

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{v}{\lambda^2+v^2}[\rho(\lambda+a)-\rho(a)]d\lambda + \rho(b)-\rho(a). \tag{3.6.6}$$

Spektral funksiya a nuqtada uzluksizligini hisobga olsak, yetarlicha kichik $\delta > 0$ son uchun

$$\left|\frac{1}{\pi}\int_{|\lambda|\le\delta}\frac{v}{\lambda^2+v^2}[\rho(\lambda+a)-\rho(a)]d\lambda\right| < \frac{\varepsilon}{2},$$

tengsizlik bajarilishi kelib chiqadi. Ushbu

$$\left|\frac{1}{\pi}\int_{|\lambda|>\delta}\frac{v}{\lambda^2+v^2}[\rho(\lambda+a)-\rho(a)]d\lambda\right| < \frac{\varepsilon}{2},$$

integral kichik v larda $\frac{\varepsilon}{2}$ dan kichik bo'lishini ko'rsatish mumkin. Xuddi shunday qolgan integrallar ham baholanadi. (3.6.6) tenglikda $v \to 0$ bo'yicha limitga o'tsak va $H(u, v)$ funksiyaning aniqlanishini e'tiborga olsak, (3.6.2) formula kelib chiqadi.■

Natija 3.6.1. Agar $[a, b]$ kesmada $m(\lambda)$ funksiyaning qutb nuqtalari bo'lmasa, u holda ixtiyoriy $\lambda \in (a, b)$ son uchun

$$\rho'(\lambda) = -\frac{1}{\pi} \lim_{v \to +0} \operatorname{Im}\{m(\lambda + iv)\}, \tag{3.6.7}$$

tenglik o'rinli bo'ladi.

Isbot. $[a, b]$ kesmada $m(\lambda)$ funksiyaning qutb nuqtalari bo'lmaganligi uchun (3.6.2) integralda integral belgisi ostida limitga o'tish mumkin:

$$\rho(b) - \rho(a) = -\frac{1}{\pi} \int_a^b \lim_{v \to +0} \operatorname{Im}\{m(u + iv)\} du . \tag{3.6.8}$$

Oxirgi tenglikdan λ, $\lambda + \Delta\lambda \in (a, b)$ nuqtalar uchun ushbu

$$\frac{\rho(\lambda + \Delta\lambda) - \rho(\lambda)}{\Delta\lambda} = -\frac{1}{\pi} \frac{1}{\Delta\lambda} \int_{\lambda}^{\lambda + \Delta\lambda} \lim_{v \to +0} \operatorname{Im}\{m(u + iv)\} du,$$

formula hosil bo'ladi. Bundan esa (3.6.7) munosabatning bajarilishi kelib chiqadi.■

Natija 3.6.2. Spektral funksiyaning xos qiymatdagi sakrash uzunligi Veyl-Titchmarsh funksiyasining shu nuqtadagi chegirmasiga teng:

$$\rho(\lambda_0 + 0) - \rho(\lambda_0 - 0) = \operatorname*{res}_{\lambda = \lambda_0} m(\lambda) . \tag{3.6.9}$$

Isbot. λ_0 nuqta $m(\lambda)$ uchun qutb nuqta bo'lsin. U holda quyidagi

$$m(\lambda) = \frac{c_{-1}}{\lambda - \lambda_0} + g(\lambda), \qquad |\lambda - \lambda_0| \leq \delta, \tag{3.6.10}$$

tasvir o'rinli. Bu yerda δ yetarlicha kichik musbat son, $g(\lambda)$ esa $|\lambda - \lambda_0| \leq \delta$ doirada uzluksiz funksiya. (3.6.10) tasvirdan ushbu

$$m(u+iv) = \frac{c_{-1} \cdot (u - \lambda_0 - iv)}{(u-\lambda_0)^2 + v^2} + g(u+iv),$$

tenglik kelib chiqadi. $\lambda_0 - \delta$ va $\lambda_0 + \delta$ nuqtalar spektral funksiyaning uzluksizlik nuqtalari ekanligidan, quyidagi

$$\rho(\lambda_0 + \delta) - \rho(\lambda_0 - \delta) =$$

$$= -\frac{1}{\pi}\lim_{v\to+0}\int\limits_{\lambda_0-\delta}^{\lambda_0+\delta}\left\{\frac{-c_{-1}v}{(u-\lambda_0)^2+v^2} + \operatorname{Im}\{g(u+iv)\}\right\}du =$$

$$= \frac{1}{\pi}\lim_{v\to+0}\int\limits_{\delta}^{-\delta}\left\{\frac{-c_{-1}v}{t^2+v^2} + \operatorname{Im}\{g(\lambda_0 - t + iv)\}\right\}dt =$$

$$= \frac{1}{\pi}\lim_{v\to+0}\int\limits_{-\delta}^{\delta}\frac{c_{-1}v}{t^2+v^2}dt - \frac{1}{\pi}\lim_{v\to+0}\int\limits_{-\delta}^{\delta}\operatorname{Im}\{g(\lambda_0 - t + iv)\}dt =$$

$$= \frac{c_{-1}}{\pi}\lim_{v\to+0}\left\{\operatorname{arctg}\left(\frac{\delta}{v}\right) - \operatorname{arctg}\left(\frac{-\delta}{v}\right)\right\} - \frac{1}{\pi}\int\limits_{-\delta}^{\delta}\operatorname{Im}\{g(\lambda_0 - t)\}dt =$$

$$= c_{-1} - \frac{1}{\pi}\int\limits_{-\delta}^{\delta}\operatorname{Im} g(\lambda_0 - t)dt,$$

tengliklar hosil bo'ladi.■

Natija 3.6.3. Veyl doirasi holida $m(\lambda)$ funksiya λ parametrning qutb nuqta bo'lmagan haqiqiy qiymatlarida uzluksiz bo'lib, haqiqiy qiymatlar qabul qilganligi uchun natija 3.6.1 ga ko'ra quyidagi

$$\rho'(\lambda) = -\frac{1}{\pi}\lim_{v\to+0}\operatorname{Im}\{m(\lambda+iv)\} = -\frac{1}{\pi}\operatorname{Im} m(\lambda) = 0,$$

tenglik o'rinli bo'ladi. Demak, bu holda uzluksiz spektr bo'lmaydi va spektr faqat xos qiymatlardan iborat bo'ladi.

(3.6.1) formuladan $|z| \to \infty$ intilganda $m(z)$ Veyl-Titchmarsh funksiyasining asimptotikasini topishda ham foydalanishimiz mumkin. Shu maqsadda quyidagi

$$-y'' + q(x)y = \lambda y, \quad (0 \le x < \infty), \tag{3.6.11}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{3.6.12}$$

chegaraviy masalasini qaraymiz. Bu yerda $q(x) \in C[0,\infty)$ haqiqiy uzluksiz funksiya, $\alpha \in R^1$ haqiqiy son.

Agar $\sin\alpha \neq 0$ bo'lsa, u holda (3.6.12) chegaraviy shartni ushbu

$$y'(0) - hy(0) = 0, \quad h = -\operatorname{ctg}\alpha, \tag{3.6.13}$$

ko'rinishda yozish mumkin. (3.6.1) formula (3.6.12) chegaraviy shart bajarilganda keltirib chiqarilgan edi. Shuning uchun (3.6.1) formuladagi $m(z)$ va $\rho(\lambda)$ funksiyalarni mos ravishda $m_\alpha(z)$ va $\rho_\alpha(\lambda)$ orqali belgilaymiz. Agar (3.6.11)÷(3.6.13) chegaraviy masalaning spektral funksiyasini $\rho_h(\lambda)$ orqali belgilasak, u holda $\rho_\alpha(\lambda)$ va $\rho_h(\lambda)$ funksiyalar o'zaro

$$\rho_\alpha(\lambda) = \frac{1}{\sin^2\alpha} \cdot \rho_h(\lambda), \tag{3.6.14}$$

formula orqali bog'langan. Shuning uchun (3.6.1) formulani quyidagi

$$m_\alpha(z) = -\operatorname{ctg}\alpha + \int\limits_{-\infty}^{\infty} \frac{d\rho_\alpha(\lambda)}{z-\lambda} = h + \frac{1}{\sin^2\alpha}\int\limits_{-\infty}^{\infty} \frac{d\rho_h(\lambda)}{z-\lambda}, \tag{3.6.15}$$

ko'rinishda yozish mumkin.

Ikkinchi tomondan $\rho_h(\lambda)$ spektral funksiya uchun ushbu

$$\rho_h(\lambda) = \frac{2}{\pi}\sqrt{\lambda} - h + \rho_h(-\infty) + \overline{\overline{o}}(1), \quad \lambda \to +\infty, \tag{3.6.16}$$

asimptotik formula o'rinli ([81], 397-bet). Spektral funksiyaning bu asimptotikasidan va (3.6.15) formuladan Veyl-Titchmarshning $m_\alpha(z)$ funksiyasi uchun asimptotik formula topishimiz mumkin.

Teorema 3.6.3. *(3.6.11), (3.6.12) chegaraviy masalaning Veyl-Titchmarsh* $m_\alpha(z)$ *funksiyasi uchun ushbu* $\delta < \arg z < \pi - \delta$ *sohada*

$$m_\alpha(z) = -\operatorname{ctg}\alpha - \frac{i}{\sqrt{z}\sin^2\alpha} + \frac{\operatorname{ctg}\alpha}{z\sin^2\alpha} + \bar{\bar{o}}\left(\frac{1}{z}\right), \quad |z| \to \infty, \tag{3.6.17}$$

asimptotik formula o'rinli bo'ladi. Bu yerda $\delta > 0$ *- biror musbat son.*

Isbot. Avvalo $\lambda = 0$ nuqta $\rho_h(\lambda)$ funksiyaning uzluksizlik nuqtasi bo'lsin va $\rho_h(0) = 0$ deb faraz qilamiz. Bu holda (3.6.15) tenglikdagi integralda bo'laklab integrallash qoidasidan foydalanib, ushbu

$$\int_{-\infty}^{0} \frac{d\rho_h(\lambda)}{z-\lambda} = \int_{-\infty}^{0} \frac{d[\rho_h(\lambda) - \rho_h(-\infty)]}{z-\lambda} = \frac{\rho_h(\lambda) - \rho_h(-\infty)}{z-\lambda}\bigg|_{-\infty}^{0} -$$

$$-\int_{-\infty}^{0} \frac{\rho_h(\lambda) - \rho_h(-\infty)}{(z-\lambda)^2} d\lambda = -\frac{\rho_h(-\infty)}{z} + \bar{\bar{o}}\left(\frac{1}{z}\right), \tag{3.6.18}$$

$$\int_{0}^{\infty} \frac{d\rho_h(\lambda)}{z-\lambda} = \int_{0}^{\infty} \frac{d[\rho_h(\lambda) - 2\sqrt{\lambda}/\pi + h - \rho_h(-\infty)]}{z-\lambda} + \frac{2}{\pi}\int_{0}^{\infty} \frac{d(\sqrt{\lambda})}{z-\lambda} =$$

$$= \frac{\rho_h(\lambda) - \frac{2}{\pi}\sqrt{\lambda} + h - \rho_h(-\infty)}{z-\lambda}\bigg|_{0}^{\infty} - \int_{0}^{\infty} \frac{\bar{\bar{o}}(1)}{(z-\lambda)^2} d\lambda - \frac{i}{\sqrt{z}} =$$

$$= \frac{-h + \rho_h(-\infty)}{z} - \frac{i}{\sqrt{z}} + \bar{\bar{o}}\left(\frac{1}{z}\right), \tag{3.6.19}$$

ifodalarni topamiz. Bu yerda quyidagi

$$\frac{2}{\pi}\int_0^\infty \frac{d(\sqrt{\lambda})}{z-\lambda} = \frac{2}{\pi}\int_0^\infty \frac{dt}{z-t^2} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{dt}{z-t^2} =$$

$$= \frac{1}{\pi}2\pi i \operatorname*{res}_{t=\sqrt{z}} \frac{1}{z-t^2} = \frac{1}{\pi}2\pi i\left(-\frac{1}{2\sqrt{z}}\right) = -\frac{i}{\sqrt{z}}$$

tenglikdan foydalanildi. Topilgan ifodalarni (3.6.15) formulaga qo'ysak,

$$m_\alpha(z) = -\operatorname{ctg}\alpha + \frac{1}{\sin^2\alpha}\left[-\frac{\rho_h(-\infty)}{z} + \bar{\bar{o}}\left(\frac{1}{z}\right) + \frac{-h+\rho_h(-\infty)}{z} -\right.$$

$$\left. -\frac{i}{\sqrt{z}} + \bar{\bar{o}}\left(\frac{1}{z}\right)\right] = -\operatorname{ctg}\alpha + \frac{1}{\sin^2\alpha}\left[\frac{\operatorname{ctg}\alpha}{z} - \frac{i}{\sqrt{z}} + \bar{\bar{o}}\left(\frac{1}{z}\right)\right] =$$

$$= -\operatorname{ctg}\alpha + \frac{\operatorname{ctg}\alpha}{z\sin^2\alpha} - \frac{i}{\sqrt{z}\sin^2\alpha} + \bar{\bar{o}}\left(\frac{1}{z}\right),$$

hosil bo'ladi.

Agar $\lambda = 0$ nuqta $\rho_h(\lambda)$ funksiyaning uzilish nuqtasi bo'lib, sakrash uzunligi $a_0 > 0$ ga teng bo'lsa u holda

$$\tilde{\rho}(\lambda) = \rho_h(\lambda) - a_0\theta(\lambda),$$

deb olamiz. Bu yerda

$$\theta(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ 1, & \lambda > 0 \end{cases}$$

Hevisayda funksiyasi. Aniqlanishiga ko'ra $\tilde{\rho}(-\infty) = \rho(-\infty)$ bo'lib, $\tilde{\rho}(\lambda)$ funksiya $\lambda = 0$ nuqtada uzluksiz. Shuning uchun (3.6.16) dan

$$\tilde{\rho}(\lambda) = \frac{2}{\pi}\sqrt{\lambda} - h + \rho_h(-\infty) - a_0 + o(1), \quad \lambda \to +\infty$$

kelib chiqadi. Xuddi yuqoridagidek,

$$\int_{-\infty}^\infty \frac{d\rho_h(\lambda)}{z-\lambda} = \int_{-\infty}^\infty \frac{d\tilde{\rho}(\lambda)}{z-\lambda} + a_0\int_{-\infty}^\infty \frac{d\theta(\lambda)}{z-\lambda} = \int_{-\infty}^\infty \frac{d\tilde{\rho}(\lambda)}{z-\lambda} + \frac{a_0}{z} =$$

$$= -\frac{i}{\sqrt{z}} - \frac{h+a_0}{z} + \frac{a_0}{z} + \bar{\bar{o}}\left(\frac{1}{z}\right) = -\frac{i}{\sqrt{z}} - \frac{h}{z} + \bar{\bar{o}}\left(\frac{1}{z}\right)$$

tenglikni topamiz. Demak, bu holda ham (3.6.17) asimptotik formula o'rinli bo'lar ekan. ■

7-§. Spektral funksiyani topishga doir masalalar yechish namunalari

Misol 1. Ushbu

$$\begin{cases} -y'' = \lambda y, & 0 \le x < \infty \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, & (\operatorname{ctg}\alpha < 0, \ \sin\alpha \ne 0), \end{cases}$$

chegaraviy masalaning Veyl-Titchmarsh va spektral funksiyalarini topamiz. Ushbu

$$-y'' = \lambda y,$$

differensial tenglamaning umumiy yechimi

$$y(x) = c_1 \cos\sqrt{\lambda}x \ + \ c_2\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

tenglik bilan beriladi. Quyidagi

$$\begin{cases} \theta(0,\lambda) = \cos\alpha, \\ \theta'(0,\lambda) = \sin\alpha\ , \end{cases} \quad \begin{cases} \varphi(0,\lambda) \ = \ \sin\alpha, \\ \varphi'(0,\lambda) = -\cos\alpha, \end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlari esa

$$\theta(x,\lambda) = \cos\alpha\ \cos\sqrt{\lambda}x + \sin\alpha\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$\varphi(x,\lambda) = \sin\alpha\ \cos\sqrt{\lambda}x - \cos\alpha\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

formulalar orqali aniqlanadi. $q(x) \equiv 0$ koeffitsiyent quyidan chegaralanganligi uchun Veylning nuqta holi o'rinli bo'ladi. Ushbu

$$\theta(x,\lambda) + m(\lambda)\varphi(x,\lambda) = \cos\alpha\ \cos\sqrt{\lambda}x + m(\lambda)\sin\alpha\cos\sqrt{\lambda}x +$$

$$+\sin\alpha\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - m(\lambda)\cos\alpha\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} =$$

$$= \frac{1}{2}\left\{\cos\alpha + m(\lambda)\sin\alpha - \frac{i}{\sqrt{\lambda}}\sin\alpha + \frac{i}{\sqrt{\lambda}}m(\lambda)\cos\alpha\right\}e^{i\sqrt{\lambda}x} +$$

$$+\frac{1}{2}\left\{\cos\alpha + m(\lambda)\sin\alpha + \frac{i}{\sqrt{\lambda}}\sin\alpha - \frac{i}{\sqrt{\lambda}}m(\lambda)\cos\alpha\right\}e^{-i\sqrt{\lambda}x}, \tag{3.7.1}$$

funksiya $L^2(0,\infty)$ fazoga tegishli bo'ladigan qilib, $m(\lambda)$ funksiyani tanlash kerak. Kompleks ildizning quyidagi

$$\sqrt{u+iv} = \sqrt{\frac{u+\sqrt{u^2+v^2}}{2}} + i\sqrt{\frac{-u+\sqrt{u^2+v^2}}{2}}, \quad v > 0,$$

shoxchasini tanlash qabul qilinganligi uchun $\operatorname{Im}\lambda > 0$ bo'lganda

$$e^{i\sqrt{\lambda}x} \in L^2(0,\infty) \text{ va } e^{-i\sqrt{\lambda}x} \notin L^2(0,\infty),$$

bo'ladi. Demak, (3.7.1) tenglikda ikkinchi qavs ichidagi ifodani nolga tenglash zarur ekan:

$$\cos\alpha + m(\lambda)\sin\alpha + \frac{i}{\sqrt{\lambda}}\sin\alpha - \frac{i}{\sqrt{\lambda}}m(\lambda)\cos\alpha = 0,$$

$$\sqrt{\lambda}\cos\alpha + m(\lambda)\sqrt{\lambda}\sin\alpha + i\sin\alpha - im(\lambda)\cos\alpha = 0,$$

$$m(\lambda)(\sqrt{\lambda}\sin\alpha - i\cos\alpha) = -i\sin\alpha - \sqrt{\lambda}\cos\alpha,$$

$$m(\lambda) = \frac{\sin\alpha - i\sqrt{\lambda}\cos\alpha}{\cos\alpha + i\sqrt{\lambda}\sin\alpha}. \tag{3.7.2}$$

Shunday qilib, biz Veyl-Titchmarsh funksiyasini topdik. U (3.7.2) formula bilan berilar ekan. $m(\lambda)$ funksiya $\operatorname{Im}\lambda \geq 0$ to'plamga uzluksizligi bo'yicha (3.7.2) formulaga binoan davom qildiriladi.

Agar $\lambda \leq 0$ bo'lsa, u holda $\sqrt{\lambda} = i\sqrt{|\lambda|}$ bo'lgani uchun ushbu

$$m(\lambda) = \frac{\sin\alpha + \sqrt{|\lambda|}\cos\alpha}{\cos\alpha - \sqrt{|\lambda|}\sin\alpha},$$

tenglik o'rinli bo'ladi. Oxirgi ifodaning maxraji $\sqrt{|\lambda|} = \operatorname{ctg}\alpha$ bo'lganda nolga aylanadi. Bu tenglama $\operatorname{ctg}\alpha \geq 0$ bo'lganda yechimga ega va $\operatorname{ctg}\alpha < 0$ bo'lganda yechimga ega emas. Biz qarayotgan masalada $\operatorname{ctg}\alpha < 0$. Demak, bu holda qutb nuqta yo'q va

$$\operatorname{Im}\{m(\lambda)\} = 0, \quad (\lambda \leq 0) \tag{3.7.3}$$

bo'ladi.

Agar $\lambda > 0$ bo'lsa, u holda $m(\lambda)$ funksiyaning maxraji nolga aylanmaydi. $\operatorname{Im}\{m(\lambda)\}$ ni topamiz:

$$m(\lambda) = \frac{\sin\alpha - i\sqrt{\lambda}\cos\alpha}{\cos\alpha + i\sqrt{\lambda}\sin\alpha} =$$

$$= \frac{[\sin\alpha - i\sqrt{\lambda}\cos\alpha]\cdot[\cos\alpha - i\sqrt{\lambda}\sin\alpha]}{\cos^2\alpha + \lambda\sin^2\alpha} =$$

$$= \frac{\sin\alpha\cos\alpha(1-\lambda) - i\sqrt{\lambda}}{\cos^2\alpha + \lambda\sin^2\alpha},$$

bo'lganligi uchun

$$\operatorname{Im}\{m(\lambda)\} = -\frac{\sqrt{\lambda}}{\cos^2\alpha + \lambda\sin^2\alpha}, \quad (\lambda > 0), \tag{3.7.4}$$

bo'ladi. (3.7.3) va (3.7.4) tengliklardan natija 3.6.1 ga asosan ushbu formula kelib chiqadi:

$$\rho'(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda}}{\pi(\cos^2\alpha + \lambda\sin^2\alpha)}, & \lambda > 0, \\ 0, & \lambda \leq 0. \end{cases} \tag{3.7.5}$$

Bu yerda $\rho(-0) = 0$ normallashtirish shartini va $\lambda = 0$ nuqta xos qiymat emasligini hisobga olib,

$$\rho(\lambda) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^\lambda \frac{\sqrt{t}}{\cos^2\alpha + t\sin^2\alpha}dt, & \lambda > 0, \\ 0, & \lambda \leq 0, \end{cases}$$

spektral funksiyani topamiz. Oxirgi integrallarni hisoblasak, ushbu

$$\rho(\lambda) = \begin{cases} \dfrac{2}{\pi \sin^2 \alpha}\sqrt{\lambda} - \dfrac{2\cos\alpha}{\pi \sin^3 \alpha} \cdot \operatorname{arctg}\left(\dfrac{\sqrt{\lambda}\sin\alpha}{\cos\alpha}\right), & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

formula hosil bo'ladi.

Misol 2. Ushbu

$$\begin{cases} -y'' = \lambda y \ , \quad 0 \le x < \infty, \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad (\operatorname{ctg}\alpha > 0, \ \sin\alpha \ne 0), \end{cases}$$

chegaraviy masalaning Veyl-Titchmarsh va spektral funksiyalarini topamiz. Qaralayotgan masalaning Veyl-Titchmarsh funksiyasi xuddi birinchi masaladagidek topiladi va (3.7.2) formula bilan beriladi, ammo $\operatorname{ctg}\alpha > 0$ bo'lganligi uchun bitta qutb nuqta hosil bo'ladi: $\lambda_0 = -\operatorname{ctg}^2\alpha$. Veyl-Titchmarsh funksiyasining shu nuqtadagi chegirmasini hisoblaymiz. Bu nuqta oddiy qutb nuqta bo'lganligi uchun

$$\operatorname*{res}_{\lambda=\lambda_0} m(\lambda) = \lim_{\lambda\to\lambda_0} (\lambda - \lambda_0) m(\lambda),$$

formula o'rinli. Demak,

$$m(\lambda) = \frac{(1-\lambda)\sin\alpha\cos\alpha}{\lambda\sin^2\alpha + \cos^2\alpha} - i\frac{\sqrt{\lambda}}{\lambda\sin^2\alpha + \cos^2\alpha},$$

$$(\lambda - \lambda_0)m(\lambda) = \frac{(1-\lambda)\cos\alpha}{\sin\alpha} - i\frac{\sqrt{\lambda}}{\sin^2\alpha} \to$$

$$\to \frac{(1+\operatorname{ctg}^2\alpha)\cos\alpha}{\sin\alpha} - i\frac{i|\operatorname{ctg}\alpha|}{\sin^2\alpha} = 2\frac{\cos\alpha}{\sin^3\alpha}, \quad (\lambda \to \lambda_0),$$

bo'lgani uchun

$$\operatorname*{res}_{\lambda=\lambda_0} m(\lambda) = 2\frac{\cos\alpha}{\sin^3\alpha},$$

tenglik bajariladi.

Natija 3.6.1 va natija 3.6.2 ga asosan, $\rho(-0) = 0$ normallashtirish shartiga ko'ra yuqoridagilarni hisobga olsak,

$$\rho(\lambda) = \begin{cases} \frac{1}{\pi}\int\limits_0^{\lambda} \frac{\sqrt{t}}{\cos^2\alpha + t\sin^2\alpha}dt, & \lambda > 0, \\ 0, & -\operatorname{ctg}^2\alpha < \lambda \le 0, \\ -\frac{2\cos\alpha}{\sin^3\alpha}, & \lambda \le -\operatorname{ctg}^2\alpha, \end{cases}$$

bo'lishi kelib chiqadi. $\lambda_0 = -\operatorname{ctg}^2\alpha$ xos qiymatga mos keluvchi xos funksiyani topishni o'quvchiga vazifa sifatida qoldiramiz.

Misol 3. Ushbu

$$\begin{cases} -y'' = \lambda y, & 0 \le x < \infty, \\ y(0) = 0, \end{cases}$$

chegaraviy masalaning Veyl-Titchmarsh va spektral funksiyalarini topamiz. Bu holda ham, birinchi misolda qo'llanilgan amallarni bajarsak,

$$m(\lambda) = -i\sqrt{\lambda},$$

bo'lishi kelib chiqadi. Bu funksiyaning qutb nuqtalari bo'lmaganligi uchun spektral funksiya uzluksiz bo'ladi, berilgan chegaraviy masala xos qiymatga ega bo'lmaydi. Ushbu

$$\operatorname{Im}\{m(\lambda)\} = \begin{cases} -\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

tenglikdan, natija 3.6.1 ga asosan

$$\rho'(\lambda) = \begin{cases} \frac{1}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

kelib chiqadi. $\rho(-0) = 0$ normallashtirish shartiga binoan

$$\rho(\lambda) = \begin{cases} \frac{2}{3\pi}\sqrt{\lambda^3}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

hosil bo'ladi.

Misol 4. Ushbu

$$\begin{cases} -y'' - \dfrac{2a^2}{ch^2(ax+b)}y = \lambda y\,, & (a > 0,\ b \in R^1), \\ y'(0) = -athb \cdot y(0), \end{cases}$$

Shturm-Liuvill masalasining spektral funksiyasini topamiz.

Agar berilgan differensial tenglamaning xususiy yechimini

$$y(x) = \cos\sqrt{\lambda}x + a(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

ko'rinishda izlasak,

$$y_1(x) = \cos\sqrt{\lambda}x - a\,th(ax+b)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

yechim kelib chiqadi. Agar xususiy yechimni ushbu

$$y(x) = \sqrt{\lambda}\sin\sqrt{\lambda}x + b(x)\cos\sqrt{\lambda}x,$$

ko'rinishda izlasak,

$$y_2(x) = \sqrt{\lambda}\sin\sqrt{\lambda}x + a\,th(ax+b)\cos\sqrt{\lambda}x,$$

yechim kelib chiqadi.

Bu yechimlar uchun

$$y_1(0) = 1, \qquad y_1'(0) = -a\,th\,b,$$

va

$$y_2(0) = a\,th\,b, \qquad y_2'(0) = \lambda + a^2 - a^2th^2\,b,$$

boshlang'ich shartlar bajariladi. Bu yechimlar yordamida tuzilgan ushbu

$$\tilde{y}(x) = \frac{y_2(x) - a\,th\,b\,y_1(x)}{\lambda + a^2}, \quad (\lambda \neq -a^2),$$

yechim esa

$$\tilde{y}(0) = 0\,, \quad \tilde{y}'(0) = 1,$$

boshlang'ich shartlarni qanoatlantiradi.

Quyidagi

$$\begin{cases} \theta(0,\lambda) = \cos\alpha \\ \theta'(0,\lambda) = \sin\alpha \end{cases} \quad \text{va} \quad \begin{cases} \varphi(0,\lambda) = \sin\alpha \\ \varphi'(0,\lambda) = -\cos\alpha \end{cases}$$

boshlang'ich shartlar bilan aniqlanadigan $\theta(x,\lambda)$ va $\varphi(x.\lambda)$ yechimlarni tuzib olamiz. Bu yerda

$$\cos\alpha = \frac{a\,th\,b}{\sqrt{1+a^2\,th^2\,b}} \quad \text{va} \quad \sin\alpha = \frac{1}{\sqrt{1+a^2\,th^2\,b}}$$

belgilash kiritildi. Bevosita

$$\sqrt{1+a^2\,th^2\,b}\,\varphi(x,\lambda) \equiv y_1(x) =$$

$$= \left[\frac{1}{2} + i\frac{a\,th(ax+b)}{2\sqrt{\lambda}}\right]e^{i\sqrt{\lambda}x} + \left[\frac{1}{2} - i\frac{a\,th(ax+b)}{2\sqrt{\lambda}}\right]e^{-i\sqrt{\lambda}x},$$

tenglikning bajarilishini ko'rish mumkin. Ushbu

$$y(x) = c_1 y_1(x) + c_2 \tilde{y}(x),$$

yechim uchun

$$y(0) = c_1, \qquad y'(0) = -a\,th\,b\cdot c_1 + c_2,$$

boshlang'ich shartlar bajariladi. Bunda

$$c_1 = \frac{a\,th\,b}{\sqrt{1+a^2\,th^2\,b}} \quad \text{va} \quad c_2 = \frac{1+a^2\,th^2\,b}{\sqrt{1+a^2\,th^2\,b}}$$

deb olsak, quyidagi

$$\theta(x,\lambda) = \frac{1}{\sqrt{1+a^2\,th^2\,b}}\left\{a\,th\,b\,y_1(x) + \frac{(1+a^2\,th^2\,b)}{\lambda+a^2}y_2(x) - \right.$$

$$\left. -\frac{(1+a^2\,th^2\,b)a\,th\,b}{\lambda+a^2}y_1(x) \right\} =$$

$$= \frac{1}{\sqrt{1+a^2\,th^2\,b}}\left\{\frac{(\lambda+a^2)a\,th\,b - (1+a^2\,th^2\,b)a\,th\,b}{\lambda+a^2}y_1(x) + \right.$$

$$\left. + \frac{(1+a^2\,th^2\,b)}{\lambda+a^2}y_2(x)\right\},$$

tenglik kelib chiqadi. $\theta(x,\lambda)$ yechimni quyidagi ko'rinishda yozamiz:

$$\theta(x,\lambda)=\frac{1}{2(\lambda+a^2)\sqrt{1+a^2\,th^2\,b}}\times$$
$$\times\left\{\left[(\lambda+a^2-1-a^2\,th^2\,b)a\,th\,b\left(1+i\frac{a\,th\,(ax+b)}{\sqrt{\lambda}}\right)+\right.\right.$$
$$\left.+(1+a^2\,th^2\,b)(a\,th(ax+b)-i\sqrt{\lambda})\quad\right]e^{i\sqrt{\lambda}x}+$$
$$+\left[(\lambda+a^2-1-a^2\,th^2\,b)a\,th\,b\left(1-i\frac{a\,th\,(ax+b)}{\sqrt{\lambda}}\right)+\right.$$
$$\left.\left.+(1+a^2\,th^2\,b)(a\,th(ax+b)+i\sqrt{\lambda})\quad\right]e^{-i\sqrt{\lambda}x}\quad\right\}.$$

Im $\lambda>0$ bo'lganda

$$\theta(x,\lambda)+m(\lambda)\varphi(x,\lambda)\in L_2(0,\infty),$$

bo'ladigan qilib $m(\lambda)$ funksiyani tanlaymiz.

Ushbu

$$2(\lambda+a^2)\sqrt{1+a^2\,th^2\,b}[\theta(x,\lambda)+m(\lambda)\varphi(x,\lambda)]=$$
$$=\left\{(\lambda+a^2-1-a^2\,\mathrm{th}^2\,b)a\,\mathrm{th}\,b\left(1+i\frac{a\,\mathrm{th}(ax+b)}{\sqrt{\lambda}}\right)+\right.$$
$$+(1+a^2\,\mathrm{th}^2\,b)(a\,\mathrm{th}(ax+b)-i\sqrt{\lambda})+$$
$$\left.+m(\lambda)(\lambda+a^2)\left(1+i\frac{a\,\mathrm{th}(ax+b)}{\sqrt{\lambda}}\right)\right\}e^{i\sqrt{\lambda}x}+$$
$$+\left\{(\lambda+a^2-1-a^2\,\mathrm{th}^2\,b)a\,\mathrm{th}\,b\left(1-i\frac{a\,\mathrm{th}(ax+b)}{\sqrt{\lambda}}\right)+\right.$$
$$+(1+a^2\,\mathrm{th}^2\,b)(a\,\mathrm{th}(ax+b)+i\sqrt{\lambda})+$$
$$\left.+m(\lambda)(\lambda+a^2)\left(1-i\frac{a\,\mathrm{th}(ax+b)}{\sqrt{\lambda}}\right)\right\}e^{-i\sqrt{\lambda}x},$$

tenglik o'ng tomonidagi birinchi qo'shiluvchi $L^2(0,\infty)$ fazoga tegishli, ikkinchi qo'shiluvchi ham shu fazoga tegishli bo'lishi uchun, xususan uning oldidagi koeffitsiyenti x cheksizlikka intilganda nolga aylanishi kerak:

$$(\lambda + a^2 - 1 - a^2 \operatorname{th}^2 b) a \operatorname{th} b \left(1 - i\frac{a}{\sqrt{\lambda}}\right) +$$

$$+(1 + a^2 \operatorname{th}^2 b)(a + i\sqrt{\lambda}) + m(\lambda)(\lambda + a^2)\left(1 - i\frac{a}{\sqrt{\lambda}}\right) = 0 ,$$

$$(\lambda + a^2 - 1 - a^2 \operatorname{th}^2 b) a \operatorname{th} b + (1 + a^2 \operatorname{th}^2 b) i\sqrt{\lambda} + m(\lambda)(\lambda + a^2) = 0 ,$$

$$m(\lambda) = -\frac{(\lambda + a^2 - 1 - a^2 \operatorname{th}^2 b) a \operatorname{th} b}{\lambda + a^2} - i\frac{(1 + a^2 \operatorname{th}^2 b)\sqrt{\lambda}}{\lambda + a^2}.$$

$m(\lambda)$ funksiyani $\operatorname{Im} \lambda \geq 0,\ \lambda \neq -a^2$ sohaga uzluksizligi bo'yicha davom qildiramiz.

Quyidagi

$$\operatorname{Im}\{m(\lambda)\} = \begin{cases} -\dfrac{(1 + a^2 \operatorname{th}^2 b)\sqrt{\lambda}}{\lambda + a^2} & \lambda > 0, \\ 0, & \lambda \leq 0,\ \lambda \neq -a^2, \end{cases}$$

tenglik bajarilishi ravshan. $\lambda_0 = -a^2$ nuqta $m(\lambda)$ funksiyaning qutb nuqtasi bo'ladi.

Veyl-Titchmarsh funksiyasining ushbu nuqtadagi chegirmasini hisoblaymiz:

$$(\lambda - \lambda_0) m(\lambda) = -(\lambda + a^2 - 1 - a^2 \operatorname{th}^2 b) a \operatorname{th} b -$$

$$-i(1 + a^2 \operatorname{th}^2 b)\sqrt{\lambda} \rightarrow (1 + a^2 \operatorname{th}^2 b)(a + a \operatorname{th} b) , \quad (\lambda \rightarrow \lambda_0).$$

Demak,

$$\operatorname*{res}_{\lambda=\lambda_0} m(\lambda) = (1 + a^2 \operatorname{th}^2 b)(a + a \operatorname{th} b),$$

bo'lar ekan.

Natija 3.6.1 va natija 3.6.2 ga asosan $\rho(-0) = 0$ normallashtirish shartiga ko'ra

$$\rho(\lambda) = \begin{cases} \int\limits_0^\lambda \dfrac{(1 + a^2 \operatorname{th}^2 b)\sqrt{t}}{\pi(t + a^2)} dt, & \lambda > 0, \\ 0, & -a^2 < \lambda \leq 0, \\ -(1 + a^2 \operatorname{th}^2 b)(a + a \operatorname{th} b), & \lambda \leq -a^2, \end{cases}$$

bo'ladi.

Izoh 3.7.1. Yuqorida ko'rilgan misolda spektr b ga bog'liq emas.

