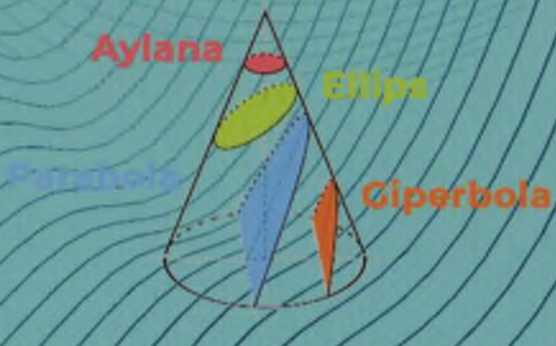
YU. P. APAKOV, B. I. JAMALOV,
A. M. TO'XTABAYEV

# OLIY MATEMATIKADAN MISOL VA MASALALAR

## I

$$\Delta = \sum_{j=1}^{n} a_{ij} A_{ij} \qquad tg\alpha = \frac{k_2 - k_1}{1 + k_1 k_2}$$

$$\int_{2}^{e+1} \frac{dx}{(x-1)\sqrt[3]{\ln(x-1)}}$$



O‘ZBEKISTON RESPUBLAKASI
OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

NAMANGAN MUHANDISLIK-QURILISH INSTITUTI

YU. P. APAKOV, B. I. JAMALOV, A. M. TO‘XTABAYEV

# OLIY MATEMATIKADAN MISOL VA MASALALAR

**Ikki jildlik**

**1-jild**

*O‘zbekiston Respublikasi Oliy va o‘rta maxsus ta’lim vazirligi tomonidan muhandis-texnika sohasidagi bakalavriat ta’lim yo‘nalishlari talabalari uchun darslik sifatida tavsiya etilgan*

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Darslik muhandis-texnika sohasidagi bakalavriat ta'lim yo‘nalishlari uchun mo‘ljallangan. Kitob ikki jilddan iborat. 1-jild 7 bobni, 2-jild 9 bobni o‘z ichiga oladi. Darslikdan kunduzgi, kechki va sirtqi bo‘lim talabalari foydalanishlari mumkin.

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# SO‘Z BOSHI

Mamlakatimizda ta’lim tizimini isloh qilishga va uni zamon talablari bilan uyg‘unlashtirishga katta ahamiyat berilmoqda.

Ayniqsa, matematika fanining o‘rni, uning barcha fanlarni o‘zlashtirishga va ularning rivojiga asos ekanligini hisobga olib, uning rivojiga alohida e’tibor qaratilmoqda. Maktabgacha ta’lim muassasasidan boshlab, o‘rta ta’lim maktablari hamda oliy ta’limning mavjud dasturi zamon talabiga javob bera olmay qolgani o‘rinli ravishda tanqid qilindi.

Darslik 16 bobga, ya’ni birinchi jild 7 bobga, ikkinchi jild 9 bobga ajratilgan. Har bir mavzuda asosiy tushunchalar va muhim formulalar keltirilib, mavzuga doir tipik misollar yechib ko‘rsatilgan. Dars jarayonida va mustaqil ishlash uchun masalalar berilgan. Masalalar osondan murakkabga prinsipi asosida joylashtirilgan. Har bir mavzuda murakkab masalalar * va ** belgisi bilan ajratilgan. Barcha masalalarning javoblari keltirilgan.

Davr talabidan kelib chiqib, darslik amaliy mashg‘ulotlarda va mustaqil o‘rganish maqsadida foydalanishga mo‘ljallangan.

Darslikdan **amaliy mashg‘ulot jarayonida** foydalanishda mavzuga doir asosiy tushunchalar amaliyot o‘qituvchisi tomonidan keltirilib, tipik misollar yechib ko‘rsatilgandan so‘ng, misollarni ishlashga malaka hosil bo‘lgach, masala ishlashga kirishish mumkin.

**Mavzuni mustaqil o‘zlashtirish**ga kirishishda, avvalo, diqqat bilan asosiy tushunchalar va formulalarni o‘zlashtirib, so‘ngra yechib ko‘rsatilgan masalani tushunib olish va uni mustaqil ravishda yechib chiqish hamda ishlanishi bilan solishtirish kerak. Agar ishlangan masala yoki misolingiz javobi kitobdagi javobga mos kelsagina, berilgan navbatdagi masalalarni ishlashga o‘tish maqsadga muvofiq.

Birinchi jild amaldagi foydalanilayotgan o‘quv adabiyotlarida ajratilgan soat kamligi uchun kiritilmagan, lekin mutaxassislik fanlarini o‘rganishda muhim ahamiyatga ega bo‘lgan quyidagi mavzular bilan to‘ldirilgan:

– Vektorlarning amaliy masalalarni yechishga qo‘llanishi;

– Bir jinsli tenglamalar sistemasini yechish;

– Ikkinchi tartibli egri chiziqlarning umumiy tenglamasini kanonik ko‘rinishga keltirish;

– Amaliy masalalarni yechishda funksiyaning eng katta va eng kichik qiymatlarini qo‘llash;

– Aniq integralning amaliy masalalarni yechishga tatbiqi.

Ikkinchi jild esa quyidagi mavzular bilan to‘ldirilgan:

– Ikki o‘lchovli integralning fizikaga tatbiqlari.

– Rikatti differensial tenglamasi.

– Eyler differensial tenglamasi.

– Differensial tenglamalar sistemasini birinchi integral yordamida yechish.

– O‘zgarmas koeffitsiyentli chiziqli bir jinsli bo‘lmagan differensial tenglamalar sistemasini integrallash usullari.

– Chegirmalarni integral hisoblashga qo‘llash.

– Birinchi tartibli xususiy hosilali differensial tenglamalarni yechish.

– Bir jinsli bo‘lmagan to‘lqin tarqalish va issiqlik tarqalish tenglamalariga doir masalalar yehish.

– Elliptik tipdagi tenglamaga Dirixle masalasini to‘g‘ri to‘rtburchakda va halqada yechish.

– O‘zgarmas koeffitsiyentli chiziqli tenglamalar sistemasini va integral tenglamani Laplas tasviri yordamida yechish va boshqalar.

Birinchi jildda 1085 ta masala va misollar keltirilgan bo‘lib, ulardan 153 tasini yechib ko‘rsatilgan. Ikkinchi jildda 1400 ta masala va misollar keltirilgan bo‘lib, ulardan 386 tasini yechib ko‘rsatilgan. Masala va misollar foydalanilgan adabiyotlar ro‘yxatida keltirilgan kitoblardan olingan yoki mualliflar tomonidan tuzilgan.

Darslikning 1-jildi B.I.Jamalov va A.M.To‘xtabayev, 2-jildi Yu.P.Apakov tomonidan yozilgan.

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Murojaat uchun e-mail: yusupjonapakov@gmail.com

*Mualliflar*

# I BOB. OLIY ALGEBRA ELEMENTLARI

## I §. Ikkinchi va uchinchi tartibli determinantlar

### Ikkinchi tartibli determinant

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ ko'rinishidagi ifoda **ikkinchi tartibli determinant**, $\Delta = a_{11}a_{22} - a_{12}a_{21}$ ayirma esa uning son qiymati deyiladi.

**1.** $\begin{vmatrix} -3 & 5 \\ -2 & -1 \end{vmatrix}$ determinantni hisoblang.

**Yechish.** $\begin{vmatrix} -3 & 5 \\ -2 & -1 \end{vmatrix} = -3\cdot(-1) - 5(-2) = 3 + 10 = 13$.

**2.** $\begin{vmatrix} x & 6 \\ 1 & -3 \end{vmatrix} = 8$ tenglamani yeching.

**Yechish.** $-3x - 6 = 8, \quad -3x = 14, \quad x = -\frac{14}{3} = -4\frac{2}{3}$ .

### Uchinchi tartibli determinant

Quyidagicha

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

belgilanadigan va son qiymati

$$\Delta = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad (1)$$

ga teng bo'lgan ifodaga **3-tartibli determinant** deyiladi.

3-tartibli determinantni hisoblash uchun uchburchak qoidasi:

Uchinchi tartibli determinantni hisoblashning Sarrius usuli:

$$\begin{array}{c} (+)\quad(+)\quad(+) \\ \left|\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right| \\ (-)\quad(-)\quad(-) \end{array}$$

Bu usulda uchinchi tartibli determinant jadvali yoniga 1- va 2- ustunlar takroriy yoziladi, 1- asosiy diagonalga parallel uchta diagonal elementlarni o'zaro ko'paytirib, ularni yig'indisi olinadi. 2-yordamchi diagonal va unga parallel uchta diagonal elementlari ko'paytirilib, ularning ayirmalari olinadi. Natijada (1) formula hosil bo'ladi.

**3.** $\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 3 & 5 & -2 \end{vmatrix}$ determinantni hisoblang.

**Yechish.**

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 3 & 5 & -2 \end{vmatrix} = 1\cdot1\cdot(-2)+2\cdot(-1)\cdot3+0\cdot5\cdot4-3\cdot1\cdot4-5\cdot(-1)\cdot1-0\cdot2\cdot(-2)=$$

$$= -2-6+0-12+5+0=-20+5=-15.$$

## Determinantning xossalari

Determinatning xossalari ularning tartibiga bog'liq bo'lmagani uchun, xossalarni asosan uchinchi tartibli determinantlar uchun keltiramiz.

$1^0$. Determinatning mos satrlari va ustunlari o'rinlari almashtirilganda uning qiymati o'zgarmaydi.

$2^0$ Agar determinantning ikki satr (ustun) elementlari o'zaro almashtirilsa, uning qiymati o'zgarmaydi, ishorasi esa qarama-qarshisiga almashadi.

$3^0$. Agar determinant ikkita bir xil elementli satrga (ustunga) ega bo'lsa, u nolga teng.

$4^0$. Determinantning biror satr (ustun) elementlarini ixtiyoriy $\alpha$ songa ko'paytirish determinantni shu songa ko'paytirishga teng kuchlidir.

$5^0$. Agar determinant nollardan iborat satrga (ustunga) ega bo'lsa, u nolga teng.

$6^0$. Agar determinantning ikkita satr (ustun) elementlari o'zaro proporsional bo'lsa, u nolga teng.

$7^0$. Agar determinantning biror satrining (ustunining) har bir elementi ikkita qo'shiluvchining yig'indisidan iborat bo'lsa, u holda bu determinant (quyidagi ko'rinishdagi) ikkita determinantlar yig'indisidan iborat bo'ladi.

Masalan:

$$\begin{vmatrix} a_{11}+b_1 & a_{12} & a_{13} \\ a_{21}+b_2 & a_{22} & a_{23} \\ a_{31}+b_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}.$$

$8^0$. Agar biror satr (ustun) elementalarini istalgan $\lambda \neq 0$, umumiy ko'paytuvchiga ko'paytirib boshqa satrning (ustunning) mos elementlariga qo'shilsa, determinant qiymati o'zgarmaydi.

Ikkinchi tartibli determinantlarni hisoblang:

**4.** $\begin{vmatrix} -3 & 2 \\ 5 & -1 \end{vmatrix}$;

**5.** $\begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix}$;

**6.** $\begin{vmatrix} \frac{1}{4} & -\frac{1}{5} \\ 2\frac{1}{2} & -4 \end{vmatrix}$;

**7.** $\begin{vmatrix} \sqrt{a}+\sqrt{b} & \sqrt{a}-\sqrt{b} \\ \sqrt{a}-\sqrt{b} & \sqrt{a}+\sqrt{b} \end{vmatrix}$;

**8.** $\begin{vmatrix} \sin 1^0 & \sin 89^0 \\ -\cos 1^0 & \cos 89^0 \end{vmatrix}$;

**9.** $\begin{vmatrix} \log_b a & 1 \\ 1 & \log_a b \end{vmatrix}$;

**10.** $\begin{vmatrix} ab & ac \\ bd & cd \end{vmatrix}$;

**11.** $\begin{vmatrix} 1 & -tg\alpha \\ ctg\alpha & 1 \end{vmatrix}$;

Tenglamalarni yeching:

**12.** $\begin{vmatrix} x & x+1 \\ -4 & x+1 \end{vmatrix} = 4$;

**13.** $\begin{vmatrix} x & 3x \\ 7 & 4 \end{vmatrix} = 34$;

**14.** $\begin{vmatrix} x & -7 \\ 2 & x \end{vmatrix} = 23$;

**15.** $\begin{vmatrix} \cos 8x & \sin 5x \\ -\sin 8x & \cos 5x \end{vmatrix} = 0$;

**16.** $\begin{vmatrix} x+3 & x+2 \\ 6-2x & x+2 \end{vmatrix} = 0$;

**17.** $\begin{vmatrix} 2x-1 & x+1 \\ x+2 & x-1 \end{vmatrix} = -6$.

Uchinchi tartibli determinantlarni hisoblang:

**18.** $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ 4 & -1 & -5 \end{vmatrix}$;

**19.** $\begin{vmatrix} 1 & -2 & 1 \\ 4 & 3 & 2 \\ 5 & 0 & 1 \end{vmatrix}$;

20. $\begin{vmatrix} -2 & 3 & 1 \\ 0 & 6 & 1 \\ 1 & 2 & 2 \end{vmatrix}$;

21. $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 1 & 3 & -2 \end{vmatrix}$;

22. $\begin{vmatrix} 18 & 9 & 27 \\ 6 & 12 & 12 \\ 13 & 26 & 39 \end{vmatrix}$;

23. $\begin{vmatrix} 1 & a & 1 \\ a & a & 0 \\ a & 0 & -a \end{vmatrix}$;

24. $\begin{vmatrix} x & -1 & x \\ 1 & x & -1 \\ x & 1 & x \end{vmatrix}$;

25. $\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix}$;

26. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$;

27. $\begin{vmatrix} \sin\alpha & \cos\alpha & 1 \\ \sin\beta & \cos\beta & 1 \\ \sin\gamma & \cos\gamma & 1 \end{vmatrix}$.

Tenglamalarni yeching:

28. $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 4 & 9 \\ x & 2 & 3 \end{vmatrix} = 0$;

29. $\begin{vmatrix} 6 & 3 & x-1 \\ 4 & x+2 & 2 \\ 2x & 1 & 0 \end{vmatrix} = 0$.

## 2 §. Yuqori tartibli determinantlar

n-tartibli (n≥4) determinantlar va ularni hisoblashni o'rganish uchun avvalo, quyidagi yordamchi tushunchalar kiritamiz:

**Algebraik to'ldiruvchi va minorlar.**

Determinantning biror elementining **minori** deb, shu element turgan satrini va ustunini o'chirishdan qolgan elementlardan hosil bo'lgan determinantga aytiladi.

Determinant $a_{ik}$ elementining minori $M_{ik}$, $(i,k=1,2,3)$ bilan belgilanadi.

Determinant $a_{ij}$ elementining **algebraik to'ldiruvchisi** deb $A_{ij}=(-1)^{i+j}M_{ij}$ songa aytiladi.

Masalan, quyidagi uchinchi tartibli determinantni olaylik:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

$a_{11}$ elementining algebraik to‘ldiruvchisi

$$A_{11}=(-1)^{1+1}M_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ bo‘ladi,}$$

$a_{32}$ elementining algebraik to‘ldiruvchisi esa

$$A_{32}=(-1)^{3+2}M_{32}=-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \text{ bo‘ladi.}$$

Bu kiritilgan tushunchalar yordamida quyidagi xossani isbotlash mumkin (xossalar tartibini saqlab qolamiz).

9°. Determinantning biror qatoridagi barcha elementlarni mos algebraik to‘ldiruvchilari bilan ko‘paytmasidan tashkil topgan yig‘indi shu determinantning qiymatiga teng.

**Yuqori tartibli determinantlar**

*n* ta satr va *n* ta ustundan tashkil topgan ushbu

$$\Delta=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \qquad (2)$$

determinantga **n -tartibli determinant** deyiladi. To‘rtinchi tartibli determinantni qaraymiz.

Determinantlarning yuqorida keltirilgan 9°-xossasini qo‘llab, ya’ni biror satr yoki ustun elementlari bo‘yicha yoyish usuli bilan 4-tartibli determinantni biror ustun yoki satr elementlari bo‘yicha yoyilganda hosil bo‘ladigan determinantlar 3-tartibli bo‘ladi. 3-tartibli determinant tushunchasi esa bizga ma’lum. n-tartibli determinantlar uchun yuqorida aytilgan barcha xossalar, jumladan, determinantning biror qator elementlari bo‘yicha yoyish formulasi ham o‘rinli bo‘ladi.

Istalgan tartibli determinantni hisoblash quyidagi usullardan biri orqali bajarilishi mumkin:

a) algebraik to‘ldiruvchilar yordamida satr yoki ustun bo‘yicha yoyish usulidan foydalanish;

b) biror satrdagi (ustundagi) bittadan boshqa barcha elementlarni nolga aylantirib, so‘ngra shu satr (ustun) bo‘yicha yoyib, ya’ni tartibini pasaytirib;

d) bosh (yordamchi) diagonaldan bir tomonda yotuvchi barcha elementlarni nolga aylantiriladi, ya'ni uchburchak ko'rinishga keltiriladi.

**30.** Determinant hisoblansin:

$$\Delta=\begin{vmatrix}2&1&4&3\\5&0&-1&0\\2&-1&6&3\\1&5&-1&2\end{vmatrix}.$$

**Yechish.** Determinantni hisoblash uchun uni ikkinchi satr elementlari bo'yicha yoyib chiqamiz. U holda

$$\Delta=a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}+a_{24}A_{24}=$$

$$=-5\begin{vmatrix}1&4&3\\-1&6&3\\5&-1&2\end{vmatrix}+0\begin{vmatrix}2&4&3\\2&6&3\\1&-1&2\end{vmatrix}+1\begin{vmatrix}2&1&3\\2&-1&3\\1&5&2\end{vmatrix}-0\begin{vmatrix}2&1&4\\2&-1&6\\1&5&-1\end{vmatrix}=$$

$$=-5(12+60+3-90+3+8)+(-4+3+30+3-30-4)=20-2=18$$

bo'ladi. Demak, yuqori tartibli determinantni hisoblash, determinant tartibini ketma-ket pasaytirish yo'li bilan amalga oshiriladi.

**31.** $\Delta=\begin{vmatrix}2&-1&0&4\\4&2&-1&3\\-2&0&3&-4\\1&1&0&-2\end{vmatrix}$

determinantni tartibini pasaytirish usuli bilan hisoblang.

**Yechish.** Buning uchun ikkita elementi nolga teng bo'lgan uchinchi ustunni tanlaymiz va uning ikkinchi satrida joylashgan elementidan boshqa barcha elementlarini nolga aylantiramiz. Buning uchun ikkinchi satr elementlarini 3 ga ko'paytirib, uchinchi satrning mos elementlariga qo'shamiz va hosil bo'lgan determinantni uchinchi ustun elementlari bo'yicha yoyamiz:

$$\Delta=\begin{vmatrix}2&-1&0&4\\4&2&-1&3\\-2&0&3&-4\\1&1&0&-2\end{vmatrix}=\begin{vmatrix}2&-1&0&4\\4&2&-1&3\\10&6&0&5\\1&1&0&-2\end{vmatrix}=(-1)\cdot(-1)^{2+3}\begin{vmatrix}2&-1&4\\10&6&5\\1&1&-2\end{vmatrix}.$$

Hosil qilingan uchinchi tartibli determinantda birinchi ustunning uchinchi satri elementidan yuqorida joylashgan elementlarini nolga aylantiramiz. Buning uchun avval uchinchi satrni $(-2)$ga ko‘paytirib, birinchi satrga qo‘shamiz, keyin uchinchi satrni $(-10)$ga ko‘paytirib, ikkinchi satrga qo‘shamiz, hosil bo‘lgan determinantni birinchi ustun elementlari bo‘yicha yoyamiz va hosil bo‘lgan ikkinchi tartibli determinantni hisoblaymiz:

$$\Delta = \begin{vmatrix} 0 & -3 & 8 \\ 0 & -4 & 25 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 8 \\ -4 & 25 \end{vmatrix} = -75 + 32 = -43.$$

**32.** $\Delta = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{vmatrix}$

determinantni uchburchak ko‘rinishga keltirib hisoblang.

**Yechish.** Determinant ustida quyidagi almashtirishlarni bajaramiz:

birinchi ustunni o‘zidan o‘ngda joylashgan ustunlar bilan ketma-ket 3 ta (toq) o‘rin almashtirib, to‘rtinchi ustunga o‘tkazamiz;

birinchi ustunning birinchi satridan pastda joylashgan elementlarini nolga aylantiramiz;

ikkinchi ustunning ikkinchi satridan pastda joylashgan elementlarini nolga aylantiramiz;

uchinchi ustunning to‘rtinchi satrida joylashgan elementini nolga aylantiramiz;

$(-1)$ ko‘paytuvchi bilan hosil bo‘lgan uchburchak ko‘rinishdagi determinantning bosh diagonalda joylashgan elementlarini ko‘paytiramiz.

$$\Delta = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{vmatrix} =$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 2 & -2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 8 \end{vmatrix} = (-1)\cdot 1\cdot 1\cdot 1\cdot 8 = -8.$$

Determinantni ikkinchi satr elementlari boyicha yoyib hisoblang:

**33.** $\begin{vmatrix} 3 & 0 & -1 & -1 \\ a & b & c & d \\ 1 & 1 & 1 & 1 \\ -1 & -3 & -2 & -4 \end{vmatrix}$; **34.** $\begin{vmatrix} 3 & 0 & 0 & -1 \\ b & a & c & d \\ 1 & 1 & 1 & 1 \\ -1 & -3 & -2 & -4 \end{vmatrix}$.

Determinantni 3-ustun elementlari bo'yicha yoyib hisoblang:

**35.** $\begin{vmatrix} 5 & 1 & x & 8 \\ -4 & -1 & y & -5 \\ 8 & -1 & z & 12 \\ 4 & -1 & t & 7 \end{vmatrix}$; **36.** $\begin{vmatrix} 1 & -1 & a & -1 \\ -1 & -2 & b & -1 \\ -2 & 0 & c & 1 \\ 0 & 1 & d & 0 \end{vmatrix}$.