Agar $a = \sqrt{c}$ va $b = -4c\sqrt{c}t$ belgilash kiritsak, $q(x)$ potensial quyidagi ko'rinishni oladi:

$$q(x) \equiv u(x,t) = -\frac{2c}{\operatorname{ch}^2 \sqrt{c}(x-4ct)}.$$

Hosil bo'lgan $u(x,t)$ funksiya

$$u_t - 6uu_x + u_{xxx} = 0$$

tenglamani qanoatlantiradi. Bu tenglamaga Korteveg-de Friz tenglamasi deyiladi. Demak, Korteveg-de Friz tenglamasining shunday yechimlari bor ekanki, ularni Shturm-Liuvill operatorining potensiali o'rniga qo'ysa, hosil bo'lgan operatorning spektri t ga bog'liq bo'lmaydi.

Misol 5. Ushbu

$$\begin{cases} -y'' + \dfrac{p^2 - \frac{1}{4}}{x^2} y = \lambda y, & 1 < x < \infty, \\ y(1) = 0, & p \geq 0,\ p \neq \frac{1}{2}; \end{cases}$$

Shturm-Liuvill chegaraviy masalasining spektral funksiyasini topamiz.

$\lambda \neq 0$ bo'lganda

$$-y'' + \frac{p^2 - \frac{1}{4}}{x^2} y = \lambda y,$$

tenglamaning umumiy yechimi

$$y(x,\lambda) = c_1\sqrt{x}J_p(x\sqrt{\lambda}) + c_2\sqrt{x}\Upsilon_p(x\sqrt{\lambda}),$$

tenglik bilan beriladi. Bu yerda qatnashayotgan

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+p}}{n!(p+n+1)},$$

va

$$\Upsilon_p(z) = \frac{J_p(z)\cos p\pi - J_{-p}(z)}{\sin p\pi},$$

funksiyalarga mos ravishda Bessel va Veber funksiyalari deyiladi. Bu funksiyalar uchun ushbu

$$J_p(z)\Upsilon_p'(z) - J_p'(z)\Upsilon(z) = \frac{2}{\pi z}$$

ayniyat o'rinli.

Im $\lambda > 0$ bo'lib, $z = x\sqrt{\lambda}$ bo'lsin. U holda $0 < \arg z < \frac{\pi}{2}$ tengsizlik bajariladi. $|\arg z| < \pi - \varepsilon$ bo'lib, $|z| \to \infty$ bo'lganda

$$J_p(z) + i\Upsilon_p(z) = H_{p,1}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right)},$$

$$J_p(z) - i\Upsilon_p(z) = H_{p,2}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{p\pi}{2} - \frac{\pi}{4}\right)}$$

asimptotikalar o'rinli bo'lishi ma'lum.

Umumiy yechimning ko'rinishidan, bu misolda, Veylning nuqta holi o'rinli bo'lishi kelib chiqadi, chunki $L^2(1,\infty)$ fazoga tegishli bo'lmagan yechimlar mavjud. $L^2(1,\infty)$ fazoga tegishli bo'lgan yechim albatta mavjud, bu yechim

$$y(x,\lambda) = \sqrt{x}H_{p,1}(x\sqrt{\lambda}),$$

formula bilan berilishi ravshan.

$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$ – Veyl yechimi bo'lsin. Bu yerda $\theta(x,\lambda)$ va $\varphi(x,\lambda)$ orqali

$$\begin{cases} \theta(1,\lambda) = 1 \\ \theta'(1,\lambda) = 0 \end{cases} \quad \text{va} \quad \begin{cases} \varphi(1,\lambda) = 0 \\ \varphi'(1,\lambda) = -1 \end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlar belgilangan. Veyl nuqtasi holi bo'lgani uchun $L^2(1,\infty)$ fazoga tegishli yechimlar o'zaro proporsional bo'ladi:

$$\psi(x,\lambda) = A\sqrt{x}H_{p,1}(x\sqrt{\lambda}).$$

Boshlang'ich shartlarga binoan

$$\psi(1,\lambda) = 1, \quad \psi'(1,\lambda) = -m(\lambda),$$

bo'lgani uchun

$$m(\lambda) = -\frac{\psi'(1,\lambda)}{\psi(1,\lambda)} = -\frac{\frac{1}{2}H_{p,1}(\sqrt{\lambda}) + H'_{p,1}(\sqrt{\lambda})\sqrt{\lambda}}{H_{p,1}(\sqrt{\lambda})},$$

ya'ni

$$m(\lambda) = -\sqrt{\lambda}\frac{H'_{p,1}(\sqrt{\lambda})}{H_{p,1}(\sqrt{\lambda})} - \frac{1}{2}$$

formulalar o'rinli bo'ladi.

Agar $\lambda \geq 0$ bo'lsa

$$\operatorname{Im}\{m(\lambda)\} = -\frac{2}{\pi}\frac{1}{J_p^2(\sqrt{\lambda}) + \Upsilon_p^2(\sqrt{\lambda})}, \tag{3.7.6}$$

bo'ladi. Oxirgi kasr maxraji nolga aylanmaydi, chunki $J_p(z)$ va $\Upsilon_p(z)$ funksiyalarning nollari o'zaro almashinib keladi. Demak, $[0,\infty)$ to'plam uzluksiz spektr bo'ladi.

$\lambda < 0$ bo'lsin. U holda

$$H_{p,1}(z) = \frac{2}{\pi i}e^{-\frac{p\pi}{2}i}K_p\left(\frac{z}{i}\right),$$

formulani ishlatib,

$$\operatorname{Im}\{m(\lambda)\} = -\operatorname{Im}\left\{\sqrt{|\lambda|}\frac{K'_p(\sqrt{|\lambda|})}{K_p(\sqrt{|\lambda|})}\right\}, \quad (\lambda < 0), \tag{3.7.7}$$

tenglikni hosil qilamiz. Bu yerdagi $K_p(x)$ Makdonald funksiyasi bo'lib, u quyidagi tengliklar bilan aniqlanadi:

$$K_p(x) = \frac{\pi}{2}\frac{I_{-p}(x) - I_p(x)}{\sin p\pi},$$

$$I_p(x) = \sum_{k=0}^{\infty}\frac{1}{k!(p+k+1)}\left(\frac{x}{2}\right)^{2k+p}.$$

$K_p(x)$ funksiya argumentning haqiqiy qiymatlarida haqiqiy qiymatlar qabul qiladi va musbat qiymatlarida musbat qiymatlar qabul qiladi. Bundan (3.7.7) tenglik o'ng tomonida qavs ichida turgan ifoda haqiqiy bo'lib, maxsuslikka ega bo'lmasligi kelib chiqadi. Demak,

$$\operatorname{Im}\{m(\lambda)\} = 0\,, \qquad (\lambda < 0), \tag{3.7.8}$$

tenglik bajariladi. Normallashtirish sharti, hamda (3.7.6) va (3.7.8) tengliklar yordamida ushbu

$$\rho(\lambda) = \begin{cases} \dfrac{2}{\pi^2}\displaystyle\int_0^{\lambda} \dfrac{du}{J_p^2(\sqrt{u}) + \Upsilon_p^2(\sqrt{u})}, & \lambda \in (0,\infty)\,, \\ 0, & \lambda \in (-\infty,\ 0]\,, \end{cases}$$

formulani keltirib chiqaramiz. Agar $u = t^2$ almashtirish bajarsak

$$\rho(\lambda) = \begin{cases} \dfrac{4}{\pi^2}\displaystyle\int_0^{\sqrt{\lambda}} \dfrac{tdt}{J_p^2(t) + \Upsilon_p^2(t)}, & \lambda > 0, \\ 0, & \lambda \leq 0, \end{cases}$$

hosil bo'ladi.

8-§. Yarim o'qda berilgan Shturm-Liuvill to'g'ri masalasiga doir mashqlar

1. Quyidagi chegaraviy masalalarning:

1) Spektral funksiyasini B.M.Levitan usulida toping;

2) Yoyilma teoremasini yozing;

3) Parseval tengligini yozing;

4) Veyl yechimlarini va Veyl-Titchmarsh funksiyasini toping;

5) Grin funksiyasini tuzing;

6) Spektral funksiyasini Veyl-Titchmarsh funksiyasi yordamida toping;

$$a)\ \begin{cases} -y'' = \lambda y, \\ y(0) = 0, \end{cases} \qquad b)\ \begin{cases} -y'' = \lambda y, \\ y'(0) = 0, \end{cases}$$

$$c)\ \begin{cases} -y'' = \lambda y, \\ y'(0) - hy(0) = 0, \quad h < 0, \end{cases} \qquad d)\ \begin{cases} -y'' = \lambda y, \\ y'(0) - hy(0) = 0, \quad h > 0. \end{cases}$$

2. Umumlashgan Parseval tengligi yordamida quyidagi integrallarni hisoblang.

$$a) \int_0^\infty \frac{\sin\sqrt{\lambda}x \sin\sqrt{\lambda}y}{\pi\lambda\sqrt{\lambda}} d\lambda, \qquad b) \int_0^\infty \frac{(\cos\sqrt{\lambda}x - 1)(\cos\sqrt{\lambda}y - 1)}{\pi\lambda\sqrt{\lambda}} d\lambda.$$

3. Quyidagi chegaraviy masalaning spektral funksiyasini toping

$(0 \le x < \infty)$.

$$a) \begin{cases} -y'' + \dfrac{2}{(x+1)^2} y = \lambda y, \\ y'(0) + y(0) = 0 \end{cases} \qquad b) \begin{cases} -y'' + \dfrac{2h^2}{(hx-1)^2} y = \lambda y, \\ y'(0) - hy(0) = 0, \quad (h < 0) \end{cases}$$

$$c) \begin{cases} -y'' - \dfrac{2}{\mathrm{ch}^2 x} y = \lambda y, \\ y(0) = 0, \end{cases} \qquad d) \begin{cases} -y'' - \dfrac{2}{\mathrm{ch}^2 x} y = \lambda y, \\ y'(0) = 0, \end{cases}$$

$$e) \begin{cases} -y'' - \dfrac{2}{\mathrm{ch}^2 (x+b)} y = \lambda y, \\ y'(0) + thb \cdot y(0) = 0, \end{cases} \qquad f) \begin{cases} -y'' - \dfrac{2a^2}{\mathrm{ch}^2 ax} y = \lambda y, \\ y(0) = 0, \end{cases}$$

$$g) \begin{cases} -y'' - \dfrac{2a^2}{ch^2 ax} y = \lambda y, \\ y'(0) = 0, \end{cases} \qquad h) \begin{cases} -y'' - \dfrac{2a^2}{ch^2 ax} y = \lambda y, \\ y'(0) - hy(0) = 0, \end{cases}$$

$$i) \begin{cases} -y'' - \dfrac{2a^2}{ch^2 (ax+b)} y = \lambda y, \\ y(0) = 0, \end{cases} \qquad j) \begin{cases} -y'' - \dfrac{2a^2}{ch^2 (ax+b)} y = \lambda y, \\ y'(0) - hy(0) = 0. \end{cases}$$

4. Quyidagi Shturm-Liuvill chegaraviy masalalari uchun Veyl-Titchmarsh funksiyasini va Veyl yechimini aniqlang, hamda yoyilma teoremasi va Parseval tengligini yozing.

$a)$ Furye-Bessel qatoriga yoyish

$$\begin{cases} -y'' + \dfrac{4p^2 - 1}{4x^2} y = \lambda y, \quad 0 < x < b < \infty, \\ y(b) \cos \beta + y'(b) \sin \beta = 0, \end{cases}$$

$$1)\ p \ge 1, \qquad 2)\ 0 < p < 1,\ p \ne \frac{1}{2}, \qquad 3)\ p = 0.$$

$b)$ Veber qatoriga yoyish

$$\begin{cases} -y'' + \dfrac{4p^2 - 1}{4x^2} y = \lambda y, \quad 0 < a < x < \infty, \\ y(a) \cos \alpha + y'(a) \sin \alpha = 0, \end{cases}$$

1) $p \geq 1$, 2) $0 < p < 1$, $p \neq \frac{1}{2}$, 3) $p = 0$.

5*. Quyidagi uzlukli koeffitsiyentli chegaraviy masalalarning

1) Spektral funksiyasini B.M.Levitan usulida toping;

2) Yoyilma teoremasini yozing;

3) Parseval tengligini yozing;

4) Veyl yechimlarini va Veyl-Titchmarsh funksiyasini toping;

5) Grin funksiyasini tuzing;

6) Spektral funksiyasini Veyl-Titchmarsh funksiyasi yordamida toping;

$$a) \begin{cases} -y'' = \lambda\rho(x)y, & 0 \leq x < \infty, \\ y(0) = 0, \end{cases}$$

$$b) \begin{cases} -y'' = \lambda\rho(x)y, & 0 \leq x < \infty, \\ y'(0) = 0, \end{cases}$$

$$c) \begin{cases} -y'' = \lambda\rho(x)y, & 0 \leq x < \infty, \\ y'(0) - hy(0) = 0, & h < 0, \end{cases}$$

$$d) \begin{cases} -y'' = \lambda\rho(x)y, & 0 \leq x < \infty, \\ y'(0) - hy(0) = 0, & h > 0. \end{cases}$$

Bu yerda

$$\rho(x) = \begin{cases} \alpha, & 0 \leq x < a, \\ 1 & a \leq x < \infty, \end{cases} \qquad \alpha > 0, \quad \alpha \neq 1.$$

IV BOB. YARIM O'QDA BERILGAN SHTURM-LIUVILL OPERATORI UCHUN TESKARI SPEKTRAL MASALALAR

1-§. Yarim o'qda berilgan Shturm-Liuvill masalasi uchun spektral funksiya

Quyidagi Shturm-Liuvill chegaraviy masalasini ko'rib chiqamiz:

$$\begin{cases} -y'' + q(x)y = \lambda y\,, \quad (0 \le x < \infty), \\ y'(0) = hy(0), \end{cases} \tag{4.1.1}$$

Bu yerda $q(x)$ haqiqiy uzluksiz funksiya, h ixtiyoriy haqiqiy son va λ kompleks parametr deb qaraladi.

Ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y\,, \quad (0 \le x < \infty), \\ y(0) = 1\,, \\ y'(0) = h\,, \end{cases} \tag{4.1.2}$$

Koshi masalasining yechimini $\varphi(x, \lambda)$ orqali belgilaymiz.

Teorema 4.1.1 (*G. Veyl, 1910 y.*). *(4.1.1) chegaraviy masala uchun butun o'qda aniqlangan, o'suvchi, chapdan uzluksiz, $\rho(-0) = 0$ shart bilan normallangan shunday $\rho(\lambda)$ funksiya mavjudki, bunda $L^2(0,\infty)$ fazodan olingan ixtiyoriy $f(x)$ funksiya uchun*

$$\int_0^\infty f^2(x)dx = \int_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda), \tag{4.1.3}$$

tenglik bajariladi. Bu yerda $F(\lambda)$ funksiya

$$F_n(\lambda) = \int_0^n f(x)\varphi(x,\lambda)dx,$$

ketma-ketlikning $L^2_{\rho(\lambda)}(-\infty, +\infty)$ fazodagi limitini bildiradi.

(4.1.3) tenglikka yarim o'qda berilgan Shturm-Liuvill chegaraviy masalasi uchun Parseval tengligi deyiladi, $\rho(\lambda)$ funksiyaga (4.1.1) chegaraviy masalaning spektral funksiyasi deyiladi, $F(\lambda)$ funksiyaga esa $f(x)$ funksiyaning $\varphi(x, \lambda)$ yechimlar bo'yicha yozilgan Furye almashtirishi deyiladi va u

$$F(\lambda) = \int_0^\infty f(x)\varphi(x, \lambda)dx,$$

kabi belgilanadi.

Izoh 4.1.1. Spektral funksiya umuman olganda yagona emas.

Ta'rif 4.1.1. Agar $\rho(\lambda)$ spektral funksiya biror λ_0 nuqtaning kichik atrofida o'zgarmas bo'lsa, λ_0 nuqta (4.1.1) chegaraviy masalaning regulyar nuqtasi deyiladi. Regulyar bo'lmagan nuqtalar to'plamiga (4.1.1) chegaraviy masalaning spektri deyiladi va u E harfi bilan belgilanadi. Spektral funksiyaning uzilish nuqtalariga (4.1.1) chegaraviy masalaning xos qiymatlari deyiladi. Spektrning ajralgan nuqtalari to'plamiga spektrning diskret qismi deyiladi, biz uni PE bilan belgilaymiz. Qolgan nuqtalar to'plamiga uzluksiz spektr deyiladi, biz uni CE bilan belgilaymiz. Uzluksiz spektrda joylashgan xos qiymatlar to'plamini esa PCE bilan belgilaymiz.

Demak, quyidagi tenglik o'rinli:

$$E = PE \cup CE \cup PCE.$$

Xos qiymat spektrning ajralgan qismi yoki chegaraviy nuqtasi yoki ichki nuqtasi bo'lishi mumkin.

Izoh 4.1.2. E spektr quyidan chegaralangan, ammo yuqoridan chegaralanmagan. Xuddi shunday hech bir $q(x)$ va h da (4.1.1) chegaraviy masalaning spektri $[a, b]$ kesmadan iborat bo'la olmaydi. Misol uchun $q(x) \equiv 0$, $h = 0$ bo'lganda spektral funksiya ushbu

$$\rho_0(\lambda) = \begin{cases} \dfrac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

ko'rinishda bo'ladi va spektr $E = [0, \infty)$ to'plamdan iborat bo'ladi. Umuman olganda (4.1.1) chegaraviy masalaning spektral funksiyasi uchun ushbu

$$\rho(\lambda) = \frac{2}{\pi}\sqrt{\lambda} + \rho(-\infty) - h + o(1), \quad \lambda \to +\infty,$$

asimptotik formula o'rinli.([82])

Agar Shturm-Liuvill chegaraviy masalasi ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \le x < \infty, \\ y(0) = 0, & q(x) \in C[0, \infty), \end{cases}$$

ko'rinishda bo'lsa, $q(x) \equiv 0$ bo'lgan holda spektral funksiya quyidagi

$$\rho_1(\lambda) = \begin{cases} \dfrac{2}{3\pi}\sqrt{\lambda^3}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

ko'rinishda bo'ladi va spektr $E = [0, \infty)$ to'plamdan iborat bo'ladi.

Ta'rif 4.1.1. (4.1.1) chegaraviy masalaning spektral funksiyasini topish masalasiga to'g'ri masala deyiladi.

2-§. V.A. Marchenko yagonalik teoremasi

Teorema 4.2.1 (*V.A.Marchenko, 1950 yil*). *$\rho(\lambda)$ funksiya ushbu*

$$\begin{cases} -y'' + q(x)y = \lambda y, & (0 \le x < \infty), \\ y'(0) = hy(0), \end{cases} \tag{4.2.1}$$

Shturm-Liuvill chegaraviy masalasining birorta spektral funksiyasi, $\tilde{\rho}(\lambda)$ funksiya esa ushbu

$$\begin{cases} -y'' + \tilde{q}(x)y = \lambda y, & (0 \le x < \infty), \\ y'(0) = \tilde{h}y(0), \end{cases} \tag{4.2.2}$$

Shturm-Liuvill chegaraviy masalasining biror spektral funksiyasi bo'lib,

$$\tilde{\rho}(\lambda) = C\rho(\lambda), \qquad C = \text{const},$$

tenglik bajarilsa, u holda $q(x) \equiv \tilde{q}(x)$ va $h = \tilde{h}$ tengliklar o'rinli bo'ladi. Bu yerda $q(x)$ va $\tilde{q}(x)$ funksiyalar $[0, \infty)$ oraliqda uzluksiz haqiqiy funksiyalar, $h, \tilde{h}$ esa haqiqiy sonlar.

Isbot. $\varphi(x, \lambda)$ orqali

$$-y'' + q(x)y = \lambda y,$$

tenglamaning quyidagi

$$\varphi(0, \lambda) = 1, \qquad \varphi'(0, \lambda) = h,$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini, $\tilde{\varphi}(x, \lambda)$ bilan esa

$$-y'' + \tilde{q}(x)y = \lambda y,$$

tenglamaning quyidagi

$$\tilde{\varphi}(0, \lambda) = 1, \qquad \varphi'(0, \lambda) = \tilde{h},$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

U holda almashtirish operatorining xossasiga ko'ra ushbu

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \int_0^x K(x,t)\varphi(t,\lambda)dt, \tag{4.2.3}$$

tenglik o'rinli bo'ladi. $b \in [0,\infty)$ ixtiyoriy son bo'lsin va $f(x)$ funksiya $x > b$ bo'lganda nolga aylanuvchi ixtiyoriy uzluksiz funksiya bo'lsin. U holda $f(x)$ funksiyaning $\tilde{\varphi}(x,\lambda)$ yechimlar bo'yicha ushbu

$$\tilde{F}(\lambda) = \int_0^b f(x)\tilde{\varphi}(x,\lambda)dx, \tag{4.2.4}$$

Furye almashtirishi mavjud bo'lib, Parseval tengligi quyidagi

$$\int_0^b f^2(x)dx = \int_{-\infty}^{\infty} \tilde{F}^2(\lambda)d\tilde{\rho}(\lambda) = C\int_{-\infty}^{\infty} \tilde{F}^2(\lambda)d\rho(\lambda), \tag{4.2.5}$$

ko'rinishda bo'ladi. (4.2.4) tenglikda $\tilde{\varphi}(x,\lambda)$ o'rniga uning (4.2.3) tenglik bilan aniqlangan ifodasini qo'yib, integrallash tartibini o'zgartirsak, quyidagi

$$\tilde{F}(\lambda) = \int_0^b g(x)\varphi(x,\lambda)dx, \tag{4.2.6}$$

tenglikka ega bo'lamiz. Bu yerda

$$g(x) = f(x) + \int_x^b K(t,x)f(t)dt.$$

Ko'rinib turibdiki, $g(x)$ funksiya ham $x > b$ bo'lganda nolga aylanadi. (4.2.6) tenglikdagi $\tilde{F}(\lambda)$ funksiya $g(x)$ funksiyaning $\varphi(x,\lambda)$ yechimlar bo'yicha yozilgan Furye almashtirishi bo'ladi. Shuning uchun Parseval tengligidan

$$\int_0^b g^2(x)dx = \int_{-\infty}^{\infty} \tilde{F}^2(\lambda)d\rho(\lambda) = \frac{1}{C}\int_{-\infty}^{\infty} \tilde{F}^2(\lambda)d\tilde{\rho}(\lambda) = \frac{1}{C}\int_0^b f^2(x)dx,$$

kelib chiqadi. Demak,

$$C\int_0^b g^2(x)dx = \int_0^b f^2(x)dx,$$

ya'ni

$$\|\sqrt{C}g\|_{L^2} = \|f\|_{L^2}.$$

Quyidagi operatorni ko'rib chiqamiz:

$$Af = \sqrt{C}\left\{ f(x) + \int_x^b K(t,x)f(t)dt \right\}.$$

Ushbu $\|Af\|_{L^2} = \|f\|_{L^2}$ tenglikka asosan bu operator $L^2(0,b)$ fazoda unitar bo'ladi. Unitar operatorlar uchun $A^*A = I$ tenglik o'rinli bo'ladi.

A^* operatorni topish qiyin emas:

$$A^*h(x) = \sqrt{C}\left\{ h(x) + \int_0^x K(x,t)h(t)dt \right\}.$$

$A^*\{Af(x)\}$ ning aniq ifodasini topamiz:

$$A^*\{Af(x)\} = \sqrt{C}\left(Af(x) + \int_0^x K(x,t)\{Af(t)\}dt \right) =$$

$$= C\cdot\left\{ f(x) + \int_x^b K(t,x)f(t)dt + \right.$$

$$\left. + \int_0^x K(x,t)\left(f(t) + \int_t^b K(s,t)f(s)ds \right) dt \right\} =$$

$$= C\cdot\left\{ f(x) + \int_x^b K(t,x)f(t)dt + \int_0^x K(x,t)f(t)dt + \right.$$

$$+\int_0^x \left(\int_t^b K(x,t)K(s,t)f(s)ds \right) dt \Bigg\}.$$

Bu yerda integrallash tartibini almashtirib quyidagi

$$A^*\{Af(x)\} =$$

$$= C \cdot \Bigg\{ f(x) + \int_0^x \left(K(x,t) + \int_0^t K(x,s)K(t,s)ds \right) f(t)dt +$$

$$+ \int_x^b \left(K(t,x) + \int_0^x K(x,s)K(t,s)ds \right) f(t)dt \Bigg\}$$

formulaga ega bo'lamiz.

$A^*\{Af(x)\} = f(x)$ ayniyatdan foydalansak,

$$(C-1)f(x) + C \cdot \int_0^x \left(K(x,t) + \int_0^t K(x,s)K(t,s)ds \right) f(t)dt +$$

$$+C \cdot \int_x^b \left(K(t,x) + \int_0^x K(x,s)K(t,s)ds \right) f(t)dt = 0$$

tenglik kelib chiqadi. Bu yerda

$$f(t) = \begin{cases} K(x,t) + \int_0^t K(x,s)K(t,s)ds, & t \in [0,x), \\ 0, & t \in [x,b], \end{cases}$$

deb olsak, ushbu

$$\int_0^x \left\{ K(x,t) + \int_0^t K(x,s)K(t,s)ds \right\}^2 dt = 0$$

tenglik kelib chiqadi. Bunga ko'ra

$$K(x,t) + \int_0^t K(x,s)K(t,s)ds = 0.$$

Oxirgi tenglik x ning har bir tayinlangan qiymatida $K(x,t)$ funksiyaga nisbatan bir jinsli Volterra integral tenglamasidir. Bunday tenglama faqat nol yechimga ega bo'lishidan $K(x,t) \equiv 0$ $(t \leq x)$ kelib chiqadi. Buni (4.2.3) formulaga qo'ysak,

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda),$$

ayniyat xosil bo'ladi. Boshlang'ich shartlarga ko'ra $h = \tilde{h}$.

Ushbu

$$-\varphi'' + q(x)\varphi = \lambda\varphi,$$

$$-\varphi'' + \tilde{q}(x)\varphi = \lambda\varphi,$$

tengliklardan foydalanib,

$$[q(x) - \tilde{q}(x)]\varphi(x,\lambda) = 0,$$

bo'lishini topamiz. Bundan va $q(x)$, $\tilde{q}(x)$ funksiyalarning uzluksizligini hamda $\varphi(x,\lambda)$ funksiyaning nollarini ajralganligini e'tiborga olsak, $q(x) = \tilde{q}(x)$ ayniyat kelib chiqadi. ■

3-§. Umumlashgan funksiyalar xossalaridan foydalanib Gelfand-Levitan integral tenglamasini keltirib chiqarish

1. Bu paragrafda biz teskari masalalar nazariyasining asosiy integral tenglamasini keltirib chiqarishning bitta usulini keltiramiz.

Quyidagi

$$-y'' + q(x)y = \lambda y, \quad 0 \leq x < \infty, \tag{4.3.1}$$

$$y'(0) = hy(0), \tag{4.3.2}$$

Shturm-Liuvill chegaraviy masalasining spektral funksiyasi $\rho(\lambda)$ bo'lsin. Bu yerda h chekli haqiqiy son bo'lib, $q(x)$ haqiqiy uzluksiz funksiya.

$\varphi(x, \lambda)$ orqali (4.3.1) tenglamaning quyidagi

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \tag{4.3.3}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimini belgilaymiz.

Lemma 4.3.1. *Ushbu*

$$\int_{-\infty}^{\infty} \varphi(x, \lambda)\varphi(t, \lambda)d\rho(\lambda) = \delta(x - t), \tag{4.3.4}$$

simvolik ayniyat o'rinli. Bu yerda $\delta(x)$ – Dirak delta-funksiyasi.

Bu ayniyatga Parseval tengligining umumlashgan funksiyalar orqali yozilishi deyiladi.

Isbot. Yoyilma haqidagi teoremaga ko'ra, ixtiyoriy ikki marta uzluksiz differensiallanuvchi finit funksiya uchun ushbu

$$f(x) = \int_{-\infty}^{\infty} F(\lambda)\varphi(x, \lambda)d\rho(\lambda), \tag{4.3.5}$$

tasvir o'rinli. Bu yerda

$$F(\lambda) = \int_{0}^{\infty} f(t)\varphi(t, \lambda)dt. \tag{4.3.6}$$

Agar (4.3.6) ifodani (4.3.5) ayniyatga qo'yib, Dirak delta-funksiyasining ushbu

$$f(x) = \int_{0}^{\infty} f(t)\delta(x - t)dt,$$

xossasidan foydalansak, quyidagi

$$\int_{0}^{\infty} f(t)\delta(x - t)dt = \int_{-\infty}^{\infty} \left\{ \int_{0}^{\infty} f(t)\varphi(x, \lambda)\varphi(t, \lambda)dt \right\} d\rho(\lambda), \tag{4.3.7}$$

tenglik kelib chiqadi. (4.3.7) tenglikning o'ng tomonida integrallash tartibini o'zgartirsak, ushbu

$$\int_0^\infty f(t)\delta(x-t)dt = \int_0^\infty f(t)\left\{\int_{-\infty}^\infty \varphi(x,\lambda)\varphi(t,\lambda)d\rho(\lambda)\right\}dt, \tag{4.3.8}$$

tenglikka ega bo'lamiz. (4.3.8) tenglikda $f(t)$ funksiya ixtiyoriy ekanligini e'tiborga olsak, (4.3.4) ayniyat kelib chiqadi.■

Natija 4.3.1. Agar (4.3.1)–(4.3.2) masalada $q(x) \equiv 0, \ \ h = 0$ desak, (4.3.4) ayniyat ushbu

$$\int_{-\infty}^\infty \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\rho_0(\lambda) = \delta(x-t), \tag{4.3.9}$$

ko'rinishga ega bo'ladi. Bu yerda

$$\rho_0(\lambda) = \begin{cases} \frac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \leq 0. \end{cases} \tag{4.3.10}$$

2. Almashtirish operatorining xossalariga ko'ra $\varphi(x,\lambda)$ yechim uchun quyidagi

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt, \tag{4.3.11}$$

tasvir o'rinli. Bu yerda $K(x,t)$ funksiya $q(x)$ potensial, hamda chegaraviy shartdagi h soni bilan

$$q(x) = 2\frac{dK(x,x)}{dx}, \quad h = K(0,0), \tag{4.3.12}$$

formulalar yordamida bog'langan.

Xuddi shunday

$$\cos\sqrt{\lambda}x = \varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)dt, \tag{4.3.13}$$

tasvir ham o'rinli ekanligini ko'rgan edik. Agar $t > x$ bo'lsa, $H(x,t) \equiv 0$ bo'lishi ma'lum.

Lemma 4.3.2. *Agar* $0 < t < x$ *bo'lsa, u holda*

$$\int_{-\infty}^{\infty} \varphi(x,\lambda) \cos\sqrt{\lambda} t d\rho(\lambda) = 0, \qquad (4.3.14)$$

ayniyat o'rinli bo'ladi.

Isbot. Bu lemmani isbotlash uchun (4.3.14) tenglik chap tomonida $\cos\sqrt{\lambda}t$ funksiya o'rniga (4.3.13) tenglikdagi ifodasini qo'yamiz:

$$\int_{-\infty}^{\infty} \varphi(x,\lambda) \cos\sqrt{\lambda} t d\rho(\lambda) =$$

$$= \int_{-\infty}^{\infty} \varphi(x,\lambda) \left\{ \varphi(t,\lambda) + \int_0^t H(t,s)\varphi(s,\lambda)ds \right\} d\rho(\lambda) =$$

$$= \int_{-\infty}^{\infty} \varphi(x,\lambda)\varphi(t,\lambda)d\rho(\lambda) + \int_{-\infty}^{\infty} \left\{ \int_0^t H(t,s)\varphi(x,\lambda)\varphi(s,\lambda)ds \right\} d\rho(\lambda).$$

Agar lemma 4.3.1 dan foydalansak va oxirgi integralda integrallash tartibini almashtirsak, quyidagi ayniyatga ega bo'lamiz:

$$\int_{-\infty}^{\infty} \varphi(x,\lambda) \cos\sqrt{\lambda} t d\rho(\lambda) =$$

$$= \delta(x-t) + \int_0^t H(t,s) \left\{ \int_{-\infty}^{\infty} \varphi(x,\lambda)\varphi(s,\lambda)d\rho(\lambda) \right\} ds =$$

$$= \delta(x-t) + \int_0^t H(t,s)\delta(x-s)ds = \delta(x-t) + H(t,x) = 0. \blacksquare$$

3. Teorema 4.3.1 (*I.M.Gelfand-B.M.Levitan, 1951 y.*). *Agar* $0 < t < x$ *bo'lsa, u holda almashtirish operatorining* $K(x,t)$

yadrosi quyidagi

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad (0 < t < x), \tag{4.3.15}$$

integral tenglamani qanoatlantiradi. Bu yerda

$$F(x,t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda), \tag{4.3.16}$$

$$\sigma(\lambda) = \rho(\lambda) - \rho_0(\lambda). \tag{4.3.17}$$

Isbot. $0 < t < x$ bo'lganda lemma 4.3.2 ga ko'ra ushbu

$$\int_{-\infty}^{\infty} \varphi(x,\lambda) \cos\sqrt{\lambda}t d\rho(\lambda) = 0, \tag{4.3.18}$$

tenglik o'rinli bo'ladi. (4.3.11) tenglikdan esa quyidagi

$$\int_{-\infty}^{\infty} \left(\cos\sqrt{\lambda}x + \int_0^x K(x,s) \cos\sqrt{\lambda}s ds \right) \cos\sqrt{\lambda}t d\rho(\lambda) = 0, \tag{4.3.19}$$

ayniyat kelib chiqadi, ya'ni

$$\int_{-\infty}^{\infty} \left\{ \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t + \int_0^x K(x,s) \cos\sqrt{\lambda}t \cos\sqrt{\lambda}s ds \right\} d\rho(\lambda) = 0,$$

ayniyat bajariladi. Oxirgi ayniyatni $\rho(\lambda) = \rho_0(\lambda) + \sigma(\lambda)$ tenglikdan foydalanib, quyidagi

$$\int_{-\infty}^{\infty} \left\{ \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t + \int_0^x K(x,s) \cos\sqrt{\lambda}t \cos\sqrt{\lambda}s ds \right\} d\rho_0(\lambda) +$$

$$+ \int_{-\infty}^{\infty} \left\{ \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t + \int_0^x K(x,s) \cos\sqrt{\lambda}t \cos\sqrt{\lambda}s ds \right\} d\sigma(\lambda) = 0, \tag{4.3.20}$$

ko'rinishda yozib olamiz. (4.3.20) tenglikda integrallash tartibini almashtiramiz:

$$\int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\rho_0(\lambda) +$$

$$+\int_0^x K(x,s)\left\{\int_{-\infty}^{\infty} \cos\sqrt{\lambda}s \cos\sqrt{\lambda}t d\rho_0(\lambda)\right\} ds +$$

$$+\int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda) +$$

$$+\int_0^x K(x,s)\left\{\int_{-\infty}^{\infty} \cos\sqrt{\lambda}s \cos\sqrt{\lambda}t d\sigma(\lambda)\right\} ds = 0.$$

Oxirgi tenglikda lemma 4.3.1 ning natijasini ishlatsak, hamda (4.3.16) belgilashdan foydalansak, quyidagi

$$\delta(x-t)+\int_0^x K(x,s)\delta(t-s)ds+F(x,t)+\int_0^x K(x,s)F(s,t)ds = 0,$$

tenglik hosil bo'ladi. Bu yerda $0 < t < x$ ekanligini e'tiborga olsak, ushbu

$$0 + K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0,$$

ayniyatga ega bo'lamiz.

Shunday qilib, biz $K(x,t)$ yadroga nisbatan ushbu

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0, \quad (0 < t < x),$$

integral tenglamaga ega bo'ldik. Bu tenglamaga teskari masalalar nazariyasining asosiy integral tenglamasi yoki Gelfand-Levitan integral tenglamasi deyiladi. Bu tenglamada x parametr sifatida qatnashadi. x parametrning har bir tayinlangan qiymatida Gelfand-Levitan integral tenglamasi Fredgolmning ikkinchi tur integral tenglamasidir.

4-§. Gelfand-Levitan integral tenglamasini keltirib chiqarishning qat'iy usuli

Oldingi paragrafdagi usul umuman olganda qat'iy emas edi, chunki u yerda qatnashayotgan integrallar oddiy ma'noda uzoqlashuvchi bo'lishi mumkin. Ammo shunga qaramasdan ko'rib chiqilgan usul qat'iy usulga yo'nalish ko'rsatadi.