Determinantni hisoblang:

**37.** $\begin{vmatrix} 1 & 3 & 0 & 4 \\ 0 & -7 & 1 & -5 \\ 0 & -1 & 1 & -2 \\ 0 & 6 & 2 & 8 \end{vmatrix}$; **42.** $\begin{vmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -4 & 7 \\ -2 & -5 & 7 & 5 \\ -2 & -5 & 2 & 3 \end{vmatrix}$;

**38.** $\begin{vmatrix} 0 & 3 & 0 & 1 \\ 7 & 1 & 2 & -2 \\ 5 & -5 & 0 & 0 \\ -4 & -6 & 0 & -2 \end{vmatrix}$; **43.** $\begin{vmatrix} -1 & 3 & 1 & 2 \\ -5 & 8 & 2 & 7 \\ 4 & -5 & 3 & -2 \\ -7 & 8 & 4 & 5 \end{vmatrix}$;

**39.** $\begin{vmatrix} 0 & 0 & 1 & -1 \\ 3 & 0 & 8 & 0 \\ -2 & -5 & 3 & 4 \\ 3 & 0 & 7 & 3 \end{vmatrix}$; **44.** $\begin{vmatrix} 1 & 5 & 7 & 2 \\ 0 & 6 & 3 & 7 \\ -2 & -8 & -7 & -3 \\ -1 & -6 & -5 & -4 \end{vmatrix}$;

**40.** $\begin{vmatrix} 7 & 0 & 1 & 0 \\ 3 & 3 & 0 & 0 \\ 2 & 10 & -2 & 3 \\ 1 & 6 & -1 & 0 \end{vmatrix}$; **45.** $\begin{vmatrix} 2 & 0 & 3 & 1 \\ -1 & -3 & 1 & 0 \\ 3 & 0 & 4 & 1 \\ 3 & 2 & 2 & 2 \end{vmatrix}$;

**41.** $\begin{vmatrix} 3 & 2 & 0 & 1 \\ -3 & 5 & 0 & 4 \\ 0 & 3 & 0 & 3 \\ 2 & -4 & 2 & 0 \end{vmatrix}$; **46.** $\begin{vmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \end{vmatrix}$.

$a$ ning qanday qiymatlarida tenglik o'rinli bo'ladi?

**47.** $\begin{vmatrix} 0 & 3 & 0 & 1 \\ 7 & 1 & a+3 & -2 \\ 5 & -5 & 0 & 0 \\ -4 & -6 & 0 & -2 \end{vmatrix} = 40$; **48.** $\begin{vmatrix} 0 & 0 & 1 & -1 \\ 3 & 0 & 8 & 0 \\ -2 & a+4 & 3 & 4 \\ 3 & 0 & 7 & 3 \end{vmatrix} = -30$;.

49. $\begin{vmatrix} 7 & 0 & 1 & 0 \\ 3 & 3 & 0 & 0 \\ 2 & 10 & -2 & a-2 \\ 1 & 6 & -1 & 0 \end{vmatrix} = 18$;

50. $\begin{vmatrix} 3 & 2 & 0 & 1 \\ -3 & 5 & 0 & 4 \\ 0 & 3 & 0 & 3 \\ 2 & -4 & 4-a & 0 \end{vmatrix} = -36$.

Quyidagi yuqori tartibli determinantlarni hisoblang:

51. $\begin{vmatrix} 1 & -2 & 3 & 4 \\ 2 & 1 & -4 & 3 \\ 3 & -4 & -1 & -2 \\ 4 & 3 & 2 & -1 \end{vmatrix}$;

54. $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1-a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1-b \end{vmatrix}$;

52. $\begin{vmatrix} -1 & -1 & -1 & -1 \\ -1 & -2 & -4 & -8 \\ -1 & -3 & -9 & -27 \\ -1 & -4 & -16 & -64 \end{vmatrix}$;

55. $\begin{vmatrix} 1 & 2 & 2 & \ldots & 2 & 2 \\ 2 & 2 & 2 & \ldots & 2 & 2 \\ 2 & 2 & 3 & \ldots & 2 & 2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 2 & 2 & 2 & \ldots & n-1 & 2 \\ 2 & 2 & 2 & \ldots & 2 & n \end{vmatrix}$.

53. $\begin{vmatrix} 10 & 2 & 0 & 0 & 0 \\ 12 & 10 & 2 & 0 & 0 \\ 0 & 12 & 10 & 2 & 0 \\ 0 & 0 & 12 & 10 & 2 \\ 0 & 0 & 0 & 12 & 10 \end{vmatrix}$;

**56*.** 1370, 1644, 2055 va 3425 sonlari 137 ga qoldiqsiz bo'linadi. $\begin{vmatrix} 1 & 3 & 7 & 0 \\ 1 & 6 & 4 & 4 \\ 2 & 0 & 5 & 5 \\ 3 & 4 & 2 & 5 \end{vmatrix}$ determinant ham 137 ga qoldiqsiz bo'linishini isbotlang.

**57*.** Determinant xossalaridan foydalanib quyidagi n-tartibli determinantlarni ko'rsatilgan qiymatlarga tengligini isbotlang:

a) $\begin{vmatrix} 1 & 1 & 1 \ldots 1 \\ 1 & 0 & 1 \ldots 1 \\ 1 & 1 & 0 \ldots 1 \\ \ldots & \ldots & \ldots \\ 1 & 1 & 1 \ldots 0 \end{vmatrix} = (-1)^{n-1}$;

b) $\begin{vmatrix} 1 & 2 & 3 & \ldots & n \\ -1 & 0 & 3 & \ldots & n \\ -1 & -2 & 0 & \ldots & n \\ -1 & -2 & -3 & \ldots & n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & -2 & -3 & \ldots & 0 \end{vmatrix} = n!$

## 3 §. Chiziqli tenglamalar sistemasini Kramer qoidasi bilan yechish

Ikkita $x_1$ va $x_2$ noma'lumli chiziqli tenglamadan iborat ushbu

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \qquad (3)$$

sistema **ikki noma'lumli chiziqli tenglamalar sistemasi** deyiladi, bunda $a_{11}, a_{12}, a_{21}, a_{22}$ - (3) sistemaning koeffitsiyentlari, $b_1, b_2$ - ozod hadlardir.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

asosiy determinant,

$$\Delta_{x_1} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad \Delta_{x_2} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

yordamchi determinantlar deb nomlanadi. Agar $\Delta \neq 0$ bo'lsa, (3) tenglamalar sistemasining yechimi quyidagicha topiladi:

$$x_1 = \frac{\Delta_{x_1}}{\Delta}, \quad x_2 = \frac{\Delta_{x_2}}{\Delta}. \qquad (4)$$

Xuddi shuningdek, uchta $x_1, x_2$ va $x_3$ noma'lumli chiziqli tenglamalardan iborat

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \qquad (5)$$

sistema **uch noma'lumli chiziqli tenglamalar sistemasi** deyiladi.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

asosiy determinant,

$$\Delta_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad \Delta_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

yordamchi determinatlar deb nomlanadi, agar $\Delta \neq 0$ bo'lsa, (5) tenglamalar sistemasining yechimi quyidagicha topiladi:

$$x_1 = \frac{\Delta_{x_1}}{\Delta}, \quad x_2 = \frac{\Delta_{x_2}}{\Delta}, \quad x_3 = \frac{\Delta_{x_3}}{\Delta}. \qquad (6)$$

(4) va (6) formulalar (3) va (5) tenglamalar sistemasini yechishning Kramer formulasi deyiladi. $\Delta \neq 0$ bo'lsa, sistema yagona yechimga ega bo'ladi.

$\Delta = 0$ hamda $\Delta x_1$, $\Delta x_2$, $\Delta x_3$ lardan hech bo'lmaganda bittasi noldan farqli bo'lsa, sistemaning yechimi mavjud emas.

$\Delta = 0$ va $\Delta x_1 = \Delta x_2 = \Delta x_3 = 0$ bo'lsa, sistema cheksiz ko'p yechimga ega bo'ladi.

**58.** $\begin{cases} x_1 + 3x_2 = -1 \\ 2x_1 - x_2 = 5 \end{cases}$ chiziqli tenglamalar sistemasi yechilsin.

**Yechish.** Asosiy va yordamchi determinantlarni hisoblaymiz.

$$\Delta = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -1 - 6 = -7,$$

$$\Delta_{x_1} = \begin{vmatrix} -1 & 3 \\ 5 & -1 \end{vmatrix} = 1 - 15 = -14, \quad \Delta_{x_2} = \begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = 5 + 2 = 7,$$

u holda, Kramer formulasiga asosan,

$$x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{-14}{-7} = 2, \quad x_2 = \frac{\Delta_{x_2}}{\Delta} = \frac{7}{-7} = -1.$$

Demak, sistemaning yechimi$(2;-1)$.

**59.** $\begin{cases} x_1 + 2x_2 = 3 \\ 3x_1 + 6x_2 = 1 \end{cases}$ sistemani yeching.

**Yechish.**

$$\Delta = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0, \quad \Delta_{x_1} = \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = 16, \quad \Delta_{x_2} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8.$$

Demak, berilgan sistemaning yechimi mavjud emas.

**60.** Quyidagi sistema yechilsin: $\begin{cases} 2x_1 + 3x_2 = 1 \\ 4x_1 + 6x_2 = 2 \end{cases}$.

**Yechish.** Bu holda asosiy va yordamchi determinantlar nolga teng:

$$\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0, \quad \Delta_{x_1} = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0, \quad \Delta_{x_2} = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0.$$

Demak, ixtiyoriy $\left(t; \frac{1-2t}{3}\right)$ ko'rinishdagi $(t \in R)$ sonlar juftligi sistemaning yechimi bo'ladi, ya'ni cheksiz ko'p yechim mavjud.

**61.** $\begin{cases} 2x_1 - x_2 + x_3 = 4 \\ 3x_1 + 2x_2 - x_3 = 1 \\ x_1 + x_2 - 2x_3 = -3 \end{cases}$ sistema yechilsin.

**Yechish.** Asosiy va yordamchi determinantlarni hisoblaymiz:

$$\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & 1 & -2 \end{vmatrix} = -10, \quad \Delta_{x_1} = \begin{vmatrix} 4 & -1 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & -2 \end{vmatrix} = -10, \quad \Delta_{x_2} = \begin{vmatrix} 2 & 4 & 1 \\ 3 & 1 & -1 \\ 1 & -3 & -2 \end{vmatrix} = 0,$$

$$\Delta_{x_3} = \begin{vmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -3 \end{vmatrix} = -20$$

u holda, Kramer formulasidan

$$x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{-10}{-10} = 1, \quad x_2 = \frac{\Delta_{x_2}}{\Delta} = \frac{0}{-10} = 0, \quad x_3 = \frac{\Delta_{x_3}}{\Delta} = \frac{-20}{-10} = 2.$$

Demak, $(1;0;2)$ sistemaning yechimi bo'ladi.

Chiziqli tenglamalar sistemasini Kramer qoidasi bilan yeching:

**62.** $\begin{cases} x_1 + 2x_2 - x_3 = 3 \\ 2x_1 + 5x_2 - 6x_3 = 1 \\ 3x_1 + 8x_2 - 10x_3 = 1 \end{cases}$;

**63.** $\begin{cases} x_1 - x_2 + 3x_3 = 7 \\ 2x_1 + x_2 - 4x_3 = -3 \\ 3x_1 + x_2 - 3x_3 = 1 \end{cases}$;

**64.** $\begin{cases} 4x_1 + x_2 - x_3 = 1 \\ 5x_1 + 3x_2 - 2x_3 = 2 \\ 3x_1 + 2x_2 - 3x_3 = 0 \end{cases}$;

**65.** $\begin{cases} 5x_1 + 2x_2 + 5x_3 = 4 \\ 3x_1 + 5x_2 - 3x_3 = -1 \\ -2x_1 - 4x_2 + 3x_3 = 1 \end{cases}$;

**66.** $\begin{cases} x_1 + 2x_2 + x_3 = 8 \\ 3x_1 + 2x_2 + x_3 = 10 \\ 4x_1 + 3x_2 - 2x_3 = 4 \end{cases}$;

**67.** $\begin{cases} 4x_1 + 2x_2 + 3x_3 = -2 \\ 3x_1 + 8x_2 - x_3 = 8 \\ 9x_1 + x_2 + 8x_3 = 0 \end{cases}$;

**68.** $\begin{cases} -2x_1 + x_2 - x_3 = 7 \\ 4x_1 + 5x_2 - 3x_3 = -5 \\ x_1 + 3x_2 - 2x_3 = 1 \end{cases}$;

**69.** $\begin{cases} x + y - 3z = -1 \\ 2x - y + z = 2 \\ 3x + 2y - 4z = 1 \end{cases}$;

**70.** $\begin{cases} x_1 + 3x_2 - 4x_3 = -1 \\ x_1 - 5x_2 + x_3 = 7 \\ 2x_1 + x_2 - 3x_3 = 3 \end{cases}$;

**71.** $\begin{cases} 3x_1 + x_2 - x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \\ x_1 - x_2 + 2x_3 = 5 \end{cases}$;

**72.** $\begin{cases} 2x_1 + 2x_2 - x_3 + x_4 = 4 \\ 4x_1 + 3x_2 - x_3 + 2x_4 = 6 \\ 8x_1 + 5x_2 - 3x_3 + 4x_4 = 12 \\ 3x_1 + 3x_2 - 2x_3 + 2x_4 = 6 \end{cases}$;

**73.** $\begin{cases} 2x_1 + 5x_2 + 4x_3 + x_4 = 20 \\ x_1 + 3x_2 + 2x_3 + x_4 = 11 \\ 2x_1 + 10x_2 + 9x_3 + 7x_4 = 40 \\ 3x_1 + 8x_2 + 9x_3 + 2x_4 = 37 \end{cases}$;

**74.** $\begin{cases} 2x - y - 6z + 3t = -1 \\ 7x - 4y + 2z - 15t = -32 \\ x - 2y - 4z + 9t = 5 \\ x - y + 2z - 6t = -8 \end{cases}$;

**75.** $\begin{cases} 2x-4y+3z=1 \\ x-2y+4z=3 \\ 3x-y+5z=2 \end{cases}$; **77.** $\begin{cases} x+2y+3z=5 \\ 2x-y-z=1 \\ x+3y+4z=6 \end{cases}$;

**76.** $\begin{cases} 2x-y+z=2 \\ 3x+2y+2z=-2 \\ x-2y+z=1 \end{cases}$; **78.** $\begin{cases} 2x_1-x_2+x_3=4 \\ 3x_1+2x_2-x_3=1 \\ x_1+x_2-2x_3=-3 \end{cases}$.

## 4 §. Matritsalar. Matritsalar ustida amallar.

Quyidagi

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{7}$$

ko‘rinishdagi jadval $(m \times n)$**-tartibli matritsa** deyiladi.

$a_{ik}$-element xuddi determinantdagi kabi $i$-satr, $k$-ustunga joylashgan bo‘ladi. Ba’zan (7) yozuv, qisqalik uchun, $\|a_{ik}\|$, $\left(i=\overline{1,m},\ k=\overline{1,n}\right)$ ko‘rinishda yoki $A=\|a_{ik}\|$ ko‘rinishda ham belgilanadi. Ravshanki, (7) matritsa $m$ ta satr va $n$ ta ustundan iborat.

Barcha elementlari nolga teng bo‘lgan matritsa **nol matritsa** deyiladi.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Xususiy holda, $m=n$ bo‘lganda,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{8}$$

ko‘rinishidagi matritsa **kvadrat matritsa** deyiladi.

$a_{11}, a_{22}, \dots, a_{nn}$ (8) matritsaning **bosh diagonal elementlari** deyiladi. Agar (8) matritsada bosh diagonalda turgan elementlardan boshqa barcha elementlari nol bo‘lsa, uni **diagonal matritsa** deyiladi:

$$\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}. \qquad (9)$$

(9) matritsada $a_{11} = a_{22} = ..... = a_{nn} = 1$ bo'lsa, yani,

$$E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

**birlik matritsa** deb ataladi.

(8) kvadrat matritsaning elementlaridan tashkil topgan determinant $A$ matritsaning determinanti deyiladi va $\det(A)$ yoki $|A|$ kabi belgilanadi. Shu o'rinda eslatib o'tamiz: matritsa sonlarning tartibli jadvali, determinant esa elementlarning ma'lum kombinatsiyasidan hosil qilingan birgina sondir.

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Agar $\det(A) = 0$ bo'lsa, bu holda $A$ matritsa **xos matritsa**, $\det(A) \neq 0$ bo'lsa, $A$ **xosmas matritsa** deyiladi. Kvadrat (8) matritsaning satr elementlarini mos ustun elementlari bilan almashtirishdan hosil bo'lgan

$$\begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

matritsa **transponirlangan matritsa** deyiladi va $A^T$ kabi belgilanadi. Determinantning 1° xossasiga asosan $\det(A) = \det(A^T)$.

Ikkita

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}, \qquad (10)$$

matritsalar berilgan bo'lib, $A$ matritsaning har bir elementi $B$ matritsaning mos elementiga teng, ya'ni $a_{ik}=b_{ik}$ bo'lsa, u holda $A$ va $B$ o'zaro teng matritsalar deyiladi va $A=B$ kabi yoziladi. Ta'rif bo'yicha ikkita (10) ko'rinishdagi $(m\times n)$ tartibli matritsalarning yig'indisi va ayirmasi mos ravshida $A\pm B$ kabi belgilanib,

$$A\pm B=\begin{pmatrix} a_{11}\pm b_{11} & a_{12}\pm b_{12} & \dots & a_{1n}\pm b_{1n} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} & \dots & a_{2n}\pm b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}\pm b_{n1} & a_{n2}\pm b_{n2} & \dots & a_{nn}\pm b_{nn} \end{pmatrix},$$

qoida bo'yicha hisoblanadi, ya'ni mos elementlari qo'shiladi yoki ayiriladi. Matritsalarni qo'shish quyidagi xossalarga ega:

1°. $A+0=A$;

2°. $A+B=B+A$.

A matritsani $\alpha\neq 0$ songa ko'paytmasi deb, uning har bir elementini α songa ko'paytirishdan hosil bo'lgan

$$\alpha A=\begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha a_{n1} & \alpha a_{n2} & \dots & \alpha a_{nn} \end{pmatrix}$$

matritsaga aytiladi. Matritsani songa ko'paytirish quyidagi xossalarga ega:

3°. $\alpha(\beta A)=(\alpha\beta)A$;

4°. $\alpha(A+B)=\alpha A+\alpha B$;

5°. $(\alpha+\beta)A=\alpha A+\beta A$.

**79.** Agar $A=\begin{pmatrix} 2 & 4 & 1 \\ -1 & 0 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ bo'lsa, $A+B,\ A-B,\ 2A-3B$ matritsalar topilsin.

**Yechish.**

$$A+B=\begin{pmatrix} 2 & 4 & 1 \\ -1 & 0 & 2 \end{pmatrix}+\begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}=\begin{pmatrix} 2+0 & 4+2 & 1+1 \\ -1+1 & 0+1 & 2+2 \end{pmatrix}=\begin{pmatrix} 2 & 6 & 2 \\ 0 & 1 & 4 \end{pmatrix},$$

$$A-B=\begin{pmatrix} 2 & 4 & 1 \\ -1 & 0 & 2 \end{pmatrix}-\begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}=\begin{pmatrix} 2-0 & 4-2 & 1-1 \\ -1-1 & 0-1 & 2-2 \end{pmatrix}=\begin{pmatrix} 2 & 2 & 0 \\ -2 & -1 & 0 \end{pmatrix},$$

$$2A-3B=2\begin{pmatrix}2&4&1\\-1&0&2\end{pmatrix}-3\begin{pmatrix}0&2&1\\1&1&2\end{pmatrix}=\begin{pmatrix}4&8&2\\-2&0&4\end{pmatrix}-\begin{pmatrix}0&6&3\\3&3&6\end{pmatrix}=$$

$$=\begin{pmatrix}4-0&8-6&2-3\\-2-3&0-3&4-6\end{pmatrix}=\begin{pmatrix}4&2&-1\\-5&-3&-2\end{pmatrix}.$$

Endi ikki matritsa ko‘paytmasi tushunchasini kiritamiz. Bunda ko‘paytiriladigan matritsalar birinchisining ustunlar soni ikkinchisining satrlar soniga teng bo‘lishi talab qilinadi.

$(m\times n)$ tartibli $A$ matritsaning $(n\times k)$ tartibli $B$ matritsaga ko‘paytmasi deb $(m\times k)$ tartibli shunday $C$ matritsaga aytiladiki, uning $c_{ij}$ elementi $A$ matritsa $i$-satri elementlarini $B$ matritsa $j$-ustinining mos elementlariga ko‘paytmalari yig‘indisiga teng, ya’ni

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+...+a_{ik}b_{kj}\ .$$

matritsalar ko‘paytmasi $C=A\cdot B$, ko‘rinishda belgilanadi.

**80.** Matritsalar ko‘paytmasi topilsin:

$$A=\begin{pmatrix}1&-1\\2&1\\1&1\end{pmatrix},\ B=\begin{pmatrix}2&1\\1&1\end{pmatrix}.$$

**Yechish.** $A\cdot B$ mavjud, chunki $A$ matritsa ikkita ustundan $B$ matritsa esa ikkita satrdan iborat:

$$A\cdot B=\begin{pmatrix}1&-1\\2&1\\1&1\end{pmatrix}\cdot\begin{pmatrix}2&1\\1&1\end{pmatrix}=\begin{pmatrix}1\cdot2-1\cdot1&1\cdot1-1\cdot1\\2\cdot2+1\cdot1&2\cdot1+1\cdot1\\1\cdot2+1\cdot1&1\cdot1+1\cdot1\end{pmatrix}=\begin{pmatrix}1&0\\5&3\\3&2\end{pmatrix}.$$

$BA$ ko‘paytma mavjud emas, chunki $B$ matritsada 2 ta ustun, $A$ matritsada esa 3 ta satr mavjud.

Agar $A$ va $B$ matritsalar bir xil tartibli kvadrat matritsalar bo‘lsa $A\cdot B$ va $B\cdot A$ ni hisoblash mumkin.

**81.** $A=\begin{pmatrix}2&-3\\5&1\end{pmatrix}$, $B=\begin{pmatrix}4&-3\\2&6\end{pmatrix}$ bo‘lsa, $A\cdot B$ va $B\cdot A$ topilsin.

**Yechish.**

$$A\cdot B=\begin{pmatrix}2&-3\\5&1\end{pmatrix}\cdot\begin{pmatrix}4&-3\\2&6\end{pmatrix}=\begin{pmatrix}2\cdot4-3\cdot2&-2\cdot3-3\cdot6\\5\cdot4+1\cdot2&-5\cdot3+1\cdot6\end{pmatrix}=\begin{pmatrix}2&-24\\22&-9\end{pmatrix},$$

$$B\cdot A=\begin{pmatrix}4&-3\\2&6\end{pmatrix}\cdot\begin{pmatrix}2&-3\\5&1\end{pmatrix}=\begin{pmatrix}4\cdot2-3\cdot5&-4\cdot3-3\cdot1\\2\cdot2+6\cdot5&-2\cdot3+6\cdot1\end{pmatrix}=\begin{pmatrix}-7&-15\\34&-0\end{pmatrix}.$$

Bu misoldan ko‘rinadiki, umuman olganda, $A\cdot B\neq B\cdot A$.

Matritsalarni ko'paytirish quyidagi xossalarga ega:

1°. $(A\cdot B)\cdot C = A\cdot(B\cdot C)$;

2°. $(A+B)\cdot C = A\cdot C + B\cdot C$;

3°. $(\lambda\cdot A)\cdot B = \lambda(A\cdot B)$;

4°. $A\cdot E = E\cdot A = A$;

5°. $A\cdot 0 = 0\cdot A = 0$. (Bunda 0-matritsa).

**82.** Agar $A=\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ bo'lsa, $f(t)=t^2-2t+3$ ni hisoblang.

**Yechish.**

$$f(A)=\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}-2\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}+3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Berilgan matritsalar ustida amallarni bajaring:

**83.** $A=\begin{pmatrix} 1 & 7 & 2 \\ -3 & 4 & -2 \\ 1 & 1 & 2 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 4 & 0 \\ 2 & 3 & 1 \\ -1 & 0 & 4 \end{pmatrix}$, $C=2A-3B$;

**84.** $A=\begin{pmatrix} 3 & -2 & 1 & 6 \\ 8 & 3 & 4 & 1 \\ -5 & 7 & 0 & 4 \end{pmatrix}$, $B=\begin{pmatrix} -2 & 1 & 9 & 0 \\ 2 & 6 & 4 & 1 \\ -3 & 4 & 5 & 2 \end{pmatrix}$, $C=A-2B$;

**85.** $A=\begin{pmatrix} 1 & 5 \\ 2 & -4 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$, $2A-B=?$

**86.** A=$\begin{pmatrix} 1 & -1 & -3 \\ 2 & 1 & 5 \end{pmatrix}$, B=$\begin{pmatrix} 0 & 3 & 2 \\ -1 & 4 & 1 \end{pmatrix}$, $3A-2B=?$

**87.** $A=\begin{pmatrix} 3 & 5 & 7 \\ 2 & -1 & 0 \\ 4 & 3 & 2 \end{pmatrix}$, $A=\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -2 \\ -1 & 0 & 1 \end{pmatrix}$, $A+B=?$

**88.** $\begin{pmatrix} 7 & 0 \\ 3 & 1 \\ -1 & 2 \end{pmatrix}-3\begin{pmatrix} 2 & \sqrt{2} \\ 1 & -1 \\ -1 & 0 \end{pmatrix}+\begin{pmatrix} 1 & \sqrt{18} \\ 4 & -5 \\ 3 & 1 \end{pmatrix}$;

**89.** A=$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, B=$\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, $AB=?$

**90.** A=$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \\ -4 & 5 & 1 \end{pmatrix}$, B=$\begin{pmatrix} 3 & 4 & 1 \\ 0 & 2 & 5 \\ 1 & -1 & 4 \end{pmatrix}$, $AB=?$

A va B matritsalarni ko'paytiring:

**91.** $A=\begin{pmatrix}1 & 2 & 3\end{pmatrix}$, $B=\begin{pmatrix}4 & -3\\ 1 & 2\\ 0 & 2\end{pmatrix}$;

**92.** $A=\begin{pmatrix}1 & 5\\ 3 & -2\end{pmatrix}$, $B=\begin{pmatrix}-3 & 1\\ 5 & 2\end{pmatrix}$;

**93.** $A=\begin{pmatrix}2 & -1\\ 7 & 3\end{pmatrix}$, $B=\begin{pmatrix}4 & 0 & -3\\ -1 & 5 & 3\end{pmatrix}$.

**94.** $A=\begin{pmatrix}1 & 8\\ 3 & -2\\ 2 & 4\end{pmatrix}$, $B=\begin{pmatrix}6 & -2\\ 1 & 3\end{pmatrix}$;

**95.** $A=\begin{pmatrix}0 & 0 & -2\\ 1 & -2 & 4\\ 2 & -2 & 5\end{pmatrix}$, $B=\begin{pmatrix}5 & 4\\ 2 & 6\\ 3 & 1\end{pmatrix}$;

**96.** $A=\begin{pmatrix}1 & 3 & 0\\ 2 & -1 & 3\end{pmatrix}$, $B=\begin{pmatrix}2 & 0\\ -1 & 5\\ 4 & 6\end{pmatrix}$;

**97.** $A=\begin{pmatrix}-2 & 4 & 1\\ 3 & -1 & 7\end{pmatrix}$, $B=\begin{pmatrix}1 & 2 & -2\\ 3 & -2 & 1\\ 1 & -2 & 2\end{pmatrix}$;

**98.** $A=\begin{pmatrix}2 & 3 & -2\\ -1 & -1 & 3\\ 0 & -2 & 4\end{pmatrix}$, $B=\begin{pmatrix}1 & 3 & 0\\ 2 & -1 & 3\\ 1 & -2 & 3\end{pmatrix}$;

**99.** $A=\begin{pmatrix}1 & 2 & -2\end{pmatrix}$, $B=\begin{pmatrix}2\\ -1\\ 4\end{pmatrix}$;

**100.** $A=\begin{pmatrix}2\\ -1\\ 4\end{pmatrix}$, $B=\begin{pmatrix}1 & 2 & -3\end{pmatrix}$;

**101.** $A=\begin{pmatrix}5 & 2 & 4\\ 1 & 1 & -3\\ 1 & 0 & 3\end{pmatrix}$, $B=\begin{pmatrix}3\\ -2\\ 5\end{pmatrix}$;

**102.** $A=\begin{pmatrix}1 & 2 & -5\end{pmatrix}$, $B=\begin{pmatrix}5 & -2 & 2\\ 7 & 0 & 1\\ 5 & 3 & 1\end{pmatrix}$;

**103.** $A=\begin{pmatrix}2 & 0 & 1\\ -2 & 3 & 2\\ 4 & -1 & 5\end{pmatrix}$, $B=\begin{pmatrix}-3 & 1 & 0\\ 0 & 2 & 1\\ 0 & -1 & 3\end{pmatrix}$

**104.** $A=\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & 5\end{pmatrix}$, $B=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$;

**105.** $A=\begin{pmatrix}1 & 3 & -1\\ 2 & 1 & 2\\ 0 & 1 & 0\end{pmatrix}$, $B=\begin{pmatrix}1 & 1\\ 2 & 3\\ 1 & 0\end{pmatrix}$;

**106.** $A=\begin{pmatrix}1 & -3 & 0\\ 2 & 5 & 1\end{pmatrix}$, $B=\begin{pmatrix}0 & -1 & 3\\ 3 & 5 & 2\\ 4 & -2 & 1\end{pmatrix}$;

**107.** $A=\begin{pmatrix}1 & 3 & 1\\ 2 & 0 & 4\\ 1 & 2 & 3\end{pmatrix}$, $B=\begin{pmatrix}2 & 1 & 0\\ 1 & -1 & 2\\ 3 & 2 & 1\end{pmatrix}$, $AB=?\ BA=?$

A va B matritsalar berilgan. Quyidagi tenglamani qanoatlantiruvchi X matritsani toping:

**108.** $A+2X-4B=0$, $A=\begin{pmatrix}2 & -4 & 0\\ 6 & -2 & 4\\ 0 & 8 & 2\end{pmatrix}$, $B=\begin{pmatrix}1 & 3 & 7\\ -2 & 0 & 5\\ 4 & 5 & 3\end{pmatrix}$;

**109.** $5A+3X-B=0$, $A=\begin{pmatrix}7 & 2 & 1 & 5\\ 3 & -2 & 4 & -3\\ 2 & 1 & 1 & 1\end{pmatrix}$, $B=\begin{pmatrix}5 & 4 & 3 & 0\\ 2 & 3 & -2 & 1\\ 1 & 0 & 2 & 4\end{pmatrix}$.

A matritsaning berilgan n-darajasini hisoblang:

**110.** $A=\begin{pmatrix}1 & -2\\ 3 & -4\end{pmatrix}$, $n=3$;

**111.** $A=\begin{pmatrix}4 & -1\\ 5 & -2\end{pmatrix}$, $n=5$;

**112.** $A=\begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}$, $n=10,\ n=15$;

**113.** $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $n=10$;

**114.** $A=\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$, $n\in N$;

**115.** A=$\begin{pmatrix}3 & 2\\ 1 & 4\end{pmatrix}$, $A^2=?$

Matritsalar ustida amallarni bajaring:

**116.** A=$\begin{pmatrix}1 & 1 & 2\\ 1 & 3 & 1\\ 4 & 1 & 1\end{pmatrix}$, E-birlik matrisa $2A^2+3A+5E=?$

**117.** A=$\begin{pmatrix}3 & 4 & 2\\ 1 & 0 & 5\end{pmatrix}$, B=$\begin{pmatrix}2 & 0\\ 1 & 3\\ 0 & 5\end{pmatrix}$, C=$\begin{pmatrix}1 & 3\\ 0 & 4\end{pmatrix}$ $AB-C^2=?$

**118.** A=$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 0 & 2 \\ 4 & 5 & 3 \end{pmatrix}$, B=$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, C=(2 0 5),

E-birlik matritsa $A\times B\times C\text{-}3E=?$

**119.** $A=\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 5 & 7 \\ -1 & 2 \end{pmatrix}$, $A^2-AB+2BA=?$

**120.** Quyidagi matritsalar berilgan: A=$\begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix}$, B=$\begin{pmatrix} 8 & 1 \\ 2 & 3 \end{pmatrix}$, C=$\begin{pmatrix} 2 & -7 \\ -8 & 1 \end{pmatrix}$.

Agar $\lambda_1=2$, $\lambda_2=-1$ $\lambda_3=1$ bo'lsa, D=$\lambda_1$A+$\lambda_2$B+$\lambda_3$C matritsani toping.

Berilganlar bo'yicha $f(A)$ ni toping:

**121.** $f(x)=x^2-3x,\ A=\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$;

**122.** $f(x)=x^2-5x+3,$ A=$\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}$;

**123.** $f(x)=x^2-x+1,$ A=$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$;

**124.** $f(x)=x^2-2x+3,$ A=$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$;

**125.** $f(x)=x^3-2x^2+x+4,$ A=$\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$;

**126.** $f(x)=3x^2-4x+1,$ A=$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$;

**127.** $f(x)=x^4-2x^2+3x-5,$ A=$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

**128.** Agar A=$\begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$, $f(x)=x^3-x^2+5x+4,$ $g(x)=x^2-2x+11$ bo'lsa, $2f(A)-3g(A)$ ni toping.

**129.** Agar B=$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$, $f(x)=x^2-2x+1,$ $g(x)=3x+5$ bo'lsa, $f(B)-2g(B)$ ni toping.

**130.** Agar A= $\begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 3 \\ 0 & -1 & -1 \end{pmatrix}$, $f(x)=x+1$ bo'lsa, $(f(A))^2$ ni toping.

B matritsa A va X matritsalarning ko'paytmasidan iborat bo'lsa ($B=AX$ yoki $B=XA$ ), X matritsani aniqlang:

**131*.** $A=\begin{pmatrix} 342 & 211 & 645 \\ 457 & 992 & 719 \\ 123 & 403 & 842 \end{pmatrix}$, $B=\begin{pmatrix} 645 & 211 & 342 \\ 719 & 992 & 457 \\ 842 & 403 & 123 \end{pmatrix}$;

**132*.** $A=\begin{pmatrix} 342 & 211 & 645 \\ 457 & 992 & 719 \\ 123 & 403 & 842 \end{pmatrix}$, $B=\begin{pmatrix} 457 & 992 & 719 \\ 342 & 211 & 645 \\ 123 & 403 & 842 \end{pmatrix}$;

**133*.** $A=\begin{pmatrix} 332 & 211 & 123 \\ 457 & 992 & 719 \\ 123 & 403 & 842 \end{pmatrix}$, $B=\begin{pmatrix} 996 & 633 & 369 \\ 457 & 992 & 719 \\ 123 & 403 & 842 \end{pmatrix}$;

**134*.** $A=\begin{pmatrix} 111 & 203 & 343 \\ 209 & 121 & 514 \\ 221 & 106 & 678 \end{pmatrix}$, $B=\begin{pmatrix} 333 & 812 & 343 \\ 627 & 484 & 514 \\ 663 & 424 & 678 \end{pmatrix}$.

## 5 §. Teskari matritsa

Ushbu kvadrat matritsani qaraylik:

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Agar $A\cdot B=B\cdot A=E$ bo'lsa, $B$ matritsa $A$ matritsaga teskari matritsa deb ataladi. $A$ matritsaga teskari matritsani $A^{-1}$ kabi belgilash qabul qilingan. $A\cdot B$ kvadrat matritsaga teskari $A^{-1}$ matritsani topish quyidagicha amalga oshiraladi:

1. $\det(A)$ hisoblanadi. Bu o'rinda $\Delta=\det(A)\neq 0$ bo'lishi kerakligini eslatib o'tamiz, aks holda teskari matritsa mavjud bo'lmaydi.

2. $A$ matritsa determinantining har bir $a_{ij}$, $(i,j=1,2,3,\dots,n)$ elementi algebraik to'ldiruvchisi $A_{ij}$ ni hisoblaymiz va $A^{-1}$ matritsani quyidagicha tuzamiz:

$$A^{-1}=\frac{1}{\det(A)}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}. \qquad (11)$$

Ishonch hosil qilish uchun $A\cdot A^{-1}=A^{-1}\cdot A=E$ ni tekshirib ko'rish yetarli.

**135.** $A=\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}$ matritsaga teskari matritsa topilsin.

**Yechish.**

$$\Delta=\begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & 0 \\ -1 & 2 & 4 \end{vmatrix}=-10\neq 0\,.$$

Demak, teskari matritsa $A^{-1}$ mavjud. Matritsa determinantining barcha elementlari algebraik to'ldiruvchilarini hisoblaymiz:

$$A_{11}=\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix}=4,\quad A_{12}=-\begin{vmatrix} 3 & 0 \\ -1 & 4 \end{vmatrix}=-12,\quad A_{13}=\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}=7,$$

$$A_{21}=-\begin{vmatrix} 0 & -2 \\ 2 & 4 \end{vmatrix}=-4,\quad A_{22}=\begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix}=2,\quad A_{23}=-\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}=-2,$$

$$A_{31}=\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix}=2,\quad A_{32}=-\begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix}=-6,\quad A_{33}=\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}=1,$$

topilganlarni (11) ga qo'ysak:

$$A^{-1}=\frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}=-\frac{1}{10}\begin{pmatrix} 4 & -4 & 2 \\ -12 & 2 & -6 \\ 7 & -2 & 1 \end{pmatrix}.$$

Tekshirish:

$$A^{-1}\cdot A=-\frac{1}{10}\begin{pmatrix} 4 & -4 & 2 \\ -12 & 2 & -6 \\ 7 & -2 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}=$$

$$\frac{-1}{10}\begin{pmatrix} 4-12-2 & 0-4+4 & -8+0+8 \\ -12+6+6 & 0+2-12 & 24+0-24 \\ 7-6-1 & 0-2+2 & -14+0+4 \end{pmatrix}=\frac{-1}{10}\begin{pmatrix} -10 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -10 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Demak, $A^{-1}$ matritsa, $A$ matritsaga teskari matritsa ekanligi kelib chiqdi.

A matritsaga teskari matritsani toping:

**136.** $A=\begin{pmatrix} 3 & 6 \\ 4 & 9 \end{pmatrix}$;

**137.** $A=\begin{pmatrix} 7 & 3 \\ 4 & 2 \end{pmatrix}$;

**138.** $A=\begin{pmatrix} 4 & 3 \\ 6 & 5 \end{pmatrix}$;

**139.** $A=\begin{pmatrix} 3 & -4 \\ 5 & -8 \end{pmatrix}$;

**140.** $A=\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$;

**141.** $A=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{pmatrix}$;

**142.** $A=\begin{pmatrix} 4 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$;

**143.** $A=\begin{pmatrix} 4 & -1 & 2 \\ 1 & 1 & -2 \\ 0 & -1 & 3 \end{pmatrix}$;

**144.** $A=\begin{pmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ 6 & -2 & 1 \end{pmatrix}$;

**145.** $A=\begin{pmatrix} 3 & 1 & 3 \\ 5 & -2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$.

Berilgan matritsaga teskari matritsani toping va ko‘paytirish bilan tekshiring.

**146.** $\begin{pmatrix} -3 & 5 \\ -1 & 2 \end{pmatrix}$

**147.** $\begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix}$

**148.** $\begin{pmatrix} 15 & -5 \\ 3 & -1 \end{pmatrix}$

**149.** $\begin{pmatrix} -5 & 4 \\ 0 & 3 \end{pmatrix}$

**150.** $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}$

**151.** $\begin{pmatrix} 3 & 1 & 1 \\ 1 & -2 & 2 \\ 4 & -3 & -1 \end{pmatrix}$

**152.** $\begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix}$

**153.** $\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{pmatrix}$

**154.** $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

**155.** $\begin{pmatrix} 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{6} & 0 & 0 \\ \frac{1}{12} & 0 & 0 & 0 \end{pmatrix}$

**156.** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$

**157.** $\begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

a, b va c parametrlarning qanday qiymatlarida A va B matritsalar o'zaro teskari matritsalar bo'ladi?

**158*.** $A=\begin{pmatrix} a-1 & -2 & 3 \\ 0 & -1 & c-2 \\ 4 & b & -3 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 0 & 1 \\ -8 & 3 & -6 \\ -4 & 2 & -3 \end{pmatrix}$;

**159*.** $A=\begin{pmatrix} a-3 & 3 & 5 \\ 0 & c & 3 \\ -5 & -1 & b-4 \end{pmatrix}$, $B=\begin{pmatrix} 1 & -2 & -1 \\ -15 & 29 & 12 \\ 10 & -19 & -8 \end{pmatrix}$;

**160*.** $A=\begin{pmatrix} a-2 & 0 & 1 \\ -8 & b+4 & -6 \\ -4 & 2 & c \end{pmatrix}$, $B=\begin{pmatrix} -3 & -2 & 3 \\ 0 & -1 & 2 \\ 4 & 2 & -3 \end{pmatrix}$;

**161*.** $A=\begin{pmatrix} a & -2 & -1 \\ -15 & b+20 & 12 \\ 10 & -19 & 2c \end{pmatrix}$, $B=\begin{pmatrix} -4 & 3 & 5 \\ 0 & 2 & 3 \\ -5 & -1 & -1 \end{pmatrix}$;

**162*.** $A=\begin{pmatrix} 2 & -3 & 1 \\ 4 & a & 2 \\ 5 & -7 & c \end{pmatrix}$, $B=\begin{pmatrix} -1 & 2 & -1 \\ b & 1 & 0 \\ -3 & -1 & 2 \end{pmatrix}$;

**163.*** $A=\begin{pmatrix} 1 & 0 & -4 \\ 3 & 1 & 1 \\ -1 & 0 & a \end{pmatrix}$, $B=\begin{pmatrix} 5 & 0 & 4 \\ b-20 & 1 & c-10 \\ 1 & 0 & 1 \end{pmatrix}$.

a ning qanday qiymatlarida berilgan matritsaga teskari matritsa mavjudligini aniqlang:

**164.** $\begin{pmatrix} 2 & 4 & -1 \\ a-2 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$; **165*.** $\begin{pmatrix} 2-a & 1 & 1 \\ 1 & 2-a & 1 \\ 1 & 1 & 2-a \end{pmatrix}$;

**166.** $\begin{pmatrix} -1 & 2 & a \\ a-1 & 5 & a \\ a & 3 & 0 \end{pmatrix}$; **167.** $\begin{pmatrix} a & 0 & a^2-1 \\ 1 & 0 & a \\ 3 & 2 & 4 \end{pmatrix}$.

## 6 §. Chiziqli tenglamalar sistemasini yechishning matritsa usuli

Bizga quyidagi uch noma'lumli uchta tenglamalar sistemasi berilgan bo'lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \quad (12)$$

Tenglamalar sistemasi koeffitsiyentlari $a_{ij}$, $(i,j=1,2,3)$, $x_1, x_2, x_3$ - noma'lumlar va $b_i$, $(i=1,2,3)$ ozod hadlardan

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

matritsalarni tuzamiz.

Matritsalarni ko'paytirish amaliga asosan, (12) sistemani, quyidagicha yozish mumkin

$$A\cdot X=B. \tag{13}$$

Bu tenglamalar sistemasining matritsalar ko'rinishida yozilishidir. Aytaylik, $A$ matritsaga teskari $A^{-1}$ matritsa mavjud bo'lsin. (13) tenglikning har ikki tomonini $A^{-1}$ ga chapdan ko'paytirib,

$A^{-1}\cdot A\cdot X=A^{-1}\cdot B$ ni hosil qilamiz.

$A^{-1}\cdot A=E$ va $E\cdot X=X$ ekanini e'tiborga olsak,

$$X=A^{-1}\cdot B \tag{14}$$

hosil bo'ladi.

(14) formula (12) tenglamalar sistemasini teskari matritsa yordamida yechish formulasidir.