Ushbu

$$F_N(x,t) = \int_{-\infty}^{N} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda), \qquad (4.4.1)$$

formula yordamida $F_N(x,t)$ funksiyalar ketma-ketligini aniqlaymiz. $F_N(x,t)$ funksiyalar ketma-ketligi mavjud, chunki bu holda spektr quyidan chegaralanganligi sababli (4.4.1) integral chekli oraliq bo'yicha olinadi.

Lemma 4.4.1. *$q(x)$ funksiyaning $[0,\infty)$ oraliqda n-tartibgacha hosilalari mavjud bo'lib, ular har bir chekli oraliqda jamlanuvchi bo'lsin. U holda ushbu*

$$\Phi_N(x) = \int_{-\infty}^{N} \cos\sqrt{\lambda}x d\sigma(\lambda), \qquad (4.4.2)$$

funksiyalar ketma-ketligi $\Phi(x) = H(x,0)$ funksiyaga chegaralanib yaqinlashadi, hamda $\Phi(x)$ funksiya $(n+1)$-tartibgacha hosilalari mavjud bo'lib, ular har bir chekli oraliqda jamlanuvchi bo'ladi.

Isbot. Avvalo $\Phi_N(x)$ ketma-ketlikni ushbu

$$\Phi_N(x) = \int_{-\infty}^{0} \cos\sqrt{\lambda}x d\rho(\lambda) + \int_{0}^{N} \cos\sqrt{\lambda}x d\sigma(\lambda), \qquad (4.4.3)$$

ko'rinishda yozib olamiz. So'ngra x o'zgaruvchining ixtiyoriy qiymatlarida

$$\int_{-\infty}^{0} \cos\sqrt{\lambda}x d\rho(\lambda), \qquad (4.4.4)$$

integral mavjud ekanligini isbotlaymiz.

Ushbu

$$\cos\sqrt{\lambda}x = \varphi(x,\lambda) + \int_{0}^{x} H(x,t)\varphi(t,\lambda)dt, \qquad (4.4.5)$$

tenglikni $(0, x)$ oraliqda integrallab, integrallash tartibini o'zgartirsak, quyidagi

$$\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} = \int_{0}^{x} \left\{1 + \int_{t}^{x} H(s,t)ds\right\} \varphi(t,\lambda)dt,$$

ayniyatga ega bo'lamiz. Ko'rinib turibdiki, $\dfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$ funksiya

$$g(t) = \begin{cases} 1 + \int_{t}^{x} H(s,t)ds, & t \leq x, \\ 0, & t > x, \end{cases}$$

funksiyaning $\varphi(t,\lambda)$ yechim bo'yicha yozilgan Furye almashtirishi bo'ladi. Shuning uchun Parseval tengligiga ko'ra

$$\int_{-\infty}^{\infty} \left(\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\right)^2 d\rho(\lambda) = \int_{0}^{x} g^2(t)dt,$$

tenglik o'rinli bo'ladi. Bu tenglikdan ushbu

$$\int_{-\infty}^{0} \left(\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right)^2 d\rho(\lambda),$$

integral x o'zgaruvchining ixtiyoriy qiymatlarida mavjud ekanligi kelib chiqadi. Bundan esa, (4.4.4) integral mavjud bo'lishi ko'rinadi. Agar

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt,$$

tenglikni e'tiborga olsak,

$$\omega(x,N) = \int_{-\infty}^{N} \varphi(x,\lambda)d\rho(\lambda), \qquad (4.4.6)$$

integral mavjudligini ko'ramiz. $\omega(x,N)$ funksiyaga Shturm-Liuvill masalasining spektral yadrosi deyiladi. $\omega_0(x,N)$ funksiya ushbu

$$-y'' = \lambda y, \quad y'(0) - hy(0) = 0,$$

masalaning spektral yadrosi bo'lsin. Ma'lumki, $h \geq 0$ bo'lganda (soddalik uchun shunday deb hisoblaymiz)

$$\omega_0(x,N) = \frac{1}{\pi}\int_0^N \left\{ \cos\sqrt{\lambda}x + h\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right\} \frac{\sqrt{\lambda}}{h^2+\lambda} d\lambda, \qquad (4.4.7)$$

bo'ladi. Boshqa tomondan, x o'zgaruvchining ixtiyoriy chekli o'zgarish oralig'ida quyidagi asimptotik formula o'rinli bo'ladi:

$$\omega(x,N) = \omega_0(x,N) + \overline{\overline{o}}(1), \ (N \to \infty). \qquad (4.4.8)$$

Shuning uchun

$$\int_0^x H(x,t)\omega(t,N)dt = \int_0^x H(x,t)\omega_0(t,N)dt + \overline{\overline{o}}(1), \ (N \to \infty), \qquad (4.4.9)$$

tenglik o'rinli. (4.4.8) va (4.4.9) tengliklarni bir-biriga qo'shsak va $\omega(x,N)$ funksiyaning (4.4.6) tenglikdagi ifodasini e'tiborga olsak,

$$\int_{-\infty}^{N}\varphi(x,\lambda)d\rho(\lambda)+\int_{-\infty}^{x}H(x,t)\left\{\int_{-\infty}^{N}\varphi(t,\lambda)d\rho(\lambda)\right\}dt=$$

$$=\omega_0(x,N)+\int_{0}^{x}H(x,t)\omega_0(t,N)dt+\bar{\bar{o}}(1),\ (N\to\infty),$$

tenglikka ega bo'lamiz.

Bu tenglikning chap tomonida integrallash tartibini o'zgartirsak, ((4.4.6) integral mavjud bo'lganligi uchun bu amalni bajarish mumkin) va (4.4.5) formulani e'tiborga olsak, ushbu

$$\int_{-\infty}^{N}\cos\sqrt{\lambda}x d\rho(\lambda)=$$

$$=\omega_0(x,N)+\int_{0}^{x}H(x,t)\omega_0(t,N)dt+\bar{\bar{o}}(1),\ (N\to\infty),\quad (4.4.10)$$

tenglikka ega bo'lamiz. Bu tenglikning har ikkala tomonidan

$$\int_{-\infty}^{N}\cos\sqrt{\lambda}x d\rho_0(\lambda),$$

funksiyani ayirsak,

$$\Phi_N(x)=\int_{-\infty}^{N}\cos\sqrt{\lambda}x d\sigma(\lambda)=\omega_0(x,N)-\frac{1}{\pi}\int_{0}^{N}\frac{\cos\sqrt{\lambda}x}{\sqrt{\lambda}}d\lambda+$$

$$+\int_{0}^{x}H(x,t)\omega_0(t,N)dt+\bar{\bar{o}}(1),\quad (N\to\infty),\qquad (4.4.11)$$

tenglikka ega bo'lamiz. (4.4.7) va (4.4.11) formulalardan ushbu

$$\Phi_N(x) = \chi_N(x) + \frac{2}{\pi}\int_0^x H(x,t)\frac{\sin\sqrt{N}t}{t}dt+$$

$$+\int_0^x H(x,t)\chi_N(t)dt + \overline{\overline{o}}(1),\ (N\to\infty), \qquad (4.4.12)$$

tenglik kelib chiqadi. Bu yerda

$$\chi_N(x) = -\frac{1}{\pi}\int_0^N \frac{h^2}{(h^2+\lambda)\sqrt{\lambda}}\cos\sqrt{\lambda}x d\lambda + \frac{h}{\pi}\int_0^N \frac{\sin\sqrt{\lambda}x}{h^2+\lambda}d\lambda.$$

$$(4.4.13)$$

Quyidagi

$$u(x) = \begin{cases} 0, & x>0, \\ -h, & x=0, \end{cases} \qquad (4.4.14)$$

belgilashni kiritaylik. $\chi_N(x)$ funksiya $N\to\infty$ da $u(x)$ funksiyaga chegaralanib yaqinlashadi. Haqiqatan ham, agar $x>0$ bo'lsa, $N\to\infty$ da

$$\chi_N(x) = -\frac{1}{\pi}\int_0^N \frac{h^2}{(h^2+\lambda)\sqrt{\lambda}}\cos\sqrt{\lambda}x d\lambda + \frac{h}{\pi}\int_0^N \frac{\sin\sqrt{\lambda}x}{h^2+\lambda}d\lambda \to$$

$$\to -he^{-hx} - \frac{d}{dx}e^{-hx} = 0.$$

Agar $x=0$ bo'lsa, $\chi_N(0)\to -h$ bo'ladi. $\chi_N(x)$ funksiyaning chegaralanganligi quyidagi

$$\int_a^N \frac{\sin\sqrt{\lambda}x}{h^2+\lambda}d\lambda = 2\int_{\sqrt{a}}^{\sqrt{N}} \frac{\mu\sin\mu x}{h^2+\mu^2}d\mu =$$

$$= 2\int_{\sqrt{a}}^{\sqrt{N}} \frac{\sin\mu x}{\mu}d\mu + \int_{\sqrt{a}}^{\sqrt{N}} \frac{2h^2}{\mu^3}d\mu + \underline{\underline{O}}\left(\frac{1}{N^2}\right)$$

baholashdan kelib chiqadi.

Endi

$$V_N(x) = \frac{2}{\pi}\int\limits_0^x H(x,t)\frac{\sin\sqrt{N}t}{t}dt =$$

$$= \frac{2}{\pi}\int\limits_0^x \frac{H(x,t)-H(x,0)}{t}\sin\sqrt{N}t dt + \frac{2}{\pi}H(x,0)\int\limits_0^x \frac{\sin\sqrt{N}t}{t}dt. \tag{4.4.15}$$

integralni qaraymiz. Bu yoyilmadan, $0 \le x \le b < \infty$ qiymatlarda ushbu

$$\left|\frac{2}{\pi}\int\limits_0^x H(x,t)\frac{\sin\sqrt{N}t}{t}dt\right| \le \frac{2}{\pi}\int\limits_0^x \left|\frac{H(x,t)-H(x,0)}{t}\right|dt +$$

$$+\frac{2}{\pi}|H(x,0)|\left|\int\limits_0^{\sqrt{N}x}\frac{\sin t}{t}dt\right| \le C(b)\,,$$

tengsizlik kelib chiqadi. Bu yerda $C(b)$ o‘zgarmasni bildiradi.

Demak, x o‘zgaruvchining ixtiyoriy chekli o‘zgarish oralig‘ida $V_N(x)$ funksiyalar ketma-ketligi chegaralangan bo‘ladi. Agar

$$\int\limits_0^\infty \frac{\sin s}{s}ds = \frac{\pi}{2},$$

tenglikni e’tiborga olsak, (4.4.15) tasvirdan

$$\lim_{N\to\infty} V_N(x) = \begin{cases} H(x,0), & x > 0, \\ 0, & x = 0, \end{cases}$$

tenglik hosil bo‘ladi.

$\chi_N(x)$ va $V_N(x)$ funksiyalar ketma-ketligi chegaralanib yaqinlashishidan (4.4.12) tenglikka asosan $\Phi_N(x)$ funksiyalar ketma-ketligi chegaralanib yaqinlashishi kelib chiqadi, hamda

$$\lim_{N\to\infty}\Phi_N(x)=\begin{cases}H(x,0), & x\neq 0,\\ -h, & x=0.\end{cases}$$

$H(0,0)=-h$ ekanini e'tiborga olsak, $\Phi_N(x)$ funksiyalar ketma-ketligi $H(x,0)$ funksiyaga chegaralanib yaqinlashishini topamiz.■

Lemma 4.4.2. *x ning har bir tayinlangan qiymatida tashuvchisi $(0,x)$ oraliqda bo'lgan ixtiyoriy $g(t)$ uzluksiz finit funksiya uchun*

$$\lim_{N\to\infty}\int_0^x F_N(x,t)g(t)dt+\int_0^x K(x,t)g(t)dt+$$

$$+\lim_{N\to\infty}\int_0^\infty g(t)\left\{\int_0^x K(x,s)F_N(s,t)ds\right\}dt=0, \tag{4.4.16}$$

tenglik o'rinli.

Isbot. Yoyilma haqidagi teoremaga asosan $g(t)$ finit funksiya bo'lganligi uchun ushbu

$$\lim_{N\to\infty}\int_0^\infty g(t)\left\{\int_{-\infty}^N \varphi(x,\lambda)\varphi(t,\lambda)d\rho(\lambda)\right\}dt=g(x)=0,$$

tenglik o'rinli bo'ladi. Ushbu

$$\cos\sqrt{\lambda}t=\varphi(t,\lambda)+\int_0^t H(t,s)\varphi(s,\lambda)ds,$$

formuladan foydalanib, quyidagi limitni hisoblaymiz:

$$\lim_{N\to\infty}\int_0^\infty g(t)\left\{\int_{-\infty}^N \varphi(x,\lambda)\cos\sqrt{\lambda}t\,d\rho(\lambda)\right\}dt=$$

$$=\lim_{N\to\infty}\int_0^\infty g(t)\left\{\int_{-\infty}^N \varphi(x,\lambda)\varphi(t,\lambda)d\rho(\lambda)\right\}dt+$$

$$+\lim_{N\to\infty}\int_0^{\infty} g(t)\left\{\int_0^t H(t,s)\left[\int_{-\infty}^{N}\varphi(x,\lambda)\varphi(s,\lambda)d\rho(\lambda)\right]ds\right\}dt=$$

$$=0+\lim_{N\to\infty}\int_0^{\infty}\left\{\int_s^{\infty} g(t)H(t,s)dt\right\}\left\{\int_{-\infty}^{N}\varphi(x,\lambda)\varphi(s,\lambda)d\rho(\lambda)\right\}ds=$$

$$=\int_x^{\infty} g(t)H(t,x)dt=0,$$

chunki $t\geq x$ qiymatlar uchun $g(t)\equiv 0$.

Quyidagi

$$\varphi(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x K(x,s)\cos\sqrt{\lambda}s ds,$$

formuladan foydalanib, ushbu

$$\int_{-\infty}^{N}\varphi(x,\lambda)\cos\sqrt{\lambda}t d\rho(\lambda)=\int_{-\infty}^{N}\cos\sqrt{\lambda}x\cos\sqrt{\lambda}t d\rho(\lambda)+$$

$$+\int_0^x K(x,s)\left\{\int_{-\infty}^{N}\cos\sqrt{\lambda}x\cos\sqrt{\lambda}s d\rho(\lambda)\right\}d\rho(\lambda)=$$

$$=\int_{-\infty}^{N}\cos\sqrt{\lambda}x\cos\sqrt{\lambda}t d\rho_0(\lambda)+$$

$$+\int_{-\infty}^{N}\cos\sqrt{\lambda}x\cos\sqrt{\lambda}t d\sigma(\lambda)+$$

$$+\int_0^x K(x,s)\left\{\int_{-\infty}^{N}\cos\sqrt{\lambda}t\cos\sqrt{\lambda}s d\rho_0(\lambda)\right\}ds+$$

$$+\int_0^x K(x,s)\left\{\int_{-\infty}^{N} \cos\sqrt{\lambda}t\cos\sqrt{\lambda}s d\sigma(\lambda)\right\}ds,$$

ayniyatni keltirib chiqaramiz. Uni $g(t)$ funksiyaga ko'paytiramiz va $[0,\infty)$ oraliqda t bo'yicha integrallab, $N\to\infty$ da limitga o'tamiz:

$$0 = g(x) + \lim_{N\to\infty}\int_0^\infty g(t)F_N(x,t)dt + \int_0^x K(x,s)g(s)ds+$$

$$+\lim_{N\to\infty}\int_0^\infty g(t)\left\{\int_0^x K(x,s)F_N(s,t)ds\right\}dt. \qquad (4.4.17)$$

Bu tenglikda $g(x)=0$ ekanligini etiborga olsak, (4.4.16) ayniyat hosil bo'ladi.■

Teorema 4.4.1. *Almashtirish operatorining $K(x,t)$ yadrosi ushbu*

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0\,, \quad (0\le t\le x), \qquad (4.4.18)$$

integral tenglamani qanoatlantiradi. Bu yerda $F(x,t)$ yadro

$$F_N(x,t) = \int_{-\infty}^{N} \cos\sqrt{\lambda}x\cos\sqrt{\lambda}t d\sigma(\lambda), \qquad (4.4.19)$$

ketma-ketlikning $N\to\infty$ dagi limitini bildiradi, bunda

$$\sigma(\lambda) = \begin{cases} \rho(\lambda) - \dfrac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ \rho(\lambda), & \lambda \le 0. \end{cases} \qquad (4.4.20)$$

Agar $q(x)$ funksiya $[0,\infty)$ oraliqda n marta uzluksiz differensiallanuvchi bo'lsa, $F(x,t)$ funksiya $0\le t\le x$ to'plamda $n+1$ marta uzluksiz differensiallanuvchi bo'ladi.

Isbot. Teoremaning birinchi qismi lemma 4.4.2 dan kelib chiqadi. Differensiallanuvchanligini isbot qilish qoladi xolos.

Almashtirish operatorini o'rganganimizda, $A(\xi, \eta)$ uchun yozilgan Volterra turidagi integral tenglamadan $q(x)$ funksiya n marta uzluksiz differensiallanuvchi bo'lsa, $K(x,t)$ yadroning $x \geq t$ to'plamda $n+1$ marta uzluksiz differensiallanuvchi bo'lishi kelib chiqadi.

Endi Gelfand-Levitan integral tenglamasida $K(x,t)$ funksiya ma'lum deb, bu tenglamani $F(x,t)$ funksiyaga nisbatan qaraymiz. U holda bu tenglama Volterra turidagi integral tenglama bo'ladi va shuning uchun unga ketma-ket yaqinlashishlar usulini qo'llash mumkin bo'ladi. Bu yo'l bilan hosil qilingan qatorni $n+1$ marta hadlab differensiallash mumkin bo'ladi.■

Izoh 4.4.1. Quyidagi

$$\Phi_N(x) = \int_{-\infty}^{N} \cos\sqrt{\lambda}d\sigma(\lambda), \quad \Phi(x) = \lim_{N\to\infty} \Phi_N(x),$$

belgilashlarni e'tiborga olib, $0 \leq t \leq x$ qiymatlar uchun

$$F(x,t) = \frac{1}{2}[\Phi(x+t) + \Phi(x-t)], \tag{4.4.21}$$

$$F(x,x) = \frac{1}{2}[\Phi(2x) + \Phi(0)], \tag{4.4.22}$$

tengliklar o'rinli bo'lishini ko'rish mumkin.

(4.4.22) tenglikka ko'ra, agar $F(x,t)$ funksiya $x \geq t$ bo'lganda $n+1$ tartibli uzluksiz hosilalarga ega bo'lsa, u holda $\Phi(x)$ funksiya $x \geq 0$ bo'lganda $n+1$ tartibli uzluksiz hosilalarga ega bo'ladi. (4.4.21) formuladan esa, aksincha, $\Phi(x)$ funksiya qancha marta uzluksiz differensiallanuvchi bo'lsa, $F(x,t)$ funksiya ham shuncha marta uzluksiz differensiallanuvchi bo'lishi kelib chiqadi.

Bizga keyinchalik spektral funksiyaning quyidagi oddiy xossasi kerak bo'ladi.

Teorema 4.4.2. *$f(x) \in L^2(0,\infty)$ ixtiyoriy uzluksiz finit funksiya bo'lib, quyidagi*

$$E(\lambda) = \int_0^\infty f(x)\cos\sqrt{\lambda}x dx,$$

tenglik o'rinli bo'lsin. Agar ushbu

$$\int_{-\infty}^{\infty} E^2(\lambda)d\rho(\lambda) = 0, \tag{4.4.23}$$

tenglik bajarilsa, $f(x) \equiv 0$ bo'ladi.

Isbot. Quyidagi

$$E(\lambda) = \int_0^\infty f(x)\left[\varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)dt\right]dx =$$

$$= \int_0^\infty f(x)\varphi(x,\lambda)dx + \int_0^\infty \left\{\int_0^x f(x)H(x,t)\varphi(t,\lambda)dt\right\}dx,$$

tenglikning o'ng tomonidagi ikkinchi integralda integrallash tartibini o'zgartiramiz:

$$E(\lambda) = \int_0^\infty f(t)\varphi(t,\lambda)dt + \int_0^\infty \left\{\int_t^\infty f(x)H(x,t)dx\right\}\varphi(t,\lambda)dt =$$

$$= \int_0^\infty \left\{f(t) + \int_t^\infty f(x)H(x,t)dx\right\}\varphi(t,\lambda)dt = \int_0^\infty g(t)\varphi(t,\lambda)dt\,.$$

Demak, $E(\lambda)$ funksiya $g(t)$ funksiyaning $\varphi(x,\lambda)$ yechimlar bo'yicha Furye almashtirishi ekan. Parseval tengligiga ko'ra ushbu

$$\int_0^\infty g^2(t)dt = \int_0^\infty E^2(\lambda)d\rho(\lambda),$$

tenglik o'rinli bo'lgani uchun (4.4.23) shartga binoan quyidagi

$$\int_0^\infty g^2(t)dt = 0,$$

tenglik o'rinli bo'ladi. Bundan esa $g(t) \equiv 0$ ekanligi kelib chiqadi, ya'ni

$$f(x) + \int_t^\infty f(x)H(x,t)dx = 0, \tag{4.4.24}$$

ayniyat bajariladi. $f(x)$ finit funksiya bo'lganligi uchun (4.4.24) tenglikdagi integral chekli oraliqda olinadi, shuning uchun (4.4.24) tenglama Volterra turidagi integral tenglama bo'ladi, bundan esa $f(x) \equiv 0$ ekanligi kelib chiqadi.■

5-§.Spektral funksiya bo'yicha teskari masala. Gelfand-Levitan integral tenglamasi yechimining mavjudligi, yagonaligi va silliqligi

Oldingi paragrafda ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \le x < \infty, \\ y'(0) = hy(0), & q(x) \in C^n[0,\infty), \end{cases} \tag{4.5.1}$$

Shturm-Liuvill masalasining $\rho(\lambda)$ spektral funksiyasi uchun quyidagi ikkita shart bajarilishi ko'rsatildi:

I. Ushbu

$$\Phi_N(x) = \int_{-\infty}^{N} \cos\sqrt{\lambda}x d\sigma(\lambda),$$

tenglik bilan aniqlanadigan funksiyalar ketma-ketligi $N \to \infty$ da biror $\Phi(x) \in C^{n+1}(0,\infty)$ funksiyaga intiladi. Bu yerda

$$\sigma(\lambda) = \begin{cases} \rho(\lambda) - \dfrac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ \rho(\lambda), & \lambda \le 0; \end{cases}$$

II. Ixtiyoriy $f(x) \in L^2(0,\infty)$ uzluksiz finit funksiya uchun quyidagi

$$\int_{-\infty}^{\infty} E^2(\lambda)d\rho(\lambda) = 0,$$

tenglik o'rinli bo'lsa, $f(x) \equiv 0$ bo'ladi. Bu yerda

$$E(\lambda) = \int_0^{\infty} f(x)\cos\sqrt{\lambda}x dx.$$

Endi, biror monoton o'suvchi $\rho(\lambda)$ funksiya uchun yuqoridagi **I** va **II** shartlar bajarilsa, u holda bu funksiya biror Shturm-Liuvill chegaraviy masalasining spektral funksiyasi bo'lib, $q(x) \in C''[0,\infty)$ bo'lishini ko'rsatamiz.

Ushbu

$$F(x,t) = \frac{1}{2}[\Phi(x+t) + \Phi(x-t)],$$

tenglik yordamida $F(x,t)$ funksiyani aniqlaymiz va quyidagi

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0\,, \quad (0 \le t \le x < \infty), \tag{4.5.2}$$

integral tenglamani tuzamiz.

Teorema 4.5.1. *Har bir tayinlangan $x > 0$ uchun (4.5.2) integral tenglamaning yechimi mavjud va yagonadir.*

Isbot. (4.5.2) integral tenglamaning yechimi mavjud va yagona ekanligini ko'rsatish uchun, Fredgolm teoremasiga ko'ra, x parametrning har bir tayinlangan musbat qiymatida, ushbu

$$g(t) + \int_0^x F(s,t)g(s)ds = 0, \tag{4.5.3}$$

bir jinsli integral tenglama faqat nol yechimga ega bo'lishini ko'rsatish yetarli.

Teskarisini faraz qilaylik, ya'ni (4.5.3) tenglamani qanoatlantiruvchi noldan farqli uzluksiz $g(t) \in L^2(0,\infty)$ funksiya mavjud deb faraz qilaylik. (4.5.3) tenglikni $g(t)$ funksiyaga ko'paytirib, $(0,x)$ oraliqda t bo'yicha integrallaymiz:

$$\int_0^x g^2(t)dt + \int_0^x \int_0^x F(s,t)g(s)g(t)dsdt = 0. \tag{4.5.4}$$

$F(s,t)$ funksiyaning ifodasini (4.5.4) tenglikka qo'yib, integrallash tartibini o'zgartirsak, ushbu

$$\int_0^x g^2(t)dt + \int_{-\infty}^{\infty} \left\{ \int_0^x \int_0^x g(s)g(t)\cos\sqrt{\lambda}s \cos\sqrt{\lambda}t ds dt \right\} d\rho(\lambda) -$$
$$- \int_{-\infty}^{\infty} \left\{ \int_0^x \int_0^x g(s)g(t)\cos\sqrt{\lambda}s \cos\sqrt{\lambda}t ds dt \right\} d\left(\frac{2}{\pi}\sqrt{\lambda}\right) = 0, \tag{4.5.5}$$

ayniyat hosil bo'ladi. Uni quyidagi

$$\int_0^x g^2(t)dt + \int_{-\infty}^{\infty} \left\{ \int_0^x g(t)\cos\sqrt{\lambda}t dt \right\}^2 d\rho(\lambda) -$$
$$- \int_{-\infty}^{\infty} \left\{ \int_0^x g(t)\cos\sqrt{\lambda}t\, dt \right\}^2 d\left(\frac{2}{\pi}\sqrt{\lambda}\right) = 0, \tag{4.5.6}$$

ko'rinishda yozish mumkin.

Parseval tengligidan foydalansak, ushbu

$$\int_{-\infty}^{\infty} \left\{ \int_0^x g(t)\cos\sqrt{\lambda}t dt \right\}^2 d\rho(\lambda) = 0,$$

tenglik hosil bo'ladi. **II** shartga ko'ra bu hol faqat $g(t) \equiv 0$ bo'lganda bajariladi. Farazimiz noto'g'ri ekan, ya'ni (4.5.3) tenglama faqat nol yechimga ega.■

Gelfand-Levitan integral tenglamasidan, $F(x,t)$ funksiya t bo'yicha qancha hosilaga ega bo'lsa, $K(x,t)$ funksiya ham t bo'yicha shuncha hosilaga ega bo'lishi ko'rinadi. Haqiqatan ham, ushbu

$$K(x,t) = -F(x,t) - \int_0^x K(x,s)F(s,t)ds = 0, \quad (0 \le t \le x),$$

tenglik o'ng tomoni t o'zgaruvchi bo'yicha qancha marta differensiallanuvchi bo'lsa uning chap tomoni ham t o'zgaruvchi bo'yicha shuncha marta differensiallanuvchi bo'ladi.

$K(x,t)$ funksiyaning x bo'yicha differensiallanuvchanligini tekshirish lemma 2.1.6 ga asoslanib amalga oshiriladi. .

6-§. Gelfand-Levitan integral tenglamasi yechimi yordamida tuzilgan $\varphi(x,\lambda)$ funksiya uchun differensial tenglamani keltirib chiqarish

Oldingi paragrafda, **I** va **II** shartlarni qanoatlantiruvchi $\rho(\lambda)$ monoton o'suvchi funksiya uchun Gelfand-Levitan integral tenglamasining yechimi mavjud va yagona bo'lishi ko'rsatildi. Bu $K(x,t)$ yechim yordamida ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt, \qquad (4.6.1)$$

funksiyani tuzib olamiz. $\varphi(x,\lambda)$ funksiya ushbu

$$\varphi(0,\lambda) = 1\,, \quad \varphi'(0,\lambda) = -F(0,0) = K(0,0) = h, \qquad (4.6.2)$$

boshlang'ich shartlarni qanoatlantirishi osongina tekshiriladi. Quyidagi teorema teskari masala yechishda muhim bosqich hisoblanadi.

Teorema 4.6.1. *$K(x,t)$ funksiya Gelfand-Levitan integral tenglamasini qanoatlantirsin. U holda (4.6.1) formula orqali* *aniqlangan $\varphi(x,\lambda)$ funksiya ushbu*

$$-y'' + q(x)y = \lambda y\,, \quad (0 \le x < \infty), \tag{4.6.3}$$

differensial tenglamani qanoatlantiradi. Bu yerda

$$q(x) = 2\frac{dK(x,x)}{dx}. \tag{4.6.4}$$

Isbot. Dastlab $F(x,t)$ funksiyani ikki marta uzluksiz differensiallanuvchi deb hisoblaymiz. U holda $K(x,t)$ ham ikki marta uzluksiz differensiallanuvchi bo'ladi. Quyidagi

$$J(x,t) = K(x,t) + F(x,t) +$$

$$+\int_0^x K(x,s)F(s,t)ds = 0, \;\; (0 \le t < x < \infty), \tag{4.6.5}$$

ayniyatdan x bo'yicha ikki marta hosila olamiz:

$$J_x = K_x + F_x + K(x,x)F + \int_0^x K_x(x,s)F(s,t)ds = 0,$$

$$J_{xx} = K_{xx} + F_{xx} + \frac{dK(x,x)}{dx}F + K(x,x)F_x +$$

$$+\left.\frac{\partial K(x,s)}{\partial x}\right|_{s=x} F + \int_0^x K_{xx}(x,s)F(s,t)ds = 0. \tag{4.6.6}$$

(4.6.5) ayniyatni t bo'yicha ikki marta differensiallaymiz:

$$J_t = K_t + F_t + \int_0^x K(x,s)F_t(s,t)ds = 0\,, \tag{4.6.7}$$

$$J_{tt} = K_{tt} + F_{tt} + \int_0^x K(x,s)F_{tt}(s,t)ds = 0\,. \tag{4.6.8}$$

$F(x,t) = \frac{1}{2}[\Phi(x+t) + \Phi(x-t)]$ bo'lganligi uchun,

$$F_t(x,t)|_{t=0} = 0, \qquad F_{tt} = F_{xx},$$

bo'ladi. (4.6.7) tenglikda $t = 0$ desak,

$$K_t(x,t)|_{t=0} = 0, \tag{4.6.9}$$

hosil bo'ladi.

(4.6.8) tenglikda integral ostidagi $F_{tt}(s,t)$ funksiya o'rniga $F_{ss}(s,t)$ funksiyani qo'yib, ikki marta bo'laklab integrallasak, quyidagi

$$J_{tt} = F_{tt}+K_{tt}+\int_0^x K(x,s)dF_s(s,t) = F_{tt}+K_{tt}+K(x,x)F_x(x,t)-$$

$$-K(x,0)F_x(x,0) - \int_0^x K_s(x,s)dF(s,t) =$$

$$= F_{tt} + K_{tt} + K(x,x)F_x(x,t) - K(x,0)F_x(x,0)-$$

$$-K_t(x,x)F(x,t) + K_t(x,0)F(0,t) + \int_0^x K_{ss}(x,s)F(s,t)ds =$$

$$= F_{tt} + K_{tt} + K(x,x)F_x(s,t)-$$

$$-K_t(x,x)F(x,t) + \int_0^x K_{ss}(x,s)F(s,t)ds = 0, \tag{4.6.10}$$

ayniyat hosil bo'ladi.

(4.6.5), (4.6.6) va (4.6.10) tengliklardan, $0 \le t \le x < \infty$ da ushbu

$$J_{xx} - J_{tt} - q(x)J = K_{xx} - K_{tt} - q(x)K+$$

$$+\int_0^x [K_{xx}(x,s) - K_{ss}(x,s) - q(x)K(x,s)]F(s,t)ds = 0, \tag{4.6.11}$$

ayniyat kelib chiqadi, ya'ni

$$\psi(t) = K_{xx}(x,t) - K_{tt}(x,t) - q(x)K(x,t),$$

funksiya quyidagi

$$\psi(t)+\int_0^x F(s,t)\psi(s)ds=0, \qquad (4.6.12)$$

bir jinsli integral tenglamani qanoatlantiradi. Bu tenglama faqat nol yechimga ega ekanligi bizga ma'lum. Demak,

$$K_{xx}-q(x)K=K_{tt},$$

ayniyat bajariladi. Bundan tashqari

$$\frac{dK(x,x)}{dx}=\frac{1}{2}q(x), \quad K(0,0)=h, \quad \frac{\partial K}{\partial t}|_{t=0}=0,$$

tengliklar ham o'rinli bo'ladi.

Shunday qilib, $K(x,t)$ funksiya quyidagi

$$\begin{cases} L_0y=-\dfrac{d^2y}{dx^2}, \\ y'(0)=0, \end{cases}$$

va

$$\begin{cases} Ly=-\dfrac{d^2y}{dx^2}+q(x)y, \\ y'(0)=hy(0), \end{cases}$$

chiziqli operatorlar uchun ushbu

$$Xf(x)=f(x)+\int_0^x K(x,t)f(t)dt,$$

almashtirish operatorining yadrosi bo'lar ekan. Shuning uchun

$$\varphi(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x K(x,t)\cos\sqrt{\lambda}tdt,$$

funksiya

$$-y''+q(x)y=\lambda y,$$

tenglamani va

$$y(0)=1, \quad y'(0)=h=-F(0,0),$$

boshlangich shartlarni qanoatlantiradi. Biz hozircha bu teoremani $F(x,t)$ ikki marta uzluksiz differensiallanuvchi bo'lgan holda isbot qildik.

Endi $F(x,t)$ funksiya faqat bir marta differensiallanuvchi bo'lgan holni ko'rib chiqamiz. Bu holda $K(x,t)$ ham faqat bir marta uzluksiz hosilaga ega bo'ladi. U holda (4.6.5) tenglikni faqat bir marta differensiallashimiz mumkin. Bunday holda, $\Phi(x)$ funksiyani har bir chekli oraliqda ikki marta uzluksiz differensiallanuvchi $\Phi_n(x)$ funksiyalar bilan tekis yaqinlashtiramiz, shu bilan birga ixtiyoriy chekli N soni uchun

$$\lim_{n\to\infty}\int_0^N |\Phi_n'(x)-\Phi'(x)|dx=0,$$

$$F_n(x,y)=\frac{1}{2}[\Phi_n(x+y)+\Phi_n(x-y)],$$

bo'lsin. U holda x parametrning har bir tayinlangan qiymatida va n ning yetarlicha katta qiymatlarida quyidagi

$$F_n(x,t)+K_n(x,t)+\int_0^x K_n(x,s)F_n(s,t)ds=0\,,\quad (0\le t\le x<\infty),$$

integral tenglama yechimga ega bo'ladi va ushbu

$$K_n(x,t)\to K(x,t),\quad \frac{\partial K_n(x,t)}{\partial x}\to\frac{\partial K(x,t)}{\partial x},$$

$$\frac{\partial K_n(x,t)}{\partial t}\to\frac{\partial K(x,t)}{\partial t},$$

munosabatlar bajariladi. Oxirgi fikrlar II bobdagi mulohazalardan kelib chiqadi.

Boshqa tomondan qaraganda, oldin isbotlaganimizdek, ushbu

$$\varphi_n(x,\lambda)=\cos\sqrt{\lambda}x+\int_0^x K_n(x,t)\cos\sqrt{\lambda}t\,dt, \qquad (4.6.13)$$

funksiya quyidagi

$$y'' + \left\{\lambda - 2\frac{d}{dx}K_n(x,x)\right\}y = 0, \qquad (4.6.14)$$

tenglamani va

$$\varphi_n(0,\lambda) = 1\,, \quad \varphi_n'(0,\lambda) = K_n(0,0), \qquad (4.6.15)$$

boshlang'ich shartlarni qanoatlantiradi.

(4.6.13), (4.6.14) va (4.6.15) tengliklarda n cheksizga intilganda limitga o'tib, (4.6.1) funksiya ushbu

$$y'' + \left\{\lambda - 2\frac{d}{dx}K(x,x)\right\}y = 0,$$

tenglamani va

$$y(0) = 1\,, \quad y(0) = K(0,0),$$

boshlang'ich shartlarni qanoatlantirishiga ishonch hosil qilamiz. ■

7-§. Parseval tengligini keltirib chiqarish

1. Oldingi paragrafda biz **I** va **II** shartlarni qanoatlantiruvchi monoton o'suvchi $\rho(\lambda)$ funksiya uchun

$$\begin{cases} -y'' + q(x)y = \lambda y, \quad (0 \le x < \infty), \\ y'(0) = hy(0), \end{cases} \qquad (4.7.1)$$

chegaraviy masala tuzgan edik.