**168.** Kramer usuli bilan yechilgan 61-misoldagi tenglamalar sistemasini teskari matritsa usulida yechilsin.

**Yechish.**

$$A=\begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix}, \quad X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

$$\Delta=\det(A)=-10\neq 0.$$

$$A_{11}=\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}=-3, \quad A_{12}=-\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}=5, \quad A_{13}=\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}=1,$$

$$A_{21}=-\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}=-1, \quad A_{22}=\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}=-5, \quad A_{23}=-\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}=-3,$$

$$A_{31}=\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}=-1, \quad A_{32}=-\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}=5, \quad A_{33}=\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}=7,$$

(1) formulaga asosan

$$A^{-1}=-0{,}1\begin{pmatrix} -3 & -1 & -1 \\ 5 & -5 & 5 \\ 1 & -3 & 7 \end{pmatrix}$$

teskari matritsani topamiz. Bundan (14) formulaga binoan

$$X = A^{-1}\cdot B = -0{,}1\cdot\begin{pmatrix}-3 & -1 & -1\\ 5 & -5 & 5\\ 1 & -3 & 7\end{pmatrix}\cdot\begin{pmatrix}4\\ 1\\ -3\end{pmatrix} = -0{,}1\cdot\begin{pmatrix}-12-1+3\\ 20-5-15\\ 4-3-21\end{pmatrix} = -0{,}1\begin{pmatrix}-10\\ 0\\ -20\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 2\end{pmatrix}$$

Javob (1; 0; 2)

Chiziqli tenglamalar sistemasini matritsa usulida yeching:

**169.** $\begin{cases} x_1 - x_2 + x_3 = -2\\ 2x_1 + x_2 - 2x_3 = 6\\ x_1 + 2x_2 + 3x_3 = 2\end{cases}$;

**170.** $\begin{cases} 3x_1 + x_2 + x_3 = 2\\ x_1 - 2x_2 + 2x_3 = -1\\ 4x_1 - 3x_2 - x_3 = 5\end{cases}$;

**171.** $\begin{cases} 3x_1 + x_2 + x_3 = 15\\ 5x_1 + x_2 + 2x_3 = 20\\ 6x_1 - x_2 + x_3 = 10\end{cases}$;

**172.** $\begin{cases} 3x_1 + 4x_2 - x_3 = 2\\ 3x_1 - 2x_2 + x_3 = 24\\ 2x_1 - x_2 + x_3 = 8\end{cases}$;

**173.** $\begin{cases} x_1 - 2x_2 + x_3 = 0\\ 2x_1 - x_2 - 1 = 0\\ 3x_1 + 2x_2 - x_3 - 4 = 0\end{cases}$;

**174.** $\begin{cases} x_1 - 2x_2 + 2x_3 = -1\\ 3x_1 + x_2 + x_3 = 2\\ 4x_1 - 3x_2 - x_3 = 5\end{cases}$;

**175.** $\begin{cases} x + y - 3z = -1\\ 2x - y + z = 2\\ 3x + 2y - 4z = 1\end{cases}$;

**176.** $\begin{cases} x_1 + 3x_2 - 4x_3 = -1\\ x_1 - 5x_2 + x_3 = 7\\ 2x_1 + x_2 - 3x_3 = 3\end{cases}$;

**177.** $\begin{cases} 3x_1 + x_2 - x_3 = 2\\ 2x_1 - 3x_2 + x_3 = -1\\ x_1 - x_2 + 2x_3 = 5\end{cases}$;

**178*.** $\begin{cases} 2x_1 + 2x_2 - x_3 + x_4 = 4\\ 4x_1 + 3x_2 - x_3 + 2x_4 = 6\\ 8x_1 + 5x_2 - 3x_3 + 4x_4 = 12\\ 3x_1 + 3x_2 - 2x_3 + 2x_4 = 6\end{cases}$.

## 7 §. Matritsaning rangi

Biror $(m\times n)$-tartibli $A=\|a_{mn}\|$ matritsa berilgan bo'lsin. $A$ matritsaning ixtiyoriy $k$ ta satrini va ixtiyoriy $k$ ta ustunini olib $(k\le \min(m,n))$, $(k\times k)-$ tartibli kvadrat matritsa tuzamiz. Bu kvadrat matritsaning determinanti $A$ matritsaning $k$- tartibli **minori** deyiladi.

**179.** Quyidagi 4x5-tartibli

$$\begin{pmatrix}2 & -4 & 3 & 1 & 0\\ 1 & -2 & 1 & -4 & 2\\ 0 & 1 & 1 & 3 & 1\\ 4 & -7 & 4 & -4 & 5\end{pmatrix}$$

matritsani qaraylik. Ushbu

$$\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = 0, \quad \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -3, \quad \begin{vmatrix} -4 & 3 & 0 \\ 1 & 1 & 1 \\ -7 & 4 & 5 \end{vmatrix} = -40, \quad \begin{vmatrix} 2 & -4 & 1 & 0 \\ 1 & -2 & -4 & 2 \\ 0 & 1 & 3 & 1 \\ 4 & -7 & -4 & 5 \end{vmatrix} = 0.$$

determinantlar qaralayotgan matritsaning mos ravishda ikkinchi, uchinchi hamda to'rtinchi tartibli minorlaridir.

$A$ matritsa yordamida hosil qilish mumkin bo'lgan barcha minorlar orasida noldan farqli bo'lgan eng yuqori tartibli minorni topish muhimdir.

Shuni ta'kidlash kerakki, agar $A$ matritsaning barcha $k$-tartibli $(k = \min(m,n))$ minorlari nolga teng bo'lsa, undan yuqori tartibli bo'lgan barcha minorlari ham nolga teng bo'ladi.

**Ta'rif.** $A$ matritsaning noldan farqli minorlarining eng yuqori tartibi uning **rangi** deyiladi va $rang\, A$ kabi belgilanadi.

**180.** $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$ matritsaning rangini toping.

**Yechish.**

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix} = 0.$$

Demak, $A$ matritsaning noldan farqli minorlarining eng katta tartibi 2 ga teng ekan. Bundan $rang\, A = 2$. Ta'rif bo'yicha 0 matritsaning rangi 0 deb olinadi.

Matritsalarning rangini topish ko'p hollarda murakkab bo'ladi. Chunki unda bir qancha turli tartibdagi determinantlarni hisoblashga to'g'ri keladi.

Quyidagi elementar almashtirishlar natijasida matritsaning rangi o'zgarmaydi:

1) ikki qator elementlarini o'zaro almashtirish;

2) biror qatorni o'zgarmas songa ko'paytirish;

3) biror qatorga boshqa qatorni o'zgarmas songa ko'paytirib qo'shish;

Bu tasdiqlar determinantlarning xossalaridan kelib chiqadi.

Agar $(m \times n)$- tartibli $A$ matritsaning $a_{11}, a_{22}, a_{33}, \dots, a_{ss}$ $(0 \le s \le \min(m.n))$ elementlarining har biri noldan farqli bo'lib, qolgan barcha elementlari nolga teng bo'lsa, u holda $A$ diagonal ko'rinishdagi matritsa deyiladi. Ravshanki, bunday diagonal ko'rinishdagi matritsaning rangi $s$ ga teng bo'ladi. Aynan shu va yuqorida aytilgan elementar almashtirishlardan foydalanib, matritsani rangini topishning ikkinchi usulini bayon qilamiz.

Bizga quyidagi $A$ matritsa berilgan bo'lsin:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

va uning rangini topish talab etilsin. Bu matritsaning rangini yuqorida aytilgan elementar almashtirishlar yordamida diagonal ko'rinishli matritsaga keltirib topamiz.

$A$ matritsaning hech bo'lmaganda bitta elementi noldan farqli bo'lsin. Bu elementning satrlari va ustunlarini o'zaro almashtirish yordamida birinchi satr birinchi ustunga chiqaramiz va birinchi satr elementlarini shu elementga bo'lib, ushbu

$$\begin{pmatrix} 1 & a'_{12} & \dots & a'_{1n} \\ a'_{21} & a'_{22} & \dots & a'_{2n} \\ \dots & \dots & \dots & \dots \\ a'_{m1} & a'_{m2} & \dots & a'_{mn} \end{pmatrix} \tag{15}$$

matritsani hosil qilamiz. (1) matritsaning birinchi ustunini $-a'_{12}$ ga ko'paytirib, ikkinchi ustunga qo'shsak, so'ng $-a'_{13}$ ga ko'paytirib uchinchi ustunga qo'shsak va h. k. Birinchi ustunni $-a'_{1n}$ ga ko'paytirib n-ustunga qo'shsak, natijada hosil bo'lgan matritsaning birinchi satrdagi elementi $a'_{11} = 1$, qolgan elemetlari esa nollar bo'lib qoladi.

Xuddi shunga o'xshash (15) matritsaning birinchi ustundagi $a'_{11} = 1$ dan boshqa elementlari nolga aylantiriladi. Natijada

$$A_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & a'_{22} & \dots & a'_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & a'_{m2} & \dots & a'_{mn} \end{pmatrix}$$

matritsaga ega bo'lamiz. Bundan $rang\,A = rang\,A_1$ kelib chiqadi.

$A_1$ matritsaga yuqoridagi elementar almashtirishlarni bir necha marta qo'llash natijasida u diagonal ko'rinishidagi matritsaga keladi. Bu diagonal ko'rinishli matritsaning rangi berilgan $A$ matritsaning rangi bo'ladi.

Matritsalarning rangini aniqlang:

**181.** $\begin{pmatrix} 2 & 5 & 1 \\ 3 & 8 & 2 \\ 1 & 2 & 0 \end{pmatrix}$;

**182.** $\begin{pmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 9 & 3 & 6 \end{pmatrix}$;

**183.** $\begin{pmatrix} 2 & 1 & 4 & -3 & 7 \\ 4 & 15 & 8 & 7 & 1 \\ 2 & 17 & 4 & 13 & -9 \end{pmatrix}$;

**184.** $\begin{pmatrix} 3 & 1 & 1 & 0 & -2 \\ 1 & 5 & 0 & 2 & -1 \\ 0 & 1 & 3 & 3 & -1 \end{pmatrix}$;

**185.** $\begin{pmatrix} 3 & 4 & 3 \\ 1 & 3 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$;

**186.** $\begin{pmatrix} 2 & 1 & -3 \\ 1 & 5 & -2 \\ 4 & 11 & -7 \\ 1 & -1 & -1 \end{pmatrix}$;

**187*.** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 6 & -3 & 3 \end{pmatrix}$;

**188.** $\begin{pmatrix} 1 & 0 & 0 & -5 \\ -1 & -3 & 0 & 1 \\ -2 & -9 & 4 & -2 \\ 5 & 18 & -8 & -1 \end{pmatrix}$;

**189.** $\begin{pmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -3 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{pmatrix}$;

**190.** $\begin{pmatrix} -1 & 1 & 1 & -2 \\ 1 & 1 & 3 & 0 \\ 2 & -1 & 0 & 3 \\ 3 & 1 & 5 & 2 \end{pmatrix}$.

Matritsalar rangini ko'rsatuvchi eng yuqori tartibli minorni aniqlang:

**191.** $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$;

**192.** $\begin{pmatrix} 1 & 2 \\ 2 & 5 \\ -1 & 3 \end{pmatrix}$;

**193.** $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$;

**194.** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$;

**195.** $\begin{pmatrix} -2 & -1 & 3 \\ 4 & 2 & -6 \\ 2 & 1 & -3 \end{pmatrix}$;

196. $\begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix}$;

198. $\begin{pmatrix} 1 & -2 & 4 & 0 \\ -1 & 3 & 5 & 1 \\ 2 & -1 & 4 & 0 \end{pmatrix}$.

197. $\begin{pmatrix} 4 & 2 & 0 \\ 20 & 10 & -40 \\ 10 & -30 & 40 \end{pmatrix}$;

Matritsalar rangini eng yuqori tartibli minor yordamida aniqlang:

199. $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 5 & 1 & 6 & 11 \\ 4 & 3 & 7 & 11 \end{pmatrix}$;

200. $\begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & 0 & 1 & 3 \\ 3 & -1 & 2 & 2 \end{pmatrix}$;

201*. $\begin{pmatrix} 1 & 2 & -3 & 4 \\ 2 & 4 & -6 & 1 \\ -4 & -8 & 12 & 1 \\ -1 & -2 & 3 & 6 \\ 3 & 6 & -9 & 12 \end{pmatrix}$;

202. $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$.

Elementar almashtirishlar yordamida matritsani rangini aniqlang:

203*. $\begin{pmatrix} -1 & 0 & 2 & 4 \\ 2 & 1 & 3 & -1 \\ 1 & 1 & 5 & 3 \\ -4 & -2 & -6 & 2 \\ 0 & 1 & 7 & 7 \end{pmatrix}$;

204*. $\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & -1 & 0 \\ 1 & 3 & 1 & 1 \\ 2 & 5 & 0 & 1 \end{pmatrix}$;

205. $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 4 & 0 \\ 1 & 8 & 6 \end{pmatrix}$;

206. $\begin{pmatrix} 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \\ 1 & 3 & -2 & 4 \\ 1 & 6 & 4 & 1 \end{pmatrix}$;

207. $\begin{pmatrix} 24 & 14 & 17 & 15 \\ 23 & 13 & 16 & 14 \\ 47 & 27 & 33 & 29 \\ 3 & 0 & 0 & 5 \end{pmatrix}$.

208. $\gamma$ ning qanday qiymatida $\begin{pmatrix} \gamma & 1 \\ 1 & 2 \end{pmatrix}$ matritsaning rangi $r=1$ ga teng bo'ladi.

A matritsa rangi $r=2$ ga teng bo'ladigan $\gamma$ ning qiymatlarini toping:

**209.** $A=\begin{pmatrix} 1 & 3 & -4 \\ \gamma & 0 & 1 \\ 4 & 3 & -3 \end{pmatrix}$;

**210.** $A=\begin{pmatrix} \gamma & 2 & 3 \\ 0 & \gamma-2 & 4 \\ 0 & 0 & 7 \end{pmatrix}$;

**211.** $A=\begin{pmatrix} 2 & -1 & 4 & 0 \\ -4 & 2 & 0 & -\gamma \\ 8 & -4 & 8 & \gamma \\ 6 & -3 & 12 & \gamma \end{pmatrix}$;

**212.** $A=\begin{pmatrix} \gamma & 0 & 1 \\ 3 & 4 & 1 \\ 1 & -1 & 2 \end{pmatrix}$.

Elementar almashtirishlar yordamida matritsaning rangini aniqlang:

**213*.** $\begin{pmatrix} 25 & 31 & 17 & 43 \\ 75 & 94 & 53 & 132 \\ 75 & 94 & 54 & 134 \\ 25 & 32 & 20 & 48 \end{pmatrix}$;

**214*.** $\begin{pmatrix} 47 & -67 & 35 & 201 & 155 \\ 26 & 98 & 23 & -294 & 86 \\ 16 & -428 & 1 & 1284 & 52 \end{pmatrix}$.

## 8 §. Chiziqli tenglamalar sistemasini yechishning Gauss usuli

$n$ ta noma'lumli $m$ ta tenglamalar sistemasini qaraymiz

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} \qquad (16)$$

Agar chiziqli tenglamalar sistemasi yechimga ega bo'lsa, u birgalikda, agar yechimga ega bo'lmasa, u birgalikda emas deyiladi. Quyidagi elementar almashtirishlar natijasida tenglamalar sistemasi o'ziga teng kuchli sistemaga almashadi:

1) Istalgan ikki tenglamaning o'rinlarini almashtirilsa;

2) Tenglamalardan istalgan birining ikkala tomonini noldan farqli songa ko'paytirilsa;

3) Tenglamalardan birini istalgan haqiqiy songa ko'paytirib, boshqa tenglamaga qo'shilsa.

Agar $n>m$ bo'lsa, $n-m$ ta bir xil noma'lumli hadlarni tengliklarning o'ng tomoniga olib o'tib, o'ng tomonidagi noma'lumlarga ixtiyoriy qiymatlarni qabul qiladi deb, tenglamalar sistemasini $n=m$

holga keltirib olish mumkin. Shuni e'tiborga olib, (16) sistemani $n=m$ holi uchun yechamiz.

Gauss usulining mohiyati noma'lumlarni ikkinchi tenglamadan boshlab, ketma-ket yo'qotib, oxirgi teglamada bitta noma'lum qolguncha davom ettiriladi va oxirgi tenglamadan yuqoriga qarab noma'lumlarni ketma-ket topib, yechim hosil qilinadi.

**1-qadam.** (16) sistemada birinchi tenglamaning har ikki tomonini $a_{11}$ ga bo'lib, teng kuchli ushbu sistemani hosil qilamiz:

$$\begin{cases} x_1+\frac{a_{12}}{a_{11}}x_2+...+\frac{a_{1n}}{a_{11}}x_n=\frac{b_1}{a_{11}} \\ a_{21}x_1+a_{22}x_2+...+a_{2n}x_n=b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1+a_{n2}x_2+...+a_{nn}x_n=b_n \end{cases}. \quad (17)$$

Birinchi tenglamani $a_{21}$ ga ko'paytirib, ikkinchi tenglamadan, $a_{31}$ ga ko'paytirib, uchinchi tenglamadan va hokazo $a_{n1}$ ga ko'paytirib, n-tenglamadan ayiramiz. Natijada yana berilgan sistemaga teng kuchli ushbu yangi sistemani hosil qilamiz:

$$\begin{cases} x_1+a'_{12}x_2+...+a'_{1n}x_n=b'_1 \\ a'_{22}x_2+...+a'_{2n}x_n=b'_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a'_{n2}x_2+...+a'_{nn}x_n=b'_n \end{cases}. \quad (18)$$

Bu sistemada quyidagicha belgilashlar kiritilgan:

$$a'_{1k}=\frac{a_{1k}}{a_{11}},\quad a'_{ik}=a_{ik}-\frac{a_{1k}}{a_{11}}a_{i1},\quad b'_1=\frac{b_1}{a_{11}},\quad b'_i=b_i-\frac{b_1}{a_{11}}a_{i1},\quad i,k=2,3,...,n.$$

Agar (4) sistemada biror tenglama chap tomonidagi barcha koeffitsiyentlar nolga teng, o'ng tomoni esa noldan farqli bo'lsa, ya'ni

$$0x_1+0x_2+...+0x_n=b_k \quad (19)$$

ko'rinishdagi tenglama hosil bo'lsa, sistema birgalikda emas bo'ladi va ishni shu yerda yakunlanadi.

Agar (19) ko'rinishdagi tenglama hosil bo'lmasa keyingi qadamga o'tiladi.

**2-qadam.** Ikkinchi tenglamani $a'_{22}$ koeffitsiyentga bo'lamiz, hosil bo'lgan sistemaning ikkinchi tenglamasini ketma-ket $a'_{32},...,a'_{n2}$

ga ko'paytirib, uchinchi, to'rtinchi va hokazo tenglamalardan ayiramiz.

Biz bu jarayonni oxirgi tenglamada $x_n$ noma'lum qolguncha davom ettirsak, dastlabki sistemaga teng kuchli

$$\begin{cases} x_1 + a'_{12}x_2 + \ldots + a'_{1n}x_n = b'_1 \\ x_2 + \ldots + a''_{2n}x_n = b''_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-1} + a''_{n-1n}x_n = b''_{n-1} \\ x_n = b''_n \end{cases} \quad (20)$$

ko'rinishdagi sistemaga ega bo'lamiz. $x_n = b''_n$ qiymatini (n-1) tenglamaga qo'yib $x_{n-1}$ ni topamiz va hokazo, bu ishni $x_1$ topilgunga qadar davom ettiramiz.

**215.** Quyidagi tenglamalar sistemasi yechilsin:

$$\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ 3x_1 + 2x_2 - 2x_3 = 1 \\ x_1 - x_2 + 2x_3 = 5 \end{cases}.$$

**Yechish.** Birinchi tenglamaning barcha hadlarini $a_{11} = 2$ ga bo'lib,

$$\begin{cases} x_1 + 0,5x_2 - 0,5x_3 = 0,5 \\ 3x_1 + 2x_2 - 2x_3 = 1 \\ x_1 - x_2 + 2x_3 = 5 \end{cases}$$

sistemani hosil qilamiz. Birinchi tenglamani 3 ga ko'paytirib ikkinchi tenglamadan, so'ngra uchinchi tenglamadan birinchi tenglamani ayiramiz:

$$\begin{cases} x_1 + 0,5x_2 - 0,5x_3 = 0,5 \\ 0,5x_2 - 0,5x_3 = -0,5 \\ -1,5x_2 + 2,5x_3 = 4,5 \end{cases}.$$

Ikkinchi tenglamani 0.5 ga bo'lib, so'ngra uni -1.5 ga ko'paytirib , uni uchinchi tenglamadan ayiramiz.
Natijada

$$\begin{cases} x_1 + 0,5x_2 - 0,5x_3 = 0,5 \\ x_2 - x_3 = -1 \\ x_3 = 3 \end{cases}$$

hosil bo'ladi.

Bundan ketma-ket $x_3 = 3,\ x_2 = -1+3 = 2,\ x_1 = 0,5 - 0,5x_2 + 0,5x_3 = 1$ larni topamiz. Shunday qilib, berilgan sistemaning yechimi $x_1 = 1,\ x_2 = 2,\ x_3 = 3$ dan iborat ekan.

**216.** $\begin{cases} x_1 + 2x_2 + 4x_3 - x_4 + 3x_5 = 7 \\ 2x_1 + x_3 + x_5 = 4 \\ x_2 + 2x_4 - x_5 = 6 \end{cases}$ sistema yechilsin.

**Yechish.** Bu sistemada uchta tenglama beshta noma'lum bo'lganligi uchun $x_4$ va $x_5$ larni o'ng tomonga olib o'tamiz.

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 7 + x_4 - 3x_5 \\ 2x_1 + x_3 = 4 - x_5 \\ x_2 = 6 - 2x_4 + x_5 \end{cases}$$

Misol uchun $x_4 = 2,\ x_5 = 1$ qiymatlarni qo'ysak

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 6 \\ 2x_1 + x_3 = 3 \\ x_2 = 3 \end{cases}$$

sistema hosil bo'ladi. $x_2 = 3$ ekanini e'tiborga olsak,

$$\begin{cases} x_1 + 4x_3 = 0 \\ 2x_1 + x_3 = 3 \end{cases}$$

sistemaga ega bo'lamiz. Birinchi tenglamani 2 ga ko'paytirib, undan ikkinchi tenglamani ayirsak

$$\begin{cases} x_1 + 4x_3 = 0 \\ 7x_3 = -3 \end{cases}$$

hosil bo'ladi. Bundan $x_3 = -\frac{3}{7},\ x_2 = 3,\ x_1 = \frac{12}{7}$.

Bu sistemada $x_4$ va $x_5$ noma'lumlarga boshqa qiymatlar berib, yangi yechim hosil qilish mumkin ekanini, boshqacha aytganda $n > m$ bo'lganda yechim yagona bo'lmay cheksiz ko'p bo'lishini eslatib o'tamiz.

Tenglamalar sistemasini birgalikda bo'lish-bo'lmasligini, uni yechmasdan turib aniqlash usuli bilan tanishamiz.

(16) tenglamalar sistemasini koeffitsiyentlaridan tuzilgan $n = m$ tartibli hamda $m \times (n+1)-$ tartibli kengaytirilgan

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad A'=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m1} & \dots & a_{m1} & b_m \end{pmatrix}$$

matritsalarni tuzib olamiz.

**Teorema.** (Kroneker-Kapelli teoremasi) (16) tenglamalar sistemasi birgalikda bo'lishi uchun $A$ va $A'$ matritsalarning ranglari teng bo'lishi, ya'ni $rang\,A = rang\,A'$ bo'lish zarur va yetarli.

Keltirilgan teoremadan quyidagi xulosalar kelib chiqadi:

**1.** Agar $rang\,A > rang\,A'$ bo'lsa, (16) sistema yechimga ega bo'lmaydi.

**2.** Agar $rang\,A = rang\,A' = k$ bo'lsa, (16) sistema yechimga ega bo'lib,

a) $k<n$ bo'lganda, tenglama cheksiz ko'p yechimga ega bo'ladi;

b) $k=n$ bo'lsa, sistema yagona yechimga ega bo'ladi.

**217.** $\begin{cases} 7x_1+3x_2=2 \\ x_1-2x_2=3 \\ 4x_1+9x_2=11 \end{cases}$ tenglamalar sistemasi yechilsin.

**Yechish.** Bu yerda $n=2, m=3$, ya'ni $m>n$.

$$A=\begin{pmatrix} 7 & 3 \\ 1 & -2 \\ 4 & 9 \end{pmatrix}, \quad A'=\begin{pmatrix} 7 & 3 & 2 \\ 1 & -2 & -3 \\ 4 & 9 & 11 \end{pmatrix},$$

$rangA=2$ chunki,

$$\begin{vmatrix} 7 & 3 \\ 1 & -2 \end{vmatrix} = -17 \neq 0, \quad \begin{vmatrix} 7 & 3 & 2 \\ 1 & -2 & -3 \\ 4 & 9 & 11 \end{vmatrix} = 0$$

bo'lishini e'tiborga olsak, $rangA'=2$, demak, bu sistemaning yechimi mavjud. Berilgan sistemaning birinchi ikki tenglamasini birgalikda yechsak, $x_1=-\frac{5}{17}$, $x_2=\frac{23}{17}$ kelib chiqadi. Bu sonlar uchinchi tenglamani ham qanoatlantiradi.

$$4x_1+9x_2=4\left(-\frac{5}{17}\right)+9\cdot\frac{23}{17}=11.$$

Demak, $\left(-\frac{5}{17};\frac{23}{17}\right)$ sistemaning yechimi bo'ladi.

Chiziqli tenglamalar sistemasini Gauss usulida yeching:

218. $\begin{cases} x_1 + 2x_2 - x_3 = 3 \\ 2x_1 + 5x_2 - 6x_3 = 1 \\ 3x_1 + 8x_2 - 10x_3 = 1 \end{cases}$;

219. $\begin{cases} x_1 - x_2 + 3x_3 = 7 \\ 2x_1 + x_2 - x_3 = -3 \\ 3x_1 + x_2 - 3x_3 = 1 \end{cases}$;

220. $\begin{cases} 4x_1 + 2x_2 - x_3 = 1 \\ 5x_1 + 3x_2 - 2x_3 = 2 \\ 3x_1 + 2x_2 - 3x_3 = 0 \end{cases}$;

221. $\begin{cases} 5x_1 + 2x_2 + 5x_3 = 4 \\ 3x_1 + 5x_2 - 3x_3 = -1 \\ -2x_1 - 4x_2 + 3x_3 = 1 \end{cases}$.

Tenglamalar sistemasini birgalikda yoki birgalikda emasligini tekshiring:

222. $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 1 \\ 2x_1 - x_2 + 2x_4 = 3 \\ 5x_2 - 6x_3 + 4x_4 = 5 \end{cases}$;

223. $\begin{cases} 3x_1 + x_2 + 5x_3 + x_4 = 2 \\ -x_1 + 4x_2 + x_3 + 3x_4 = 1 \\ -5x_1 + 7x_2 - 3x_3 + 5x_4 = 2 \end{cases}$;

224*. $\begin{cases} 4x_1 + x_2 + 3x_3 + 2x_4 + x_5 = 3 \\ -2x_1 + x_2 + 2x_4 + 3x_5 = 0 \\ x_1 - 4x_2 + 3x_3 + x_4 = 2 \\ 3x_1 - 2x_2 + 6x_3 + 5x_4 + 4x_5 = 4 \end{cases}$.

a parametrning qanday qiymatlarida tenglamalar sistemasi birgalikda bo'ladi?

225. $\begin{cases} x_1 - x_2 + 2x_3 = 3 \\ 2x_1 + 5x_3 = 7 \\ x_1 + x_2 + (a+9)x_3 = 6 \end{cases}$;

226. $\begin{cases} x_1 + 2x_2 - x_3 = 3 \\ 2x_1 + 6x_2 - 5x_3 = 7 \\ x_1 + 6x_2 - 7x_3 = a + 3 \end{cases}$;

227. $\begin{cases} x_1 + 3x_2 - 2x_3 = 1 \\ 2x_1 + x_2 - 4x_3 = 4 \\ x_1 + 8x_2 + (a-7)x_3 = 5 \end{cases}$;

228. $\begin{cases} 2x_1 - x_3 + 3x_4 = 10 \\ 3x_1 - x_2 + 2x_3 + x_4 = 8 \\ 8x_1 - 2x_2 + 3x_3 + 4x_4 = 18 \\ 3x_1 - x_2 + 2x_3 = a \end{cases}$;

229. $\begin{cases} x_1 + 3x_2 - 2x_3 + 4x_4 = 1 \\ -x_1 + 2x_2 + x_3 + x_4 = 2 \\ 3x_1 + 2x_2 = 1 \\ 3x_1 + 5x_2 + x_3 + ax_4 = 5 \end{cases}$;

230. $\begin{cases} x_1 + 5x_2 - 2x_3 + 3x_4 = 2 \\ 4x_1 + x_2 - x_3 - 3x_4 = 2 \\ -11x_1 + 2x_2 + x_3 + 12x_4 = a \end{cases}$;

231. $\begin{cases} -x_1 - 4x_2 + 2x_3 + 3x_4 = 3 \\ -4x_1 + 3x_2 + 2x_3 + x_4 = -2 \\ x_1 - 15x_2 + 4x_3 + 8x_4 = a \end{cases}$;

232. $\begin{cases} x_1 - 3x_2 + 4x_3 + 2x_4 = 5 \\ -3x_1 + 2x_2 + 3x_3 + x_4 = -4 \\ 8x_1 - 10x_2 + 2x_3 + 2x_4 = a \end{cases}$;

233. $\begin{cases} -x_1 + 3x_2 - 2x_3 + 4x_4 = 5 \\ 2x_1 + 3x_2 - 4x_3 + 3x_4 = -4 \\ 4x_1 + 15x_2 - 16x_3 + 17x_4 = a \end{cases}$.

Kroneker-Kapelli teoremasi yordamida tekshiring va birgalikda bo'lganlarini yeching:

234. $\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 1 \\ 3x_1 - x_2 + 2x_4 = 0 \\ x_1 - 2x_2 + x_3 + x_4 = 2 \end{cases}$;

235. $\begin{cases} x_1 - x_2 + 3x_3 = 1 \\ 4x_1 + x_2 + 4x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$;

236. $\begin{cases} 2x_1 + 3x_2 - x_3 = 4 \\ x_1 - 2x_2 + 2x_3 = 1 \\ 3x_1 + x_2 + 3x_3 = 7 \\ 2x_1 - x_2 + 2x_3 = 2 \end{cases}$;

**237.** $\begin{cases} x_1 + 2x_2 + 3x_3 = 2 \\ 2x_1 + 3x_2 - x_3 = 1 \\ 3x_1 + 5x_2 + 2x_3 = 3 \end{cases};$

**238.** $\begin{cases} x_1 + 3x_2 - x_3 = 3 \\ 2x_1 + x_2 + 2x_3 = 5 \\ 3x_1 + 4x_2 - 7x_3 = 0 \\ x_1 - x_2 + 3x_3 = 3 \end{cases};$

**239.** $\begin{cases} x_1 + 2x_2 + x_3 = -1 \\ 2x_1 + 3x_2 + 4x_3 = 3 \\ 3x_1 + 5x_2 + 5x_3 = 0 \end{cases};$

**240.** $\begin{cases} x_1 - 2x_2 + 3x_3 + 4x_4 = -2 \\ 2x_1 - 4x_2 + x_3 - x_4 = 3 \end{cases};$

**241.** $\begin{cases} x_1 - 3x_2 + 4x_3 - 4x_4 = 2 \\ 2x_1 + 3x_2 + x_3 + 5x_4 = 1 \\ 3x_1 + 5x_3 + 4x_4 = 10 \end{cases};$

**242.** $\begin{cases} x_1 - 5x_2 + 3x_3 - x_4 = 1 \\ 2x_1 - 10x_2 + 3x_4 = 0 \\ 4x_1 - 20x_2 + 6x_3 + x_4 = 2 \end{cases};$

**243.** $\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 + x_5 = 1 \\ 2x_1 - 3x_2 + 2x_3 + x_4 - x_5 = 3 \end{cases}.$

## 9 §. *n* ta noma'lumli chiziqli tenglamalar sistemasi

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ ...................................... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases} \qquad (21)$$

(21) tenglamalar sistemasi $n$ **ta noma'lumli chiziqli tenglamar sistemasi** deyiladi.

(21) tenglamalar sistemasini yechishning Kramer va matritsalar usullarini ko'ramiz. Noma'lumlar oldidagi koeffitsiyentlardan asosiy determinantni

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{vmatrix}$$

va uch noma'lumli chiziqli tenglamalar sistemasidagiga o'xshash yordamchi $\Delta x_1, \Delta x_1, ..., \Delta x_n$ determinantlarni tuzamiz. Ular yuqori tartibli determinantlarni hisoblash usuli bilan hisoblanadi. Bu yerda ham agar $\Delta \neq 0$ bo'lsa, Kramer formulasiga asosan

$$x_1 = \frac{\Delta x_1}{\Delta}, \ x_2 = \frac{\Delta x_2}{\Delta}, ..., x_n = \frac{\Delta x_n}{\Delta}.$$

yechimni hosil qilamiz. Bu yechim yagona yechim bo'ladi. Agar (21) sistemaning koeffitsiyentlari va noma'lumlaridan $A, B$ va $X$ matritsalar

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},\quad B=\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix},\quad X=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

ko'rinishida tuzilsa, (21) sistemasini $AX = B$ holda yozish mumkin.

Agar $\Delta \neq 0$ bo'lsa, $A$ ga teskari $A^{-1}$ matritsa mavjud bo'ladi va berilgan sistemaning matritsalar ko'rinishidagi yechimi $X = A^{-1}B$ bo'ladi.

Agar $\Delta = 0$ bo'lib, $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ lardan aqalli birortasi noldan farqli bo'lsa, (21) sistema yechimga ega bo'lmaydi.

Agar $\Delta = 0$ bo'lib, $\Delta x_1 = \Delta x_2 = \dots = \Delta x_n = 0$ bo'lsa, (21) sistema cheksiz ko'p yechimga ega bo'ladi.

**244.** $\begin{cases} 2x_1 + x_2 - 5x_3 + x_4 = 8 \\ x_1 - 3x_2 - 6x_4 = 9 \\ 2x_2 - x_3 + 2x_4 = -5 \\ x_1 + 4x_2 - 7x_3 + 6x_4 = 0 \end{cases}$ tenglamalar sistemasi yechilsin.

**Yechish.** Sistemani Kramer usulida yechamiz.

$$\Delta = \begin{vmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{vmatrix} = 2\cdot\begin{vmatrix} -3 & 0 & -6 \\ 2 & -1 & 2 \\ 4 & -7 & 6 \end{vmatrix} - \begin{vmatrix} 1 & -5 & 1 \\ 2 & -1 & 2 \\ 4 & -7 & 6 \end{vmatrix} - \begin{vmatrix} 1 & -5 & 1 \\ -3 & 0 & -6 \\ 2 & -1 & 2 \end{vmatrix} = 27$$

Xuddi shu usul bilan hisoblashni davom ettirib, quyidagilarni topamiz: $\Delta x_1 = 81,\ \Delta x_2 = -108,\ \Delta x_3 = -27,\ \Delta x_4 = 27$.

Demak, sistema yagona yechimga ega, chunki $\Delta \neq 0$. Bu yechim esa

$$x_1 = \frac{\Delta x_1}{\Delta} = \frac{81}{27} = 3,\quad x_2 = \frac{\Delta x_2}{\Delta} = \frac{-108}{27} = -4,\quad x_3 = \frac{\Delta x_3}{\Delta} = \frac{-27}{27} = -1,\quad x_4 = \frac{\Delta x_4}{\Delta} = \frac{27}{27} = 1$$

$(3;-4;-1;1)$ bo'ladi.

(21) tenglamalar sistemasining $b_1 = b_2 = \dots = b_n = 0$ bo'lgan holini ko'ramiz.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0 \end{cases} \qquad (22)$$

(22) tenglamalar sistemasini bir jinsli, chiziqli tenglamalar sistemasi deyiladi.

Osonlik bilan ishonch hosil qilish mumkinki, $x_1 = x_2 = ... = x_n = 0$ (22) sistemaning yechimlari bo'ladi va bu yechimni trivial yechim deb ataladi. Agar (22) bir jinsli sistemanig asosiy determinanti $\Delta$ noldan farqli bo'lsa, bu sistema faqat trivial yechimga ega bo'ladi. Chunki bu holda $\Delta x_1 = \Delta x_2 = ... = \Delta x_n = 0$ va Kramer formulasiga asosan $x_1 = x_2 = ... = x_n = 0$ bo'ladi .

Demak, (22) sistemaning notrivial yani noldan farqli yechimi mavjud bo'lishi uchun $\Delta = 0$ bo'lishi zarur ekan.

**245.** $\begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$ tenglamalar sistemasi yechilsin.

**Yechish.** $x_1 = x_2 = 0$ trivial yechim ekani ravshan.

$$\Delta = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0,$$

bundan ko'rinadiki, sistemaning notrivial yechimi bo'lishi mumkin. Haqiqatdan ham $x_1 = x_2 = t$ ($t$ -ixtiyoriy haqiqiy son) sistemaning notrivial yechimi bo'ladi.

Chiziqli tenglamalar sistemasini yeching:

**246.** $\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 3x_2 + 4x_3 = 0 \\ 3x_1 + 4x_2 + 5x_3 = 0 \end{cases}$;

**247.** $\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \\ 3x_1 - x_2 - x_3 = 0 \end{cases}$;

**248.** $\begin{cases} 6x_1 - 8x_2 + 2x_3 + 3x_4 = 0 \\ 3x_1 - 4x_2 + x_3 - x_4 = 0 \end{cases}$;

**249.** $\begin{cases} x_1 - 2x_2 + 3x_3 - x_4 = 0 \\ x_1 + x_2 - x_3 + 2x_4 = 0 \\ 4x_1 - 5x_2 + 8x_3 + x_4 = 0 \end{cases}$;

**250.** $\begin{cases} x_1 + 4x_2 - 7x_3 = 0 \\ 3x_1 - 2x_2 + x_3 = 0 \\ 2x_1 + x_2 - 3x_3 = 0 \end{cases}$;

**251.** $\begin{cases} x_1 + x_2 - x_3 - x_4 = 0 \\ 3x_1 - x_2 + 2x_3 + x_4 = 0 \\ 5x_1 - x_2 - x_4 = 0 \end{cases}$;

**252.** $\begin{cases} x_1 + x_2 + x_3 = 0 \\ 5x_1 - x_2 - x_3 = 0 \\ 3x_1 + x_2 + x_3 = 0 \\ 2x_1 + 3x_2 + 2x_3 = 0 \end{cases}$;

**253.** $\begin{cases} x_1 + x_2 + 3x_3 + x_4 = 0 \\ 2x_1 - x_2 + x_3 + 3x_4 = 0 \\ 4x_1 + x_2 + 7x_3 + 5x_4 = 0 \\ 5x_1 - x_2 + 5x_3 + 7x_4 = 0 \end{cases}$;

**254.** $\begin{cases} 2x_1 - 4x_2 + 5x_3 + 3x_4 = 0, \\ 3x_1 - 6x_2 + 4x_3 + 2x_4 = 0, \\ 4x_1 - 8x_2 + 17x_3 + 11x_4 = 0. \end{cases}$

**255*.** $\begin{cases} 2x_1 - x_2 + 3x_3 - 2x_4 + 4x_5 = 0, \\ 4x_1 - 2x_2 + 5x_3 + x_4 + 7x_5 = 0, \\ 2x_1 - x_2 + x_3 + 8x_4 + 2x_5 = 0. \end{cases}$

# II BOB. VEKTORLAR ALGEBRASI

## 10 §. Vektor. Vektorlar ustida amallar.

**Vektorlar**. Fizika, mexanika, texnika va matematikada, asosan, ikki xil kattaliklar bilan ish ko'riladi. Ulardan biri o'zining son qiymati bilan to'la xarakterlanib, skalyar miqdorlar yoki skalyarlar deb ataladi.

Ikkinchi tur kattaliklarni to'la xarakterlash uchun ularning son qiymatlarigina yetarli bo'lmay, balki yo'nalishi ham berilgan bo'lishi kerak.

O'zining son qiymati bilan birga yo'nalishi ma'lum bo'lganda to'la xarakterlanadigan kattaliklar vektor miqdorlar yoki vektorlar deb ataladi.

**1-ta'rif**. Yo'nalishga ega bo'lgan kesma vektor deb ataladi. Vektorlar boshlanish va tugash nuqtalari orqali $\overline{AB}$, $\overline{BC}$... kabi yoki $\bar{a}, \bar{b}, \bar{c}$ ko'rinishida belgilanadi.

Vektorning son qiymati uning moduli yoki uzunligi deyiladi va $|\bar{a}|$ bilan belgilanadi.

Ikki $\bar{a}$ va $\bar{b}$ vektorlar bir to'g'ri chiziqda yoki parallel to'g'ri chiziqlarda yotsa, bunday vektorlar **kollinear vektorlar** deb ataladi.

1-shakl

Bitta tekislikda yoki parallel tekisliklarda yotgan $\bar{a}$ va $\bar{b}$ vektorlar **komplanar vektorlar** deb ataladi.

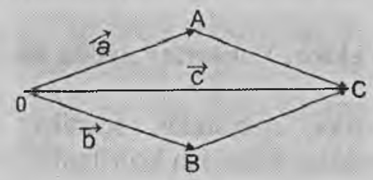
$\bar{a}$ va $\bar{b}$ kollinear, bir xil yo'nalishli, uzunliklari teng vektorlar bo'lsa o'zaro teng vektorlar bo'ladi.

1- shaklda o'zaro teng vektorlar tasvirlangan.

### Vektorlar ustida amallar

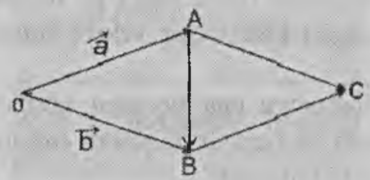
1.**Vektorlarni qo'shish**. $\bar{a}, \bar{b}, \bar{c}$ vektorlar berilgan bo'lsin. Bu vektorlarni boshini biror O nuqtaga ko'chiramiz. Ikki $\bar{a}$ va $\bar{b}$

vektorlarni yig'indisi deb tomonlari $\bar{a}$ va $\bar{b}$ vektorlardan iborat parallelogrammning O uchidan chiqqan OC dioganaliga teng $\bar{c}$ vektorga aytiladi (parallelogramm qoidasi) va $\bar{a}+\bar{b}=\bar{c}$ .




**2-shakl**

**2. Vektorlarni ayirish**. Parallelogramning ikkinchi dioganali $\overline{AB}$ ga teng $\bar{d}$ vektor $\bar{a}$ va $\bar{b}$ vektorlarni ayirmasi deyiladi.




**3-shakl**

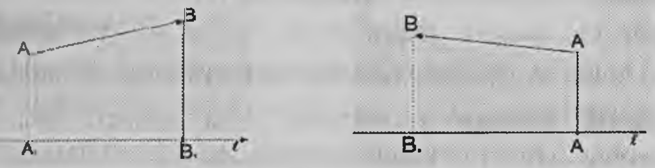
**3. Vektorni songa ko'paytirish**.

$\bar{a}$ vektorni $k$ ($k$-const) soniga ko'paytmasi deb, uzunligi $|k\bar{a}|$ ga teng bo'lgan $\bar{c}$ vektorga aytiladi. $\bar{c}=k\bar{a}$. Agar $k$>0 bo'lsa, $\bar{c}$ vektor yo'nalishi $\bar{a}$ vektor yo'nalishi bilan bir xil, aks holda esa $\bar{c}$ vektor yo'nalishiga qarama-qarshi bo'ladi. Agar $k=0$ bo'lsa, $\bar{c}=k\bar{a}=0\bar{a}=0$ bo'ladi.

**4. Vektorlarning proyeksiyasi.**

$M$ nuqtaning berilgan o'qdagi proyeksiyasi deb, shu $M$ nuqtadan o'qqa tushirilgan perpendikulyarning asosiga aytiladi.

$\overline{AB}$ vektor boshining (A nuqta) proyeksiyasini uning oxirining (B nuqta) proyeksiyasi bilan tutashtiruvchi $\overline{A_1B_1}$ vektor $\overline{AB}$ vektorning o'qdagi tashkil etuvchisi yoki komponentasi deyiladi. $\overline{AB}$ vektorning $\ell$ o'qqa proeksiyasi deb uning $\overline{A_1B_1}$ tashkil etuvchisining $\ell$ o'q yo'nalishida yoki unga qarama – qarshi yo'nalganligiga qarab, musbat yoki manfiy ishora bilan olingan uzunligiga aytiladi (4-shakl).