Endi, $\rho(\lambda)$ funksiya haqiqatan ham (4.7.1) chegaraviy masalaning spektral funksiyasi bo'lishini ko'rsatamiz. Buning uchun Parseval tengligi bajarilishini ko'rsatish yetarli, ya'ni ixtiyoriy $f(x)$ silliq finit funksiya uchun

$$\int_0^\infty f^2(x)dx = \int_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda), \qquad (4.7.2)$$

tenglik bajarilishini ko'rsatish yetarli. Bu yerda

$$F(\lambda) = \int_0^\infty f(x)\varphi(x,\lambda)dx. \tag{4.7.3}$$

2. $F(x,t) = \frac{1}{2}[\Phi(x+t) + \Phi(x-t)]$ bo'lib, $K(x,t)$ Gelfand-Levitan integral tenglamasining yechimi bo'lsin. Bundan tashqari

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x,t)\cos\sqrt{\lambda}t dt,$$

$$\cos\sqrt{\lambda}x = \varphi(x,\lambda) + \int_0^x H(x,t)\varphi(t,\lambda)dt.$$

bo'lsin.

Lemma 4.7.1. *Quyidagi ayniyat o'rinli*

$$H(x,t) = F(x,t) + \int_0^t K(t,s)F(x,s)ds, \quad (0 \le t \le x). \tag{4.7.4}$$

Isbot. Dastlab $F(x,t)$ funksiya ikki marta uzluksiz differensiallanuvchi bo'lgan holni ko'rib chiqamiz. Bu holda $K(x,t)$ funksiya ham ikki marta uzluksiz differensiallanuvchi bo'ladi va ushbu

$$K_{xx} - q(x)K = K_{tt}, \tag{4.7.5}$$

tenglamani hamda

$$K_t|_{t=0} = 0,$$

$$K(x,x) = h + \frac{1}{2}\int_0^x q(s)ds, \tag{4.7.6}$$

shartlarni qanoatlantiradi.

$H(x,t)$ funksiya esa

$$H_{xx} = H_{tt} - q(t)H, \tag{4.7.7}$$

tenglamani va ushbu

$$(H_t - hH)\,|_{t=0} = 0, \tag{4.7.8}$$

$$H(x,x) = -h - \frac{1}{2}\int_0^x q(s)ds = -K(x,x), \tag{4.7.9}$$

shartlarni qanoatlantiradi.

Shunday qilib, lemmani isbotlash uchun (4.7.4) tenglik o'ng tomonidagi funksiya (4.7.7)+ (4.7.8)+(4.7.9) masalaning yechimi ekanligini ko'rsatish yetarli.

Ushbu

$$Z(x,t) = F(x,t) + \int_0^t K(t,s)F(x,s)ds,$$

begilashni kiritamiz. U holda

$$Z(x,0) = F(x,0),$$

bo'ladi. Bundan tashqari

$$F_t|_{t=0} = 0,$$

va

$$Z_t(x,t) = F_t(x,t) + K(t,t)F(x,t) + \int_0^t K_t(t,s)F(x,s)ds,$$

ayniyatlardan quyidagi

$$Z_t(x,t)|_{t=0} = K(0,0)F(x,0) = hF(x,0) = hZ(x,0),$$

tenglik kelib chiqadi, ya'ni $Z(x,t)$ funksiya (4.7.8) shartni qanoatlantirar ekan.

$Z(x,t)$ funksiya uchun (4.7.9) tenglik bajarilishini ko'rsatamiz. Buning uchun ushbu

$$Z(x,x) = F(x,x) + \int_0^x K(x,s)F(x,s)ds,$$

tenglik va Gelfand-Levitan integral tenglamasi yordamida hosil qilingan

$$F(x,x)+\int_0^x K(x,s)F(s,x)ds=-K(x,x),$$

tenglikdan foydalanish kifoya.

Endi $Z(x,t)$ funksiya (4.7.7) tenglamani qanoatlantirishini ko‘rsatamiz:

$$Z_{tt}=F_{tt}+\frac{dK(t,t)}{dt}F+K(t,t)\cdot F_t+K_t(t,s)|_{s=t}F+$$

$$+\int_0^t K_{tt}(t,s)F(x,s)ds,$$

$$Z_x=F_x+\int_0^t K(t,s)F_x(x,s)ds,$$

$$Z_{xx}=F_{xx}+\int_0^t K(t,s)\,F_{xx}\,(x,s)\,ds.$$

Oxirgi ayniyatda $F_{xx}(x,s)=F_{ss}(x,s)$ tenglikni ishlatib, bo‘laklab integrallashni qo‘llasak, quyidagi

$$Z_{xx}=F_{xx}+\int_0^t K(t,s)dF_s(x,s)=F_{xx}+K(t,t)F_t(x,t)-$$

$$-K(t,0)F_t(x,0)-\int_0^t K_s(t,s)dF(x,s)=F_{xx}+K(t,t)F_t-$$

$$-K_s(t,s)|_{s=t}F+K_s(t,s)|_{s=0}F(x,0)+\int_0^t K_{ss}(t,s)F(x,s)ds,$$

ya'ni

$$Z_{xx} = F_{xx} + K(t,\, t)F_t - K_s(t,\, s)|_{s=t}F + \int\limits_0^t K_{ss}(t,\, s)F(x,\, s)ds,$$

hosil bo'ladi.

Demak,

$$Z_{xx} - Z_{tt} = -q(t)F(x,t) + \int\limits_0^t [K_{ss}(t,s) - K_{tt}(t,s)]F(x,s)ds,$$

tenglik o'rinli. Bundan esa,

$$Z_{xx} - Z_{tt} + q(t)Z =$$

$$= \int\limits_0^t [K_{ss}(t,s) - K_{tt}(t,s) + q(t)K(t,s)]F(x,s)ds = 0,$$

chunki,

$$K_{ss}(t,s) - K_{tt}(t,s) + q(t)K(t,s) = 0.$$

Shunday qilib, $F(x,t)$ funksiya ikki marta uzluksiz differensiallanuvchi bo'lgan holda lemma isbotlandi. $F(x,t)$ funksiya faqat bir marta uzluksiz differensiallanuvchi bo'lgan hol limitga o'tish usulidan foydalanib ko'rsatiladi.■ .

3. Endi Parseval tengligini isbot qilishga o'tamiz. $f(t)$ ixtiyoriy silliq finit funksiya bo'lib,

$$F(\lambda) = \int\limits_0^\infty f(t)\varphi(t,\lambda)dt =$$

$$= \int\limits_0^\infty f(t)\left[\cos\sqrt{\lambda}t + \int\limits_0^t K(t,s)\cos\sqrt{\lambda}s ds\right]dt =$$

$$= \int\limits_0^\infty g(t)\cos\sqrt{\lambda}t dt, \qquad (4.7.10)$$

bo'lsin. Bu yerda

$$g(t) = f(t) + \int_t^{\infty} K(s,t)f(s)ds. \tag{4.7.11}$$

$g(t)$ funksiya ham finit va silliq bo'lishi ravshan. Bundan tashqari, $F(\lambda)$ funksiya $g(t)$ funksiyaning kosinuslar bo'yicha yozilgan Furye almashtirishi bo'ladi. Demak,

$$\int_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda) = \int_{-\infty}^{\infty} F^2(\lambda)d\rho_0(\lambda) + \int_{-\infty}^{\infty} F^2(\lambda)d\sigma(\lambda) = \int_0^{\infty} g^2(t)dt +$$

$$+ \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} g(t)\cos\sqrt{\lambda}t dt \right\}^2 d\sigma(\lambda) = \int_0^{\infty} g^2(t)dt +$$

$$+ \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} g(t)\cos\sqrt{\lambda}t dt \right\} \left\{ \int_0^{\infty} g(x)\cos\sqrt{\lambda}x dx \right\} d\sigma(\lambda) =$$

$$= \int_0^{\infty} g^2(t)dt + \int_{-\infty}^{\infty} \left\{ \int_0^{\infty}\int_0^{\infty} g(x)g(t)\cos\sqrt{\lambda}x\cos\sqrt{\lambda}t dx dt \right\} d\sigma(\lambda) =$$

$$= \int_0^{\infty} g^2(t)dt + \int_0^{\infty}\int_0^{\infty} g(x)g(t)F(x,t)dxdt. \tag{4.7.12}$$

Quyidagi integralning shaklini o'zgartiramiz:

$$\int_0^{\infty} F(x,t)g(t)dt = \int_0^{\infty} F(x,t)\left\{ f(t) + \int_t^{\infty} K(s,t)f(s)ds \right\} dt =$$

$$= \int_0^{\infty} F(x,t)f(t)dt + \int_0^{\infty} F(x,t)\left\{ \int_t^{\infty} K(s,t)f(s)ds \right\} dt. \tag{4.7.13}$$

(4.7.13) tenglikdagi oxirgi integralda integrallash tartibini almashtiramiz:

$$\int_0^\infty F(x,t)g(t)dt =$$

$$= \int_0^\infty F(x,s)f(s)ds + \int_0^\infty \left\{ \int_0^s F(x,t)K(s,t)dt \right\} f(s)ds =$$

$$= \int_0^\infty \left\{ F(x,s) + \int_0^s K(s,t)F(x,t)dt \right\} f(s)ds.$$

Oxirgi integralni ikkiga ajratib yozib, lemma 4.7.1 va Gelfand-Levitan integral tenglamasidan foydalansak, quyidagi

$$\int_0^\infty F(x,t)g(t)dt = \int_0^x \left\{ F(x,s) + \int_0^s K(s,t)F(x,t)dt \right\} f(s)ds+$$

$$+\int_x^\infty \left\{ F(x,s) + \int_0^s K(s,t)F(x,t)dt \right\} f(s)ds =$$

$$= \int_0^x H(x,s)f(s)ds - \int_x^\infty K(s,x)f(s)ds, \qquad (4.7.14)$$

tenglik hosil bo'ladi.

Endi (4.7.12) va (4.7.14) ayniyatlardan foydalanamiz:

$$\int_{-\infty}^\infty F^2(\lambda)d\rho(\lambda) = \int_0^\infty g^2(t)dt+$$

$$+\int_0^\infty g(x)\left\{ \int_0^x H(x,s)f(s)ds - \int_x^\infty K(s,x)f(s)ds \right\} dx =$$

$$= \int_0^\infty g^2(t)dt + \int_0^\infty g(x)\left\{ \int_0^x H(x,s)f(s)ds \right\} dx-$$

$$-\int_0^\infty g(x)\left\{\int_x^\infty K(s,x)f(s)ds\right\}dx. \qquad (4.7.15)$$

(4.7.15) tenglik oxiridagi ikkinchi integralda integrallash tartibini o'zgartirib yozamiz:

$$\int_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda) = \int_0^\infty g^2(t)dt + \int_0^\infty f(s)\left\{\int_s^\infty H(x,s)g(x)dx\right\}ds-$$

$$-\int_0^\infty g(x)\left\{\int_x^\infty K(s,x)f(s)ds\right\}dx. \qquad (4.7.16)$$

$g(x)$ funksiyaning tuzilishidan ushbu

$$\int_x^\infty K(s,x)f(s)ds = g(x) - f(x),$$

tenglik kelib chiqadi.

Demak,

$$\int_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda) = \int_0^\infty f(s)\left\{g(s) + \int_s^\infty H(x,s)g(x)dx\right\}ds. \qquad (4.7.17)$$

Ushbu

$$g(s) + \int_s^\infty H(x,s)g(x)dx = f(s), \qquad (4.7.18)$$

tenglik bajarilishini ko'rsatamiz. Buning uchun

$$f(t) + \int_0^t K(t,s)f(s)ds = (I+K)f,$$

simvolik yozuv kiritib olamiz. Almashtirish operatorining ta'rifiga ko'ra

$$(I+K)^{-1} = I + H,$$

bo'ladi. Bundan esa,

$$(I+K^*)^{-1} = I + H^*,$$

tenglik kelib chiqadi. Bu yerda "yulduzcha" operatorning qo'shmasini bildiradi.

Simvolik belgilash yordamida $g(x)$ va $f(x)$ funksiyalar orasidagi bog'lanishni

$$g = f + K^* f,$$

ko'rinishda yozish mumkin. Undan esa

$$f = (I+K^*)^{-1}g = (I+H^*)g = g + H^*g,$$

kelib chiqadi, ya'ni (4.7.18) tenglik bajariladi. Buni hisobga olib, (4.7.17) tenglikni ushbu

$$\int_{-\infty}^{\infty} F^2(\lambda)d\rho(\lambda) = \int_{0}^{\infty} f^2(s)ds,$$

ko'rinishda yozishimiz mumkin, ya'ni Parseval tengligi o'rinli ekan.

4. Xulosa qilib, olingan natijalarni bitta teorema ko'rinishida yozishimiz mumkin.

Teorema 4.7.1. *Monoton o'suvchi $\rho(\lambda)$ funksiya biror*

$$\begin{cases} -y'' + q(x)y = \lambda y, & 0 \le x < \infty, \\ y'(0) = hy(0), & h < \infty,\ h \in R^1,\ q(x) \in C^n[0,\infty), \end{cases}$$

haqiqiy $q(x)$ koeffitsiyentli Shturm-Liuvill chegaraviy masalasining spektral funksiyasi bo'lishi uchun quyidagi ikkita shart bajarilishi zarur va yetarli:

I. $\Phi_N(x) = \int\limits_{-\infty}^{N} \cos\sqrt{\lambda}x d\sigma(\lambda)$ *funksiyalar ketma-ketligi biror $\Phi(x)$ funksiyaga chegaralanib yaqinlashadi, hamda $\Phi(x)$ funksiya $[0,\infty)$ oraliqda $n+1$ marta uzluksiz differensiallanuvchi bo'lib,*

$\Phi(+0) = -h$ *tenglik o'rinli. Bu yerda*

$$\sigma(\lambda) = \begin{cases} \rho(\lambda), & \lambda \le 0, \\ \rho(\lambda) - \frac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \end{cases}$$

II. *Ixtiyoriy silliq finit $f(x)$ funksiya uchun*

$$\int_{-\infty}^{\infty} E^2(\lambda) d\rho(\lambda) = 0,$$

tenglikdan $f(x) \equiv 0$ tenglik kelib chiqadi. Bu yerda

$$E(\lambda) = \int_0^{\infty} f(\lambda) \cos\sqrt{\lambda} x dx.$$

Lemma 4.7.2. *Agar $\rho(\lambda)$ funksiya o'sish nuqtalari to'plami chekli limitik nuqtaga ega bo'lsa,* **II** *shart bajariladi.*

Isbot. $\rho(\lambda)$ funksiyaning o'sish nuqtalari to'plamini P bilan belgilaylik. Shartga ko'ra P to'plam chekli limitik nuqtaga ega. Demak, $\lambda_n \in P$ ketma-ketlik topilib,

$$\lambda_n \to \lambda_0, \; n \to \infty$$

bo'ladi. Bu yerda λ_0 chekli son. U holda ushbu

$$\int_{\lambda_n - \varepsilon}^{\lambda_n + \varepsilon} E^2(\lambda) d\rho(\lambda) \le \int_{-\infty}^{\infty} E^2(\lambda) d\rho(\lambda) = 0, \quad (n = 1, 2, ...),$$

tengsizlikdan

$$E(\lambda_n) = 0, \quad (n = 1, 2, ...),$$

kelib chiqadi. $E(\lambda)$ butun funksiya bo'lib, chekli limitik nuqtaga ega bo'lgan to'plamda nolga aylanadi. Kompleks o'zgaruvchili funksiyalar nazariyasidagi yagonalik teoremasiga ko'ra $E(\lambda) \equiv 0$ bo'ladi. Parseval tengligidan

$$\int_0^{\infty} f^2(x) dx = \int_{-\infty}^{\infty} E^2(\lambda) d\rho(\lambda) = 0,$$

ya'ni $f(x) \equiv 0$ kelib chiqadi.■

8-§.Gelfand-Levitan integral tenglamasiga doir misollar

1-misol. Spektral funksiyasi

$$\rho(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \frac{2}{\pi}\sqrt{\lambda} + \frac{1}{a}, & \lambda > 0, (a > 0), \end{cases} \tag{4.8.1}$$

bo'lgan Shturm-Liuvill chegaraviy masalasini tuzamiz. Bu holda

$$\sigma(\lambda) = \rho(\lambda) - \rho_0(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \frac{1}{a}, & \lambda > 0, \end{cases}$$

bo'ladi, ya'ni $\sigma(\lambda) = \frac{1}{a}\theta(\lambda)$. Gelfand-Levitan integral tenglamasining yadrosi $F(x,t)$ uchun ushbu

$$F(x,t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda) =$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t\delta(\lambda)d\lambda = \frac{1}{a}$$

tenglik kelib chiqadi. Uni Gelfand-Levitan integral tenglamasiga qo'yib, hosil bo'lgan tenglamani yechamiz:

$$K(x,t) + \frac{1}{a} + \frac{1}{a}\int_0^x K(x,s)ds = 0,$$

$$K(x,t) = -\frac{1}{a} - \frac{1}{a}\int_0^x K(x,s)ds.$$

Quyidagi

$$\int_0^x K(x,s)ds = B(x), \tag{4.8.2}$$

belgilashni kiritsak, oxirgi tenglik ushbu

$$K(x,t) = -\frac{1}{a} - \frac{1}{a}B(x),$$

ko'rinishni oladi. Buni (4.8.2) belgilashga qo'ysak,

$$\int_0^x \left[-\frac{1}{a} - \frac{1}{a}B(x)\right] ds = B(x),$$

bo'ladi. Oxirgi tenglamadan $B(x)$ funksiyani topamiz:

$$B(x) = \frac{-x}{x+a}.$$

Demak,

$$K(x,t) = -\frac{1}{a} + \frac{x}{ax+a^2} = \frac{-a}{ax+a^2} = -\frac{1}{x+a},$$

bo'lar ekan. Undan

$$q(x) = 2\frac{dK(x,x)}{dx} = \left(\frac{-2}{x+a}\right)' = \frac{2}{(x+a)^2},$$

$$h = K(0,0) = -\frac{1}{a},$$

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x - \frac{1}{x+a}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

kelib chiqadi. $\lambda = 0$ xos qiymatga mos keluvchi xos funksiya ushbu

$$\varphi(x,0) = 1 - \frac{x}{x+a} = \frac{a}{x+a},$$

tenglik bilan beriladi.

Shunday qilib, biz izlagan Shturm-Liuvill chegaraviy masalasi

$$\begin{cases} -y'' + \dfrac{2}{(x+a)^2}y = \lambda y, & (0 \le x < \infty), \\ y'(0) = -\dfrac{1}{a}y(0), \end{cases}$$

ko'rinishda bo'ladi. Bu chegaraviy masalaning spektri $E = PE \cup CE \cup PCE$ uchun quyidagi munosabatlar bajariladi:

$$PE = \emptyset, \;\; CE = [0,\infty), \;\; PCE = \{0\}.$$

2-misol. Spektral funksiyasi

$$\rho(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \dfrac{2}{3\pi}\sqrt{\lambda^3} + a, & \lambda > 0, \ (a > 0), \end{cases} \tag{4.8.3}$$

bo'lgan Shturm-Liuvill chegaraviy masalasini tuzamiz. Bu hol chegaraviy shart nol bo'lgan holga to'g'ri kelishi spektral funksiya asimptotikasining bosh qismidan ko'rinadi. Avvalo $\sigma(\lambda)$ funksiyani quyidagicha aniqlaymiz:

$$\sigma(\lambda) = \rho(\lambda) - \rho_1(\lambda) = \begin{cases} 0, & \lambda \le 0 \\ a, & \lambda > 0 \end{cases}, \quad \rho_1(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \dfrac{2}{3\pi}\sqrt{\lambda^3}, & \lambda > 0. \end{cases}$$

$\sigma(\lambda) = a\theta(\lambda)$ bo'lgani uchun Gelfand-Levitan integral tenglamasining yadrosi

$$F(x,t) = \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} d\sigma(\lambda) =$$

$$= \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} a\delta(\lambda)\, d\lambda = axt,$$

bo'ladi. Gelfand-Levitan integral tenglamasining o'zi esa,

$$K(x,t) + axt + \int_0^x K(x,s)astds = 0,$$

ko'rinishni oladi. Uni ushbu

$$K(x,t) = -axt - a\left(\int_0^x K(x,s)sds\right)t,$$

tarzda yozib olib,

$$\int_0^x K(x,s)sds = B(x),$$

belgilash kiritamiz. U holda

$$K(x,t) = -axt - aB(x)t,$$

bo'lgani uchun

$$\int_0^x \left[-axs^2 - aB(x)s^2\right] ds = B(x),$$

bo'ladi. Oxirgi tenglamadan $B(x)$ funksiyani topamiz:

$$B(x) = -\frac{ax^4}{3+ax^3}.$$

Demak,

$$K(x,t) = -axt + \frac{a^2x^4t}{ax^3+3} = -\frac{3axt}{ax^3+3},$$

$$q(x) = 2\frac{dK(x,x)}{dx} = \left(\frac{-6ax^2}{ax^3+3}\right)' = \frac{-12ax(ax^3+3)+18a^2x^4}{(ax^3+3)^2} =$$

$$= \frac{-12a^2x^4 - 36ax + 18a^2x^4}{(ax^3+3)^2} = \frac{6a^2x^4 - 36ax}{(ax^3+3)^2}.$$

Bu holda $\varphi(x,\lambda)$ yechim ushbu

$$\varphi(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{3ax}{ax^3+3}\int_0^x t\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt,$$

formula bilan aniqlanadi. Bundan $\lambda = 0$ xos qiymatga mos keluvchi xos funksiya ushbu

$$\varphi(x,\lambda) = x - \frac{3ax}{ax^3+3}\cdot\frac{x^3}{3} = \frac{3ax}{ax^3+3},$$

ko'rinishda bo'lishi kelib chiqadi.

Shunday qilib, biz izlagan Shturm-Liuvill chegaraviy masalasi ushbu

$$\begin{cases} -y'' + \dfrac{6a^2x^4 - 36ax}{(ax^3+3)^2}y = \lambda y, \quad (0 \le x < \infty), \\ y(0) = 0, \end{cases}$$

ko'rinishda bo'ladi. Bu chegaraviy masalaning spektri uchun quyidagi munosabatlar bajariladi:

$$PE = \emptyset, \; CE = [0, \infty), \; PCE = \{0\}.$$

3-misol. Spektral funksiyasi ushbu

$$\rho(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \dfrac{2}{\pi}\sqrt{\lambda}, & 0 < \lambda \le \lambda_1, \\ \dfrac{2}{\pi}\sqrt{\lambda} + a, & \lambda > \lambda_1, (a > 0), \end{cases} \tag{4.8.4}$$

tenglik bilan aniqlangan Shturm-Liuvill chegaraviy masalasini tuzamiz.

$\rho_0(\lambda)$ funksiya ushbu

$$\rho_0(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \dfrac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \end{cases}$$

formula bilan berilganligi uchun quyidagi

$$\sigma(\lambda) = \rho(\lambda) - \rho_0(\lambda) = \begin{cases} 0, & \lambda \le \lambda_1, \\ a, & \lambda > \lambda_1, \end{cases}$$

tenglik o'rinli bo'ladi, ya'ni $\sigma(\lambda) = a\theta\,(\lambda - \lambda_1)$.

$d\sigma(\lambda) = a\delta(\lambda - \lambda_1)d\lambda$ bo'lishidan foydalanib, Gelfand-Levitan integral tenglamasining yadrosini hisoblaymiz:

$$F(x,t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda) =$$

$$= \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t a\delta(\lambda - \lambda_1)d\lambda =$$

$$= a\cos\sqrt{\lambda_1}x \cos\sqrt{\lambda_1}t.$$

Bu holda, Gelfand-Levitan integral tenglamasi ushbu

$$K(x,t) + a\cos\sqrt{\lambda_1}x\cos\sqrt{\lambda_1}t +$$

$$+\int_0^x K(x,s)a\cos\sqrt{\lambda_1}s\cos\sqrt{\lambda_1}t\,ds=0,$$

ko'rinishda bo'ladi. Bu tenglamani yechish uchun avvalo uni quyidagi

$$K(x,t)=-a\cos\sqrt{\lambda_1}x\cos\sqrt{\lambda_1}t-$$

$$-a\left(\int_0^x K(x,s)\cos\sqrt{\lambda_1}s\,ds\right)\cos\sqrt{\lambda_1}t, \qquad (4.8.5)$$

shaklda yozib olamiz. Oxirgi tenglikda qavs ichidagi integralni $B(x)$ deb belgilab olamiz:

$$\int_0^x K(x,s)\cos\sqrt{\lambda_1}s\,ds=B(x). \qquad (4.8.6)$$

U holda (4.8.5) tenglik quyidagicha

$$K(x,t)=-a\cos\sqrt{\lambda_1}x\cos\sqrt{\lambda_1}t-aB(x)\cos\sqrt{\lambda_1}t, \qquad (4.8.7)$$

yoziladi. (4.8.7) formuladan ushbu

$$K(x,s)=-a\cos\sqrt{\lambda_1}x\cos\sqrt{\lambda_1}s-aB(x)\cos\sqrt{\lambda_1}s,$$

tenglik kelib chiqadi. Bu ifodani (4.8.6) tenglikka qo'ysak, ushbu

$$\int_0^x\left[-a\cos\sqrt{\lambda_1}x\cos^2\sqrt{\lambda_1}s-aB(x)\cos^2\sqrt{\lambda_1}s\right]ds=B(x),$$

tenglik hosil bo'ladi. Bu tenglikdan $B(x)$ funksiyani topamiz:

$$B(x)=\frac{-a\left(\int_0^x\cos\sqrt{\lambda_1}s\,ds\right)\cos\sqrt{\lambda_1}x}{1+a\int_0^x\cos^2\sqrt{\lambda_1}s\,ds}.$$

Demak, bu holda Gelfand-Levitan integral tenglamasining yechimi ushbu

$$K(x,t)=-a\cos\sqrt{\lambda_1}x\cos\sqrt{\lambda_1}t+$$

$$+\frac{a^2\left(\int\limits_0^x \cos^2\sqrt{\lambda_1}s ds\right)\cos\sqrt{\lambda_1}x\cos\sqrt{\lambda_1}t}{1+a\int\limits_0^x \cos^2\sqrt{\lambda_1}s ds},$$

ya'ni

$$K(x,t)=\frac{-a\cos\sqrt{\lambda_1}x\cos\sqrt{\lambda_1}t}{1+a\int\limits_0^x \cos^2\sqrt{\lambda_1}s ds},$$

formula bilan beriladi. Bundan

$$h=K(0,0)=-a,$$

va

$$q(x)=2\frac{dK(x,x)}{dx}=2\left(\frac{-a\cos^2\sqrt{\lambda_1}x}{1+a\int\limits_0^x \cos^2\sqrt{\lambda_1}s ds}\right)'=$$

$$=2\frac{2a\sqrt{\lambda_1}\cos\sqrt{\lambda_1}x\sin\sqrt{\lambda_1}x\left(1+a\int\limits_0^x \cos^2\sqrt{\lambda_1}s ds\right)+a^2\cos^4\sqrt{\lambda_1}x}{\left(1+a\int\limits_0^x \cos^2\sqrt{\lambda_1}s ds\right)^2}.$$

Demak,

$$q(x)=\frac{a^2+2a\sqrt{\lambda_1}\left(1+\frac{ax}{2}\right)\sin 2\sqrt{\lambda_1}x+a^2\cos 2\sqrt{\lambda_1}x}{\left(1+\frac{ax}{2}+\frac{a}{4\sqrt{\lambda_1}}\sin 2\sqrt{\lambda_1}x\right)^2}.$$

Masalan, $a=2$ va $\lambda_1=\frac{1}{4}$ bo'lsa, $h=-2$ va

$$q(x)=\frac{4+2(1+x)\sin x+4\cos x}{(1+x+\sin x)^2},$$

bo'ladi. Ushbu

$$\varphi(x,\lambda)=\cos\sqrt{\lambda}x+\int\limits_0^x K(x,t)\cos\sqrt{\lambda}t dt,$$

formuladan foydalanib, tuzilgan differensial tenglamaning yechimini ham topishimiz mumkin. Xususan, $a = 2$, $\lambda_1 = \frac{1}{4}$ bo'lgan holda $\lambda = \frac{1}{4}$ xos qiymatga mos keluvchi xos funksiya ushbu

$$\varphi(x) = \frac{\cos\frac{x}{2}}{1 + x + \sin x},$$

formula bilan beriladi.

Demak, topilgan

$$\begin{cases} -y'' + \left\{\dfrac{4 + 2(1+x)\sin x + 4\cos x}{(1 + x + \sin x)^2}\right\} y = \lambda y, \quad (0 \le x < \infty), \\ y'(0) = -2y(0), \end{cases}$$

chegaraviy masalaning spektri uchun quyidagi munosabatlar o'rinli:

$$PE = \emptyset, \ \ CE = [0, \infty), \ \ PCE = \left\{\frac{1}{4}\right\}.$$

4-misol. Spektral funksiyasi

$$\rho(\lambda) = \begin{cases} -a, & \lambda \le \lambda_1, \ (\lambda_1 < 0, \ a > 0), \\ 0, & \lambda_1 < \lambda \le 0, \\ \dfrac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \end{cases} \tag{4.8.8}$$

bo'lgan Shturm-Liuvill chegaraviy masalasini tuzamiz. Bu holda

$$\sigma(\lambda) = \rho(\lambda) - \rho_0(\lambda) = \begin{cases} -a, & \lambda \le \lambda_1, \\ 0, & \lambda > \lambda_1, \end{cases}$$

bo'ladi, ya'ni $\sigma(\lambda) = a\theta(\lambda - \lambda_1) - a$. Bundan $F(x,t)$ yadro uchun ushbu

$$F(x,t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda) =$$

$$= \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t a\delta(\lambda - \lambda_1) d\lambda =$$

$$= ach\sqrt{|\lambda_1|}xch\sqrt{|\lambda_1|}t\,,$$

tenglik kelib chiqadi.

Bu holda, Gelfand-Levitan integral tenglamasi

$$K(x,t) + ach\sqrt{|\lambda_1|}x ch\sqrt{|\lambda_1|}t +$$

$$+\int_0^x K(x,s)\, ach\sqrt{|\lambda_1|}s \cdot ch\sqrt{|\lambda_1|}tds = 0,$$

ko'rinishni oladi. Bu integral tenglamani yechamiz:

$$K(x,t) = -ach\sqrt{|\lambda_1|}x ch\sqrt{|\lambda_1|}t -$$

$$-a\left(\int_0^x K(x,s)\, a_1 ch\sqrt{|\lambda_1|}sds\right) ch\sqrt{|\lambda_1|}t.$$

Bu tenglikda

$$\int_0^x K(x,s) ch\sqrt{|\lambda_1|}sds = B(x), \qquad (4.8.9)$$

belgilash kiritib,

$$K(x,t) = -ach\sqrt{|\lambda_1|}x ch\sqrt{|\lambda_1|}t - aB(x) ch\sqrt{|\lambda_1|}t, \qquad (4.8.10)$$

ifodani hosil qilamiz. $K(x,t)$ funksiyaning (4.8.9) tenglikdagi ifodasini (4.8.8) tenglikka qo'yamiz va $B(x)$ funksiyani topamiz:

$$B(x) = \frac{-a_1\left(\int_0^x ch^2\sqrt{|\lambda_1|}sds\right) ch\sqrt{|\lambda_1|}x}{1 + a_1\int_0^x ch^2\sqrt{|\lambda_1|}sds}.$$

Demak,

$$K(x,t) = \frac{-ach\sqrt{|\lambda_1|}x ch\sqrt{|\lambda_1|}t}{1 + a\int_0^x ch^2\sqrt{|\lambda_1|}sds},$$

$$h = K(0,0) = -a\,,$$

$$q(x) = -2a\left(\frac{ch^2\sqrt{|\lambda_1|}x}{1+a\int\limits_0^x ch^2\sqrt{|\lambda_1|}sds}\right)'$$

bo'ladi.

Masalan, $a_1 = 2$ va $\lambda_1 = -\frac{1}{4}$ bo'lsa, $h = -2$ va

$$q(x) = \frac{4-2(1+x)shx+4chx}{(1+x+shx)^2},$$

bo'ladi. Ushbu

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \int\limits_0^x K(x,t)\cos\sqrt{\lambda}tdt,$$

formuladan foydalanib, tuzilgan differensial tenglamaning yechimini ham topishimiz mumkin. Xususan, $a_1 = 2$, $\lambda_1 = -\frac{1}{4}$ bo'lgan holda $\lambda = -\frac{1}{4}$ xos qiymatga mos keluvchi xos funksiya ushbu

$$\varphi(x,\lambda) = \frac{\operatorname{ch}\frac{x}{2}}{1+x+\operatorname{sh}x}$$

formula bilan beriladi.

Demak, biz topgan

$$\begin{cases} -y'' - \left\{\dfrac{4-2(1+x)\,shx+4\operatorname{ch}x}{(1+x+\operatorname{sh}x)^2}\right\}y = \lambda y, \quad (0 \le x < \infty), \\ y'(0) = -2y(0), \end{cases}$$

chegaraviy masalaning E spektri uchun quyidagi tengliklar o'rinli:

$$PE = \left\{-\frac{1}{4}\right\}, \quad CE = [0,\infty), \quad PCE = \emptyset.$$

Mustaqil yechish uchun mashqlar

1. Spektral funksiyasi quyidagi formula bilan berilgan Shturm-Liuvill chegaraviy masalasini tuzing.

a) $$\rho(\lambda) = \begin{cases} 0, & \lambda \le 0, \\ \dfrac{2}{\pi}(\sqrt{\lambda} + \operatorname{arctg}\sqrt{\lambda}), & \lambda > 0, \end{cases}$$

b) $$\rho(\lambda) = \begin{cases} \dfrac{1}{\pi}\int\limits_0^\lambda \dfrac{t + a^2}{(t + a^2 cth^2 b)\sqrt{t}} dt, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

c) $$\rho(\lambda) = \begin{cases} \dfrac{1}{\pi}\int\limits_0^\lambda \dfrac{\sqrt{t}}{t + a^2} dt, & \lambda > 0, \ (a > 0), \\ 0, & -a^2 < \lambda \le 0, \\ -a, & \lambda \le -a^2, \end{cases}$$

Ko'rsatma. Bu masalalarni yechishda quyidagi formuladan foydalaniladi:

$$\int\limits_0^\infty \frac{\cos \alpha z}{z^2 + 1} dz = \frac{\pi}{2} e^{-|\alpha|}.$$

V BOB. BUTUN O‘QDA BERILGAN SHTURM-LIUVILL OPERATORI UCHUN TO‘G‘RI VA TESKARI SPEKTRAL MASALALAR

1-§. Parseval tengligi

Ushbu

$$-y'' + q(x)y = \lambda y\,, \quad -\infty < x < \infty, \tag{5.1.1}$$

Shturm-Liuvill masalasi uchun Parseval tengligini keltirib chiqaramiz. Bu yerda $q(x)$ haqiqiy uzluksiz funksiya, λ esa ixtiyoriy parametr.

$\theta(x,\lambda)$ va $\varphi(x,\lambda)$ orqali (5.1.1) tenglamaning ushbu

$$\begin{cases} \theta(0,\lambda) = 1, \\ \theta'(0,\lambda) = 0, \end{cases} \qquad \begin{cases} \varphi(0,\lambda) = 0, \\ \varphi'(0,\lambda) = 1, \end{cases}$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimlarini belgilaymiz.

Butun o‘qda berilgan Shturm-Liuvill masalasiga mos keluvchi Parseval tengligini keltirib chiqarish maqsadida $(a,\ b)$ chekli oraliqda berilgan Shturm-Liuvill chegaraviy masalasi uchun olingan Parseval tengligida biror $a_k \to -\infty$, $b_k \to \infty$ ketma-ketliklar bo‘yicha limitga o‘tish mumkinligini ko‘rsatamiz. Bu usul B.M.Levitan usuli deb yuritiladi.