**4-shakl**

Vektorning $\ell$ o'qqa proyeksiyasi bunday belgilanadi: $np_\ell \overrightarrow{AB}$. Demak,

$$np_\ell \overrightarrow{AB} = \pm\left|\overrightarrow{A_1B_1}\right|.$$

## Proyeksiyalarning asosiy xossalari

1. $a$ vektorning $\ell$ o'qqa proyeksiyasi $\vec{a}$ vektor modulining bu vektor bilan o'q orasidagi burchak kosinusiga ko'paytmasiga teng, ya'ni

$$np_\ell \vec{a} = \left|\vec{a}\right|\cos\varphi, \qquad \varphi = \left(\vec{a},^\wedge \ell\right).$$

2. Vektorning o'qdagi proyeksiyasi skalyar miqdordir.

3. Ikki vektor yig'indisi uning o'qqa proyeksiyasi qo'shiluvchi vektorlarning shu o'qqa proyeksiyalari yig'indisiga teng, ya'ni

$$\Pi p_l(\vec{a}+\vec{b}) = \Pi p_l\vec{a} + \Pi p_l\vec{b}.$$

4. $\lambda$ o'zgarmas sonni proyeksiyadan tashqariga chiqarish mumkin:

$$\Pi p_l(\lambda\vec{a}) = \lambda \Pi p_l \vec{a}.$$

**2-ta'rif.** $\alpha_1, \alpha_2, ..., \alpha_n$ sonlarning mos $\overrightarrow{a_1}, \overrightarrow{a_2}, ... \overrightarrow{a_n}$ vektorlarga ko'paytmalari yig'indisiga, ya'ni

$$\alpha_1\overrightarrow{a_1} + \alpha_2\overrightarrow{a_2} + \alpha_3\overrightarrow{a_3} + ... + \alpha_n\overrightarrow{a_n}$$

ifodaga vektorlarning chiziqli kombinatsiyasi deb ataladi.

**3-ta'rif.** Orasida noldan farqlilari ham bo'lgan shunday $\alpha_1, \alpha_2, ..., \alpha_n$ sonlar mavjud bo'lsaki, ular uchun vektorlarning chiziqli kombinatsiyasi nolga teng, ya'ni

$$\alpha_1\overline{a_1}+\alpha_2\overline{a_2}+\alpha_3\overline{a_3}+...+\alpha_n\overline{a_n}=0 \quad (1)$$

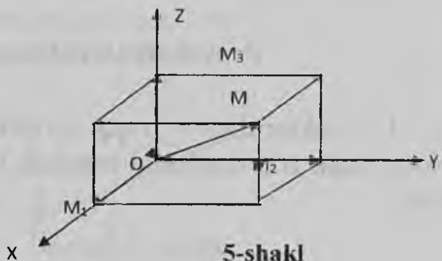
bo‘lsa, $\alpha_1,\alpha_2,...,\alpha_n$ vektorlar **chiziqli bog‘liq** deb ataladi.

Agar (1) tenglik faqat $\alpha_1=\alpha_2=...=\alpha_n=0$ bo‘lganda o‘rinli bo‘lsa, u holda $\alpha_1,\alpha_2,...,\alpha_n$ vektorlar **chiziqli erkli** deb ataladi.

**4-ta’rif.** Istalgan $\bar{a}$ vektorni $n$ ta chiziqli erkli $\bar{e}_1,\bar{e}_2,...,\bar{e}_n$ vektorlarning chiziqli kombinatsiyasi orqali ifodalash mumkin bo‘lsa, u holda bu vektorlar fazoning **bazisi** deb ataladi.

Bazisni hosil qiladigan vektorlar soni fazoning o‘lchami deyiladi. Bazisga kiruvchi vektorlar **bazis vektorlar** deyiladi.

To‘g‘ri burchakli koordinatalar sistemasida boshi 0 dan chiqqan $\overline{OM}$ vektor berilgan bo‘lsin (5-shakl).




5-shakl

Bu vektorning OX, OY, OZ o‘qlaridagi proyeksiyalarini topish uchun, OM vektor oxiridan YOZ tekisligiga parallel tekislik o‘tkazib, bu tekislikni OX o‘qi bilan kesishgan nuqtasini $M_1$, XOY tekisligiga parallel tekislik o‘tkazib bu tekislikni OY o‘qi bilan kesishgan nuqtasini $M_2$, OXY tekisligiga parallel tekislik o‘tkazib, bu tekislikning OZ nuqtasini $M_3$ deb belgilaymiz.

$\overline{OM}$ vektorning OX, OY, OZ koordinata o‘qlaridagi proyeksiyalari mos ravishda $OM_1$, $OM_2$, $OM_3$ ga teng bo‘ladi.

Bu proyeksiyalarning har biri $\overline{OM}$ vektorning o‘qlardagi komponentalaridir. Chizmadan ko‘rinib turibdiki, $\overline{OM}$ vektor $\overline{OM_1}$, $\overline{OM_2}$, $\overline{OM_3}$ vektorlarning yig‘indisiga teng:

$$\overline{OM}=\overline{OM_1}+\overline{OM_2}+\overline{OM_3}. \quad (2)$$

Ko‘pincha koordinata o‘qlariga mos keluvchi asosiy birlik vektorlarni tanlab olish qulay bo‘ladi.

OX, OY, OZ o‘qlaridagi birlik vektorlarni mos ravishda $i,j,k$ lar bilan belgilaylik.

$\overline{OM_1},\overline{OM_2},\overline{OM_3}$ vektorlar $\overline{OM}$ vektorning bazis vektorlari bo‘ladi. $|\overline{OM_1}|=x$, $|\overline{OM_2}|=y$, $|\overline{OM_3}|=z$ larni (2) ga qo‘yib

$\overline{OM}=\bar{i}x+\bar{j}y+\bar{k}z$ tenglikni hosil qilamiz. 5-shakldan ko‘rinadiki, OM vektorning uzunligi parallelopiped diogonalining uzunligiga teng bo‘lganidan $|\overline{OM}|=\sqrt{x^2+y^2+z^2}$ bo‘ladi.

**256.** $\bar{a}=\{6;3;-2\}$ vektorning uzunligini toping.

**257.** Vektorning ikki koordinatasi $x=4$, $y=-12$ berilgan. $|\bar{a}|=13$ bo‘lsa, uning uchinchi z koordinatasini toping.

**258.** $A(3;\ —1;\ 2)$ va $B(—\ 1;\ 2;\ 1)$ nuqtalarning koordinatalari berilgan bo‘lsa, $\overline{AB}$ va $\overline{BA}$ vektorlarning koordinatalarini toping.

**259.** $\bar{a}=\{3;-1;4\}$ vektorning boshi $M(1;\ 2;-3)$ nuqtada bo‘lsa, uning oxiri $N$ nuqtaning koordinatalarini toping.

**260.** $\bar{a}=\{2;\ —3;\ —1\}$ vektorning oxiri (1; —1; 2) nuqtada bo‘lsa, uning boshining koordinatalarini toping.

**261.** $\bar{a}$ vektorning uzunligi berilgan. $|\bar{a}|=2$ va $\alpha=45°$, $\beta=60°$, $\gamma=120°$. $\bar{a}$ ning koordinata o‘qlaridagi proyeksiyalarini toping.

**262.** $\bar{a}$ $=\{12;\ —15;\ —16\}$ vektorning yo‘naltiruvchi kosinuslarini aniqlang.

**263.** $\bar{a}=\left\{\frac{3}{13};\frac{4}{13};\frac{12}{13}\right\}$ vektorning yo‘naltiruvchi kosinuslarini aniqlang.

**264.** Quyidagi burchaklar vektorning koordinata o‘qlari bilan hosil qilgan burchaklari bo‘ladimi?

1) $\alpha=45°$, $\beta=60°$, $\gamma=120°$; 2) $\alpha=45°$, $\beta=135°$, $\gamma=60°$;
3) $\alpha=90°$, $\beta=150°$; $\gamma=60°$?

**265.** Vektor *Ox* va *Oz* o‘qlari bilan mos ravishda $\alpha=120°$ va $\gamma=45°$ burchaklarni hosil qiladi. Uning *Oy* o‘qi bilan hosil qilgan burchakni toping.

**266.** $\bar{a}$ vektor uzunligi $|\bar{a}|=2$ bo‘lib, *Ox* va *Oy* o‘qlari bilan hosil qilgan burchaklari $\alpha=60°$, $\beta=120°$ bo‘lsa, uning koordinatalarini aniqlang.

**267.** Uzunligi 3 ga teng bo‘lib, radius vektori koordinata o‘qlari bilan bir xil burchak hosil qilgan M nuqtaning koordinatalarini toping.

**268.** Berilgan $\bar{a}$ va $\bar{b}$ vektorlar bo‘yicha quyidagi vektorlarni yasang:

1) $\bar{a}+\bar{b}$; 2) $\bar{a}-\bar{b}$; 3) $\bar{b}-\bar{a}$; 4) $-\bar{a}-\bar{b}$.

**269.** Quyidagilar berilgan: $|\vec{a}|=13$, $|\vec{b}|=19$ va $|\vec{a}+\vec{b}| = 24$. $|\vec{a}-\vec{b}|$ ni hisoblang.

**270.** $|\vec{a}|=11$, $|\vec{b}|=23$ va $|\vec{a}-\vec{b}|=30$ bo'lsa, $|\vec{a}+\vec{b}|$ ni hisoblang.

**271.** $\vec{a}$ va $\vec{b}$ vektorlar o'zaro perpendikulyar bo'lib $|\vec{a}|=5$ va $|\vec{b}|=12$ bo'lsa, $|\vec{a}+\vec{b}|$ va $|\vec{a}-\vec{b}|$ ni toping.

**272.** $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak $\varphi = 60°$ bo'lib, $|\vec{a}|=5$ va $|\vec{b}|=8$ bo'lsa, $|\vec{a}+\vec{b}|$ va $|\vec{a}-\vec{b}|$ ni aniqlang.

**273.** $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak $\varphi=120°$ va $|\vec{a}|=3$, $|\vec{b}|=5$ bo'lsa, $|\vec{a}+\vec{b}|$ va $|\vec{a}-\vec{b}|$ larni aniqlang.

**274*.** Quyidagi tengsizliklar bajarilishi uchun $\vec{a}$ va $\vec{b}$ vektorlar qanday shartlarni bajarishi kerak:

1) $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, 2) $|\vec{a}+\vec{b}|>|\vec{a}-\vec{b}|$, 3) $|\vec{a}+\vec{b}|<|\vec{a}-\vec{b}|$.

**275.** $\vec{a}$ va $\vec{b}$ vektorlar qanday bo'lganda $\vec{a}+\vec{b}$ vektor $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchakni teng ikkiga bo'ladi?

**276.** *O* nuqta ABC uchburchakning og'irlik markazi bo'lsa, quyidagi tenglikni isbotlang: $\overline{OA}+\overline{OB}+\overline{OC}=0$.

**277.** α, β ning qanday qiymatlarida $\vec{a}= -2\vec{i}+3\vec{j}+\beta\vec{k}$ и $\vec{b}=\alpha\vec{i}-6\vec{j}+2\vec{k}$ vektorlar kollinear bo'ladi.

**278.** $\vec{a}=\{3;\ -5;\ 8\}$ va $\vec{b}=\{-1;\ 1;\ -4\}$ vektorlarning yig'indisi va ayirmasi uzunliklarini aniqlang.

## 11 §. Ikki vektorning skalyar ko'paytmasi

**Ta'rif.** Ikki $\vec{a}$ va $\vec{b}$ vektorlarning **skalyar ko'paytmasi** deb, bu vektorlar uzunliklari bilan ular orasidagi burchak kosinusining ko'paytmasiga teng bo'lgan songa aytiladi.

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi \qquad (3)$$

Bu yerda $\varphi$ – $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchak

Skalyar ko'paytmaning ba'zi xossalarini keltiramiz.

$1^0$. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$;

$2^0$. $(\lambda\vec{a})\cdot\vec{b}=\lambda\cdot(\vec{a}\cdot\vec{b})$;

$3^0$. $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$;

$4^0$. $\vec{a}\cdot\vec{a}=|\vec{a}|^2$;

$5^0$. Agar $\vec{a}\perp\vec{b}$ bo'lsa, $\vec{a}\cdot\vec{b}=0$ va aksincha .

$4^0$ va $5^0$ xossalardan $\vec{i},\vec{j}$ va $\vec{k}$ bazis vektorlar uchun quyidagi tengliklarni topamiz.

$$\vec{i}^2=\vec{j}^2=\vec{k}^2=1,\quad \vec{i}\cdot\vec{j}=\vec{i}\cdot\vec{k}=\vec{j}\cdot\vec{i}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=\vec{k}\cdot\vec{j}=0.$$

**Teorema.** Agar $\vec{a}$ va $\vec{b}$ vektorlar o'zlarining koordinatalari bilan berilgan bo'lsa, $\vec{a}=\{x_1,y_1,z_1\}$, $\vec{b}=\{x_2,y_2,z_2\}$ u holda ularning skalyar ko'paytmasi quyidagi formula bilan aniqlanadi

$$(\vec{a}\cdot\vec{b})=x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2. \qquad (4)$$

Teoremadan quyidagi ikkita muhim natija kelib chiqadi:

1. $\vec{a}$ va $\vec{b}$ vektorlar orasidagi burchakni

$$\cos\varphi=\frac{x_1\cdot x_2+y_1\cdot y_2+z_1\cdot z_2}{\sqrt{x_1^2+y_1^2+z_1^2}\cdot\sqrt{x_2^2+y_2^2+z_2^2}} \qquad (5)$$

formula bilan aniqlanadi.

**4.** $x_1x_2+y_1y_2+z_1z_2=0$ vektorlarning perpendikulyarlik sharti.

**279.** $|\vec{a}|=3,\ |\vec{b}|=4,\ \varphi=60^\circ,\ \vec{a}\cdot\vec{b}=?$

**Yechish.** (3) formulaga asosan $\vec{a}\cdot\vec{b}=3\cdot 4\cdot\cos 60^\circ=6.$

**280.** $\vec{a}(2;-3;4),\ \vec{a}(5;1;-6),\ \vec{a}\cdot\vec{b}=?.$

**Yechish.** (4) formulaga asosan $\vec{a}\cdot\vec{b}=2\cdot 5-3\cdot 1-4\cdot 6=-17.$

**281.** $\vec{c}=2\vec{a}+3\vec{b},\ |\vec{a}|=4,\ |\vec{b}|=5,\ \varphi=60^\circ,\ |\vec{c}|=?.$

**Yechish.** 4-xossaga asosan

$$|\vec{c}|=\sqrt{(\vec{c})^2}=\sqrt{(2\vec{a}+3\vec{b})^2}=\sqrt{4\vec{a}^2+12\vec{a}\cdot\vec{b}+9\vec{b}^2}.$$

$\vec{a}^2=16,\ \vec{b}^2=25,\ \vec{a}\cdot\vec{b}=4\cdot 5\cdot\cos 60^\circ=10$ bo'lgani uchun

$$|\vec{c}|=\sqrt{4\cdot 16+12\cdot 10+9\cdot 25}=\sqrt{409}\approx 20,22.$$

**282.** $\vec{a}=3\vec{i}+\vec{j}-\vec{k},\quad \vec{b}=2\vec{i}+2\vec{j}+\vec{k},\quad \cos\varphi=?$

**Yechish.** (3) - formulaga asosan

$$\cos\varphi=\frac{3\cdot 2+1\cdot 2+(-1)\cdot 1}{\sqrt{3^2+1^2+(-1)^2}\sqrt{2^2+2^2+1^2}}=\frac{7}{3\sqrt{11}}\approx 0,703.$$

**283.** $m$ ning qanday qiymatida $\vec{a}=2\vec{i}+3\vec{j}-\vec{k},\quad \vec{b}=\vec{i}-5\vec{j}+m\vec{k}$ vektorlar perpendikulyar bo'ladi.

**Yechish.** Vektorlarning perpendikulyarlik shartiga asosan

$$2\cdot 1+3\cdot(-5)+(-1)m=0,\quad m=-13.$$

**284.** $\vec{a}$ va $\vec{b}$ vektorlar $\varphi=\frac{2}{3}\pi$ burchak hosil qiladi. $|\vec{a}|=3$, $|\vec{b}|=4$ bo'lsa, quyidagilarni hisoblang:

1) $\vec{a}\vec{b}$; 2) $\vec{a}^2$; 3) $\vec{b}^2$; 4) $(\vec{a}+\vec{b})^2$; 5) $(3\vec{a}-2\vec{b})(\vec{a}+2\vec{b})$;

6) $(\vec{a}-\vec{b})^2$; 7) $(3\vec{a}+2\vec{b})^2$.

**285.** $\vec{a}$ va $\vec{b}$ vektorlar o'zaro perpendikulyar; $\vec{c}$ vektor ular bilan $\frac{\pi}{3}$ burchak hosil qiladi. Agar $|\vec{a}|=3$, $|\vec{b}|=5$, $|\vec{c}|=8$ bo'lsa, quyidagilarni hisoblang:

1) $(3\vec{a}-2\vec{b})(\vec{b}+3\vec{c})$; 2) $(\vec{a}+\vec{b}+\vec{c})^2$; 3) $(\vec{a}+2\vec{b}-3\vec{c})^2$.

**286*.** Tenglikni isbotlang va geometrik ma'nosini aniqlang:

$$(\vec{a}+\vec{b})^2+(\vec{a}-\vec{b})^2=2(\vec{a}^2+\vec{b}^2).$$

**287*.** $\vec{a}$, $\vec{b}$ va $\vec{c}$ birlik vektorlar uchun $\vec{a}+\vec{b}+\vec{c}=0$ tenglik bajarilsa, $\vec{a}\vec{b}+\vec{a}\vec{b}+\vec{c}\vec{a}$ ni hisoblang.

**288.** $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar uchun $\vec{a}+\vec{b}+\vec{c}=0$ tenglik bajarilib, $|\vec{a}|=3$, $|\vec{b}|=1$ va $|\vec{c}|=4$ bo'lsa, $\vec{a}\vec{b}+\vec{b}\vec{c}+\vec{c}\vec{a}$ ni hisoblang.

**289.** $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar bir-biri bilan o'zaro 60° burchak hosil qiladi. $|\vec{a}|=4$, $|\vec{b}|=2$ va $|\vec{c}|=6$ bo'lsa, $\vec{p}=\vec{a}+\vec{b}+\vec{c}$ vektorning uzunligini toping.

**290.** $|\vec{a}|=3$, $|\vec{b}|=5$ bo'lsa $\alpha$ ning qanday qiymatlarida $\vec{a}+\alpha\vec{b}$ va $\vec{a}-\alpha\vec{b}$ vektorlar perpendikulyar bo'ladi.

**291.** $\vec{a}=\vec{i}+2\vec{j}+\vec{k}-\frac{4(\vec{i}+2\vec{j})+3\vec{k}}{5}$ vektorning modulini hisoblang va yo'naltiruvchi kosinuslarini toping

**292.** $(\vec{a}+\vec{b})\vec{c}$ ni hisoblang, agar

$$|\vec{a}|=4,\ |\vec{b}|=\sqrt{2},\ |\vec{c}|=3,\ (\vec{a},\hat{}\ \vec{b})=120^\circ,\ (\vec{b},\hat{}\ \vec{c})=45^\circ.$$

**293.** $\vec{a}$ va $\vec{b}$ vektorlar $\varphi=\frac{\pi}{6}$ burchak hosil qiladi va $|\vec{a}|=\sqrt{3}$, $|\vec{b}|=1$. $\vec{p}=\vec{a}+\vec{b}$ va $\vec{q}=\vec{a}-\vec{b}$ vektorlar orasidagi burchakni toping.

**294.** $\vec{a}=\{4;\ -2;\ -4\}$, $\vec{b}=\{6;\ -3;\ 2\}$ bo'lsa, quyidagilarni hisoblang:

1) $\vec{a}\vec{b}$; 2) $\sqrt{\vec{a}^2}$; 3) $\sqrt{\vec{b}^2}$; 4) $(2\vec{a}-\vec{b})(\vec{a}+2\vec{b})$; 5) $(\vec{a}+\vec{b})^2$; 6) $(\vec{a}-\vec{b})^2$.

**295.** $A(-1;3;-7)$, $B(2;-1;5)$ va C (0; 1; -5) nuqtalar berilgan. Quyidagilarni aniqlang:

1) $(2\overline{AB}-\overline{CB})(2\overline{BC}+\overline{BA})$; 2) $\sqrt{\overline{AB}^2}$; 3) $\sqrt{\overline{AC}^2}$.

**296.** To‘rtburchak uchlarining koordinatalari berilgan: $A$ (1; -2; 2), $B$(1; 4; 0), $C$(-4; 1; 1) va D(-5;-5; 3). $AC$ va $BD$ diognallarning o‘zaro perpendikulyar ekanini ko‘rsating.

**297.** $\alpha$ ning qanday qiymatlarida $\bar{a}=\alpha\bar{i}-3\bar{j}+2\bar{k}$ va $\bar{b}=\bar{i}+2\bar{j}-\alpha\bar{k}$ vektorlar o‘zaro perpendikulyar bo‘ladi.

**298.** Uchburchak uchlari koordinatalari berilgan: $A$(-1; -2; 4), $B$(-4;-2; 0) va C(3;-2; 1). $B$ uchidagi burchakni aniqlang.

**299.** Uchburchak uchlari koordinatalari berilgan: A(3; 2;-3), B(5; 1; -1) va C(1;-2; 1). A uchidagi tashqi burchakni aniqlang.

**300.** Uchlari A(1; 2; 1), B(3;-1; 7), C(7; 4;-2) nuqtalarda bo‘lgan uchburchakni ichki burchaklarini aniqlang va teng yonli ekanini ko‘rsating.

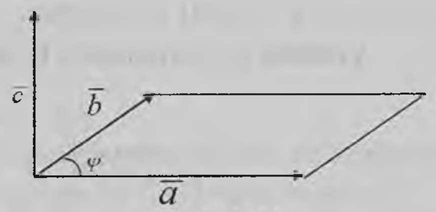
**301.** $\bar{a}=\{3;-1;5\}$ va $\bar{b}=\{1;2;-3\}$ vektorlar berilgan. $\bar{x}\bar{a}=9$, $\bar{x}\bar{b}=-4$ shartlarni qanoatlantiruvchi va *Oz* o‘qiga perpendikulyar $\bar{x}$ vektorni toping.

**302.** $\bar{a}=2\bar{i}-\bar{j}+3\bar{k}$, $\bar{b}=\bar{i}-3\bar{j}+2\bar{k}$ va $\bar{c}=3\bar{i}+2\bar{j}-4\bar{k}$ vektorlar berilgan. $\bar{x}\bar{a}=-5$, $\bar{x}\bar{b}=-11$, $\bar{x}\bar{c}=20$ shartlarni qanoatlantiruvchi $\bar{x}$ vektorni toping.

## 12 §. Vektorlarning vektor va aralash ko‘paytmasi

### Ikki vektorning vektor ko‘paytmasi

**Ta’rif.** Ikki $\bar{a}$ va $\bar{b}$ vektorlarning **vektor ko‘paytmasi** deb shunday $\bar{c}$ vektorga aytiladiki, bu vektor $\bar{a}$ va $\bar{b}$ vektorlarga perpendikulyar, uzunligi tomonlari $\bar{a}$ va $\bar{b}$ vektorlardan tuzilgan parallelogramm yuziga teng, yo‘nalishi $\bar{c}$ vektorning uchidan qaraganda $\bar{a}$ vektordan $\bar{b}$ vektorga o‘tishning eng qisqa yo‘l soat strelkasi harakati yo‘nalishiga qarama-qarshi bo‘lishi kerak. (6-shakl).



6-shakl

Vektor ko‘paytma $\bar{a}\times\bar{b}$ yoki $[\bar{a}\,\bar{b}]$ ko‘rinishda yoziladi.

Ta’rifga ko‘ra,

$$|\bar{c}|=\left|\left[\bar{a}\cdot\bar{b}\right]\right|=|\bar{a}|\cdot|\bar{b}|\sin\varphi,$$

bu yerda $\varphi$ $-\bar{a}$ va $\bar{b}$ vektorlar orasidagi burchak.

Vektor ko‘paytma quyidagi xossalarga ega:

1. $\left[\bar{a}\bar{b}\right]=-\left[\bar{b}\bar{a}\right]$;
2. $\lambda\left[\bar{a}\bar{b}\right]=\left[\lambda\bar{a}\bar{b}\right]=\left[\bar{a}\lambda\bar{b}\right]$;
3. $(\bar{a}+\bar{b})\cdot\bar{c}=[\bar{a}\bar{c}]+[\bar{b}\bar{c}]$;
4. Agar $\bar{a}\|\bar{b}$ bo‘lsa, $\left[\bar{a}\bar{b}\right]=0$.

1 va 4 xossalardan foydalanib $\bar{i},\bar{j},\bar{k}$ bazis vektorlar uchun quyidagi formulalarni topamiz:

$$[\bar{i}\bar{i}]=0;\ [\bar{i}\bar{j}]=\bar{k};\ [\bar{i}\bar{k}]=-\bar{j};$$
$$[\bar{j}\bar{i}]=-\bar{k};\ [\bar{j}\bar{j}]=0;\ [\bar{j}\bar{k}]=\bar{i};$$
$$[\bar{k}\bar{i}]=\bar{j};\ [\bar{k}\bar{j}]=-\bar{i};\ [\bar{k}\bar{k}]=0.$$

**1-teorema.** $\bar{a}$ va $\bar{b}$ vektorlar o‘zlarining koordinatalari bilan berilgan bo‘lsin:

$$\bar{a}=\{X_1,Y_1,Z_1\},\ \bar{b}=\{X_2,Y_2,Z_2\}.$$

U holda $\bar{a}$ va $\bar{b}$ vektorlarning vektor ko‘paytmasi quyidagi formula bilan topiladi:

$$\left[\bar{a}\,\bar{b}\right]=\begin{vmatrix}Y_1 & Z_1\\ Y_2 & Z_2\end{vmatrix}\bar{i}+\begin{vmatrix}Z_1 & X_1\\ Z_2 & X_2\end{vmatrix}\bar{j}+\begin{vmatrix}X_1 & Y_1\\ X_2 & Y_2\end{vmatrix}\bar{k}=\begin{vmatrix}i & j & k\\ X_1 & Y_1 & Z_1\\ X_2 & Y_2 & Z_2\end{vmatrix}. \qquad (6)$$

**303.** $\bar{a}$ va $\bar{b}$ vektorlar o‘zlarining koordinatalari bilan berilgan $\bar{a}=(2;5;7)$ va $\bar{b}=[1;2;4]$ $[\bar{a}\,\bar{b}]$ topilsin.

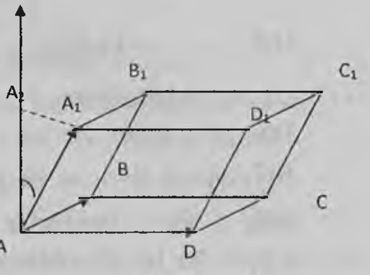
**Yechish.** (6) formuladan foydalansak,

$$\left[\bar{a}\bar{b}\right]=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & 5 & 7\\ 1 & 2 & 4\end{vmatrix}=6\bar{i}-\bar{j}-\bar{k}$$

natijani olamiz.

## Uch vektorning aralash ko‘paytmasi.

Biz ikki $\bar{a}$ va $\bar{b}$ vektorlar uchun skalyar va vektor ko‘paytma tushunchalari bilan tanishdik. Bizga ixtiyoriy 3 ta $\bar{a},\bar{b}$ va $\bar{c}$ vektorlar berilgan bo‘lsin. $\bar{a}$ vektorni $\bar{b}$ vektorga vektor ko‘paytirib, $[\bar{a}\bar{b}]$ vektorni hosil qilamiz. $[\bar{a}\bar{b}]$ vektorni $\bar{c}$ vektorga skalyar ko‘paytirib, $[\bar{a}\bar{b}]\bar{c}$ sonni hosil qilamiz. Berilgan $\bar{a},\bar{b}$ va $\bar{c}$ vektorlarni bunday tartibda ko‘paytirish vektor-skalyar yoki aralash ko‘paytma deb ataladi.



**7-shakl**

**2-teorema.** $\bar{a},\bar{b}$ va $\bar{c}$ vektorlarning aralash ko‘paytmasining absolyut qiymati, shu vektorlarga qurilgan parallelopipedning hajmiga teng.

**Natija.** $V_{pir} = \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$.

$\bar{a},\bar{b}$ va $\bar{c}$ vektorlar koordinatalar bilan berilgan bo‘lsin.

$\bar{a}=\{X_1,Y_1,Z_1\}$, $\bar{b}=\{X_2,Y_2,Z_2\}$ va $\bar{c}=\{X_3,Y_3,Z_3\}$

$$[\bar{a}\bar{b}]=\begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}\bar{i}+\begin{vmatrix} Z_1 & X_1 \\ Z_2 & X_2 \end{vmatrix}\bar{j}+\begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix}\bar{k}$$

vektorni $\bar{c}=X_3\bar{i}+Y_3\bar{j}+Z_3\bar{k}$ vektorlarga skalyar ko‘paytiramiz.

$$([\bar{a}\bar{b}]\bar{c})=X_3\begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}+Y_3\begin{vmatrix} Z_1 & X_1 \\ Z_2 & X_2 \end{vmatrix}+Z_3\begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix}=\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}$$

Demak,

$$\left[\vec{a}\vec{b}\right]\vec{c}=\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} \qquad (7)$$

$\bar{a}=\{X_1,Y_1,Z_1\}$, $\bar{b}=\{X_2,Y_2,Z_2\}$ va $\bar{c}=\{X_3,Y_3,Z_3\}$ vektorlarning aralash ko‘paytmasi nolga teng bo‘lishi uchun ular komplanar bo‘lishi zarur va yetarli.

**304.** $\bar{a}=\{1;2;3\}$, $\bar{b}=\{-1;3;4\}$ va $\bar{c}=\{2;5;2\}$ vektorlarga qurilgan parallelopipedning hajmini toping.

**Yechish.** (7) formulaga asosan

$$V=\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}=\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & 5 & 2 \end{vmatrix}=|6-15+16-18-20+4|=|-27|=27 \text{ kub birlik.}$$

**305.** $\vec{a}$ va $\vec{b}$ vektorlar $\varphi=\frac{\pi}{6}$ burchak hosil qiladi. $|\vec{a}|=6$, $|\vec{b}|=5$ bo'lsa, $|[\vec{a}\vec{b}]|$ ni hisoblang.

**306.** $|\vec{a}|=10$, $|\vec{b}|=2$ va $\vec{a}\vec{b}=12$ bo'lsa, $|[\vec{a}\vec{b}]|$ ni hisoblang.

**307.** $|\vec{a}|=3$, $|\vec{b}|=26$ va $\left|[\vec{a}\vec{b}]\right|=72$ bo'lsa, $\vec{a}\vec{b}$ ni hisoblang.

**308.** $\vec{a}$ va $\vec{b}$ vektorlar o'zaro perpendikulyar, shuningdek, $|\vec{a}|=3$, $|\vec{b}|=4$ bo'lsa, hisoblang:

1) $\left|\left[(\vec{a}+\vec{b})(\vec{a}-\vec{b})\right]\right|$;

2) $\left|\left[(3\vec{a}-\vec{b})(\vec{a}-2\vec{b})\right]\right|$.

**309.** $\vec{a}$ va $\vec{b}$ vektorlar $\varphi=\frac{2\pi}{3}$ burchak hosil qiladi. $|\vec{a}|=1$, $|\vec{b}|=2$ bo'lsa, hisoblang:

1) $[\vec{a}\,\vec{b}]^2$; 2) $[(2\vec{a}+\vec{b})(\vec{a}+2\vec{b})]^2$;

3) $[(\vec{a}+3\vec{b})(3\vec{a}-\vec{b})]^2$.

**310.** $\vec{a}+\vec{b}$ va $\vec{a}-\vec{b}$ vektorlar kollinear bo'lsa, $\vec{a}$ va $\vec{b}$ vektorlar qanday joylashgan bo'ladi?

**311*.** Tenglikni isbotlang: $[\vec{a}\,\vec{b}]^2+(\vec{a}\,\vec{b})^2=\vec{a}^2\vec{b}^2$.

**312*.** Tengsizlikni isbotlang

$$[\vec{a}\,\vec{b}]^2\le\vec{a}^2\vec{b}^2;$$

Qanday holda tenglik o'rinli bo'ladi?

**313*.** $\vec{a}$, $\vec{b}$ va $\vec{c}$ vektorlar uchun $\vec{a}+\vec{b}+\vec{c}=0$ tenglik o'rinli bo'lsa,

$[\vec{a}\vec{b}]=[\vec{b}\vec{c}]=[\vec{c}\vec{a}]$ ni isbotlang.

**314.** $\vec{a}$, $\vec{b}$, $\vec{c}$ va $\vec{d}$ vektorlar uchun $[\vec{a}\vec{b}]=[\vec{c}\vec{d}]$; $[\vec{a}\vec{c}]=[\vec{b}\vec{d}]$ munosabatlar o'rinli bo'lsa, $\vec{a}-\vec{d}$ va $\vec{b}-\vec{c}$ vektorlar kolleniar bo'lishini isbotlang.

**315.** $\vec{a}=\{3;-1;-2\}$ va $\vec{b}=\{1;2;-1\}$ vektorlar berilgan. Quyidagilarni aniqlang:

1) $[\vec{a}\vec{b}]$; 2) $[(2\vec{a}+\vec{b})\vec{b}]$; 3) $[(2\vec{a}-\vec{b})(2\vec{a}+\vec{b})]$.

**316.** A(2; -1; 2), B(1:2; — 1) va C(3; 2; 1) nuqtalar berilgan. Quyidagi vektor ko'paytmalarni aniqlang:

1) $[\overline{AB}\ \overline{BC}]$; 2) $[(\overline{BC}-2\overline{CA})\ \overline{CB}]$.

**317.** $\bar{f}=\{3; 2; -4\}$ kuch va uning qo'yilish nuqtasi A(2; -1; 1) berilgan. Koordinatalar boshiga nisbatan kuch momentini toping.

**318.** $\bar{P}=\{2; -4; 5\}$ kuch va uning qo'yilish nuqtasi $M_0(4;-2;3)$ berilgan. A(3; 2; -1) nuqtaga nisbatan kuch momentini toping.

**319.** $\bar{Q}=\{3; 4; -2\}$ kuch va uning qo'yilish nuqtasi C(2;-1;-2) berilgan. Koordinatalar boshiga nisbatan kuch momentini va moment bilan koordinata o'qlari orasidagi burchaklarning kosinusini toping.

**320.** *A*(1; 2; 0), *B*(3; 0; — 3) и *C*(5; 2; 6) nuqtalar berilgan. *ABC* uchburchakning yuzini toping.

**321.** Shunday $\bar{x}$ vektorni topingki, u $\bar{a}=\{2; -3; 1\}$ va $\bar{b}=\{1; -2; 3\}$ vektorlarga perpendikulyar va $\bar{x}(\bar{i}+2\bar{j}-7\bar{k})=10$ tenglik o'rinli bo'ladi.

**322.** $\bar{a}, \bar{b}, \bar{c}$ vektorlar qanday uchlik(o'ng yoki chap) hosil qiladi:

1) $\bar{a}=\bar{k}, \bar{b}=\bar{i}, \bar{c}=\bar{j}$; 2) $\bar{a}=\bar{i}, \bar{b}=\bar{k}, \bar{c}=\bar{j}$;

3) $\bar{a}=\bar{j}, \bar{b}=\bar{i}, \bar{c}=\bar{k}$; 4) $\bar{a}=\bar{i}+\bar{j}, \bar{b}=\bar{j}, \bar{c}=\bar{k}$;

5) $\bar{a}=\bar{i}+\bar{j}, \bar{b}=\bar{i}-\bar{j}, \bar{c}=\bar{j}$; 6) $\bar{a}=\bar{i}+\bar{y}, \bar{b}=\bar{i}-\bar{j}, \bar{c}=\bar{k}$.

**323.** $\bar{a}, \bar{b}, \bar{c}$ o'zaro perpendikulyar va o'ng uchlikni tashkil etadi. Agar $|\bar{a}|=4$, $|\bar{b}|=2$, $|\bar{c}|=3$, bo'lsa $\bar{a}\bar{b}\bar{c}$ ni hisoblang.

**324.** $\bar{c}$ vektor $\bar{a}$ va $\bar{b}$ vektorlarga perpendikulyar, $\bar{a}$ va $\bar{b}$ vektorlar esa 30° li burchak hosil qiladi. Agar $|\bar{a}|=6$, $|\bar{b}|=3$, $|\bar{c}|=3$ bo'lsa, $\bar{a}\bar{b}\bar{c}$ ni hisoblang.

**325*.** $\bar{a}, \bar{b}, \bar{c}$ vektorlar $[\bar{a}\bar{b}]+[\bar{b}\bar{c}]+[\bar{c}\bar{a}]=0$ shart bajarilganda komplanar bo'lishini isbotlang.

**326.** $\bar{a}=\{1; -1; 3\}$, $\bar{b}=\{-2; 2; 1\}$, $\bar{c}=\{3; -2; 5\}$ bo'lsa, $\bar{a}\bar{b}\bar{c}$ ni hisoblang.

**327.** $\bar{a}, \bar{b}, \bar{c}$ vektorlar komplanar bo'ladimi?

1) $\vec{a}=\{2;3;-1\}$, $\vec{b}=\{1;-1;3\}$, $\vec{c}=\{1;9;-11\}$;

2) $\vec{a}=\{3;-2;1\}$, $\vec{b}=\{2;1;2\}$, $\vec{c}=\{3;-1;-2\}$;

3) $\vec{a}=\{2;-1;2\}$, $\vec{b}=\{1;2;-3\}$, $\vec{c}=\{3;-4;7\}$.

**328.** A(1; 2; -1), B (0; 1; 5), C (-1; 2; 1), D (2; 1; 3) nuqtalar bir tekislikda yotishini ko‘rsating.

**329.** Uchlari $A(2;2;2)$, $B(4;3;3)$, $C(4;5;4)$ va $D(5;5;6)$ nuqtalarda bo‘lgan uchburchakli piramidaning hajmini toping.

**330.** Uchlari A (2; -1; 1), B (5; 5; 4), C (3; 2; -1) va D (4; 1; 3) nuqtalarda bo‘lgan tetraedrni hajmini aniqlang.

# III BOB. TEKISLIKDA VA FAZODA ANALITIK GEOMETRIYA

## 13 §. Tekislikda analitik geometriyaning sodda masalalari

Analitik geometriya kursida asosan figuralarni ularning algebraik tenglamalari yordamida o'rganiladi. Bu o'rganishning asosini koordinatalar metodi tashkil qiladi. Analitik geometriya fani tekislikda va fazoda analitik geometriyaga bo'lib o'rganiladi. Biz asosan tenglamalari birinchi va ikkinchi darajali algebraik tenglamalar bilan ifodalaniladigan geometrik figuralar bilan shug'ullanamiz. Biz o'rganadigan geometrik figuralar sinfi fan va texnikada juda muhim rol o'ynaydi.

O nuqtada kesishuvchi o'zaro perpendikulyar $X'X$ va Y'Y to'g'ri chiziqlarni qaraylik (1-shakl). $OX$ va $OY$ nurlarni musbat, $OX'$ va $OY'$ nurlarni manfiy yo'nalish deb qabul qilamiz va bu nurlarda yagona birlik kesma tanlab masshtab kiritamiz. Hosil bo'lgan sistema to'g'ri burchakli Dekart koordinatalar sistemasi deb nomlanadi. Bunda $OX$ nurni $x$ o'qi yoki absissa o'qi, $OY$ nurni $y$ o'qi yoki ordinata o'qi deyiladi. Tekislikdagi ixtiyoriy $M$ nuqta olaylik va bu nuqtadan $x$ va $y$ o'qlariga parallel to'g'ri chiziqlar o'tkazaylik. Bu to'g'ri chiziqlar $x$ va $y$ o'qni mos ravishda $x=a, y=b$ nuqtalarda kesib o'tsin. Bu nuqtalarni $P$ va $Q$ bilan belgilaylik. $M$ nuqtaga $(a,b)$ sonlar juftini mos qo'yamiz va bu juftlikni $M$ nuqtaning koordinatasi deb ataymiz, $M(a,b)$ kabi belgilaymiz. Shu tarzda tekislikdagi barcha nuqtalar va $(x,y)$ haqiqiy sonlar juftligi orasida moslik o'rnatib chiqish mumkin.

Endi ba'zi muhim formulalarni keltiraylik.

Dekart koordinatalari sistemasida biror $M(a,b)$ nuqta berilgan bo'lsin. Bu nuqtaning $Ox$ o'qidagi proyeksiyasi $M_{ox}(a,0)$, $Oy$ o'qidagi proyeksiyasi $M_{oy}(0,b)$, $Ox$ o'qiga nisbatan simmetrik bo'lgan nuqtasi $M'_{ox}(a,-b)$, $Oy$ o'qiga nisbatan simmetrik bo'lgan

$$\begin{cases} x' = x - a \\ y' = y - b \end{cases} \quad (8)$$

$OXY$ Dekart koordinatalar sistemao‘qlarini O nuqta atrofida $\varphi$ burc Koordinata boshiga nisbatan burishda yangi $OX'Y'$ koordinatalar sistemasi hosil bo‘lsin. $OXY$ koordinatalar sistemasidagi $M(x, y)$ nuqtaning $OX'Y'$ koordinatalar sistemasidagi koordinatalarini quyidagi formula orqali topiladi:

$$\begin{cases} x' = x\cos\varphi + y\sin\varphi \\ y' = -x\sin\varphi + y\cos\varphi \end{cases} \quad (9)$$

**331.** A(3;8) B(-5;14) nuqtalar orasidagi masofasini toping.

**Yechish.** (1) formuladan foydalanib,

$d = \sqrt{(-5-3)^2 + (14-8)^2} = \sqrt{64+36} = 10$ ga ega bo‘lamiz.

**332.** A(-1;3) B(3;-2) AB kesmani 1.5 nisbatda bo·luvchi C nuqtani va AB ning o‘rtasi D nuqtani koordinatalarini toping.

**Yechish.** (2) formula orqali C nuqtani, (3) formula orqali D nuqtani koordinatalarini topamiz:

$$\begin{cases} x = \dfrac{-1+4.5}{1+1.5} = 1.4 \\ y = \dfrac{3-3}{1+1.5} = 0 \end{cases}, \quad \begin{cases} x = \dfrac{-1+3}{2} = 1 \\ y = \dfrac{3-2}{2} = 0.5 \end{cases}.$$

Demak, C(1.4;0), D(1;0.5) ekan.

**333.** Uchlari A(1;-2), B(3;4),C(-7;6) nuqtada bo‘lgan uchburchak yuzini toping.

**Yechish.** (6) formuladan foydalanamiz.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & -7 \\ -2 & 4 & 6 \end{vmatrix} = 18 + 14 + 4 + 6 + 28 - 6 = 64, \; S = \frac{1}{2} \cdot 64 = 32.$$

**334.** A(2;135⁰) nuqtani Dekart, $B(\frac{-3}{2}; \frac{-\sqrt{3}}{2})$ nuqtani qutb koordinatalar sistemasidagi koordinatalarini toping.

**Yechish**. (7) formuladan $\begin{cases} x = 2\cos 135^0 = -\sqrt{2} \\ y = 2\sin 135^0 = \sqrt{2} \end{cases}$ A($-\sqrt{2}$; $\sqrt{2}$),

$$\begin{cases} r=\sqrt{(-\frac{3}{2})^2+(-\frac{\sqrt{3}}{2})^2}=\sqrt{3} \\ \varphi = arctg\frac{-\frac{\sqrt{3}}{2}}{-\frac{3}{2}} = arctg\frac{\sqrt{3}}{3} \end{cases}$$ ekanligini topamiz. B nuqtaning ikkala koordinatalari ham manfiy bo'lgani uchun 3- chorak burchagi ya'ni $\varphi=210^0$ ekan. Bundan $B(\sqrt{3};210^0)$ ekani kelib chiqadi.

**335.** Koordinata o'qlarini $60^0$ ga burish natijasida $M(4,-2)$ nuqtaning yangi koordinatalar sistemasidagi koordinatalarini toping.

**Yechish.** (9) formulaga ko'ra

$$\begin{cases} x'=4\cos60^0-2\sin60^0 \\ y'=-4\sin60^0-2\cos60^0 \end{cases}, \begin{cases} x'=2-\sqrt{3} \\ y'=-2\sqrt{3}-1 \end{cases}$$ ni hosil qilamiz.