Yuqorida zikr etilgan g‘oyani amalga oshirish maqsadida quyidagi chekli oraliqda berilgan Shturm-Liuvill masalasini ko‘rib chiqamiz:

$$-y'' + q(x)y = \lambda y\,, \quad a < x < b, \qquad (a < 0,\ b > 0), \tag{5.1.2}$$

$$\begin{cases} y(a)\cos\alpha + y'(a)\sin\alpha = 0, \\ y(b)\cos\beta + y'(b)\sin\beta = 0. \end{cases} \tag{5.1.3}$$

(5.1.2)+(5.1.3) masalaning xos qiymatlarini λ_1, λ_2, ..., λ_k, ... orqali, ularga mos keluvchi ortonormallangan xos funksiyalarini esa $u_1(x)$, $u_2(x)$, ..., $u_k(x)$, ... orqali belgilaymiz. $\theta(x,\lambda)$ va $\varphi(x,\lambda)$ yechimlar chiziqli erkli bo'lganligi uchun (5.1.2) tenglamaning ixtiyoriy yechimi ular orqali chiziqli ifodalanadi:

$$u_k(x) = c_k\theta(x,\lambda_k) + d_k\varphi(x,\lambda_k). \tag{5.1.4}$$

(5.1.2)+(5.1.3) masala uchun Parseval tengligi ushbu

$$\int_a^b f^2(x)dx = \sum_{k=1}^{\infty}\left\{\int_a^b f(x)u_k(x)dx\right\}^2, \tag{5.1.5}$$

ko'rinishda bo'ladi. (5.1.4) ifodani (5.1.5) tenglikka qo'yamiz:

$$\int_a^b f^2(x)dx = \sum_{k=1}^{\infty}\left\{c_k\int_a^b f(x)\theta(x,\lambda_k)dx + d_k\int_a^b f(x)\varphi(x,\lambda_k)dx\right\}^2. \tag{5.1.6}$$

Ushbu

$$F(\lambda) = \int_a^b f(x)\theta(x,\lambda)dx, \quad G(\lambda) = \int_a^b f(x)\varphi(x,\lambda)dx,$$

belgilashlarni kiritib, (5.1.6) tenglikni quyidagicha yozamiz:

$$\int_a^b f^2(x)dx = \sum_{k=1}^{\infty} c_k^2F^2(\lambda_k) + 2\sum_{k=1}^{\infty} c_kd_kF(\lambda_k)G(\lambda_k) + \sum_{k=1}^{\infty} d_k^2G^2(\lambda_k). \tag{5.1.7}$$

Bu tenglikdagi qatorlarni Stiltes integrali orqali yozish maqsadida quyidagi sakrash funksiyalarini kiritib olamiz:

$$\xi_{a,b}(\lambda) = \begin{cases} \sum\limits_{0\le\lambda_k<\lambda} c_k^2, & \lambda > 0, \\ -\sum\limits_{\lambda<\lambda_k\le 0} c_k^2, & \lambda \le 0, \end{cases}$$

$$\eta_{a,\,b}(\lambda)=\begin{cases}\sum\limits_{0\le\lambda_k<\lambda} c_k d_k, & \lambda>0,\\ -\sum\limits_{\lambda<\lambda_k\le 0} c_k d_k, & \lambda\le 0,\end{cases}$$

$$\zeta_{a,\,b}(\lambda)=\begin{cases}\sum\limits_{0\le\lambda_k<\lambda} d_k^2, & \lambda>0,\\ -\sum\limits_{\lambda<\lambda_k\le 0} d_k^2, & \lambda\le 0.\end{cases}$$

Stiltes integrali ta'rifidan foydalanib, (5.1.7) tenglikni quyidagi ko'rinishda yozamiz:

$$\int\limits_a^b f^2(x)dx=\int\limits_{-\infty}^{\infty} F^2(\lambda)d\xi_{a,\,b}(\lambda)+2\int\limits_{-\infty}^{\infty} F(\lambda)G(\lambda)d\eta_{a,\,b}(\lambda)+$$

$$+\int\limits_{-\infty}^{\infty} G^2(\lambda)d\zeta_{a,\,b}(\lambda). \tag{5.1.8}$$

Lemma 5.1.1. *$h\in(0,\ b)$ biror son bo'lsin. U holda ushbu*

$$g(x)=\begin{cases}\dfrac{x^3(h-x)^3}{\int\limits_a^b t^3(h-t)^3dt}, & x\in[0,h],\\ 0, & x\notin[0,h],\end{cases}$$

funksiya quyidagi shartlarni qanoatlantiradi:

1) $g(x)\in C^2[a,\ b]$;

2) $g(x)\ge 0$;

3) $\int\limits_a^b g(x)dx=1$.

Bu lemma isbotini o'quvchiga vazifa sifatida qoldiramiz.

Lemma 5.1.2. $P=\{(x,\lambda):\ 0\le x\le b,\ \ |\lambda|\le N\}$ *bo'lsin. U holda ixtiyoriy $\varepsilon>0$ son uchun shunday $h=h_N(\varepsilon)>0$ son topiladiki, bunda $0\le x\le h$, $|\lambda|\le N$ shartlarni qanoatlantiruvchi*

(x, λ) *juftliklar uchun ushbu*

$$|\theta(x, \lambda) - 1| < \varepsilon, \quad |\varphi(x, \lambda)| < \varepsilon, \tag{5.1.9}$$

$$|\theta'(x, \lambda)| < \varepsilon, \quad |\varphi'(x, \lambda) - 1| < \varepsilon, \tag{5.1.10}$$

tengsizliklar bajariladi.

Isbot. $\theta(x, \lambda)$, $\theta'(x, \lambda)$, $\varphi(x, \lambda)$, $\varphi'(x, \lambda)$ funksiyalar P yopiq to'g'ri to'rtburchakda uzluksiz bo'lganliklari uchun Kantor teoremasiga ko'ra tekis uzluksiz bo'ladilar, ya'ni ixtiyoriy $\varepsilon > 0$ son uchun shunday $h = h_N(\varepsilon) > 0$ son topiladiki, bunda ushbu

$$|x - \tilde{x}| < h, \quad \left|\lambda - \tilde{\lambda}\right| < h,$$

shartlarni qanoatlantiruvchi (x, λ), $(\tilde{x}, \tilde{\lambda}) \in P$ juftliklar uchun quyidagi tengsizliklar o'rinli bo'ladi:

$$\left|\theta(x, \lambda) - \theta(\tilde{x}, \tilde{\lambda})\right| < \varepsilon, \quad \left|\varphi(x, \lambda) - \varphi(\tilde{x}, \tilde{\lambda})\right| < \varepsilon,$$

$$\left|\theta'(x, \lambda) - \theta'(\tilde{x}, \tilde{\lambda})\right| < \varepsilon, \quad \left|\varphi'(x, \lambda) - \varphi'(\tilde{x}, \tilde{\lambda})\right| < \varepsilon.$$

Bu yerda $\tilde{x} = 0$, $\tilde{\lambda} = \lambda$ desak va boshlang'ich shartlarni inobatga olsak, (5.1.9) va (5.1.10) tengsizliklar kelib chiqadi.■

Lemma 5.1.3. *Ixtiyoriy $\varepsilon > 0$ va $N > 0$ sonlar uchun shunday $h = h_N(\varepsilon) > 0$ son mavjudki, bunda $|\lambda| \le N$ bo'lganda quyidagi tengsizliklar o'rinli bo'ladi:*

$$1 - \varepsilon < F(\lambda) < 1 + \varepsilon, \quad -\varepsilon < G(\lambda) < \varepsilon, \tag{5.1.11}$$

$$-\varepsilon < F_1(\lambda) < \varepsilon, \quad 1 - \varepsilon < G_1(\lambda) < 1 + \varepsilon. \tag{5.1.12}$$

Bu yerda

$$F(\lambda) = \int_0^h g(x)\theta(x, \lambda)dx, \quad G(\lambda) = \int_0^h g(x)\varphi(x, \lambda)dx,$$

$$F_1(\lambda) = \int_0^h (-g'(x))\theta(x, \lambda)dx, \quad G_1(\lambda) = \int_0^h (-g'(x))\varphi(x, \lambda)dx.$$

Isbot. Lemma 5.1.2 dagi $h = h_N(\varepsilon) > 0$ sonini olamiz va lemma 5.1.1 dan foydalanib, quyidagi baholashlarni amalga oshiramiz:

$$|F(\lambda) - 1| = \left| \int_0^h g(x)\theta(x,\lambda)dx - \int_0^h g(x)dx \right| =$$

$$= \left| \int_0^h g(x)[\theta(x,\lambda) - 1]dx \right| \leq$$

$$\leq \int_0^h g(x)|\theta(x,\lambda) - 1|dx < \varepsilon \int_0^h g(x)\, dx = \varepsilon,$$

$$|G(\lambda)| = \left| \int_0^h g(x)\varphi(x,\lambda)dx \right| \leq$$

$$\leq \int_0^h g(x)|\varphi(x,\lambda)|dx < \varepsilon \int_0^h g(x)dx = \varepsilon,$$

$$|F_1(\lambda)| = \left| \int_0^h (-g'(x))\theta(x,\lambda)dx \right| =$$

$$= \left| g(x)\theta(x,\lambda)\Big|_0^h - \int_0^h g(x)\theta'(x,\lambda)dx \right| \leq$$

$$\leq \int_0^h g(x)|\theta'(x,\lambda)|dx < \varepsilon,$$

$$|G_1(\lambda) - 1| = \left| \int_0^h (-g'(x))\varphi(x,\lambda)dx - 1 \right| =$$

$$= \left| \int_0^h g(x)\varphi'(x.\lambda)dx - \int_0^h g(x)dx \right| \leq$$

$$\leq \int_0^h g(x)|\varphi'(x,\lambda) - 1|dx < \varepsilon. \blacksquare$$

Lemma 5.1.4. *Har qanday $N > 0$ son uchun quyidagi tengsizlik o'rinli bo'ladi:*

$$\bigvee_{-N}^{N} \eta_{a,\,b}(\lambda) \leq \frac{1}{2} \cdot [\xi_{a,\,b}(N) - \xi_{a,\,b}(-N)] + \frac{1}{2} \cdot [\zeta_{a,\,b}(N) - \zeta_{a.\,b}(-N)]. \tag{5.1.13}$$

Isbot. Sakrash funksiyasining variatsiyasi sakrash uzunliklarining yig'indisiga teng bo'lishini inobatga olib, ushbu

$$\bigvee_{-N}^{N} \{\eta_{a,\,b}(\lambda)\} = \sum_{-N\leq\lambda_k\leq N} |c_k d_k| \leq \frac{1}{2} \sum_{-N\leq\lambda_k\leq N} c_k^2 + \frac{1}{2} \sum_{-N\leq\lambda_k\leq N} d_k^2,$$

baholashga ega bo'lamiz. Bundan (5.1.13) tengsizlik kelib chiqadi.■

Lemma 5.1.5. *Ixtiyoriy $\varepsilon \in (0, \ \sqrt{5} - 2)$ va $N > 0$ sonlar uchun shunday $h = h_N(\varepsilon) > 0$ son topiladiki, bunda ushbu*

$$[\xi_{a,\,b}(N) - \xi_{a,\,b}(-N)] + [\zeta_{a,\,b}(N) - \zeta_{a,\,b}(-N)] \leq$$

$$\leq \frac{1}{1 - 4\varepsilon - \varepsilon^2} \cdot \int_0^h [g^2(x) + g'^2(x)]dx, \tag{5.1.14}$$

tengizlik bajariladi.

Isbot. Lemma 5.1.3 dan va (5.1.7) Parseval tengligidan foydalanib, quyidagi baholashlarni olamiz:

$$\int_0^h g^2(x)dx = \sum_{k=1}^{\infty} \{c_k F(\lambda_k) + d_k G(\lambda_k)\}^2 \geq$$

$$\geq \sum_{|\lambda_k|\leq N} \{c_k F(\lambda_k) + d_k G(\lambda_k)\}^2 =$$

$$= \sum_{|\lambda_k|\le N} c_k^2 F^2(\lambda_k) + 2\sum_{|\lambda_k|\le N} c_k d_k F(\lambda_k) G(\lambda_k) + \sum_{|\lambda_k|\le N} d_k^2 G^2(\lambda_k) \ge$$

$$= \sum_{|\lambda_k|\le N} c_k^2 F^2(\lambda_k) - 2\sum_{|\lambda_k|\le N} |c_k d_k||F(\lambda_k)||G(\lambda_k)| +$$

$$+ \sum_{|\lambda_k|\le N} d_k^2 G^2(\lambda_k) \ge \sum_{|\lambda_k|\le N} c_k^2 (1-\varepsilon)^2 - 2\sum_{|\lambda_k|\le N} |c_k d_k| \cdot (1+\varepsilon)\varepsilon \ge$$

$$\ge (1 - 2\varepsilon + \varepsilon^2) \sum_{|\lambda_k|\le N} c_k^2 - (\varepsilon + \varepsilon^2) \cdot \sum_{|\lambda_k|\le N} (c_k^2 + d_k^2) =$$

$$= (1 - 3\varepsilon) \sum_{|\lambda_k|\le N} c_k^2 - (\varepsilon + \varepsilon^2) \cdot \sum_{|\lambda_k|\le N} d_k^2. \qquad (5.1.15)$$

Xuddi shu tarzda quyidagi tengsizlikni keltirib chiqaramiz:

$$\int_0^h g'^2(x)dx \ge \sum_{|\lambda_k|\le N} c_k^2 F_1^2(\lambda_k) - 2\sum_{|\lambda_k|\le N} |c_k d_k| \cdot |F_1(\lambda_k)| \cdot |G_1(\lambda_k)| +$$

$$+ \sum_{|\lambda_k|\le N} d_k^2 G_1^2(\lambda_k) \ge -2\sum_{|\lambda_k|\le N} |c_k d_k| \cdot \varepsilon(1+\varepsilon) + \sum_{|\lambda_k|\le N} d_k^2 (1-\varepsilon)^2 \ge$$

$$\ge -(\varepsilon + \varepsilon^2) \cdot \sum_{|\lambda_k|\le N} (c_k^2 + d_k^2) + (1 - 2\varepsilon + \varepsilon^2) \sum_{|\lambda_k|\le N} d_k^2 =$$

$$= -(\varepsilon + \varepsilon^2) \cdot \sum_{|\lambda_k|\le N} c_k^2 + (1 - 3\varepsilon) \cdot \sum_{|\lambda_k|\le N} d_k^2. \qquad (5.1.16)$$

(5.1.15) va (5.1.16) tengsizliklarni qo'shsak, ushbu

$$\int_0^h [g^2(x) + g'^2(x)]dx \ge (1 - 4\varepsilon - \varepsilon^2) \left\{ \sum_{|\lambda_k|\le N} c_k^2 + \sum_{|\lambda_k|\le N} d_k^2 \right\}, \qquad (5.1.17)$$

tengsizlik hosil bo'ladi. Agar $\varepsilon \in (0, \sqrt{5}-2)$ bo'lsa, $1-4\varepsilon-\varepsilon^2 > 0$ bo'lishi ravshan. Buni hisobga olib, (5.1.17) tengsizlikdan (5.1.14) baholashni keltirib chiqaramiz. ■

Lemma 5.1.6. *Har qanday musbat N soni uchun a va b larga bog'liq bo'lmagan shunday musbat $A = A(N)$ son topiladiki, bunda* *ushbu*

$$\overset{N}{\underset{-N}{V}}\{\xi_{a,\,b}(\lambda)\}\le A,\quad \overset{N}{\underset{-N}{V}}\{\eta_{a.\,b}(\lambda)\}\le A,\quad \overset{N}{\underset{-N}{V}}\{\zeta_{a.b}(\lambda)\}\le A, \tag{5.1.18}$$

tengsizliklar bajariladi, ya'ni har bir chekli $[-N, N]$ *oraliqda* $\xi_{a.\,b}(\lambda)$, $\eta_{a.\,b}(\lambda)$, $\zeta_{a.b}(\lambda)$ *funksiyalarning variatsiyasi* a *va* b *parametrlarga nisbatan tekis chegaralangan.*

Isbot. Lemma 5.1.5 da ε sonining aniq bitta qiymatini olib,

$$A = A(N) = \frac{1}{1-4\varepsilon-\varepsilon^2}\cdot\int\limits_0^{h_N(\varepsilon)}[g^2(x)+g'^2(x)]dx,$$

desak, lemma 5.1.4 va lemma 5.1.5 dan (5.1.18) baholashlar o'rinli bo'lishi kelib chiqadi. ■

Natija 5.1.1. $\xi_{a,\,b}(\lambda)$, $\eta_{a.\,b}(\lambda)$, $\zeta_{a,\,b}(\lambda)$ funksiyalarning ta'riflari va $[-N, N]$ kesmada variatsiyalari a va b ga nisbatan tekis chegaralangan bo'lishidan, bu funksiyalarning o'zlari ham shu kesmada a va b ga nisbatan tekis chegaralanganligi kelib chiqadi. Bu fikrni isbotlash o'quvchiga vazifa sifatida qoldiriladi.

Lemma 5.1.7. *Agar* $f(x)\in C^2(a,\ b)$ *haqiqiy funksiya biror* $[-n,\ n]\subset(a,\ b)$ *oraliqdan tashqarida nolga aylanadigan bo'lsa, u holda bu funksiya uchun quyidagi tengsizlik o'rinli bo'ladi:*

$$\Bigg|\int\limits_a^b f^2(x)dx-\int\limits_{-N}^{N}\big\{F^2(\lambda)d\xi_{a.\,b}(\lambda)+2F(\lambda)G(\lambda)d\eta_{a,\,b}(\lambda)+$$

$$+G^2(\lambda)d\zeta_{a,\,b}(\lambda)\big\}\Bigg|\le\frac{1}{N^2}\int\limits_a^b[f''(x)-q(x)f(x)]^2dx. \tag{5.1.19}$$

Isbot. $f(x)$ funksiya uchun (5.1.5) Parseval tengligini yozamiz:

$$\int\limits_{-n}^{n} f^2(x)dx=\sum_{-\infty<\lambda_k<\infty}\left\{\int\limits_a^b f(x)u_k(x)dx\right\}^2.$$

Bundan ushbu

$$\left|\int_{-n}^{n} f^2(x)dx - \sum_{|\lambda_k|\le N}\left\{\int_a^b f(x)u_k(x)dx\right\}^2\right| =$$

$$= \sum_{|\lambda_k|>N}\left\{\int_a^b f(x)u_k(x)\right\}^2, \qquad (5.1.20)$$

tenglik kelib chiqadi. (5.1.20) tenglikning o'ng tomonini baholaymiz. Buning uchun ushbu

$$u_k(x) = -\frac{1}{\lambda_k}[u_k''(x) - q(x)u_k(x)],$$

ayniyatdan foydalanamiz. Bunga ko'ra

$$\int_a^b f(x)u_k(x)dx = -\frac{1}{\lambda_k}\int_a^b f(x)[u_k''(x) - q(x)u_k(x)]dx =$$

$$= -\frac{1}{\lambda_k}\int_a^b [f''(x) - q(x)f(x)]\, u_k(x)dx.$$

Demak,

$$\left|\int_{-n}^{n} f^2(x)dx - \sum_{|\lambda_k|\le N}\left\{\int_a^b f(x)u_k(x)dx\right\}^2\right| =$$

$$= \sum_{|\lambda_k|>N}\frac{1}{\lambda_k^2}\left\{\int_a^b [f''(x) - q(x)f(x)]\, u_k(x)dx\right\}^2 \le$$

$$\le \frac{1}{N^2}\cdot\sum_{|\lambda_k|>N}\left\{\int_a^b [f''(x) - q(x)f(x)]\, u_k(x)dx\right\}^2 \le$$

$$\le \frac{1}{N^2}\cdot\sum_{k=1}^{\infty}\left\{\int_a^b [f''(x) - q(x)f(x)]\, u_k(x)dx\right\}^2 =$$

$$= \frac{1}{N^2} \cdot \int_a^b [f''(x) - q(x)f(x)]^2 dx.$$

Oxirgi tenglikni yozishda ushbu $f''(x) - q(x)f(x)$ funksiya uchun Parseval tengligi qo'llanildi. Agar modul ichidagi yig'indini oldin ko'rganimizdek qilib integral orqali yozsak, (5.1.19) baholash kelib chiqadi.■

Teorema 5.1.1 (*G.Veyl, 1910 yil*). $f(x) \in L^2(-\infty, \infty)$ *ixtiyoriy haqiqiy funksiya bo'lsin. $f(x)$ funksiyaga bog'liq bo'lmagan monoton o'suvchi, har bir chekli oraliqda chegaralangan $\xi(\lambda)$, $\zeta(\lambda)$ funksiyalar va har bir chekli oraliqda chegaralangan variatsiyali $\eta(\lambda)$ funksiya mavjudki, bunda ushbu*

$$\int_{-\infty}^{\infty} f^2(x)dx = \int_{-\infty}^{\infty} F^2(\lambda)d\xi(\lambda) + 2\int_{-\infty}^{\infty} F(\lambda)G(\lambda)d\eta(\lambda) +$$

$$+ \int_{-\infty}^{\infty} G^2(\lambda)d\zeta(\lambda), \qquad (5.1.21)$$

tenglik o'rinli bo'ladi. Bu yerda

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\theta(x,\lambda)dx, \quad G(\lambda) = \int_{-\infty}^{\infty} f(x)\varphi(x,\lambda)dx.$$

Izoh. (5.1.21) tenglikka butun o'qda berilgan Shturm-Liuvill masalasi uchun Parseval tengligi deyiladi. Quyidagi matritsa-funksiyaga esa

$$\Re(\lambda) = \begin{pmatrix} \xi(\lambda) & \eta(\lambda) \\ \eta(\lambda) & \xi(\lambda) \end{pmatrix}, \quad -\infty < \lambda < \infty,$$

butun o'qda berilgan Shturm-Liuvill masalasining spektral matritsa-funksiyasi deyiladi. Agar $\lambda = \lambda_0$ nuqtaning biror atrofida spektral matritsa-funksiya o'zgarmas bo'lsa, λ parametrning bu qiymatiga Shturm-Liuvill masalasining regulyar qiymati deyiladi. Haqiqiy o'qning regulyar bo'lmagan barcha nuqtalaridan tuzilgan to'plamga Shturm-Liuvill masalasining spektri deyiladi. Agar spektrning $\lambda = \lambda_0$ nuqtasida spektral matritsa-funksiya uzilishga ega bo'lsa, bu qiymatga Shturm-Liuvill masalasining xos qiymati deyiladi. Spektrning spektral matritsa-funksiya uzluksiz bo'ladigan nuqtalaridan tuzilgan to'plamga Shturm-Liuvill masalasining uzluksiz spektri deyiladi. Yechim tilida aytadigan bo'lsak, bu quyidagilarni bildiradi: agar $\lambda = \lambda_0$ bo'lganda Shturm-Liuvill tenglamasi noldan farqli, $L^2(-\infty,\infty)$ fazoga qarashli $y(x,\lambda_0)$ yechimga ega bo'lsa, bu $\lambda = \lambda_0$ qiymat xos qiymat bo'ladi, ana shu $y(x,\lambda_0)$ yechim esa xos funksiya bo'ladi; agar $\lambda = \lambda_0$ bo'lganda Shturm-Liuvill tenglamasi noldan farqli, $L^2(-\infty,\infty)$ fazoga qarashli bo'lmagan, chegaralangan yechimga ega bo'lsa, bu qiymat uzluksiz spektrga tegishli bo'ladi.

Agar $\lambda = \lambda_0$ xos qiymatga ikkita chiziqli erkli xos funksiya mos kelsa, bu xos qiymat ikki karrali deyiladi. Agar $\lambda = \lambda_0$ uzluksiz spektrning nuqtasi bo'lib, unga ikkita chiziqli erkli, noldan farqli, chegaralangan yechimlar mos kelsa, bu qiymat uzluksiz spektrning ikki karrali nuqtasi deyiladi. Xos qiymat, uzluksiz spektrning ichida ham, chetida ham, tashqarisida ham joylashishi mumkin (bunday hollarga misol tuzish o'quvchiga vazifa). Spektr faqat diskret, ya'ni faqat xos qiymatlardan iborat bo'lishi, yoki aksincha faqat uzluksiz bo'lishi, yoki aralash bo'lishi mumkin. Bularning hammasi $q(x)$ potensialning xossalariga bog'liq.

Berilgan $q(x)$ potensial bo'yicha Shturm-Liuvill operatorining $\Re(\lambda)$ spektral matritsa-funksiyasini topish masalasiga, to'g'ri spektral masala deyiladi, aksincha $\Re(\lambda)$ spektral matritsa-funksiya bo'yicha $q(x)$ potensialni tiklash masalasiga teskari spektral masala deyiladi.

Veyl teoremasining isboti. 1-hol. Nolni o'z ichiga olgan, ixtiyoriy chekli oraliqni (a, b) orqali belgilaymiz. Dastlab biror $[-n, n] \subset (a, b)$ chekli oraliqdan tashqarida nolga aylanadigan $f_n(x) \in C^2(-\infty, \infty)$ funksiyalar uchun (5.1.21) tenglikni isbotlaymiz.

$\xi_{a,b}(\lambda)$, $\eta_{a,b}(\lambda)$, $\zeta_{a,b}(\lambda)$ funksiyalarning $[-N, N]$ kesmada o'zlari ham, variatsiyalari ham tekis chegaralanganligi bois, Xellining birinchi teoremasiga ko'ra, shunday $\xi_{a_k, b_k}(\lambda)$, $\eta_{a_k, b_k}(\lambda)$, $\zeta_{a_k,b_k}(\lambda)$ qismiy ketma-ketlik mavjudki, ular mos ravishda biror $\xi(\lambda)$, $\eta(\lambda)$, $\zeta(\lambda)$ funksiyalarga intiladilar. Bunda $\lambda \in [-N, N]$, $a_k \to -\infty$, $b_k \to \infty$.

Xellining ikkinchi teoremasiga ko'ra $\xi(\lambda)$, $\eta(\lambda)$, $\zeta(\lambda)$ funksiyalar $[-N, N]$ kesmada chegaralangan variatsiyaga ega bo'ladi va (5.1.19) tengsizlikda $a_k \to -\infty$, $b_k \to \infty$ limitga o'tish mumkin:

$$\left|\int_{-\infty}^{\infty} f_n^2(x)dx - \int_{-N}^{N} \{F^2(\lambda)d\xi(\lambda) + 2F(\lambda)G(\lambda)d\eta(\lambda)+\right.$$
$$\left. + G^2(\lambda)d\zeta(\lambda)\}\right| \leq \frac{1}{N^2}\int_{-\infty}^{\infty} [f''_n(x) - q(x)f_n(x)]^2 dx. \tag{5.1.22}$$

(5.1.22) tengsizlikda $N \to \infty$ limitga o'tsak, quyidagi tenglik kelib chiqadi:

$$\int_{-\infty}^{\infty} f_n^2(x)dx = \int_{-\infty}^{\infty} F_n^2(\lambda)d\xi(\lambda) + 2\int_{-\infty}^{\infty} F_n(\lambda)G_n(\lambda)d\eta(\lambda)+$$
$$+\int_{-\infty}^{\infty} G_n^2(\lambda)d\zeta(\lambda). \tag{5.1.23}$$

2-hol. Endi ixtiyoriy $f(x) \in L^2(-\infty, \infty)$ funksiya uchun (5.1.21) Parseval tengligi o'rinli ekanini ko'rsatamiz. Buning uchun dastlab $f(x)$ funksiyaga $L^2(-\infty,\infty)$ fazoning normasi bo'yicha yaqinlashuvchi $f_n(x) \in C^2(-\infty,\infty)$ finit funksiyalar ketma-ketligini olamiz:

1) $f_n(x) = 0,\ x \notin [-n,\ n],$

2) $\lim\limits_{n\to\infty}\int\limits_{-\infty}^{\infty} [f(x) - f_n(x)]^2 dx = 0.$

$f_n(x) - f_m(x)$ funksiya 1-holdagi shartlarni qanoatlantirgani bois bu funksiya uchun (5.1.23) tenglik o'rinli bo'ladi:

$$\int\limits_{-\infty}^{\infty} [f_n(x) - f_m(x)]^2 dx = \int\limits_{-\infty}^{\infty} [F_n(\lambda) - F_m(\lambda)]^2 d\xi(\lambda) +$$

$$+2\int\limits_{-\infty}^{\infty} [F_n(\lambda) - F_m(\lambda)]\cdot[G_n(\lambda) - G_m(\lambda)] d\eta(\lambda) +$$

$$+\int\limits_{-\infty}^{\infty} [G_n(\lambda) - G_m(\lambda)]^2 d\zeta(\lambda). \qquad (5.1.24)$$

Bundan tashqari, $f_n(x)$ yaqinlashuvchi bo'lgani uchun, u fundamental bo'ladi, ya'ni

$$\lim_{n,\, m\to\infty}\int\limits_{-\infty}^{\infty} [f_n(x) - f_m(x)]^2 dx = 0. \qquad (5.1.25)$$

(5.1.24) va (5.1.25) tengliklardan $\begin{pmatrix} F_n(\lambda) \\ G_n(\lambda) \end{pmatrix}$ ketma-ketlik ushbu $L^2_{\Re(\lambda)}(-\infty,\ \infty)$ vaznli fazoda fundamental ekanligi kelib chiqadi. Bu fazo to'la bo'lgani uchun $\begin{pmatrix} F_n(\lambda) \\ G_n(\lambda) \end{pmatrix}$ ketma-ketlikning limiti mavjud. Bu limitni $\begin{pmatrix} F(\lambda) \\ G(\lambda) \end{pmatrix}$ orqali belgilaymiz. Endi (5.1.23) tenglikda $n \to \infty$ da limitga o'tamiz. Buning uchun normaning ushbu $|\,\|f_n\| - \|f\|\,| \leq \|f_n - f\|$ xossasidan foydalanamiz. Bunga ko'ra (5.1.23) tenglikning o'ng tomoni (5.1.21) tenglikning o'ng tomoniga, chap tomoni esa (5.1.21) tenglikning chap tomoniga intiladi. ■

Misol. Quyidagi

$$-y'' = \lambda y, \quad (-\infty < x < \infty), \tag{5.1.26}$$

Shturm-Liuvill masalaning spektral matritsa-funksiyasini topamiz.

$\theta(x, \lambda)$ va $\varphi(x, \lambda)$ orqali berilgan tenglamaning ushbu

$$\begin{cases} \theta(0, \lambda) = 1, \\ \theta'(0, \lambda) = 0, \end{cases} \qquad \begin{cases} \varphi(0, \lambda) = 0, \\ \varphi'(0, \lambda) = 1, \end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini belgilaymiz. Bu holda

$$\theta(x, \lambda) = \cos\sqrt{\lambda}x, \quad \varphi(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

bo'lishi ravshan. Ushbu

$$\begin{cases} -y'' = \lambda y, \quad a < x < b, \\ y(a) = 0, \\ y(b) = 0, \end{cases} \tag{5.1.27}$$

regulyar chegaraviy masalaning xos qiymatlari va ortonormallangan xos funksiyalarini topamiz. (5.1.26) tenglamaning ushbu

$$y(x. \lambda) = \frac{\sin\sqrt{\lambda}(x - a)}{\sqrt{\lambda}},$$

yechimi birinchi chegaraviy shartni qanoatlantiradi. Uni ikkinchi chegaraviy shartga qo'ysak, xarakteristik tenglama kelib chiqadi:

$$\frac{\sin\sqrt{\lambda}(b - a)}{\sqrt{\lambda}} = 0.$$

Bunga ko'ra xos qiymatlar ushbu

$$\lambda_n = \frac{\pi^2 n^2}{(b - a)^2}, \quad n = 1,\ 2,\ \dots,$$

sonlardan iborat bo‘ladi. Normallovchi o‘zgarmaslarni topamiz:

$$\alpha_n^2 = \int_a^b \frac{\sin^2\sqrt{\lambda_n}(x-a)}{\lambda_n}dx =$$

$$= \frac{1}{2\lambda_n}\int_a^b [1-\cos 2\sqrt{\lambda_n}(x-a)]dx = \frac{b-a}{2\lambda_n}.$$

Demak, ortonormallashtirilgan xos funksiyalar quyidagilar bo‘ladi:

$$u_n(x) = \sqrt{\frac{2}{b-a}}\sin\frac{\pi n(x-a)}{b-a}, \quad n = 1,\ 2,\ \dots .$$

Bu funksiyalarni $\theta(x,\lambda)$ va $\varphi(x,\lambda)$ orqali chiziqli ifodalaymiz:

$$u_n(x) = \sqrt{\frac{2}{b-a}}\sin\sqrt{\lambda_n}(x-a) =$$

$$= \sqrt{\frac{2}{b-a}}\sin\sqrt{\lambda_n}x\cos\sqrt{\lambda_n}a - \sqrt{\frac{2}{b-a}}\sin\sqrt{\lambda_n}a\cos\sqrt{\lambda_n}x,$$

$$u_n(x) = -\sqrt{\frac{2}{b-a}}\sin\sqrt{\lambda_n}a\cdot\theta(x,\lambda_n)+$$

$$+\sqrt{\frac{2\lambda_n}{b-a}}\cos\sqrt{\lambda_n}a\cdot\varphi(x,\lambda_n).$$

Bunga ko‘ra

$$c_n^2 = \frac{2}{b-a}\sin^2\sqrt{\lambda_n}a,$$

$$c_n d_n = -\frac{2}{b-a}\sqrt{\lambda_n}\sin\sqrt{\lambda_n}a\cos\sqrt{\lambda_n}a,$$

$$d_n^2 = \frac{2}{b-a}\lambda_n\cos^2\sqrt{\lambda_n}a,$$

bo‘ladi. $f(x)$ ixtiyoriy finit funksiya bo‘lib,

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\theta(x.\lambda)dx, \quad G(\lambda) = \int_{-\infty}^{\infty} f(x)\varphi(x,\lambda)dx,$$

bo'lsin. (5.1.27) masala uchun Parseval tengligini yozsak, u ushbu

$$\int_a^b f^2(x)dx = \sum_{n=1}^{\infty} c_n^2 F^2(\lambda_n) + 2\sum_{n=1}^{\infty} c_n d_n F(\lambda_n) G(\lambda_n) + \sum_{n=1}^{\infty} d_n^2 G^2(\lambda_n), \tag{5.1.28}$$

ko'rinishda bo'ladi. Qulaylik uchun ushbu

$$s_n = \sqrt{\lambda_n} = \frac{\pi n}{b-a},$$

belgilashni kiritib olamiz. Agar $s_{n+1} - s_n = \frac{\pi}{b-a}$ ekanini hisobga olsak,

$$\sum_{n=1}^{\infty} c_n^2 F^2(\lambda_n) = \sum_{n=1}^{\infty} \frac{2}{b-a} \sin^2 \sqrt{\lambda_n} a F^2(\lambda_n) =$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \sin^2 s_n a \cdot F^2(s_n^2)(s_{n+1} - s_n),$$

kelib chiqadi. Bu yerda $b \to \infty$ da limitga o'tsak, hamda integral ta'rifidan foydalansak,

$$\sum_{n=1}^{\infty} c_n^2 F^2(\lambda_n) \to \frac{2}{\pi} \int_0^{\infty} \sin^2 sa F^2(s^2) ds =$$

$$= \int_0^{\infty} F^2(s) d\left(\frac{1}{\pi} s\right) - \frac{1}{\pi} \int_0^{\infty} F^2(s) \cos 2sa ds,$$

kelib chiqadi. Eng oxirgi integral Riman-Lebeg lemmasiga ko'ra $a \to -\infty$ da nolga intiladi. Demak, $b \to \infty$, $a \to -\infty$ bo'lganda limitga o'tsak,

$$\sum_{n=1}^{\infty} c_n^2 F^2(\lambda_n) \to \int_0^{\infty} F^2(s^2) d\left(\frac{1}{\pi} s\right) = \int_0^{\infty} F^2(\lambda) d\left(\frac{1}{\pi}\sqrt{\lambda}\right),$$

kelib chiqadi. Demak,

$$\xi(\lambda) = \begin{cases} \frac{1}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases}$$

(5.1.28) tenglikdagi ikkinchi yig'indini ko'rib chiqamiz:

$$2\sum_{n=1}^{\infty} c_n d_n F(\lambda_n) G(\lambda_n) = -2\sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n}}{b-a} \sin 2\sqrt{\lambda_n} a \cdot F(\lambda_n)\, G(\lambda_n) =$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty} s_n F(s_n^2) G(s_n^2) \sin 2s_n a \cdot (s_{n+1} - s_n).$$

Bu yerda $b \to \infty$ da limitga o'tsak, hamda integral ta'rifidan foydalansak,

$$2\sum_{n=1}^{\infty} c_n d_n F(\lambda_n) G(\lambda_n) - \frac{2}{\pi}\int_0^{\infty} sF(s^2) G(s^2) \sin 2sa\, ds,$$

kelib chiqadi. Eng oxirgi integral Riman-Lebeg lemmasiga ko'ra $a \to -\infty$ da nolga intiladi. Demak, $b \to \infty$, $a \to -\infty$ bo'lganda limitga o'tsak,

$$2\sum_{n=1}^{\infty} c_n d_n F(\lambda_n) G(\lambda_n) \to 0,$$

bo'ladi. Bunga ko'ra $\eta(\lambda) \equiv 0$, $\lambda \in (-\infty, \infty)$ bo'ladi.

Nihoyat, (5.1.28) tenglikdagi uchinchi yig'indini ko'rib chiqamiz:

$$\sum_{n=1}^{\infty} d_n^2 G^2(\lambda_n) = \sum_{n=1}^{\infty} \frac{2}{b-a} \lambda_n \cos^2 \sqrt{\lambda_n} a\, G^2(\lambda_n) =$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} s_n^2 \cos^2 s_n a\, G^2(s_n^2)(s_{n+1} - s_n).$$

Bu yerda $b \to \infty$ da limitga o'tsak, hamda integral ta'rifidan foydalansak,

$$\sum_{n=1}^{\infty} d_n^2 G^2(\lambda_n) \to \frac{2}{\pi}\int_0^{\infty} s^2 \cos^2 sa \cdot G^2(s^2) ds =$$

$$= \int_0^{\infty} (1 - \cos 2sa) \cdot G^2(s^2) d\left(\frac{1}{3\pi} s^3\right) =$$

$$= \int_0^\infty G^2(s^2) d\left(\frac{1}{3\pi}s^3\right) - \int_0^\infty \cos 2sa \cdot G^2(s^2) d\left(\frac{1}{3\pi}s^3\right),$$

kelib chiqadi. Eng oxirgi integral Riman-Lebeg lemmasiga ko'ra $a \to -\infty$ da nolga intiladi. Demak, $b \to \infty$, $a \to -\infty$ bo'lganda limitga o'tsak,

$$\sum_{n=1}^{\infty} d_n^2 G^2(\lambda_n) \to \int_0^\infty G^2(\lambda) d\left(\frac{1}{3\pi}\sqrt{\lambda^3}\right),$$

kelib chiqadi. Demak,

$$\zeta(\lambda) = \begin{cases} \frac{1}{3\pi}\sqrt{\lambda^3}, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases}$$

Shunday qilib, agar $\lambda < 0$ bo'lsa,

$$\Re(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

agar $\lambda \ge 0$ bo'lsa,

$$\Re(\lambda) = \begin{pmatrix} \frac{1}{\pi}\sqrt{\lambda} & 0 \\ 0 & \frac{1}{3\pi}\sqrt{\lambda^3} \end{pmatrix}.$$

Bu holda spektr uzluksiz bo'lib, ushbu $E = [0, \infty)$ to'plamdan iborat. Spektrning chetki nuqtasi bir karrali, ya'ni oddiy, ichki nuqtalari esa ikki karrali. Bular yechimning tuzilishidan kelib chiqadi.

2-§. Yoyilma teoremasi

Ushbu

$$-y'' + q(x)y = \lambda y\,, \quad -\infty < x < \infty, \tag{5.2.1}$$

Shturm-Liuvill masalasi uchun yoyilma teoremasini Parseval tengligi yordamida isbotlaymiz. Bu yerda $q(x)$ haqiqiy uzluksiz funksiya, λ esa ixtiyoriy parametr.

$\theta(x, \lambda)$ va $\varphi(x, \lambda)$ orqali (5.2.1) tenglamaning ushbu

$$\begin{cases} \theta(0, \lambda) = 1, \\ \theta'(0, \lambda) = 0, \end{cases} \qquad \begin{cases} \varphi(0, \lambda) = 0, \\ \varphi'(0, \lambda) = 1, \end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini belgilaymiz.

Teorema 5.2.1 (*Yoyilma haqida*). *Quyidagi*

$$\Re(\lambda) = \begin{pmatrix} \xi(\lambda) & \eta(\lambda) \\ \eta(\lambda) & \xi(\lambda) \end{pmatrix}, \quad (-\infty < \lambda < \infty),$$

matritsa-funksiya (5.2.1) masalaning spektral matritsa-funksiyasi bo'lsin. U holda ixtiyoriy $f(x)$ finit, uzluksiz funksiya uchun quyidagi yoyilma o'rinli:

$$f(x) = \int_{-\infty}^{\infty} \{F(\lambda)\theta(x, \lambda)d\xi(\lambda) +$$

$$+[F(\lambda)\varphi(x, \lambda) + G(\lambda)\theta(x, \lambda)]d\eta(\lambda) + G(\lambda)\varphi(x, \lambda)d\zeta(\lambda)\}. \tag{5.2.2}$$

Bu yerda

$$F(\lambda) = \int_{-\infty}^{\infty} f(t)\theta(t, \lambda)dt, \quad G(\lambda) = \int_{-\infty}^{\infty} f(t)\varphi(t, \lambda)dt. \tag{5.2.3}$$

Isbot. $g(x)$ ixtiyoriy finit, uzluksiz funksiya bo'lsin. $f(x) + g(x)$ va $f(x) - g(x)$ funksiyalar uchun Parseval tengligini yozib, birinchisidan ikkinchisini ayiramiz:

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} \{F(\lambda)F_1(\lambda)d\xi(\lambda)+$$

$$+[F(\lambda)G_1(\lambda) + F_1(\lambda)G(\lambda)]\, d\eta(\lambda) + G(\lambda)G_1(\lambda)d\zeta(\lambda)\}. \quad (5.2.4)$$

Bu yerda

$$F_1(\lambda) = \int_{-\infty}^{\infty} g(t)\theta(t,\lambda)dt, \quad G_1(\lambda) = \int_{-\infty}^{\infty} g(t)\varphi(t,\lambda)dt. \quad (5.2.5)$$

(5.2.4) tenglikka Parsevalning umumlashgan tengligi deyiladi.

Agar (5.2.5) ifodalarni (5.2.4) tenglikka qo'ysak va integrallash tartibini o'zgartirsak, (bu mumkin chunki $f(x)$ va $g(x)$ silliq finit funksiyalar) quyidagi tenglik kelib chiqadi:

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \{F(\lambda)\theta(x,\lambda)d\xi(\lambda)+ \right.$$

$$+[F(\lambda)\varphi(x,\lambda) + G(\lambda)\theta(x,\lambda)]d\eta(\lambda) + G(\lambda)\varphi(x,\lambda)d\xi(\lambda)g(x)dx\}\}$$

Bu tenglikda $f(x)$ va $g(x)$ funksiyalarning uzluksizligini, hamda $g(x)$ funksiyaning ixtiyoriyligini hisobga olsak, (5.2.2) formula kelib chiqadi.■

Izoh. Umumlashgan funksiyalar yordamida yoyilma formulasini ishlatib, simvolik ko'rinishda yozish mumkin:

$$\int_{-\infty}^{\infty} \{\theta(x,\lambda)\theta(t,\lambda)d\xi(\lambda) + [\theta(x,\lambda)\varphi(t,\lambda) + \varphi(x,\lambda)\theta(t,\lambda)]d\eta(\lambda)+$$

$$+\varphi(x,\lambda)\varphi(t,\lambda)d\zeta(\lambda)\} = \delta(x-t). \quad (5.2.6)$$

Bu yerda $\delta(x)$ Dirak delta-funksiyasi.

Misol. Quyidagi

$$-y'' = \lambda y, \quad (-\infty < x < \infty), \quad (5.2.7)$$

Shturm-Liuvill masalasi uchun yoyilma teoremasini yozamiz. $f(x)$ ixtiyoriy uzluksiz finit funksiya bo'lsin. Bu holda

$$\theta(x,\lambda)=\cos\sqrt{\lambda}x, \quad \varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

bo'lgani uchun

$$F(\lambda)=\int_{-\infty}^{\infty} f(x)\cos\sqrt{\lambda}x dx, \quad G(\lambda)=\int_{-\infty}^{\infty} f(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}dx,$$

bo'ladi. (5.2.7) masalaning spektral matritsa-funksiyasining elementlari

$$\xi(\lambda)=\begin{cases} 0, & \lambda\le 0, \\ \frac{1}{\pi}\sqrt{\lambda}, & \lambda>0, \end{cases} \quad \eta(\lambda)\equiv 0, \quad \zeta(\lambda)=\begin{cases} 0, & \lambda\le 0, \\ \frac{1}{3\pi}\sqrt{\lambda^3}, & \lambda>0, \end{cases}$$

bo'lganligi uchun yoyilma teoremasiga ko'ra ushbu

$$f(x)=\int_0^{\infty}\left\{F(\lambda)\cos\sqrt{\lambda}x d\left(\frac{1}{\pi}\sqrt{\lambda}\right)+G(\lambda)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}d\left(\frac{1}{3\pi}\sqrt{\lambda^3}\right)\right\}, \tag{5.2.8}$$

tenglik o'rinli bo'ladi. Bu tenglikni sodda ko'rinishda yozamiz:

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\{\,F(\lambda)\cos\sqrt{\lambda}x+G(\lambda)\sqrt{\lambda}\sin\sqrt{\lambda}x\,\}\,d(\sqrt{\lambda}). \tag{5.2.9}$$

Agar

$$a(k)=\int_{-\infty}^{\infty} f(x)e^{ikx}dx, \quad k=\sqrt{\lambda},$$

almashtirish bajarsak,

$$F(\lambda)=\frac{1}{2}[a(k)+\overline{a(k)}], \quad G(\lambda)\sqrt{\lambda}=\frac{1}{2i}[a(k)-\overline{a(k)}],$$

bo'ladi. Bu esa (5.2.9) yoyilma formulasini yanada soddaroq ko'rinishda yozishga imkon beradi:

$$f(x) = \frac{1}{2\pi}\int_0^{\infty} a(k)e^{-ikx}dk + \frac{1}{2\pi}\int_0^{\infty} a(-k)e^{ikx}dk =$$

$$= \frac{1}{2\pi}\int_0^{\infty} a(k)e^{-ikx}dk + \frac{1}{2\pi}\int_{-\infty}^{0} a(k)e^{-ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{-ikx}dk.$$

Shunday qilib, bu holda yoyilma formulasi quyidagi ko'rinishga keltirilar ekan:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{-ikx}dk,$$

bunda

$$a(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx.$$

Agar $q(x) \equiv 0$ bo'lsa, u holda (5.2.6) tenglik quyidagi ko'rinishda bo'ladi:

$$\frac{1}{2\pi}\int_0^{\infty} \{\cos\sqrt{\lambda}x\cos\sqrt{\lambda}t + \sin\sqrt{\lambda}x\sin\sqrt{\lambda}t\}\frac{d\lambda}{\sqrt{\lambda}} = \delta(x-t).$$

3-§. Shturm-Liuvill masalasining spektral matritsa-funksiyasi haqida teoremalar

Ushbu

$$-y'' + q(x)y = \lambda y\,, \quad -\infty < x < \infty, \qquad (5.3.1)$$

Shturm-Liuvill masalasini ko'rib chiqamiz. Bu yerda $q(x)$ haqiqiy uzluksiz funksiya, λ esa kompleks parametr. (5.3.1) masalaning spektral matritsa-funksiyasi

$$\Re(\lambda) = \begin{pmatrix} \xi(\lambda) & \eta(\lambda) \\ \eta(\lambda) & \zeta(\lambda) \end{pmatrix}, \quad (-\infty < \lambda < \infty), \qquad (5.3.2)$$

bo'lsin.

$m^+(z)$ va $m^-(z)$ orqali mos ravishda quyidagi masalalarning Veyl-Titchmarsh funksiyalarini belgilaymiz:

$$\begin{cases} -y'' + q(x)y = zy, & (0 \le x < \infty), \\ y(0) = 0, \end{cases} \qquad (5.3.3)$$

$$\begin{cases} -y'' + q(x)y = zy, & (-\infty < x \le 0), \\ y(0) = 0\,. \end{cases} \qquad (5.3.4)$$

Teorema 5.1.3. *Spektral matritsa-funksiyaning elementlari uchun quyidagi formulalar o'rinli:*

$$\xi(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int\limits_0^\lambda \operatorname{Im}\left\{\frac{1}{m^+(x+iy) - m^-(x+iy)}\right\}dx,$$

$$\eta(\lambda) = \lim_{y\to 0}\frac{1}{\pi}\int\limits_0^\lambda \operatorname{Im}\left\{\frac{1}{2}\cdot\frac{m^+(x+iy) + m^-(x+iy)}{m^+(x+iy) - m^-(x+iy)}\right\}dx,$$

$$\zeta(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int\limits_0^\lambda \operatorname{Im}\left\{\frac{m^+(x+iy)\cdot m^-(x+iy)}{m^+(x+iy) - m^-(x+iy)}\right\}dx.$$

Bu formulalarga Titchmarsh-Kodaira formulalari deyiladi.

Teorema 5.1.4. *Agar $q(x)$ potensial juft funksiya bo'lsa, spektral matritsa-funksiya diagonal ko'rinishda bo'ladi, ya'ni $\eta(\lambda) \equiv 0$ bo'ladi.*

Isbot. Agar biz ushbu

$$\begin{cases} -y'' + q(x)y = \lambda y, & (-\infty < x \le 0), \\ y(0) = 1, \\ y'(0) = 0, \end{cases} \tag{5.3.5}$$

$$\begin{cases} -y'' + q(x)y = \lambda y, & (-\infty < x \le 0), \\ y(0) = 0, \\ y'(0) = 1, \end{cases} \tag{5.3.6}$$

masalalarda $x = -t$ almashtirish bajarsak va $q(x)$ potensial juft funksiya ekanligini hisobga olsak, quyidagi masalalar kelib chiqadi:

$$\begin{cases} -\ddot{y} + q(t)y = \lambda y, & (0 \le t < \infty), \\ y(0) = 1, \\ \dot{y}(0) = 0, \end{cases} \tag{5.3.7}$$

$$\begin{cases} -\ddot{y} + q(t)y = \lambda y, & (0 \le t < \infty), \\ y(0) = 0, \\ \dot{y}(0) = -1. \end{cases} \tag{5.3.8}$$

$\theta(x, \lambda)$ va $\varphi(x, \lambda)$ yechimlar uchun yozilgan boshlang'ich shartlarga ko'ra, $\theta(t, \lambda)$ va $-\varphi(t, \lambda)$ funksiyalar mos ravishda (5.3.7) va (5.3.8) masalalarning yechimlari bo'ladi. Ikkinchi tomondan, (5.3.5), (5.3.6) masalalar, hamda $x = -t$ almashtirishga ko'ra $\theta(-t, \lambda)$ va $\varphi(-t, \lambda)$ funksiyalar ham mos ravishda (5.3.7) va (5.3.8) masalalarning yechimi bo'ladi. Koshi masalasining yechimi yagona ekanligidan $\theta(-t, \lambda) = \theta(t, \lambda)$ va $\varphi(-t, \lambda) = -\varphi(t, \lambda)$ kelib chiqadi.

Veyl-Titchmarsh funksiyasining ta'rifiga ko'ra

$$\theta(x, \lambda) + m^+(\lambda)\varphi(x, \lambda) \in L^2(0, \infty)\,,$$

bo'lganligidan $\theta(t,\lambda) - m^+(\lambda)\varphi(t,\lambda) \in L^2(-\infty, 0)$ ekanligi kelib chiqadi. Bunga ko'ra $m^-(\lambda) = -m^+(\lambda)$ bo'ladi. Titchmarsh-Kodaira formulasiga binoan $\eta(\lambda) \equiv 0$ bo'lishi kelib chiqadi.■

4-§. Shturm-Liuvill masalasi uchun almashtirish operatori

Ushbu

$$-y'' + q(x)y = \lambda y\,, \quad -\infty < x < \infty, \tag{5.4.1}$$

Shturm-Liuvill masalasini ko'rib chiqamiz. Bu yerda $q(x)$ haqiqiy uzluksiz funksiya, λ esa kompleks parametr.

Teorema 5.4.1. $\theta(x,\lambda)$ va $\varphi(x,\lambda)$ *yechimlarni quyidagicha tasvirlash mumkin:*

$$\theta(x,\lambda) = \cos\sqrt{\lambda}x + \int_{-x}^{x} K(x,t)\cos\sqrt{\lambda}t dt, \tag{5.4.2}$$

$$\varphi(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_{-x}^{x} K(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt. \tag{5.4.3}$$

Bunda $K(x,t)$ yadro λ ga bog'liq emas va quyidagi Gursa masalasining yechimi bo'ladi:

$$\frac{\partial^2 K}{\partial t^2} = \frac{\partial^2 K}{\partial x^2} - q(x)K, \tag{5.4.4}$$

$$K(x,x) = \frac{1}{2}\int_{0}^{x} q(s)ds, \tag{5.4.5}$$

$$K(x,-x) = 0. \tag{5.4.6}$$

Isbot. (5.4.4)+(5.4.5)+(5.4.6) masala klassik Gursa masalasi bo'lganligi uchun uning yechimi mavjud va yagona bo'ladi. Bu yechimni $K(x,t)$ orqali belgilaymiz. (5.4.2) tenglik yordamida tuzilgan funksiyani $y(x,\lambda)$ deb belgilab, u (5.4.1) differensial tenglamani qanoatlantirishini ko'rsatamiz:

$$y'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + K(x,x)\cos\sqrt{\lambda}x + K(x,-x)\cos\sqrt{\lambda}x +$$
$$+\int_{-x}^{x}\frac{\partial K(x,t)}{\partial x}\cos\sqrt{\lambda}t dt.$$

(5.4.5) va (5.4.6) tengliklarni inobatga olsak,

$$y'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \left\{\frac{1}{2}\int_0^x q(s)ds\right\}\cdot\cos\sqrt{\lambda}x +$$
$$+\int_{-x}^{x}\frac{\partial K(x,t)}{\partial x}\cos\sqrt{\lambda}t dt. \tag{5.4.7}$$

kelib chiqadi. Bundan esa ushbu

$$y''(x,\lambda) = -\lambda\cos\sqrt{\lambda}x + \frac{1}{2}q(x)\cos\sqrt{\lambda}x -$$
$$-\left\{\frac{1}{2}\int_0^x q(s)ds\right\}\sqrt{\lambda}\sin\sqrt{\lambda}x + \left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x}\cdot\cos\sqrt{\lambda}x +$$
$$+\left.\frac{\partial K(x,t)}{\partial x}\right|_{t=-x}\cdot\cos\sqrt{\lambda}x + \int_{-x}^{x}\frac{\partial^2 K(x,t)}{\partial x^2}\cos\sqrt{\lambda}t dt, \tag{5.4.8}$$

tenglik hosil bo'ladi. Ikkinchi tomondan, ikki marta bo'laklab integrallash natijasida

$$\lambda y(x,\lambda) = \lambda\cos\sqrt{\lambda}x + \lambda\int_{-x}^{x}K(x,t)\cos\sqrt{\lambda}t dt =$$
$$= \lambda\cos\sqrt{\lambda}x + K(x,t)\cdot\sqrt{\lambda}\sin\sqrt{\lambda}t\Big|_{-x}^{x} -$$
$$-\int_{-x}^{x}\frac{\partial K(x,t)}{\partial t}\cdot\sqrt{\lambda}\sin\sqrt{\lambda}t dt =$$

$$= \lambda \cos\sqrt{\lambda}x + \left\{\frac{1}{2}\int\limits_0^x q(s)ds\right\} \cdot \sqrt{\lambda}\sin\sqrt{\lambda}x +$$

$$+\int\limits_{-x}^{x} \frac{\partial K(x,t)}{\partial t} \cdot d(\cos\sqrt{\lambda}t) =$$

$$= \lambda \cos\sqrt{\lambda}x + \left\{\frac{1}{2}\int\limits_0^x q(s)ds\right\} \cdot \sqrt{\lambda}\sin\sqrt{\lambda}x + \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x} \cdot \cos\sqrt{\lambda}x -$$

$$- \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=-x} \cdot \cos\sqrt{\lambda}x - \int\limits_{-x}^{x} \frac{\partial^2 K(x,t)}{\partial t^2} \cdot \cos\sqrt{\lambda}t dt, \quad (5.4.9)$$

kelib chiqadi. (5.4.8) va (5.4.9) tegliklardan (5.4.4), (5.4.5) va (5.4.6) ifodalarni hisobga olib, quyidagilarni keltirib chiqaramiz:

$$y''(x,\lambda) + \lambda y(x,\lambda) = \frac{1}{2}q(x)\cos\sqrt{\lambda}x +$$

$$+ \left.\left\{\frac{\partial K(x,t)}{\partial x} + \frac{\partial K(x,t)}{\partial t}\right\}\right|_{t=x} \cdot \cos\sqrt{\lambda}x +$$

$$+ \left.\left\{\frac{\partial K(x,t)}{\partial x} - \frac{\partial K(x,t)}{\partial t}\right\}\right|_{t=-x} \cdot \cos\sqrt{\lambda}x +$$

$$+ \int\limits_{-x}^{x} \left\{\frac{\partial^2 K(x,t)}{\partial x^2} - \frac{\partial^2 K(x,t)}{\partial t^2}\right\} \cos\sqrt{\lambda}t dt,$$

ya'ni

$$y''(x,\lambda) + \lambda y(x,\lambda) = \frac{1}{2}q(x)\cos\sqrt{\lambda}x + \frac{dK(x,x)}{dx} \cdot \cos\sqrt{\lambda}x +$$

$$+ \frac{dK(x,-x)}{dx} \cdot \cos\sqrt{\lambda}x + \int\limits_{-x}^{x} q(x)K(x,t)\cos\sqrt{\lambda}t dt =$$

$$= q(x)\cos\sqrt{\lambda}x + \int\limits_{-x}^{x} q(x)K(x,t)\cos\sqrt{\lambda}t dt = q(x)y(x,\lambda).$$

Shunday qilib, (5.4.2) tenglik yordamida tuzilgan $y(x,\lambda)$ funksiya (5.4.1) tenglamani va $y(0,\lambda) = 1$, $y'(0,\lambda) = 0$ boshlang'ich shartlarni qanoatlantiradi. Demak, bu yechim $\theta(x,\lambda)$ bilan ustma-ust tushadi. (5.4.3) tenglik ham shu tarzda isbot qilinadi. ■

5-§. Gelfand-Levitan-Blox integral tenglamasi

Ushbu

$$-y'' + q(x)y = \lambda y\,, \quad -\infty < x < \infty, \qquad (5.5.1)$$

Shturm-Liuvill masalasini ko'rib chiqamiz. Bu yerda $q(x)$ haqiqiy uzluksiz funksiya, λ esa kompleks parametr.

$\theta(x,\lambda)$ va $\varphi(x,\lambda)$ orqali (5.5.1) tenglamaning ushbu

$$\begin{cases} \theta(0,\lambda) = 1, \\ \theta'(0,\lambda) = 0, \end{cases} \qquad \begin{cases} \varphi(0,\lambda) = 0, \\ \varphi'(0,\lambda) = 1, \end{cases}$$

boshlang'ich shartlarni qanoatlantiruvchi yechimlarini belgilaymiz. (5.5.1) masalaning spektral matritsa-funksiyasi

$$\Re(\lambda) = \begin{pmatrix} \xi(\lambda) & \eta(\lambda) \\ \eta(\lambda) & \xi(\lambda) \end{pmatrix}, \quad (-\infty < \lambda < \infty), \qquad (5.5.2)$$

bo'lsin. U holda Parseval tengligini simvolik tarzda quyidagicha yozish mumkin:

$$\int_{-\infty}^{\infty} \{\theta(x,\lambda)\theta(t,\lambda)d\xi(\lambda) + [\theta(x,\lambda)\varphi(t,\lambda) + \varphi(x,\lambda)\theta(t,\lambda)]d\eta(\lambda) +$$

$$+\varphi(x,\lambda)\varphi(t,\lambda)d\zeta(\lambda)\} = \delta(x-t). \qquad (5.5.3)$$

Bu yerda $\delta(x)$ - Dirak delta-funksiyasi.

Mazkur bobning oldingi paragrafida $\theta(x,\lambda)$, $\varphi(x,\lambda)$ yechimlar uchun ushbu

$$\theta(x,\lambda) = \cos\sqrt{\lambda}x + \int_{-x}^{x} K(x,t)\cos\sqrt{\lambda}t dt, \qquad (5.5.4)$$

$$\varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_{-x}^{x}K(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt, \qquad (5.5.5)$$

integral tasvirlar olingan edi.

Avval aytganimizdek, $\Re(\lambda)$ spektral matritsa-funksiya bo'yicha $q(x)$ potensialni tiklash masalasiga teskari spektral masala deyiladi.

Bu paragrafda (5.5.3) tenglik va (5.5.4), (5.5.5) integral tasvirlardan foydalanib, teskari spektral masalani yechish jarayonidagi asosiy tenglamani, ya'ni Gelfand-Levitan integral tenglamasini keltirib chiqaramiz.

Agar, (5.5.4) va (5.5.5) tengliklarni $\cos\sqrt{\lambda}t$ va $\dfrac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}$ funksiyalarga nisbatan qarasak, u Volterraning ikkinchi tur integral tenglamalari bo'ladi. Bu tenglamalarni yechsak, quyidagilar kelib chiqadi:

$$\cos\sqrt{\lambda}t=\theta(t,\lambda)+\int_{-t}^{t}L(t,s)\theta(s,\lambda)ds, \qquad (5.5.6)$$

$$\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}=\varphi(t,\lambda)+\int_{-t}^{t}L(t,s)\varphi(s,\lambda)ds. \qquad (5.5.7)$$

Lemma 5.5.1. *$\theta(x,\lambda)$, $\varphi(x,\lambda)$ yechimlar va spektral matritsa-funksiyaning elementlari quyidagi tenglikni qanoatlantiradi:*

$$\int_{-\infty}^{\infty}\theta(x,\lambda)\cos\sqrt{\lambda}t\,d\xi(\lambda)+\int_{-\infty}^{\infty}\varphi(x,\lambda)\cos\sqrt{\lambda}t\,d\eta(\lambda)+$$

$$+\int_{-\infty}^{\infty}\theta(x,\lambda)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,d\eta(\lambda)+$$

$$+\int_{-\infty}^{\infty} \varphi(x,\lambda)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, d\zeta(\lambda) = 0, \quad (|t| < |x|). \qquad (5.5.8)$$

Isbot. (5.5.8) tenglikning chap tomoniga (5.5.6), (5.5.7) ifodalarni qo'yib, integrallash tartibini almashtiramiz, hamda Parseval tengligining (5.5.3) simvolik yozuvidan foydalanamiz:

$$\int_{-\infty}^{\infty} \{\theta(x,\lambda)\theta(t,\lambda)\, d\xi(\lambda) + \varphi(x,\lambda)\theta(t,\lambda)d\eta(\lambda) +$$

$$+\theta(x,\lambda)\varphi(t,\lambda)d\eta(\lambda) + \varphi(x,\lambda)\varphi(t,\lambda)d\zeta(\lambda)\} +$$

$$+\int_{-t}^{t} L(t,s)\int_{-\infty}^{\infty} \{\theta(x.\lambda)\theta(s,\lambda)\, d\xi(\lambda) + \varphi(x,\lambda)\theta(s,\lambda)d\eta(\lambda) +$$

$$+\theta(x,\lambda)\varphi(s,\lambda)d\eta(\lambda) + \varphi(x,\lambda)\varphi(s,\lambda)d\zeta(\lambda)\}ds =$$

$$= \delta(x-t) + \int_{-t}^{t} L(t,s)\delta(x-s)ds = \delta(x-t) + L(t,x)\cdot \operatorname{sign}(t) = 0.$$

Bu yerda $|t| < |x|$ bo'lganda $L(t,x) = 0$ ekanligidan foydalanildi.■

Teorema 5.5.1 (*Gelfand-Levitan-Blox, 1953 yil*). *$\theta(x,\lambda)$, $\varphi(x,\lambda)$ yechimlarning integral tasviridagi $K(x,t)$ yadro quyidagi integral tenglamani qanoatlantiradi:*

$$K(x,t)\operatorname{sign} x + F(x,t) + \int_{-x}^{x} K(x,s)F(s,t)ds = 0, \quad (|t| < |x|). \qquad (5.5.9)$$

Bu yerda

$$F(x,t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda) +$$

$$+\int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(x+t)}{\sqrt{\lambda}} d\eta(\lambda) + \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}x \sin\sqrt{\lambda}t}{\lambda} d\tau(\lambda), \qquad (5.5.10)$$

bo'lib, $\sigma(\lambda)=\xi(\lambda)-\xi_0(\lambda)$, $\tau(\lambda)=\zeta(\lambda)-\zeta_0(\lambda)$,

$$\xi_0(\lambda)=\begin{cases} 0, & \lambda\le 0,\\ \frac{1}{\pi}\sqrt{\lambda}, & \lambda>0,\end{cases}\qquad \zeta_0(\lambda)=\begin{cases} 0, & \lambda\le 0,\\ \frac{1}{3\pi}\sqrt{\lambda^3}, & \lambda>0.\end{cases}$$

Isbot. (5.5.8) tenglikka (5.5.5) va (5.5.6) integral tasvirlarni qo'yamiz:

$$\int_{-\infty}^{\infty}\left\{\cos\sqrt{\lambda}x\cos\sqrt{\lambda}t+\int_{-x}^{x}K(x,s)\cos\sqrt{\lambda}s\cos\sqrt{\lambda}tds\right\}d\sigma(\lambda)+$$

$$+\int_{-\infty}^{\infty}\left\{\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\cos\sqrt{\lambda}t+\int_{-x}^{x}K(x,s)\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}\cos\sqrt{\lambda}tds\right\}d\eta(\lambda)+$$

$$+\int_{-\infty}^{\infty}\left\{\cos\sqrt{\lambda}x\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}+\int_{-x}^{x}K(x,s)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\cos\sqrt{\lambda}sds\right\}d\eta(\lambda)+$$

$$+\int_{-\infty}^{\infty}\left\{\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}+\int_{-x}^{x}K(x,s)\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}ds\right\}d\tau(\lambda)+$$

$$+\frac{1}{2\pi}\int_{0}^{\infty}\left\{\cos\sqrt{\lambda}x\cos\sqrt{\lambda}t+\sin\sqrt{\lambda}x\sin\sqrt{\lambda}t\right\}\frac{d\lambda}{\sqrt{\lambda}}+$$

$$+\int_{-x}^{x}K(x,s)\frac{1}{2\pi}\int_{0}^{\infty}\left\{\cos\sqrt{\lambda}s\cos\sqrt{\lambda}t+\sin\sqrt{\lambda}s\sin\sqrt{\lambda}t\right\}\frac{d\lambda}{\sqrt{\lambda}}ds=0.$$

Bu yerda integrallash tartibini almashtirib, $F(x,t)$ belgilashdan, hamda $q(x)\equiv 0$ bo'lgandagi Parseval tengligining simvolik yozuvidan foydalansak, quyidagi ayniyatni keltirib chiqaramiz:

$$F(x,t)+\int_{-x}^{x}K(x,s)F(s,t)ds+\delta(x-t)+\int_{-x}^{x}K(x,s)\delta(s-t)ds=0.$$

Agar $|t| < |x|$ ekanligini hisobga olsak, ushbu

$$K(x,t)\cdot \operatorname{sign} x + F(x,t) + \int_{-x}^{x} K(x,s)F(s,t)ds = 0,$$

tenglik kelib chiqadi. ■

Teorema 5.5.2. *Biror $\Re(\lambda)$ matritsa-funksiya musbat aniqlangan bo'lib, o'sish nuqtalar to'plami chekli limitik nuqtaga ega bo'lsa, (5.5.9) integral tenglamaning yechimi mavjud va yagona bo'ladi.*

Izoh. Gelfand-Levitan-Blox integral tenglamasining $K(x,t)$ yechimi $K(x,-x) = 0$ shartni qanoatlantirgandagina u yordamida tuzilgan $\theta(x,\lambda)$ va $\varphi(x,\lambda)$ funksiyalar, potensiali $q(x) = 2\dfrac{dK(x,x)}{dx}$ bo'lgan Shturm-Liuvill tenglamasining yechimi bo'ladi. Aks holda, $\Re(\lambda)$ matritsa-funksiya hech bir Shturm-Liuvill masalasining spektral matritsa-funksiyasi bo'la olmaydi.

Izoh. $K(x,-x) = 0$ shart hozirgi kunga qadar $\Re(\lambda)$ spektral matritsa-funksiya tilida ifodalanmagan.

6-§. Rofe-Beketov teoremasi

$\Re(\lambda)$ matritsa-funksiya qanday shartlarni qanoatlantirganda u biror Shturm-Liuvill masalasining spektral matritsa-funksiyasi bo'ladi degan tabiiy savol tug'iladi. Bu muammoning yechimi Ukrainalik matematik Rofe-Beketov tomonidan berilgan. Bunda, butun o'qdagi teskari masala $(-\infty,\ 0]$ va $[0,\ \infty)$ oraliqlardagi teskari masalalarga keltiriladi.

Rofe-Beketov teoremasini bayon qilishdan oldin, quyidagi funksiyalarni kiritib olamiz:

$$M_{11}(z) = \frac{1}{m^+(z) - m^-(z)}, \quad M_{12}(z) = \frac{1}{2}\cdot\frac{m^+(z) + m^-(z)}{m^+(z) - m^-(z)},$$

$$M_{22}(z) = \frac{m^+(z)m^-(z)}{m^+(z) - m^-(z)}. \qquad (5.6.1)$$

Bu funksiyalar ushbu

$$M_{11}(z)M_{22}(z) - M_{12}^2(z) = -\frac{1}{4}, \qquad (5.6.2)$$

Rofe-Beketov ayniyati deb yuritiladigan ayniyatni qanoatlantirishlari ravshan. (5.6.1) tengliklardan quyidagi formulalar osongina kelib chiqadi:

$$m^+(z) = \frac{-M_{22}(z)}{\frac{1}{2} - M_{12}(z)} = \frac{\frac{1}{2} + M_{12}(z)}{M_{11}(z)}, \qquad (5.6.3)$$

$$m^-(z) = \frac{M_{22}(z)}{\frac{1}{2} + M_{12}(z)} = \frac{-\frac{1}{2} + M_{12}(z)}{M_{11}(z)}. \qquad (5.6.4)$$

Titchmarsh-Kodaira formulalarini (5.6.1) funksiyalar yordamida yozadigan bo'lsak, ular quyidagi ko'rinishni oladi:

$$\xi(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int_0^{\lambda} \operatorname{Im}\{M_{11}(x+iy)\}dx,$$

$$\eta(\lambda) = \lim_{y\to 0}\frac{1}{\pi}\int_0^{\lambda} \operatorname{Im}\{M_{12}(x+iy)\}dx,$$

$$\zeta(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int_0^{\lambda} \operatorname{Im}\{M_{22}(x+iy)\}dx.$$

Oxirgi formulalarga Stiltes almashtirishini qo'llasak, quyidagi integral tasvirlar kelib chiqadi:

$$M_{11}(z) = \int_{-\infty}^{\infty} \frac{d\xi(\lambda)}{z-\lambda}, \quad M_{12}(z) = \int_{-\infty}^{\infty} \frac{d\eta(\lambda)}{z-\lambda},$$

$$M_{22}(z) = \int_{-\infty}^{\infty} \left(\frac{1}{z-\lambda} + \frac{\lambda}{\lambda^2+1}\right) d\zeta(\lambda) + a, \qquad (5.6.5)$$

bu yerdagi a o'zgarmas son ushbu

$$M_{22}(z) = -\frac{i}{2} \cdot \sqrt{z} + \overline{\overline{o}}(1), \quad (z \to i\infty),$$

tenglik yordamida aniqlanadi.

Teorema 5.6.1 (*Rofe-Beketov, 1967 y.*). *$\Re(\lambda)$ matritsa-funksiya biror $-y'' + q(x)y = \lambda y$, $x \in R^1$ Shturm-Liuvill masalasining spektral matritsa-funksiyasi bo'lishi uchun quyidagi shartlarning bajarilishi zarur va yetarli:*

1) $\operatorname{Im} z > 0$ *bo'lganda (5.6.5) funksiyalar mavjud va (5.6.2) Rofe-Beketov ayniyatini qanoatlantiradi.*

2) *Ushbu*

$$m^+(z) = \frac{\frac{1}{2} + M_{12}(z)}{M_{11}(z)}, \quad m^-(z) = \frac{-\frac{1}{2} + M_{12}(z)}{M_{11}(z)}, \qquad (5.6.6)$$

funksiyalar mos ravishda biror

$$\begin{cases} -y'' + q_+(x)y = \lambda y, & 0 \le x < \infty, \\ y(0) = 0, \end{cases}$$

va

$$\begin{cases} -y'' + q_-(x)y = \lambda y, & -\infty < x \le 0, \\ y(0) = 0, \end{cases}$$

Shturm-Liuvill masalalarining Veyl-Titchmarsh funksiyalari bo'ladi.

Isbot. $\Re(\lambda)$ spektral matritsa-funksiya bo'lsa, bu fikrlar o'rinli ekanligi yuqoridagi mulohazalardan ko'rinib turibdi. Shuning uchun yetarlilik qismini isbotlaymiz.

(5.6.5) formulalar yordamida $M_{11}(z)$, $M_{22}(z)$, $M_{12}(z)$ funksiyalarni, ular orqali (5.6.6) formulalardan $m^+(z)$, $m^-(z)$ funksiyalarni topamiz. $m^+(z)$, $m^-(z)$ orqali $q_+(x)$ va $q_-(x)$ potensiallar bir qiymatli topiladi. Ushbu

$$q(x) = \begin{cases} q_+(x), & x \ge 0, \\ q_-(x), & x < 0, \end{cases}$$

funksiyani tuzib olib, quyidagi Shturm-Liuvill masalasini ko'rib chiqamiz:

$$-y'' + q(x)y = \lambda y, \quad -\infty < x < \infty. \tag{5.6.7}$$

Bu masalaning spektral matritsa-funksiyasi $\Re(\lambda)$ ekanini isbotlaymiz. $m^+(z)$ va $m^-(z)$ funksiyalar (5.6.7) masalaning Veyl-Titchmarsh funksiyalari ekanligi ravshan. Bu masalaning spektral matritsa-funksiyasini $\tilde{\Re}(\lambda)$ orqali belgilasak, u holda uning elementlari uchun quyidagi tengliklar o'rinli bo'ladi:

$$\tilde{\xi}(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int_0^\lambda \operatorname{Im}\left\{\frac{1}{m^+(x+iy) - m^-(x+iy)}\right\} dx,$$

$$\tilde{\eta}(\lambda) = \lim_{y\to 0}\frac{1}{\pi}\int_0^\lambda \operatorname{Im}\left\{\frac{1}{2}\cdot\frac{m^+(x+iy) + m^-(x+iy)}{m^+(x+iy) - m^-(x+iy)}\right\} dx,$$

$$\tilde{\zeta}(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int_0^\lambda \operatorname{Im}\left\{\frac{m^+(x+iy)\cdot m^-(x+iy)}{m^+(x+iy) - m^-(x+iy)}\right\} dx.$$

Agar (5.6.6) tengliklarni inobatga olsak, bu formulalar ushbu

$$\tilde{\xi}(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int_0^\lambda \operatorname{Im}\left\{M_{11}(x+iy)\right\} dx,$$

$$\tilde{\eta}(\lambda) = \lim_{y\to 0}\frac{1}{\pi}\int^\lambda \operatorname{Im}\left\{M_{12}(x+iy)\right\} dx,$$

$$\tilde{\zeta}(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int_0^\lambda \operatorname{Im}\left\{M_{22}(x+iy)\right\} dx,$$

ko'rinishni oladi. (5.6.5) tengliklarga Stiltes teskari almashtirishini qo'llasak, ushbu

$$\xi(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int_0^{\lambda} \operatorname{Im}\{M_{11}(x+iy)\}dx,$$

$$\eta(\lambda) = \lim_{y\to 0}\frac{1}{\pi}\int_0^{\lambda} \operatorname{Im}\{M_{12}(x+iy)\}dx,$$

$$\zeta(\lambda) = -\lim_{y\to 0}\frac{1}{\pi}\int_0^{\lambda} \operatorname{Im}\{M_{22}(x+iy)\}dx,$$

tengliklarning o'rinli bo'lishi kelib chiqadi. Demak, $\tilde{\Re}(\lambda) = \Re(\lambda)$, ya'ni $\Re(\lambda)$ matritsa-funksiya (5.6.7) Shturm-Liuvill masalasining spektral matritsa-funksiyasi bo'lar ekan. ■

Izoh 5.6.1. Rofe-Beketov teoremasiga ko'ra teskari masala yechsak, umuman olganda $q(x)$ potensial $x = 0$ nuqtada uzilishga ega bo'lishi mumkin.

Izoh 5.6.2. Rofe-Beketov teoremasining shartlari bajarilsa, $q(x)$ potensialni birdaniga butun o'qda Gelfand-Levitan-Blox integral tenglamasidan foydalanib topish mumkin.

Teorema 5.6.2. *$\Re(\lambda)$ matritsa-funksiya Rofe-Beketov teoremasidagi shartlarni qanoatlantirsa va u diagonal ko'rinishda bo'lsa, ya'ni $\eta(\lambda) \equiv 0$ bo'lsa, u holda $q(x)$ potensial juft funksiya bo'ladi, ya'ni $q(-x) \equiv q(x)$ bo'ladi.*

Isbot. Rofe-Beketov teoremasiga ko'ra $q(x)$ mavjud. Bundan tashqari, u Gelfand-Levitan-Blox integral tenglamasining yechimi orqali ushbu

$$q(x) = \frac{2dK(x,x)}{dx}, \tag{5.6.8}$$

formula yordamida topiladi. $K(x,x)$ funksiya toq ekanligini ko'rsatamiz. Buning uchun Gelfand-Levitan-Blox integral tenglamasini ko'rib chiqamiz:

$$K(x,t)\operatorname{sign}x + F(x,t) + \int_{-x}^{x} K(x,s)F(s,t)ds = 0, \quad (|t| < |x|), \tag{5.6.9}$$

bu yerda

$$F(x,t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t d\sigma(\lambda) +$$

$$+\int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(x+t)}{\sqrt{\lambda}} d\eta(\lambda) + \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}x \sin\sqrt{\lambda}t}{\lambda} d\tau(\lambda). \tag{5.6.10}$$

Agar $\eta(\lambda) \equiv 0$ ekanini hisobga olsak, Gelfand-Levitan-Blox integral tenglamasining yadrosi ushbu

$$F(x,t) = \int_{-\infty}^{\infty} \cos\sqrt{\lambda}x \cos\sqrt{\lambda}t \, d\sigma(\lambda) + \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}x \sin\sqrt{\lambda}t}{\lambda} d\tau(\lambda), \tag{5.6.11}$$

ko'rinishni oladi. Bunga ko'ra $F(-x,-t) = F(x,t)$ bo'ladi. (5.6.9) tenglamada $x \to -x$, $t \to -t$ almashtirish bajarsak, ushbu

$$-K(-x,-t)\operatorname{sign}x + F(x,t) + \int_{x}^{-x} K(-x,s)F(s,-t)ds = 0,$$

tenglama kelib chiqadi. Agar integral ostida $s = -u$ almashtirish bajarib,

$$-K(-x,-t) = A(x,t)$$

desak, u holda quyidagi tenglama kelib chiqadi:

$$A(x,t)\operatorname{sign}x + F(x,t) + \int_{-x}^{x} A(x,u)F(u,t)\,du = 0.$$

Nihoyat, Gelfand-Levitan-Blox integral tenglamasining yechimi yagonaligidan foydalansak, $A(x,t) \equiv K(x,t)$ ekanligi, ya'ni $-K(-x,-t) \equiv K(x,t)$ ekanligi kelib chiqadi. Xususan, $K(-x,-x) \equiv -K(x,x)$, ya'ni $K(x,x)$ funksiya toq. Bunga ko'ra, $q(x) \equiv q(-x)$, ya'ni $q(x)$ potensial juft funksiya ekan.■

Izoh 5.6.3. $\Re(\lambda)$ matritsa-funksiya uchun Rofe-Bektov teoremasining shartlari bajarilib, u diagonal ko'rinishda bo'lsa, u holda $q(x)$ potensial $x = 0$ nuqtada uzluksiz bo'ladi. Bu fikr $q(x)$ funksiyaning juft bo'lishidan kelib chiqadi.

7-§. Masalalar yechish namunalari

1-masala. Ushbu

$$-y'' - \frac{2}{ch^2x}y = \lambda y, \quad -\infty < x < \infty,$$

Shturm-Liuvill masalasi uchun quyidagilarni toping:

a) $\theta(0,\lambda) = 1$, $\theta'(0,\lambda) = 0$; $\varphi(0,\lambda) = 0$, $\varphi'(0,\lambda) = 1$ boshlang'ich shartlarni qanoatlantiruvchi yechimlarni toping.

b) $m^+(\lambda)$, $m^-(\lambda)$ Veyl-Titchmarsh funksiyalarini va $\psi_-(x,\lambda)$, $\psi_+(x,\lambda)$ Veyl yechimlarini toping.

c) $m^+(\lambda)$, $m^-(\lambda)$ Veyl-Titchmarsh funksiyalari bo'yicha $M_{11}(\lambda)$, $M_{12}(\lambda)$, $M_{22}(\lambda)$ funksiyalarni toping.

d) $\Re(\lambda) = \begin{pmatrix} \xi(\lambda) & \eta(\lambda) \\ \eta(\lambda) & \zeta(\lambda) \end{pmatrix}$, $\lambda \in R^1$ spektral matritsa-funksiyani toping.

e) Spektrini, xos qiymatlarini va xos funksiyalarini toping.

Yechish. *a*) Ushbu

$$y_1(x,\lambda) = \cos\sqrt{\lambda}x - thx\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$y_2(x,\lambda) = \sqrt{\lambda}\sin\sqrt{\lambda}x - thx\cos\sqrt{\lambda}x,$$

funksiyalar, berilgan differensial tenglamaning yechimlari bo'lishi va ular ushbu

$$y_1(0) = 1, \; y_1'(0) = 0,$$

$$y_2(0) = 0, \quad y_2'(0) = \lambda + 1,$$

boshlang'ich shartlarni qanoatlantirishi osongina tekshiriladi. $\lambda \neq -1$ bo'lganda bu yechimlar chiziqli erkli bo'ladi. Bu holda

$$\theta(x, \lambda) = \cos\sqrt{\lambda}x - thx\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$\varphi(x, \lambda) = \frac{\lambda}{\lambda + 1}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{1}{\lambda + 1}thx\cos\sqrt{\lambda}x,$$

bo'ladi. $\lambda = -1$ bo'lsin. Bu holda

$$y_1(x, -1) = \cos ix - thx\frac{1}{i}\sin ix =$$

$$= \frac{e^{-x} + e^{x}}{2} - thx\frac{1}{i}\frac{e^{-x} - e^{x}}{2i} = chx - \frac{shx}{chx}shx = \frac{1}{chx},$$

$y_2(x, -1) \equiv 0$ bo'ladi. $y_1(x) = \dfrac{1}{chx}$ yechimga proporsional bo'lmagan yechimni tuzish uchun Vronskiy determinanti x ga bog'liq emasligidan foydalanamiz:

$$\begin{vmatrix} y_1 & y \\ y_1' & y' \end{vmatrix} = 1,$$

$$y'chx + yshx = ch^2x,$$

$$(ychx)' = \frac{ch2x + 1}{2},$$

$$y = \frac{1}{2}shx + \frac{1}{2}\frac{x}{chx} + \frac{c}{chx}.$$

Bu yerda $c = 0$ deb olamiz. Bu yechim uchun ushbu $y(0) = 0$, $y'(0) = 1$ boshlang'ich shartlar bajarilishi ravshan. Demak, $\lambda = -1$ bo'lganda

$$\theta(x, -1) = \frac{1}{chx}, \quad \varphi(x, -1) = \frac{1}{2}\frac{x}{chx} + \frac{1}{2}shx,$$

bo'lar ekan. Shunday qilib,

$$\theta(x, \lambda) = \begin{cases} \cos\sqrt{\lambda}x - thx\dfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}, & \lambda \neq -1, \\ \dfrac{1}{chx}, & \lambda = -1, \end{cases}$$

$$\varphi(x,\lambda)=\begin{cases}\dfrac{\lambda}{\lambda+1}\dfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}-\dfrac{1}{\lambda+1}thx\cos\sqrt{\lambda}x, & \lambda\neq-1,\\ \dfrac{1}{2}\dfrac{x}{chx}+\dfrac{1}{2}shx, & \lambda=-1.\end{cases}$$

b) Butun o'qda berilgan Shturm-Liuvill masalasining Veyl-Titchmarsh funksiyalari va Veyl yechimlarining ta'rifiga ko'ra ushbu

$$\psi_+(x,\lambda)=\theta(x,\lambda)+m^+(\lambda)\varphi(x,\lambda)\in L^2(0,\infty),\quad (\operatorname{Im}(\lambda)\neq 0),$$

$$\psi_-(x,\lambda)=\theta(x,\lambda)+m^-(\lambda)\varphi(x,\lambda)\in L^2(-\infty,0),\quad (\operatorname{Im}(\lambda)\neq 0),$$

shartlar bajarilishi lozim. $\psi_\pm(x,\lambda)$ yechimni quyidagicha yozib olamiz:

$$\psi_\pm(x,\lambda)=\left(1+\frac{m^\pm(\lambda)}{\lambda+1}thx\right)\cos\sqrt{\lambda}x+$$

$$+\left(\frac{\lambda m^\pm(\lambda)}{\lambda+1}-thx\right)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}=$$

$$=\left\{\frac{1}{2}\left(1+\frac{m^\pm(\lambda)}{\lambda+1}thx\right)+\frac{1}{2i\sqrt{\lambda}}\left(\frac{\lambda m^\pm(\lambda)}{\lambda+1}-thx\right)\right\}e^{i\sqrt{\lambda}x}+$$

$$+\left\{\frac{1}{2}\left(1+\frac{m^\pm(\lambda)}{\lambda+1}thx\right)-\frac{1}{2i\sqrt{\lambda}}\left(\frac{\lambda m^\pm(\lambda)}{\lambda+1}-thx\right)\right\}e^{-i\sqrt{\lambda}x}.\tag{5.7.1}$$

Agar biz $\operatorname{Im}\sqrt{\lambda}>0$ va $e^{i\sqrt{\lambda}x}\in L^2(0,\infty)$, $e^{-i\sqrt{\lambda}x}\notin L^2(0,\infty)$ bo'lishini hisobga olsak, $\psi_+(x,\lambda)\in L^2(0,\infty)$ shart bajarilishi uchun (5.7.1) tenglikda oxirgi qavs $x\to\infty$ da nolga intilishi kerakligi kelib chiqadi:

$$\frac{1}{2}\left(1+\frac{m^+(\lambda)}{\lambda+1}\right)-\frac{1}{2i\sqrt{\lambda}}\left(\frac{\lambda m^+(\lambda)}{\lambda+1}-1\right)=0.\tag{5.7.2}$$

(5.7.2) tenglikdan $m^+(\lambda)$ funksiyani topamiz:

$$\frac{1}{2}\left(1+\frac{m^+(\lambda)}{\lambda+1}\right)+\frac{i\sqrt{\lambda}m^+(\lambda)}{\lambda+1}+\frac{1}{2i\sqrt{\lambda}}=0,$$

$$m^+(\lambda)\frac{i\sqrt{\lambda}+1}{\lambda+1}=-\frac{i\sqrt{\lambda}+1}{i\sqrt{\lambda}},$$

$$m^+(\lambda) = i\frac{\lambda + 1}{\sqrt{\lambda}}. \quad (5.7.3)$$

(5.7.3) ifodani (5.7.1) tenglikka qo'yamiz:

$$\psi_+(x, \lambda) = \left(1 + \frac{i}{\sqrt{\lambda}} thx\right) \cos\sqrt{\lambda}x + (i\sqrt{\lambda} - thx)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} =$$

$$= \left(1 + \frac{i}{\sqrt{\lambda}} thx\right) \cos\sqrt{\lambda}x + i\left(1 + \frac{i}{\sqrt{\lambda}} thx\right) \sin\sqrt{\lambda}x,$$

$$\psi_+(x, \lambda) = \left(1 + \frac{i}{\sqrt{\lambda}} thx\right) e^{i\sqrt{\lambda}x}. \quad (5.7.4)$$

Agar biz $e^{-i\sqrt{\lambda}x} \in L^2(-\infty, 0)$, $e^{-i\sqrt{\lambda}x} \notin L^2(-\infty, 0)$ bo'lishini hisobga olsak, (5.7.1) tenglikdan quyidagilarni topamiz:

$$m^-(\lambda) = -i\frac{\lambda + 1}{\sqrt{\lambda}}, \quad (5.7.6)$$

$$\psi_-(x, \lambda) = \left(1 - \frac{i}{\sqrt{\lambda}} thx\right) e^{-i\sqrt{\lambda}x}. \quad (5.7.7)$$

c) $M_{11}(\lambda), M_{12}(\lambda), M_{22}(\lambda)$ funksiyalarning ta'riflariga ko'ra

$$M_{11}(\lambda) = \frac{1}{m^+(\lambda) - m^-(\lambda)} = \frac{\sqrt{\lambda}}{2i(\lambda + 1)} = \frac{-i\sqrt{\lambda}}{2(\lambda + 1)},$$

$$M_{12}(\lambda) = \frac{1}{2} \cdot \frac{m^+(\lambda) + m^-(\lambda)}{m^+(\lambda) - m^-(\lambda)} = 0,$$

$$M_{22}(\lambda) = \frac{m^+(\lambda)m^-(\lambda)}{m^+(\lambda) - m^-(\lambda)} = \frac{(\lambda + 1)^2}{\lambda} \cdot \frac{-i\sqrt{\lambda}}{2(\lambda + 1)} = \frac{-i(\lambda + 1)}{2\sqrt{\lambda}},$$

bo'ladi.

d) Agar $\lambda < 0$, $\lambda \neq -1$ bo'lsa,

$$\text{Im}\,\{M_{11}(\lambda)\} = 0, \ \ \text{Im}\,\{M_{12}(\lambda)\} = 0, \ \ \text{Im}\,\{M_{22}(\lambda)\} = 0,$$

bo'ladi. Agar $\lambda > 0$ bo'lsa,

$$\text{Im}\,\{M_{11}(\lambda)\} = -\frac{\sqrt{\lambda}}{2(\lambda + 1)}, \quad \text{Im}\,\{M_{22}(\lambda)\} = -\frac{\lambda + 1}{2\sqrt{\lambda}},$$

bo'ladi. $\lambda = -1$ va $\lambda = 0$ nuqtalardagi chegirmalarni hisoblaymiz:

$$\operatorname*{res}_{\lambda=-1} M_{11}(\lambda) = \lim_{\lambda\to-1} (\lambda+1)\frac{-i\sqrt{\lambda}}{2(\lambda+1)} = \frac{1}{2},$$

$$\operatorname*{res}_{\lambda=0} M_{22}(\lambda) = \lim_{\lambda\to 0} \lambda\frac{-i(\lambda+1)}{2(\lambda+1)} = 0.$$

Endi spektral matritsa-funksiyani topish uchun quyidagi formulalardan foydalanamiz:

$$\xi(\lambda) = -\frac{1}{\pi}\lim_{y\to 0}\int_0^\lambda \operatorname{Im}\{M_{11}(x+iy)\}dx,$$

$$\eta(\lambda) = \frac{1}{\pi}\lim_{y\to 0}\int_0^\lambda \operatorname{Im}\{M_{12}(x+iy)\}dx,$$

$$\zeta(\lambda) = -\frac{1}{\pi}\lim_{y\to 0}\int_0^\lambda \operatorname{Im}\{M_{22}(x+iy)\}dx.$$

Bularga ko'ra $\eta(\lambda) \equiv 0, \;\; \lambda \in R^1$. Agar $\lambda > 0$ bo'lsa,

$$\xi(\lambda) = \frac{1}{2\pi}\int_0^\lambda \frac{\sqrt{t}}{t+1}dt = \left\{ \begin{array}{l} t = x^2 \\ dt = 2xdx \end{array} \right\} = \frac{1}{\pi}\int_0^{\sqrt{\lambda}} \frac{x^2}{x^2+1} = dx$$

$$= \frac{1}{\pi}\int_0^{\sqrt{\lambda}} (1 - \frac{1}{x^2+1})dx = \frac{1}{\pi}\sqrt{\lambda} - \frac{1}{\pi}\operatorname{arctg}\sqrt{\lambda},$$

$$\zeta(\lambda) = \frac{1}{2\pi}\int_0^\lambda \frac{t+1}{\sqrt{t}}dt = \left\{ \begin{array}{c} t = x^2 \\ dt = 2xdx \end{array} \right\} =$$

$$= \frac{1}{\pi}\int_0^{\sqrt{\lambda}} (x^2+1)dx = \frac{1}{3\pi}\sqrt{\lambda^3} + \frac{1}{\pi}\sqrt{\lambda},$$

bo'ladi. Agar $\lambda \leq 0$, $\lambda \neq -1$ bo'lsa, $\operatorname{Im}\{M_{11}(\lambda)\} = 0$ va $\operatorname*{res}_{\lambda=-1} M_{11}(\lambda) = \frac{1}{2}$ bo'lgani uchun $\xi(0) = 0$ ekanligini hisobga olib,

$$\xi(\lambda) = \begin{cases} -\dfrac{1}{2}, & \lambda \leq -1, \\ 0, & -1 < \lambda \leq 0, \\ \dfrac{1}{\pi}\sqrt{\lambda} - \dfrac{1}{\pi}\operatorname{arctg}\sqrt{\lambda}, & \lambda > 0, \end{cases}$$

bo'lishini topamiz. $\lambda < 0$ bo'lganda $\operatorname{Im}\{M_{22}(\lambda)\} = 0$ bo'lishidan $\zeta(0) = 0$ normallashtirish shartini inobatga olsak, ushbu

$$\zeta(\lambda) = \begin{cases} 0, & \lambda \leq 0, \\ \dfrac{1}{3\pi}\sqrt{\lambda^3} + \dfrac{1}{\pi}\sqrt{\lambda}, & \lambda > 0, \end{cases}$$

tenglik kelib chiqadi.

e) $\lambda > 0$ bo'lganda ushbu

$$\theta(x,\lambda) = \cos\sqrt{\lambda}x - thx\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$\varphi(x,\lambda) = \frac{\lambda}{\lambda+1}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{1}{\lambda+1}thx\cos\sqrt{\lambda}x,$$

yechimlar chegaralanganligi va $L^2(-\infty,\infty)$ fazoga tegishli bo'lmasligi ravshan. $\lambda = 0$ bo'lsa, bu yechimlar quyidagi ko'rinishda bo'ladi:

$$\theta(x,\lambda) = 1 - xthx, \quad \varphi(x,\lambda) = thx.$$

Bu yechimlardan bittasi $\varphi(x,\lambda) = thx$ chegaralangan bo'lib, u $L^2(-\infty,\infty)$ fazoga tegishli emas. Agar $\lambda < 0$, $\lambda \neq -1$ bo'lsa,

$$\theta(x,\lambda) = ch\sqrt{|\lambda|}x - thx\frac{sh\sqrt{|\lambda|}x}{\sqrt{|\lambda|}},$$

$$\varphi(x,\lambda) = \frac{\lambda}{\lambda+1}\frac{sh\sqrt{|\lambda|}x}{\sqrt{|\lambda|}} - \frac{1}{\lambda+1}thxch\sqrt{|\lambda|}x,$$

yechimlar chegaralanmagan. Bu yechimlarning noldan farqli chiziqli kombinatsiyasi ham chegaralanmagan bo'lishi ravshan. Agar $\lambda = -1$ bo'lsa,

$$\theta(x,-1) = \frac{1}{chx} \in L^2(-\infty,\infty)$$

bo'lib,

$$\varphi(x,-1) = \frac{1}{2}\frac{x}{chx} + \frac{1}{2}shx$$

yechim chegaralanmagan. Demak, $\lambda_0 = -1$ xos qiymat ekan. Normallovchi o'zgarmasni hisoblaymiz:

$$\alpha_0^2 = \int_{-\infty}^{\infty} \frac{1}{ch^2x}dx = thx|_{-\infty}^{\infty} = 2.$$

Shunday qilib, berilgan masalaning spektri $E = \{-1\} \cup [0,\infty)$ to'plamdan iborat ekan.

2-masala. $\Re(\lambda) = \begin{pmatrix} \xi(\lambda) & \eta(\lambda) \\ \eta(\lambda) & \zeta(\lambda) \end{pmatrix}$, $\lambda \in R^1$ spektral matritsa-funksiya berilgan. Bu yerda

$$\xi(\lambda) = \begin{cases} -\dfrac{1}{2}, & \lambda \le -1, \\ 0, & -1 < \lambda \le 0, \\ \dfrac{1}{\pi}\sqrt{\lambda} - \dfrac{1}{\pi}\operatorname{arctg}\sqrt{\lambda}, & \lambda > 0, \end{cases}$$

$$\eta(\lambda) \equiv 0, \quad \lambda \in R^1,$$

$$\zeta(\lambda) = \begin{cases} 0, & \lambda \le -1, \\ \dfrac{1}{3\pi}\sqrt{\lambda^3} + \dfrac{1}{\pi}\sqrt{\lambda}, & \lambda > 0. \end{cases}$$

Quyidagilarni toping:

a) Rofe-Beketov teoremasidagi $M_{11}(z)$, $M_{12}(z)$, $M_{22}(z)$ funksiyalarni toping va Rofe-Beketov ayniyatini tekshiring.

b) $M_{11}(z)$, $M_{12}(z)$, $M_{22}(z)$ funksiyalar bo'yicha $m^+(\lambda)$, $m^-(\lambda)$ Veyl-Titchmarsh funksiyalarini toping.

c) $m^+(\lambda)$, $m^-(\lambda)$ bo'yicha Roffe-Beketov teoremasidagi $\rho_+(\lambda)$, $\rho_-(\lambda)$ spektral funksiyalarni toping.

Yechish. *a)* $\xi(\lambda)$, $\eta(\lambda)$, $\zeta(\lambda)$ bo'yicha $M_{11}(z)$, $M_{12}(z)$, $M_{22}(z)$ funksiyalarni topamiz. Buning uchun ushbu

$$M_{11}(z) = \int_{-\infty}^{\infty} \frac{d\xi(\lambda)}{z-\lambda}, \quad M_{12}(z) = \int_{-\infty}^{\infty} \frac{d\eta(\lambda)}{z-\lambda},$$

$$M_{22}(z) = \int_{-\infty}^{\infty} \left(\frac{1}{z-\lambda} + \frac{\lambda}{1+\lambda^2}\right) d\zeta(\lambda) + a,$$

$$M_{22}(z) = -\frac{i}{2}\sqrt{z} + \bar{\bar{o}}(1), \quad z \to +i\infty,$$

formulalardan foydalanamiz. Bunga ko'ra

$$M_{11}(z) = \int_{-\infty}^{0} \frac{1}{z-\lambda} d(\frac{1}{2}\theta(\lambda+1) - \frac{1}{2}) +$$

$$+ \int_{0}^{\infty} \frac{1}{z-\lambda} d(\frac{1}{\pi}\sqrt{\lambda} - \frac{1}{\pi} arctg\sqrt{\lambda}) =$$

$$= \frac{1}{2}\int_{-\infty}^{0} \frac{1}{z-\lambda}\delta(\lambda+1)d\lambda + \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{z-\lambda}\left(\frac{1}{2\sqrt{\lambda}} - \frac{1}{\lambda+1}\cdot\frac{1}{2\sqrt{\lambda}}\right) d\lambda =$$

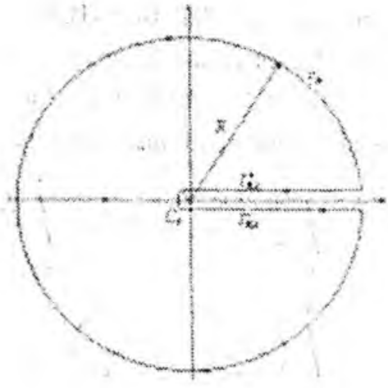
$$= \frac{1}{2}\frac{1}{z+1} - \frac{1}{2\pi}\int_{0}^{\infty} \frac{\sqrt{\lambda}}{(\lambda-z)(\lambda+1)} d\lambda.$$

Bu yerda $\theta(\lambda)$ – Xevisayda funksiyasi.

Oxirgi integralni chegirmalar yordamida hisoblaymiz. Buning uchun ushbu

$$f(\lambda) = \frac{\sqrt{\lambda}}{(\lambda-z)(\lambda+1)}.$$

funksiyani quyidagi kontur bo'yicha integrallaymiz.



4-rasm.

Koshining chegirmalar haqidagi teoremasiga ko'ra quyidagi tenglik o'rinli bo'ladi:

$$\int\limits_{C_R} f(\lambda)d\lambda + \int\limits_{C_\varepsilon} f(\lambda)d\lambda + \int\limits_{l^-_{R,\varepsilon}} f(\lambda)d\lambda + \int\limits_{l^+_{R,\varepsilon}} f(\lambda)d\lambda =$$

$$= 2\pi i(\operatorname*{res}_{\lambda=-1} f(\lambda) + \operatorname*{res}_{\lambda=z} f(\lambda)). \qquad (5.7.8)$$

Agar ushbu

$$\int\limits_{C_R} f(\lambda)d\lambda =$$

$$\int\limits_0^{2\pi} \frac{\sqrt{R}e^{\frac{it}{2}}}{(Re^{it} - z)(Re^{it} + 1)} iRe^{it}dt = \underline{\underline{O}}\left(\frac{1}{\sqrt{R}}\right) \to 0, \ (R \to \infty),$$

$$\int\limits_{C_\varepsilon} f(\lambda)d\lambda = \int\limits_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{\varepsilon}e^{\frac{it}{2}}}{(\varepsilon e^{it} - z)(\varepsilon e^{it} + 1)} i\varepsilon e^{it}dt = \underline{\underline{O}}(\varepsilon\sqrt{\varepsilon}) \to 0, (\varepsilon \to 0),$$

$$f(\lambda)|_{l^-_{R,\varepsilon}} = -f(\lambda)|_{l^+_{R,\varepsilon}},$$

$$\operatorname*{res}_{\lambda=-1} f(\lambda) = \frac{-i}{1+z}, \quad \operatorname*{res}_{\lambda=z} f(\lambda) = \frac{\sqrt{z}}{1+z},$$

munosabatlarni hisobga olsak va (5.7.8) tenglikda $R \to \infty$, $\varepsilon \to 0$ da limitga o'tsak,

$$2\int_0^\infty f(\lambda)d\lambda = 2\pi i\left(\frac{-i}{1+z}+\frac{\sqrt{z}}{1+z}\right),$$

kelib chiqadi. Bunga ko'ra

$$\int_0^\infty \frac{\sqrt{\lambda}}{(\lambda-z)(\lambda+1)} = \pi\left(\frac{1+i\sqrt{\lambda}}{z+1}\right).$$

Demak,

$$M_{11} = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1+i\sqrt{z}}{z+1} = -\frac{i\sqrt{z}}{2(z+1)}$$

Endi $M_{22}(z)$ ni hisoblaymiz:

$$M_{22}(z) = \int_0^\infty \left(\frac{\lambda}{\lambda^2+1}-\frac{1}{\lambda-z}\right)\left(\frac{1}{2\pi}\sqrt{\lambda}+\frac{1}{2\pi\sqrt{\lambda}}\right)d\lambda + a =$$

$$= -\frac{1}{2\pi}\int_0^\infty \frac{(\lambda z+1)(\lambda+1)}{(\lambda^2+1)(\lambda-z)\sqrt{\lambda}}d\lambda + a.$$

Oxirgi integralni ham oldingi konturdan foydalanib hisoblaymiz. Bu holda

$$f(\lambda) = \frac{(\lambda z+1)(\lambda+1)}{(\lambda^2+1)(\lambda-z)\sqrt{\lambda}},$$

deb olsak, ushbu

$$\int_{C_R} f(\lambda)d\lambda + \int_{C_\varepsilon} f(\lambda)d\lambda + \int_{l^-_{R,\varepsilon}} f(\lambda)d\lambda + \int_{l^+_{R,\varepsilon}} f(\lambda)d\lambda =$$

$$= 2\pi i(\operatorname*{res}_{\lambda=i} f(\lambda) + \operatorname*{res}_{\lambda=-i} f(\lambda) + \operatorname*{res}_{\lambda=z} f(\lambda)), \tag{5.7.9}$$

tenglik hosil bo'ladi. Bu holda

$$\int_{C_R} f(\lambda)d\lambda =$$

$$= \int_0^{2\pi} \frac{(Rze^{it}+1)(Re^{it}+1)iRe^{it}}{(R^2e^{2it}+1)(Re^{it}-z)\sqrt{R}e^{\frac{it}{2}}}dt = \underline{\underline{O}}\left(\frac{1}{\sqrt{R}}\right) \to 0, \quad (R \to \infty),$$

$$\int_{C_\varepsilon} f(\lambda)d\lambda = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{(\varepsilon ze^{it}+1)(\varepsilon e^{it}+1)i\varepsilon e^{it}}{(\varepsilon^2e^{2it}+1)(\varepsilon e^{it}-z)\sqrt{\varepsilon}e^{\frac{it}{2}}} = \underline{\underline{O}}(\sqrt{\varepsilon}) \to 0, \quad (\varepsilon \to 0),$$

$$f(\lambda)|_{l^-_{R,\varepsilon}} = -f(\lambda)|_{l^+_{R,\varepsilon}},$$

$$\operatorname*{res}_{\lambda=i} f(\lambda) = \frac{(iz+1)(i+1)}{2i(i-z)\sqrt{i}} = \frac{(iz+1)(i+1)}{-2(1-iz)(\frac{1+i}{\sqrt{2}})} = -\frac{1}{\sqrt{2}},$$

$$\operatorname*{res}_{\lambda=-i} f(\lambda) = \frac{(-iz+1)(-i+1)}{-2i(-i-z)\sqrt{-i}} = \frac{(-iz+1)(-i+1)}{-2(1-iz)(\frac{-1+i}{\sqrt{2}})} = \frac{1}{\sqrt{2}},$$

$$\operatorname*{res}_{\lambda=z} f(\lambda) = \frac{z+1}{\sqrt{z}}.$$

Bu munosabatlarni hisobga olib, (5.7.9) tenglikda $R \to \infty$, $\varepsilon \to 0$ limitga o'tsak,

$$\int_0^\infty \frac{(\lambda z+1)(\lambda+1)}{(\lambda^2+1)(\lambda-z)\sqrt{\lambda}}d\lambda = i\pi\frac{z+1}{\sqrt{z}},$$

kelib chiqadi. Bunga ko'ra

$$M_{22}(z) = -\frac{i}{2}\cdot\frac{z+1}{\sqrt{z}} + a,$$

bo'lishini topamiz. Bu tenglikdan, xususan,

$$M_{22}(z) = -\frac{i}{2}\sqrt{z} + a + \overline{\overline{o}}(1), \quad z \to +i\infty,$$

kelib chiqadi. Demak, $a = 0$ ekan, ya'ni

$$M_{22}(z) = -\frac{i}{2}\frac{z+1}{\sqrt{z}}.$$

Endi Rofe-Beketov ayniyatini tekshiramiz:

$$M_{11}(z)M_{22}(z) - M_{12}^2(z) = \frac{-i\sqrt{z}}{2(z+1)}\cdot\frac{-i(z+1)}{2\sqrt{z}} - 0 = -\frac{1}{4}.$$

b) Ushbu

$$m^+(\lambda) = \frac{M_{12}(\lambda) + \frac{1}{2}}{M_{11}(\lambda)}, \quad m^-(\lambda) = \frac{M_{12}(\lambda) - \frac{1}{2}}{M_{11}(\lambda)},$$

formulalarga oldingi bandda topilgan ifodalarni qo'yamiz:

$$m^+(\lambda) = \frac{i(\lambda+1)}{\sqrt{\lambda}}, \quad m^-(\lambda) = -\frac{i(\lambda+1)}{\sqrt{\lambda}}.$$

c) $m^+(\lambda)$ funksiyani qutb maxsusligi bo'lmagani uchun

$$\rho'_+(\lambda) = \frac{1}{\pi}\operatorname{Im}\{m^+(\lambda)\} = \begin{cases} \frac{1}{\pi}\cdot\frac{\lambda+1}{\sqrt{\lambda}}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

$$\rho_+(\lambda) = \begin{cases} \frac{2}{3\pi}\sqrt{\lambda^3} + \frac{2}{\pi}\sqrt{\lambda}, & \lambda > 0, \\ 0, & \lambda \le 0, \end{cases}$$

bo'ladi. $(-\infty, 0)$ oraliqni $x = -t$ almashtirish yordamida $(0, \infty)$ oraliqqa o'tkazib olamiz. Bunda hosil bo'ladigan masalaning Veyl-Titchmarsh funksiyasi $(-m^-(\lambda))$ bo'ladi. Biz qarayotgan holda $-m^-(\lambda) = m^+(\lambda)$ bo'ladi, ya'ni $\rho_-(\lambda)$ ning hojati yo'q.

3-masala. Ushbu

$$-y'' + \frac{2}{(|x|+1)^2}y = \lambda y, \quad -\infty < x < \infty$$

Shturm-Liuvill masalasi uchun quyidagilarni toping:

a) $\theta(0,\lambda) = 1$, $\theta'(0,\lambda) = 0$, $\varphi(0,\lambda) = 0$, $\varphi'(0,\lambda) = 1$ boshlang'ich shartlarni qanoatlantiruvchi yechimlarni toping.

b) $m^+(\lambda)$, $m^-(\lambda)$ Veyl-Titchmarsh funksiyalarini va $\psi_-(x,\lambda)$, $\psi_-(x,\lambda)$ Veyl yechimlarini toping.

c) $m^+(\lambda)$, $m^-(\lambda)$ Veyl-Titchmarsh funksiyalari bo'yicha $M_{11}(\lambda)$, $M_{12}(\lambda)$, $M_{22}(\lambda)$ funksiyalarni toping.

d) $\Re(\lambda) = \begin{pmatrix} \xi(\lambda) & \eta(\lambda) \\ \eta(\lambda) & \zeta(\lambda) \end{pmatrix}$, $\lambda \in R^1$ spektral matritsa-funksiyani toping.

e) Spektrini, xos qiymatlarini va xos funksiyalarini toping.

Yechish. a) 1) Ushbu

$$y_1(x,\lambda) = \cos\sqrt{\lambda}x - \frac{1}{x+1}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$y_2(x,\lambda) = \sqrt{\lambda}\sin\sqrt{\lambda}x + \frac{1}{x+1}\cos\sqrt{\lambda}x$$

funksiyalar

$$y'' + \lambda y = \frac{2}{(x+1)^2}y, \quad 0 < x < \infty$$

differentsial tenglamaning yechimlari bo‘lishi va ular quyidagi

$$y_1(0) = 1, \ \ y_1'(0) = -1$$

$$y_2(0) = 1, \ \ y_2'(0) = \lambda - 1$$

boshlang‘ich shartlarni qanoatlantirishi osongina tekshiriladi. $\lambda \neq 0$ bo‘lganda bu yechimlar chiziqli erkli bo‘ladi. Bu holda

$$\theta(x,\lambda) = \left(\lambda - 1 + \frac{1}{x+1}\right)\frac{\cos\sqrt{\lambda}x}{\lambda} + \left(\lambda - \frac{\lambda-1}{x+1}\right)\frac{\sin\sqrt{\lambda}x}{\lambda\sqrt{\lambda}};$$

$$\varphi(x,\lambda) = \left(\lambda + \frac{1}{x+1}\right)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{x}{x+1}\frac{\cos\sqrt{\lambda}x}{\lambda}$$

bo‘ladi. $\lambda = 0$ bo‘lsin. Bu holda

$$y'' = \frac{2}{(x+1)^2}y,$$

$$(x+1)^2 y'' - 2y = 0$$

Eyler tenglamasiga ega bo‘lamiz. Bu tenglamaning umumiy yechimi

$$y = c_1(x+1)^2 + c_2\frac{1}{x+1}$$

bo‘ladi. Endi $\theta(x,0)$ va $\varphi(x,0)$ yechimlarni topamiz:

$$\theta(x,0) = \frac{1}{3}(x+1)^2 + \frac{2}{3}\cdot\frac{1}{x+1},$$

$$\varphi(x,0)=\frac{1}{3}(x+1)^2-\frac{1}{3}\cdot\frac{1}{x+1}.$$

Shunday qilib, $x>0$ da

$$\theta(x,\lambda)=\begin{cases}\frac{1}{3}(x+1)^2+\frac{2}{3}\frac{1}{x+1}, & \lambda=0,\\ \left(\lambda-1+\frac{1}{x+1}\right)\frac{\cos\sqrt{\lambda}x}{\lambda}+\left(\lambda-\frac{\lambda-1}{x+1}\right)\frac{\sin\sqrt{\lambda}x}{\lambda\sqrt{\lambda}}, & \lambda\neq 0,\end{cases}$$

$$\varphi(x,\lambda)=\begin{cases}\frac{1}{3}(x+1)^2-\frac{1}{3}\frac{1}{x+1}, & \lambda=0,\\ -\frac{x}{x+1}\frac{\cos\sqrt{\lambda}x}{\lambda}+\left(\lambda+\frac{1}{x+1}\right)\frac{\sin\sqrt{\lambda}x}{\lambda\sqrt{\lambda}}, & \lambda\neq 0.\end{cases}$$

2) Ushbu

$$y_1(x,\lambda)=\cos\sqrt{\lambda}x-\frac{1}{x-1}\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},$$

$$y_2(x,\lambda)=\sqrt{\lambda}\sin\sqrt{\lambda}x+\frac{1}{x-1}\cos\sqrt{\lambda}x$$

funksiyalar

$$y''+\lambda y=\frac{2}{(x-1)^2}y,\quad -\infty<x<0$$

differentsial tenglamaning yechimlari bo'lishini va ular

$$y_1(0)=0,\quad y_1'(0)=1,$$

$$y_2(0)=-1,\quad y_2'(0)=\lambda-1$$

boshlang'ich shartlarni qanoatlantirishini ko'rsatish mumkin. $\lambda\neq 0$ bo'lganda bu yechimlar chiziqli erkli bo'lib, ular

$$\theta(x.\lambda)=\left(\lambda-1+\frac{1}{-x+1}\right)\frac{\cos\sqrt{\lambda}x}{\lambda}+\left(\lambda-\frac{\lambda-1}{-x+1}\right)\frac{\sin\sqrt{\lambda}(-x)}{\lambda\sqrt{\lambda}},$$

$$\varphi(x,\lambda)=\left(\lambda+\frac{1}{-x+1}\right)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}-\frac{x}{-x+1}\frac{\cos\sqrt{\lambda}x}{\lambda}$$

ko'rinishda bo'ladi.

Endi $\lambda=0$ bo'lsin. U holda

$$y''=\frac{2}{(x-1)^2}y$$

yoki

$$(x-1)^2y''-2y=0$$

Eyler tenglamasiga ega bo'lamiz. Bu tenglamaning umumiy yechimi

$$y=c_1(x-1)^2+c_2\frac{1}{x-1}.$$

Bu yerdagi c_1 va c_2 o'zgarmaslarni topish uchun

$$\theta(0,\lambda)=1,\ \theta'(0.\lambda)=0,\ \varphi(0,\lambda)=0,\ \varphi'(0,\lambda)=1$$

boshlang'ich shartlardan foydalanamiz:

$$\theta(x,0)=\frac{1}{3}(x-1)^2-\frac{2}{3}\cdot\frac{1}{x-1};$$

$$\varphi(x,0)=-\frac{1}{3}(x-1)^2-\frac{1}{3}\cdot\frac{1}{x-1}.$$

Shunday qilib, biz izlagan yechimlar $x<0$ da

$$\theta(x.\lambda)=\begin{cases}\frac{1}{3}(x-1)^2-\frac{2}{3}\cdot\frac{1}{x-1}, & \lambda=0,\\ \left(\lambda-1-\frac{1}{x-1}\right)\frac{\cos\sqrt{\lambda}x}{\lambda}+\left(\lambda+\frac{\lambda-1}{x-1}\right)\frac{\sin\sqrt{\lambda}(-x)}{\lambda\sqrt{\lambda}}, & \lambda\neq 0,\end{cases}$$

$$\varphi(x,\lambda)=\begin{cases}-\frac{1}{3}(x-1)^2-\frac{1}{3}\cdot\frac{1}{x-1}, & \lambda=0,\\ \frac{x}{x-1}\frac{\cos\sqrt{\lambda}x}{\lambda}+\left(\lambda-\frac{1}{x-1}\right)\frac{\sin\sqrt{\lambda}x}{\lambda\sqrt{\lambda}}, & \lambda\neq 0\end{cases}$$

ko'rinishda bo'lar ekan. $x\in(-\infty,\infty)$ bo'lganda $\theta(x,\lambda)$ va $\varphi(x,\lambda)$ yechimlarni quyidagicha tasvirlash mumkin:

$$\theta(x,\lambda)=\begin{cases}\frac{1}{3}(|x|+1)^2+\frac{2}{3}\frac{1}{|x|+1}, & \lambda=0,\\ \left(\lambda-1+\frac{1}{|x|+1}\right)\frac{\cos\sqrt{\lambda}x}{\lambda}+\left(\lambda-\frac{\lambda-1}{|x|+1}\right)\frac{\sin\sqrt{\lambda}|x|}{\lambda\sqrt{\lambda}}, & \lambda\neq 0\end{cases}$$

$$\varphi(x,\lambda)=\begin{cases}\left(\frac{1}{3}(|x|+1)^2-\frac{1}{3}\frac{1}{|x|+1}\right)\operatorname{sign}x, & \lambda=0,\\ -\frac{x}{|x|+1}\frac{\cos\sqrt{\lambda}x}{\lambda}+\left(\lambda+\frac{1}{|x|+1}\right)\frac{\sin\sqrt{\lambda}|x|}{\lambda\sqrt{\lambda}}\operatorname{sign}x, & \lambda\neq 0.\end{cases}$$

b) Butun o'qda berilgan Shturm-Liuvill chegaraviy masalasining Veyl-Titchmarsh funksiyalari va Veyl yechimlarining ta'rifiga ko'ra ushbu

$$\psi_+(x,\lambda) = \theta(x,\lambda) + m^+(\lambda)\varphi(x,\lambda) \in L^2(0,\infty), \quad \operatorname{Im}\lambda \neq 0,$$

$$\psi_-(x,\lambda) = \theta(x,\lambda) + m^-(\lambda)\varphi(x,\lambda) \in L^2(-\infty,0), \quad \operatorname{Im}\lambda \neq 0$$

shartlarning bajarilishi lozim. Endi $\psi_\pm(x,\lambda)$ yechimlarni quyidagicha yozib olamiz:

$$\psi_\pm(x,\lambda) = \left(\lambda - 1 + \frac{1 - m^\pm(\lambda)x}{x+1}\right)\frac{\cos\sqrt{\lambda}x}{\lambda} +$$
$$+\left(\lambda m^\pm(\lambda) + \lambda + \frac{m^\pm(\lambda) - \lambda + 1}{x+1}\right)\frac{\sin\sqrt{\lambda}x}{\lambda\sqrt{\lambda}} =$$
$$= \left\{\frac{1}{2\lambda}\left(\lambda - 1 + \frac{1 - m^\pm(\lambda)x}{x+1}\right) +\right.$$
$$\left. + \frac{1}{2i\lambda\sqrt{\lambda}}\left(\lambda m^\pm(\lambda) + \lambda + \frac{m^\pm(\lambda) - \lambda + 1}{x+1}\right)\right\} e^{i\sqrt{\lambda}x} +$$
$$+\left\{\frac{1}{2\lambda}\left(\lambda - 1 + \frac{1 - m^\pm(\lambda)x}{x+1}\right) -\right.$$
$$\left. - \frac{1}{2i\lambda\sqrt{\lambda}}\left(\lambda m^\pm(\lambda) + \lambda + \frac{m^\pm(\lambda) - \lambda + 1}{x+1}\right)\right\} e^{-i\sqrt{\lambda}x}. \quad (5.7.10)$$

Agar biz $\operatorname{Im}\sqrt{\lambda} > 0$ da $e^{i\sqrt{\lambda}x} \in L^2(0,\infty)$, $e^{-i\sqrt{\lambda}x} \notin L^2(0,\infty)$ bo'lishini hisobga olsak, $\psi_+(x,\lambda) \in L^2(0,\infty)$ shart bajarilishi uchun (5.7.10) tenglikdagi oxirgi qavs $x \to \infty$ da nolga intilishi kerak. Shuning uchun

$$\frac{1}{2\lambda}\left(\lambda - 1 - m^+(\lambda)\right) - \frac{1}{2i\lambda\sqrt{\lambda}}\left(\lambda m^+(\lambda) + \lambda\right) = 0. \quad (5.7.11)$$

Bundan $m^+(\lambda)$ funksiyani topamiz:

$$m^+(\lambda) = \frac{i\lambda\sqrt{\lambda} - 1}{\lambda + 1}. \quad (5.7.12)$$

(5.7.12) ifodani (5.7.10) tenglikka qo'ysak,

$$\psi_+(x,\lambda) = \left(\frac{\lambda}{\lambda+1} + \frac{1 - ix\sqrt{\lambda}}{(x+1)(\lambda+1)}\right) e^{i\sqrt{\lambda}x} \tag{5.7.13}$$

kelib chiqadi. Xuddi shunday, $\operatorname{Im}\sqrt{\lambda} > 0$ da

$$e^{-i\sqrt{\lambda}x} \in L^2(-\infty, 0), \quad e^{i\sqrt{\lambda}x} \notin L^2(-\infty, 0)$$

bo'lishini hisobga olsak, u holda (5.7.10) tenglikdan quyidagilarni topamiz:

$$m^-(\lambda) = -\frac{i\lambda\sqrt{\lambda} - 1}{\lambda + 1},$$

$$\psi_-(x,\lambda) = \left(\frac{1}{x+1} + \frac{1 - xi\sqrt{\lambda}}{(x-1)(\lambda+1)}\right) e^{-i\sqrt{\lambda}x}.$$

c) $M_{11}(\lambda)$, $M_{12}(\lambda)$, $M_{22}(\lambda)$ funksiyalarning ta'riflariga ko'ra

$$M_{11}(\lambda) = \frac{1}{m^+(\lambda) - m^-(\lambda)} = -\frac{1}{2}\frac{i\lambda\sqrt{\lambda} + 1}{\lambda^2 - \lambda + 1},$$

$$M_{12}(\lambda) = \frac{1}{2} \cdot \frac{m^+(\lambda) + m^-(\lambda)}{m^+(\lambda) - m^-(\lambda)} = 0,$$

$$M_{22}(\lambda) = \frac{m^+(\lambda)m^-(\lambda)}{m^+(\lambda) - m^-(\lambda)} = -\frac{1}{2}\frac{i\lambda\sqrt{\lambda} - 1}{\lambda + 1}.$$

d) Agar $\lambda < 0$, $\lambda \neq -1$ bo'lsa,

$$\operatorname{Im}\{M_{11}(\lambda)\} = 0, \ \operatorname{Im}\{M_{12}(\lambda)\} = 0, \ \operatorname{Im}\{M_{22}(\lambda)\} = 0$$

bo'ladi. Agar $\lambda > 0$ bo'lsa,

$$\operatorname{Im}\{M_{11}(\lambda)\} = -\frac{1}{2}\frac{\lambda\sqrt{\lambda}}{\lambda^2 - \lambda + 1}, \quad \operatorname{Im}\{M_{22}(\lambda)\} = -\frac{1}{2}\frac{\lambda\sqrt{\lambda}}{\lambda + 1}$$

bo'ladi. $\lambda = -1$ va $\lambda = \dfrac{1 \pm i\sqrt{3}}{2}$ nuqtalardagi chegirmalarni hisoblaymiz:

$$\operatorname*{res}_{\lambda = \frac{1+i\sqrt{3}}{2}} M_{11}(\lambda) = -\lim_{\lambda \to \frac{1+i\sqrt{3}}{2}} \left(\lambda - \frac{1+i\sqrt{3}}{2}\right) \frac{i\lambda\sqrt{\lambda} + 1}{2(\lambda - \frac{1+i\sqrt{3}}{2})(\lambda - \frac{1-i\sqrt{3}}{2})} = 0,$$

$$\operatorname*{res}_{\lambda=\frac{1-i\sqrt{3}}{2}} M_{11}(\lambda) = -\lim_{\lambda\to\frac{1-i\sqrt{3}}{2}} (\lambda-\frac{1-i\sqrt{3}}{2})\frac{i\lambda\sqrt{\lambda}+1}{2(\lambda-\frac{1+i\sqrt{3}}{2})(\lambda-\frac{1-i\sqrt{3}}{2})} = 0,$$

$$\operatorname*{res}_{\lambda=-1} M_{22} = -\lim_{\lambda\to-1}(\lambda+1)\frac{i\lambda\sqrt{\lambda}+1}{2(\lambda+1)} = 0.$$

Endi spektral matritsa-funksiyani topish uchun quyidagi Titchmarsh-Kodaira formulalaridan foydalanamiz:

$$\xi(\lambda) = -\frac{1}{\pi}\lim_{y\to0}\int_0^\lambda \operatorname{Im}\{M_{11}(x+iy)\}dx,$$

$$\eta(\lambda) = \frac{1}{\pi}\lim_{y\to0}\int_0^\lambda \operatorname{Im}\{M_{12}(x+iy)\}dx,$$

$$\zeta(\lambda) = -\frac{1}{\pi}\lim_{y\to0}\int_0^\lambda \operatorname{Im}\{M_{22}(x+iy)\}dx.$$

Bularga ko'ra $\eta(\lambda)\equiv 0,\ \ \lambda\in R^1$. Agar $\lambda>0$ bo'lsa, u holda

$$\xi(\lambda) = \frac{1}{2\pi}\int_0^\lambda \frac{x\sqrt{x}}{x^2-x+1}dx = \left\{\begin{array}{l} x=t^2 \\ dx=2tdt \end{array}\right\} = \frac{1}{\pi}\int_0^{\sqrt{\lambda}} \frac{t^4}{t^4-t^2+1}dt =$$

$$= \frac{1}{\pi}\sqrt{\lambda} + \frac{1}{2\pi\sqrt{3}}\int_0^{\sqrt{\lambda}}\left(\frac{1}{t+\frac{1}{t}-\sqrt{3}} - \frac{1}{t+\frac{1}{t}+\sqrt{3}}\right)d(t+\frac{1}{t}) =$$

$$= \frac{1}{\pi}\sqrt{\lambda} + \frac{1}{2\pi\sqrt{3}}\ln\frac{\lambda-\sqrt{3\lambda}+1}{\lambda+\sqrt{3\lambda}+1},$$

$$\zeta(\lambda) = \frac{1}{2\pi}\int_0^\lambda \frac{x\sqrt{x}}{x+1}dx = \frac{1}{3\pi}\sqrt{\lambda^3} - \frac{1}{\pi}\sqrt{\lambda} + \frac{1}{\pi}\operatorname{arctg}\sqrt{\lambda}.$$

Agar $\lambda\le 0$ bo'lsa, $\operatorname{Im}\{M_{11}(\lambda)\}=0$ bo'lgani uchun, $\xi(0)=0$ ni hisobga olib,

$$\xi(\lambda)=\begin{cases} 0, & \lambda\le 0,\\ \dfrac{1}{\pi}\sqrt{\lambda}+\dfrac{1}{2\pi\sqrt{3}}\ln\dfrac{\lambda-\sqrt{3\lambda}+1}{\lambda+\sqrt{3\lambda}+1}, & \lambda>0,\end{cases}$$

tenglikni topamiz. $\lambda<0$ bo'lganda $\operatorname{Im}\{M_{22}(\lambda)\}=0$ bo'lgani uchun, $\zeta(0)=0$ normallashtirish shartini inobatga olsak, ushbu

$$\zeta(\lambda)=\begin{cases} 0, & \lambda\le 0,\\ \dfrac{1}{3\pi}\sqrt{\lambda^3}-\dfrac{1}{\pi}\sqrt{\lambda}+\dfrac{1}{\pi}artg\sqrt{\lambda}, & \lambda>0,\end{cases}$$

tenglik kelib chiqadi.

Shunday qilib, berilgan masala faqat uzluksiz spektrga ega bo'lib, u $E=[0,\infty)$ to'plamdan iborat ekan.

4-masala. $\Re(\lambda)=\begin{pmatrix}\xi(\lambda) & \eta(\lambda)\\ \eta(\lambda) & \zeta(\lambda)\end{pmatrix}$, $\lambda\in R^1$ spektral matritsa-funksiya berilgan. Bu yerda

$$\xi(\lambda)=\begin{cases} 0, & \lambda\le 0,\\ \dfrac{1}{\pi}\sqrt{\lambda}+\dfrac{1}{2\pi\sqrt{3}}\ln\dfrac{\lambda-\sqrt{3\lambda}+1}{\lambda+\sqrt{3\lambda}+1}, & \lambda>0,\end{cases}$$

$$\eta(\lambda)\equiv 0,\ \ \lambda\in R^1,$$

$$\zeta(\lambda)=\begin{cases} 0, & \lambda\le 0,\\ \dfrac{1}{3\pi}\sqrt{\lambda^3}-\dfrac{1}{\pi}\sqrt{\lambda}+\dfrac{1}{\pi}arctg\sqrt{\lambda}, & \lambda>0.\end{cases}$$

Quyidagilarni toping:

a) Rofe-Beketov teoremasidagi $M_{11}(z)$, $M_{12}(z)$, $M_{22}(z)$ funksiyalarni toping va Rofe-Beketov ayniyatini tekshiring.

b) $M_{11}(z)$, $M_{12}(z)$, $M_{22}(z)$ funksiyalar bo'yicha $m^+(\lambda)$, $m^-(\lambda)$ Veyl-Titchmarsh funksiyalarini toping.

c) $m^+(\lambda)$, $m^-(\lambda)$ bo'yicha Rofe-Beketov teoremasidagi $\rho_+(\lambda)$, $\rho_-(\lambda)$ spektral funksiyalarni toping.

Yechish. a) Berilgan $\xi(\lambda)$, $\eta(\lambda)$, $\zeta(\lambda)$ funksiyalar bo'yicha $M_{11}(z)$, $M_{12}(z)$, $M_{22}(z)$ funksiyalarni topamiz. Buning uchun ushbu

$$M_{11}(z)=\int\limits_{-\infty}^{\infty}\frac{d\xi(\lambda)}{z-\lambda},\quad M_{12}(z)=\int\limits_{-\infty}^{\infty}\frac{d\eta(\lambda)}{z-\lambda},$$

$$M_{22}(z)=\int\limits_{-\infty}^{\infty}\left(\frac{1}{z-\lambda}+\frac{\lambda}{1+\lambda^2}\right)d\zeta(\lambda)+a,$$

$$M_{22}(z)=-\frac{i}{2}\sqrt{z}+\overline{o}(1),\quad z\to+i\infty$$

formulalardan foydalanamiz. Bunga ko'ra

$$M_{11}(z)=\int\limits_{-\infty}^{\infty}\frac{1}{z-\lambda}d\left(\frac{1}{\pi}\sqrt{\lambda}+\frac{1}{2\pi\sqrt{3}}\ln\frac{\lambda-\sqrt{3\lambda}+1}{\lambda+\sqrt{3\lambda}+1}\right)=$$

$$=-\frac{1}{2\pi}\int\limits_{0}^{\infty}\frac{\lambda\sqrt{\lambda}}{(\lambda-z)(\lambda^2-\lambda+1)}d\lambda.$$

Oxirgi integralni chegirmalar yordamida hisoblaymiz. Buning uchun ushbu

$$f(\lambda)=\frac{\lambda\sqrt{\lambda}}{(\lambda-z)(\lambda^2-\lambda+1)}$$

funksiyani 4-rasmdagi kontur bo'yicha integrallaymiz.

Koshining chegirmalar haqidagi teoremasiga ko'ra quyidagi tenglik o'rinli bo'ladi:

$$\int\limits_{C_R}f(\lambda)d\lambda+\int\limits_{C_\varepsilon}f(\lambda)d\lambda+\int\limits_{l^-_{R,\varepsilon}}f(\lambda)d\lambda+\int\limits_{l^+_{R,\varepsilon}}f(\lambda)d\lambda=$$

$$=2\pi i\left(\operatorname*{res}_{\lambda=\frac{1+i\sqrt{3}}{2}}f(\lambda)+\operatorname*{res}_{\lambda=\frac{1-i\sqrt{3}}{2}}f(\lambda)+\operatorname*{res}_{\lambda=z}f(\lambda)\right).\qquad(5.7.14)$$

Ushbu

$$\int\limits_{C_R}f(\lambda)d\lambda=$$

$$= \int_0^{2\pi} \frac{Re^{it}\sqrt{R}e^{\frac{it}{2}}}{(Re^{it}-z)(R^2e^{2it}-Re^{it}+1)} iRe^{it}dt =$$

$$= \underline{\underline{O}}\left(\frac{1}{\sqrt{R}}\right) \to 0, \ (R \to \infty).$$

$$\int_{C_\varepsilon} f(\lambda)d\lambda = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{\varepsilon e^{it}\sqrt{\varepsilon}e^{\frac{it}{2}}}{(\varepsilon e^{it}-z)(\varepsilon^2e^{2it}-\varepsilon e^{it}+1)} i\varepsilon e^{it}dt =$$

$$= \underline{\underline{O}}(\varepsilon\sqrt{\varepsilon}) \to 0, \quad (\varepsilon \to 0),$$

$$f(\lambda)|_{l^-_{R,\varepsilon}} = -f(\lambda)|_{l^+_{R,\varepsilon}},$$

$$\operatorname*{res}_{\lambda=z} f(\lambda) = \frac{z\sqrt{z}}{z^2-z+1}, \quad \operatorname*{res}_{\lambda=\frac{1+i\sqrt{3}}{2}} f(\lambda) = \frac{2}{\sqrt{3}(1+i\sqrt{3}-2z)},$$

$$\operatorname*{res}_{\lambda=\frac{1-i\sqrt{3}}{2}} f(\lambda) = \frac{2}{\sqrt{3}(1-i\sqrt{3}-2z)}$$

munosabatlarni hisobga olsak va (5.7.14) tenglikda $R \to \infty$, $\varepsilon \to 0$ limitga o'tsak,

$$2\int_0^{\infty} f(\lambda)d\lambda = 2\pi i\left(\frac{z\sqrt{z}}{z^2-z+1} + \frac{2}{\sqrt{3}(1+i\sqrt{3}-2z)} +\right.$$

$$\left. + \frac{2}{\sqrt{3}(1-i\sqrt{3}-2z)}\right)$$

kelib chiqadi. Bunga ko'ra

$$\int_0^{\infty} \frac{\sqrt{\lambda}}{(\lambda-z)(\lambda+1)}d\lambda = \pi\left(\frac{1+iz\sqrt{z}}{z^2-z+1}\right).$$

Demak,

$$M_{11}(z) = -\frac{1}{2}\cdot\frac{iz\sqrt{z}+1}{z^2-z+1}.$$

Endi $M_{22}(z)$ ni hisoblaymiz:

$$M_{22}(z) = \int_0^\infty \left(\frac{\lambda}{\lambda^2+1} - \frac{1}{\lambda - z} \right) d(\frac{1}{3\pi}\sqrt{\lambda^3} -$$

$$-\frac{1}{\pi\sqrt{\lambda}} + \frac{1}{\pi}\operatorname{arctg}\sqrt{\lambda}) + a =$$

$$= -\frac{1}{2\pi}\int_0^\infty \frac{(\lambda z+1)\lambda\sqrt{\lambda}}{(\lambda^2+1)(\lambda-z)(\lambda+1)} d\lambda + a.$$

Oxirgi integralni ham oldingi konturdan foydalanib hisoblaymiz. Bu holda

$$f(\lambda) = \frac{(\lambda z+1)\lambda\sqrt{\lambda}}{(\lambda^2+1)(\lambda-z)(\lambda+1)}$$

deb olsak, ushbu

$$\int_{C_R} f(\lambda)d\lambda + \int_{C_\varepsilon} f(\lambda)d\lambda + \int_{l_{R,\varepsilon}^-} f(\lambda)d\lambda + \int_{l_{R,\varepsilon}^+} f(\lambda)d\lambda =$$

$$= 2\pi i(\operatorname*{res}_{\lambda=i} f(\lambda) + \operatorname*{res}_{\lambda=-i} f(\lambda) + \operatorname*{res}_{\lambda=z} f(\lambda) + \operatorname*{res}_{\lambda=-1} f(\lambda)) \qquad (5.7.15)$$

tenglik hosil bo'ladi. Oxirgi tenglikning chap tarafidagi integrallarni quyidagicha baholaymiz va o'ng tarafidagi $f(\lambda)$ funksiyaning chegirmalarini hisoblaymiz:

$$\int_{C_R} f(\lambda)d\lambda =$$

$$= \int_0^{2\pi} \frac{(Rze^{it}+1)Re^{it}\sqrt{R}e^{\frac{it}{2}}}{(R^2e^{2it}+1)(Re^{it}-z)(Re^{it}+1)} dt =$$

$$= \underline{\underline{O}}\left(\frac{1}{\sqrt{R^3}}\right) \to 0, \quad (R \to \infty),$$

$$\int_{C_\varepsilon} f(\lambda)d\lambda =$$

$$\int\limits_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{(\varepsilon z e^{it}+1)\varepsilon e^{it}\sqrt{\varepsilon}e^{\frac{it}{2}}}{(\varepsilon^2 e^{2it}+1)(\varepsilon e^{it}-z)(\varepsilon e^{it}+1)}dt = \underline{\underline{O}}(\varepsilon\sqrt{\varepsilon}) \to 0, \quad (\varepsilon \to 0),$$

$$\operatorname*{res}_{\lambda=i} f(\lambda) = \frac{i(iz+1)\sqrt{i}}{2i(i-z)(i+1)} = \frac{-i(i-z)\frac{i+1}{\sqrt{2}}}{2(i-z)(i+1)} = -\frac{i\sqrt{2}}{4},$$

$$\operatorname*{res}_{\lambda=-i} f(\lambda) = \frac{-i(-iz+1)\sqrt{-i}}{-2i(-i-z)(-i+1)} = \frac{-i(i+z)\frac{i-1}{\sqrt{2}}}{2(i+z)(i-1)} = -\frac{i\sqrt{2}}{4},$$

$$\operatorname*{res}_{\lambda=z} f(\lambda) = \frac{z\sqrt{z}}{z+1};$$

$$\operatorname*{res}_{\lambda=-1} f(\lambda) = \frac{-(1-z)\sqrt{-1}}{2(-1-z)} = \frac{i(1-z)}{2(1+z)}.$$

Ushbu munosabatlarni va

$$f(\lambda)|_{l^-_{R,\varepsilon}} = -f(\lambda)|_{l^+_{R,\varepsilon}}$$

ni hisobga olib, (5.7.15) tenglikda limitga o'tsak,

$$\int\limits_0^\infty \frac{(\lambda z+1)\lambda\sqrt{\lambda}}{(\lambda^2+1)(\lambda-z)(\lambda+1)}d\lambda = i\pi\left(\frac{z\sqrt{z}}{z+1} + \frac{i(1-z)}{2(1+z)} - \frac{i\sqrt{2}}{2}\right)$$

kelib chiqadi. Bunga ko'ra

$$M_{22}(z) = -\frac{1}{2}\left(\frac{iz\sqrt{z}}{z+1} - \frac{1-z}{2(1+z)} + \frac{\sqrt{2}}{2}\right) + a$$

bo'lishini topamiz. Bu tenglikdan

$$M_{22}(z) = -\frac{i}{2}\sqrt{z} - \frac{1}{4} - \frac{\sqrt{2}}{4} + a + \bar{\bar{o}}(1), \quad z \to +i\infty$$

kelib chiqadi. Demak, $a = \frac{1}{4} + \frac{\sqrt{2}}{4}$ ekan, ya'ni

$$M_{22}(z) = -\frac{1}{2} \cdot \frac{iz\sqrt{z}+1}{z+1}.$$

Endi Rofe-Beketov ayniyatini tekshiramiz:

$$M_{11}(z)M_{22}(z) - M_{12}^2(z) = \frac{iz\sqrt{z}+1}{2(z^2-z+1)} \cdot \frac{iz\sqrt{z}-1}{2(z+1)} - 0 = -\frac{1}{4}.$$

b) Ushbu

$$m^+(\lambda) = \frac{M_{12}(\lambda) + \frac{1}{2}}{M_{11}(\lambda)}; \quad m^-(\lambda) = \frac{M_{12}(\lambda) - \frac{1}{2}}{M_{11}(\lambda)}$$

formulalarga oldingi bandda topilgan ifodalarni qo'yamiz:

$$m^+(\lambda) = -\frac{\lambda^2 - \lambda + 1}{i\lambda\sqrt{\lambda} + 1} = \frac{i\lambda\sqrt{\lambda} - 1}{\lambda + 1},$$

$$m^-(\lambda) = \frac{\lambda^2 - \lambda + 1}{i\lambda\sqrt{\lambda} + 1} = -\frac{i\lambda\sqrt{\lambda} - 1}{\lambda + 1}.$$

c) $m^+(\lambda)$ funksiyani qutb maxsusligi bo'lmagani uchun

$$\rho'_+(\lambda) = \frac{1}{\pi} \operatorname{Im}\left\{m^+(\lambda)\right\} = \begin{cases} \frac{1}{\pi} \cdot \frac{\lambda\sqrt{\lambda}}{\lambda + 1}, & \lambda > 0 \\ 0, & \lambda \leq 0, \end{cases}$$

$$\rho_+(\lambda) = \begin{cases} \frac{2}{3\pi}\sqrt{\lambda^3} - \frac{2}{\pi}\sqrt{\lambda} + \frac{2}{\pi}\operatorname{arctg}\sqrt{\lambda}, & \lambda > 0 \\ 0, & \lambda \leq 0. \end{cases}$$

$(-\infty, 0)$ oraliqni $x = -t$ almashtirish yordamida $(0, \infty)$ oraliqqa o'tkazib olamiz. Bunda hosil bo'ladigan masalaning Veyl-Titchmarsh funksiyasi $(-m^-(\lambda))$ bo'ladi. Biz qarayotgan holda $-m^-(\lambda) = m^+(\lambda)$ bo'ladi, ya'ni $\rho_-(\lambda)$ ni topishning xojati yo'q. Aniqrog'i bu holda izlanayotgan $q(x)$ potensial juft funksiya bo'ladi.

Mustaqil yechish uchun mashqlar

Quyidagi Shturm-Liuvill chegaraviy masalalari uchun Veyl-Titchmarsh funksiyasini va Veyl yechimini aniqlang, hamda yoyilma teoremasini va Parseval tengligini yozing.

1. Furye-Xankel qatoriga yoyish

$$-y'' + \frac{4p^2 - 1}{4x^2}y = \lambda y, \quad 0 < x < \infty,$$

$$a)\ p \geq 1, \qquad b)\ 0 < p < 1,\ p \neq \frac{1}{2}, \qquad c)\ p = 0.$$

2. Chebishev-Ermit ortogonal funksiyalari bo'yicha yoyish

$$-y'' + x^2 y = \lambda y, \quad -\infty < x < \infty .$$

3. Lejandr ko'phadlari bo'yicha yoyish

$$-y'' - \left(\frac{1}{4}\operatorname{tg}^2 x + \frac{1}{4}\right) y = \lambda y, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$y\left(-\frac{\pi}{2}\right) = y\left(\frac{\pi}{2}\right) = 0.$$

4. Lejandrning ergashgan ko'phadlari bo'yicha yoyish

$$-y'' - \left(\frac{1}{4} tg^2 x + \frac{1}{4} - \frac{m^2}{\cos^2 x}\right) y = \lambda y, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$y\left(-\frac{\pi}{2}\right) = y\left(\frac{\pi}{2}\right) = 0 .$$

5. Chebishev-Lagerr ko'phadlari bo'yicha yoyish

$$-y'' + \left(\frac{x^2}{16} - \frac{1}{4x^2} - \frac{1}{2}\right) y = \lambda y, \quad 0 < x < \infty .$$

6. Lagerr-Sonin ko'phadlari bo'yicha yoyish

$$-y'' + \left(x^2 + \frac{4p^2 - 1}{4x^2}\right) y = \lambda y, \quad 0 < x < \infty,$$

7. Chebishev-Lagerr ergashgan ko'phadlari bo'yicha yoyish

$$-(xy')' + \left(\frac{x}{4} + \frac{m^2}{4x} + \frac{m-1}{2}\right) y = \lambda y, \quad 0 < x < \infty.$$

8. "Vodorod atomi" tenglamasi

$$-y'' + \left(\frac{m(m+1)}{x^2} - \frac{c}{x}\right) y = \lambda y, \quad 0 < x < \infty.$$

9. Gipergeometrik funksiyalar tenglamasi $x \in R^1$

$$-y'' + \frac{2(\alpha - 1)(2\gamma - 1 - \alpha)\operatorname{ch} x + 2\alpha^2 - 4\gamma\alpha + (1 - 2\gamma)^2}{4\operatorname{sh}^2 x} y = \lambda y.$$

10. $$-y'' + \frac{\alpha(1-\alpha)}{4\,\mathrm{ch}^2 x} y = \lambda y, \qquad x \in R^1.$$

11. $$-y'' + \left(\frac{Ae^{-x}}{e^{-x}+1} + \frac{Be^{-x}}{(e^{-x}+1)^2} \right) y = \lambda y, \qquad x \in R^1.$$

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