**336.** Quyidagi nuqtalarni Dekart koordinatalar sistemasida belgilang.

$$A(2;\ 3), B(-5;1), C(-2;\ -3), D(0;\ 3), E(-5;\ 0), F(-\frac{1}{3};\frac{2}{3})$$

**337.** Quyidagi nuqtalarni absissa va ordinata o'qidagi proyeksiyalarini toping.

$$A(2;-3), B(3;-1), C(-5;1), D(-3;-2), E(-5;-1).$$

**338.** Quyidagi nuqtalarga absissa va ordinata o'qlariga va koordinata boshiga nisbatan simmetrik nuqtalarni toping.

1) $A(2;\ 3)$; 2) $B(-3;\ 2)$; 3) $C(-1;\ -1)$;

4) $D(-3;\ -5)$; 5) $E(-4;\ 6)$; 6) $F(a;b)$.

**339.** Uchburchakning tomonlari o'rtalarining koordinalari $M(1;-1), N(-1;\ 4)$ va $P(-2;\ 2)$. Uchlarini koordinatalarini toping.

**340*.** Uchlari $A(1;-3)$ va $B(4;3)$ nuqtalarda bo'lgan kesma teng 5 qismga bo'lingan. Bo'linish nuqtalarining koordinatalarini toping.

**341.** Uchlarining koordinatalari berilgan uchburchakning yuzasini hisoblang:

1) $A(2;\ -3), B(3;2)$ va $C(-2;\ 5)$;

2) $M_1(-3;\ 2), M_2(5;-2), M_3(1;\ 3)$;

3) $M(3;\ -4), N(-2;3), P(4;\ 5)$.

**342*.** Uchlarining koordinatalari $A(3;6), B(-1;3), C(2;-1)$ bo'lgan uchburchakning C uchidan tushirilgan balandligini toping.

**343.** Uchta uchining koordinatalari berilgan parallelogramm yuzasini hisoblang:

$A(-2;\ 3), B(4;\ -5), C(-3;\ 1)$.

**344*.** Uchburchakning yuzi $S=4$, ikkita uchi $A(2;\ 1), B(3;-2)$ bo‘lib, C uchi $Ox$ o‘qida yotadi. C uchini koordintasini toping.

**345*.** Qutb koordinatalari sistemasida quyidagi $A(3;-\frac{4}{9}\pi)$, $B(5;\ \frac{3}{13}\pi)$ nuqtalar berilgan. $ABCD$ parallelogrammning diagonallar kesishgan nuqtasi qutb boshi bilan ustma-ust tushadi. $C$ va $D$ nuqta koordinalarini toping.

**346.** Qutb koordinatalari sistemasida quyidagi $A(\ 8;\ -\frac{2}{3}\pi)$ va $B(6;\ \frac{\pi}{3})$ nuqtalar berilgan. AB kesma o‘rtasini koordinatasini toping.

**347.** Koordinata o‘qlarini parallel ko‘chirish natijasida koordinata boshi O nuqta

$O'$(-1;3) nuqtaga o‘tsa, M(2;-1) nuqtaning yangi sistemadagi koordinatalarini toping.

**348.** Koordinata o‘qlarini $60^0$ ga burish natijasida qanday nuqta (-2;4) nuqtaga o‘tadi.

## 14 §. Tekislikda to‘g‘ri chiziq tenglamalari. Tekislikda to‘g‘ri chiziqqa doir turli masalalar.

Maktab geometriya kursidan ma’lumki, to‘g‘ri chiziq eng sodda geometrik shakllardan biri bo‘lib, u ta’riflanmaydi.

To‘g‘ri chiziqning turli tenglamalarini keltiramiz.

**To‘g‘ri chiziqning burchak koeffitsiyentli tenglamasi.**

$$y = kx + b \qquad (10)$$

(10) tenglama to‘g‘ri chiziqning burchak koeffitsiyentli tenglamasi deb ataladi. Bunda to‘g‘ri chiziqning $OX$ o‘qi musbat yo‘nalishi bilan hosil qilgan burchagi $\alpha$, to‘g‘ri chiziqning ordinatalar o‘qidan ajratgan kesmasining kattaligi $b$ ga teng va $tg\alpha = k$ munosabat o‘rinli.

**To‘g‘ri chiziqning umumiy tenglamasi**

$$Ax + By + C = 0 \qquad (11)$$

(11) tenglama to‘g‘ri chiziqning umumiy tenglamasi deyiladi. Bunda $A^2+B^2>0$, $\bar{n}(A;B)$ to‘g‘ri chiziqqa perpendikulyar bo‘lgan ixtiyoriy vektor (normal vektor).

1) $A\neq 0,\ B\neq 0,\ C=0$ bo‘lsa, $Ax+By=0$ bo‘lib, to‘g‘ri chiziq koordinatlar boshidan o‘tadi, chunki $O(0,\ 0)$ nuqtaning koordinatlari tenglamani qanoatlantiradi.

2) $A=0,\ B\neq 0,\ C\neq 0$, bo‘lsa, $y=-\frac{C}{B}$ bo‘lib, $OY$ o‘qdan $-\frac{C}{B}$ kesma ajratib, $OX$ o‘qiga parallel to‘g‘ri chiziq tenglamasi bo‘ladi.

3) $B=0,\ A\neq 0,\ C\neq 0$ bo‘lsa, $x=-\frac{C}{A}$ bo‘lib, $OX$ o‘qdan $-\frac{C}{A}$ kesma ajratib, $OY$ o‘qiga parallel to‘g‘ri chiziq tenglamasi bo‘ladi.

4) $A=0,\ C=0,\ B\neq 0$ bo‘lsa, $y=0$ bo‘lib, $OX$ o‘qining tenglamasi hosil bo‘ladi.

5) $B=0,\ C=0,\ A\neq 0$ bo‘lsa, $x=0$ bo‘lib, $OY$ o‘qining tenglamasi hosil bo‘ladi.

**To‘g‘ri chiziqning kanonik tenglamasi**

$$\frac{x-x_1}{m}=\frac{y-y_1}{n} \qquad (12)$$

(12) tenglama to‘g‘ri chiziqning kanonik tenglamasi deyiladi. Bunda $(x_1;y_1)$to‘g‘ri chiziqqa tegishli biror nuqtaning koordinatalari, $s(m;n)$ shu to‘g‘ri chiziqqa parallel bo‘lgan biror vektor.

**To‘g‘ri chiziqning paramertik tenglamasi**

$$\begin{cases} x=x_1+mt \\ y=y_1+nt \end{cases} \qquad (13)$$

(13) tenglama to‘g‘ri chiziqning parametrik tenglamasi deyiladi. Bu tenglama (12) tenglamani $t$ ga tenglab hosil qilinadi.

**To‘g‘ri chiziqning kesmalarga nisbatan tenglamasi**

To‘g‘ri chiziq koordinat o‘qlaridan mos ravishda $a$ va $b$ kesmalar ajratib o‘tsin ya’ni to‘g‘ri chiziq $A(a,0)$ va $B(0,\ b)$ nuqtalardan o‘tsin. U holda bu to‘g‘ri chiziq tenglamasi quyidagi

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (14)$$

tenglama orqali ifodalaniladi. Bu tenglamaga to‘g‘ri chiziqning kesmalarga nisbatan tenglamasi deyiladi.

**Ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi**

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad (15)$$

(15) tenglama berilgan ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi deyiladi.

Bunda to'g'ri chiziq $(x_1;y_1), (x_2;y_2)$ turli nuqtalar orqali o'tadi. (15) quyidagicha ham ifodalash mumkin:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0. \qquad (16)$$

**To'g'ri chiziqning normal tenglamasi**

$$x\cos\alpha+y\sin\alpha-p=0 \qquad (17)$$

(17) tenglama to'g'ri chiziqning normal tenglamasi hosil bo'ladi. (11) tenglamada $A^2+B^2+C^2>0$ shart bajarilsa, bu tenglamani (17) ko'rinishida yozish mumkin.

Bunda $p$-koordinata boshidan to'g'ri chiziqqa o'tkazilgan perpendikulyar uzunligi, $\alpha$-perpendikulyar bilan OX o'qning musbat yo'nalishi orasidagi burchak.

**Ikki to'g'ri chiziq orasidagi burchak**

$y=k_1x+b_1$, $y=k_2x+b_2$ to'g'ri chiziqlar orasidagi burchak

$$tg\alpha=\left|\frac{k_2-k_1}{1+k_1\cdot k_2}\right| \qquad (18)$$

formula orqali aniqlanadi. Agar to'g'ri chiziqlar $A_1x+B_1y+C_1=0$, $A_2x+B_2y+C_2=0$ umumiy tenglamalari orqali berilgan bo'lsa,

$$\cos\alpha=\frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\cdot\sqrt{A_2^2+B_2^2}} \qquad (19)$$

formula orqali aniqlanadi.

**Nuqtadan to'g'ri chiziqqacha bo'lgan masofa** $M_0(x_0;y_0)$ nuqtadan $l: Ax+By+C=0$ to'g'ri chiziqqacha bo'lgan masofa

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}} \qquad (20)$$

formula orqali aniqlanadi.

**349.** $A(-1;1), B(1;3)$ nuqtalar orqali o'tuvchi to'g'ri chiziqning turli tenglamalarini yozing, $Ox$ o'qi bilan hosil qilgan burchakni toping, $O(0;0)$ nuqtadan to'g'ri chiziqqacha masofani hisoblang.

**Yechish.** (15) tenglamadan $\frac{x+1}{2}=\frac{y-1}{2}$ kanonik tenglamani hosil qilamiz. Bundan shu to'g'ri chiziqning $\begin{cases} x=t-1 \\ y=t+1 \end{cases}$ parametrik,

$x-y+2=0$ umumiy, $\frac{x}{-2}+\frac{y}{2}=1$ kesmalar bo'yicha va $y=x+2$ burchak koeffitsiyentli tenglamalarini hosil qilamiz. $Ox$ o'qi bilan hosil qilgan burchagi $\alpha=arctg1=45^0$ ekanini topamiz. (20) formula orqali $O(0;0)$ nuqtadan to'g'ri chiziqqacha bo'lgan masofa $d=\sqrt{2}$ ekanligini topamiz. $x-y+2=0$ umumiy tenglamani $-\sqrt{A^2+B^2}=-\sqrt{2}$ ga bo'lib, $-\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y-\sqrt{2}=0$ yoki $x\cos135^0+y\sin135^0-\sqrt{2}=0$ to'g'ri chiziqning normal tenglamasini hosil qilamiz.

**350.** $y=-3x+7, y=2x+1$ to'g'ri chiziqlar orasidagi burchakni aniqlang.

**Yechish.** $k_1=-3, k_2=2$ bo'lgani uchun (18) formuladan $tg\alpha=\left|\frac{2-(-3)}{1+(-3)\cdot 2}\right|=1$, yoki $\alpha=45$ ekanini aniqlaymiz.

**351.** Berilgan nuqtalardan qaysilari $2x-3y-3=0$ to'g'ri chiziqqa tegishli?

$M_1(3; 1)$, $M_2(2; 3)$, $M_3(6; 3)$, $M_4(-3; -3)$. $M_5(3; -1)$, $M_6(-2; 1)$.

**352.** $2x-3y-12=0$ to'g'ri chiziqni koordinata o'qlari bilan kesishish nuqtalarini toping. Grafigini yasang.

**353.** To'g'ri chiziqlar kesishish nuqtasini va orasidagi burchagini toping:

$$3x-y-14=0,\ 2x+y-6=0.$$

**354.** Uchburchakning tomonlari tenglamalari $x-3y=0$, $x-y+2=0$, $3x+y-10=0$ bo'lgan to'g'ri chiziqlarda yotsa, uchburchakning uchlari koordinatalarini, yuzini va burchaklarini toping.

**355.** To'g'ri chiziqlar orasidagi burchakni aniqlang.

1) $3x-y+5=0,\ 2x+y-7=0;$

2) $x\sqrt{2}-y\sqrt{3}-5=0$, $(3+\sqrt{2})x+(\sqrt{6}-\sqrt{3})y+7=0$;

3) $x\sqrt{3}+y\sqrt{2}-2=0$, $x\sqrt{6}-3y+3=0$.

**356*.** Agar uchburchakda A(2;–1), B uchidan o‘tkazilgan balandlik $3x-4y+27=0$, C uchidan o‘tkazilgan bissektrissa $x+2y-5=0$, tenglamalari ma’lum bo‘lsa, tomonlari tenglamalarini tuzing.

**357*.** *m* va *n* ning qanday qiymatlarida quyidagi to‘g‘ri chiziqlar

$$mx+8y+n=0,\ 2x+my-1=0$$

1) Parallel bo‘ladi; 2) Usma-ust tushadi; 3) Perpendikulyar bo‘ladi?

**358.** $A(-1;3)$ nuqta va $3x-y-4=0$ to‘g‘ri chiziq berilgan.

1) Nuqtadan to‘g‘ri chiziqqacha bo‘lgan masofani toping;

2) Nuqtaning to‘g‘ri chiziqdagi proyeksiyasini toping;

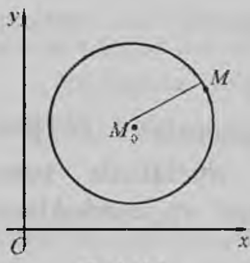
3) To‘g‘ri chiziqqa nisbatan nuqtaga simmetrik nuqtaning koordinatasini toping.

**359*.** *M*(4;3) nuqta orqali o‘tuvchi va koordinata o‘qlari bilan kesishib 3 birlik yuza ajratuvchi to‘g‘ri chiziqning koordinata o‘qlari bilan kesishgan nuqtalari koordinatalarini toping.

**360.** Umumiy tenglama bilan berilgan to‘g‘ri chiziqning turli tenglamalarini tuzing.

1) $3x-4y-10=0$ 2) $5x-12y+26=0$ 3) $24x-10y+39=0$.

## 15 §. Ikkinchi tartibli chiziqlar. Aylana, ellips, giperbola va parabola.



**5-shakl.**

Tekislikda

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \qquad (21)$$

tenglama bilan aniqlanuvchi chiziqlar **ikkinchi tartibli chiziqlar** deb ataladi. Bunda $A^2+B^2+C^2>0$.

**Aylana.** Markaz deb ataluvchi nuqtadan teng uzoqlikda yotuvchi tekislik nuqtalarining geometrik o‘rniga **aylana** deyiladi (5-shakl). Aylananing kanonik tenglamasi

$$(x-a)^2+(y-b)^2=R^2 \qquad (22)$$

(22) tenglama bilan aniqlanadi. Bunda $M_0(a;b)$ nuqta aylana **markazi**, $R$ masofa aylana **radiusi** deb ataladi.

Xususan, $a=0, b=0$ da (22) tenglamadan quyidagini topamiz:

$$x^2+y^2=R^2. \tag{23}$$

Bu tenglama markazi koordinatalar boshida yotuvchi va radiusi $R$ ga teng aylanani aniqlaydi.

$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \ t\in[0;2\pi] \tag{24}$$

(24) tenglamalar sistemasiga **aylananing parametrik tenglamalari** deyiladi.

Aylana bilan bitta $N_0(x_0;y_0)$ umumiy nuqtaga ega bo'lgan to'g'ri chiziq aylanaga shu nuqtadan o'tkazilgan urinma deb ataladi. $x^2+y^2=R^2$ aylananing $N_0(x_0;y_0)$ nuqtasidan o'tuvchi urinma tenglamasi quyidagicha bo'ladi:

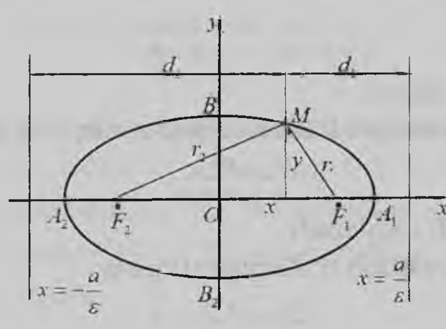
$$xx_0+yy_0-R^2=0. \tag{25}$$

$M_0(a;b)$ markazli aylanaga $N_0(x_0;y_0)$ nuqtada o'tkazilgan urinma esa

$$(x-a)(x_0-a)+(y-b)(y_0-b)=R^2 \tag{26}$$

ko'rinishda bo'ladi.

**Ellips.** Fokuslar deb ataluvchi berilgan ikki nuqtagacha bo'lgan masofalarning yig'indisi o'zgarmas miqdorga teng bo'lgan tekislik nuqtalarining geometrik o'rniga **ellips** deyladi (6-shakl). $F_1$ va $F_2$ ellipsning fokuslari, $M$ ellipsning ixtiyoriy nuqtasi bo'lsin.




**6-shakl**

$|F_1F_2|=2c$, $|F_1M|=r_1$, $|F_2M|=r_2$ bo'lsa, ellipsning ta'rifiga ko'ra

$$r_1 + r_2 = 2a, \qquad (27)$$

bu yerda $a$ – o‘zgarmas musbat son $(2a > 2c)$ va $b^2 = a^2 - c^2$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (28)$$

(28) tenglamaga **ellipsning kanonik tenglamasi** deyiladi.

Ellipsda $A_1(a;0)$, $A_2(-a;0)$, $B_1(0;b)$, $B_2(0;-b)$ nuqtalarga uchlar, $|A_1A_2|$, $|B_1B_2|$ kesmalarning $2a$, $2b$ uzunliklariga mos ravishda katta va kichik o‘qlar, $a$, $b$ sonlarga mos ravishda katta va kichik yarim o‘qlar, $|F_1M|$, $|F_2M|$ kesmalarning $r_1, r_2$ uzunliklariga fokal radiuslar deyiladi.

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, \ t \in [0;2\pi] \qquad (29)$$

tenglamalar sistemasiga **ellipsning parametrik tenglamalari** deyiladi.

$\varepsilon = \frac{c}{a}$ kattalikka **ellipsning ekssentrisiteti** deyiladi. Bunda $0 < \varepsilon < 1$, chunki $0 < c < a$. $M$ nuqtadan $d_1$, $d_2$ masofada o‘tuvchi va tenglamalari $x = \pm\frac{a}{\varepsilon}$ dan iborat bo‘lgan to‘g‘ri chiziqlar **ellipsning direktrisalari** deb ataladi. Direktrisalar ushbu

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon \qquad (30)$$

tengliklarni qanoatlantiradi. Bu tengliklardan ellipsning fokal radiuslari uchun

$$r_1 = a - \varepsilon x, \ r_2 = a + \varepsilon x \qquad (31)$$

formulalar kelib chiqadi.

Ellipsning qutb koordinatalar sistemasidagi tenglamasi

$$\rho = \frac{p}{1 - \varepsilon\cos\varphi} \qquad (32)$$

(32) formula orqali aniqlanadi.

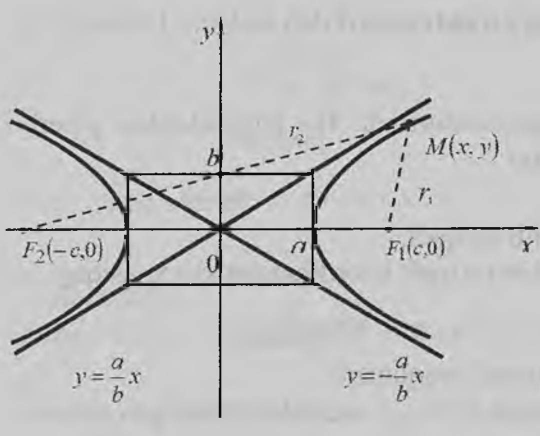
Ellipsga $M_0(x_0;y_0)$ nuqtada o‘tkazilgan urinma

$$\frac{x \cdot x_0}{a^2} + \frac{y \cdot y_0}{b^2} = 1 \qquad (33)$$

formula orqali aniqlanadi.

**Giperbola.** Fokuslar deb ataluvchi berilgan ikki nuqtagacha bo'lgan masofalar ayirmasining moduli o'zgarmas miqdorga teng bo'lgan tekislik nuqtalarining geometrik o'rniga **giperbola** deyiladi (7-shakl).

$F_1$ va $F_2$ ellipsning fokuslari, $M$ giperbolaning ixtiyoriy nuqtasi bo'lsin.




**7-shakl**

$|F_1F_2|=2c$, $|F_1M|=r_1$, $|F_2M|=r_2$ bo'lsa, ellipsning ta'rifiga ko'ra

$$|r_1-r_2|=2a, \tag{34}$$

bu yerda $a-$ o'zgarmas musbat son $(2a<2c)$ va $b^2=c^2-a^2$.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \tag{35}$$

(35) tenglamaga **giperbolaning kanonik tenglamasi** deyiladi.

Giperbolada $A_1(a;0)$, $A_2(-a;0)$, $B_1(0;b)$, $B_2(0;-b)$ nuqtalarga uchlar, $|A_1A_2|$, $|B_1B_2|$ kesmalarning $2a$, $2b$ uzunliklariga mos ravishda haqiqiy va mavhum $|F_1M|$, $|F_2M|$ kesmalarning $r_1, r_2$ uzunliklariga fokal radiuslar deyiladi.

$$\begin{cases} x=a\,\mathrm{ch}\,t \\ y=b\,\mathrm{sh}\,t \end{cases} \tag{36}$$

tenglamalar sistemasiga **giperbolaning parametrik tenglamalari** deyiladi.

$\varepsilon = \dfrac{c}{a}$ kattalikka **giperbolaning ekssentrisiteti** deyiladi. Bunda $\varepsilon > 1$ chunki $c > a$. $M$ nuqtadan $d_1$, $d_2$ masofada o'tuvchi va tenglamalari $x = \pm\dfrac{a}{\varepsilon}$ dan iborat bo'lgan to'g'ri chiziqlar **giperbolaning direktrisalari** deb ataladi. Direktrisalar ushbu

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon \qquad (37)$$

tengliklarni qanoatlantiradi. Bu tengliklardan giperbolaning fokal radiuslari uchun

$$r_1 = a - \varepsilon x, \; r_2 = a + \varepsilon x \qquad (38)$$

formulalar kelib chiqadi.

Giperbolaning qutb koordinatalar sistemasidagi tenglamasi

$$\rho = \frac{p}{1 - \varepsilon \cos\varphi} \qquad (39)$$

(39) formula orqali aniqlanadi.

Giperbolaga $M_0(x_0; y_0)$ nuqtada o'tkazilgan urinma

$$\frac{x \cdot x_0}{a^2} - \frac{y \cdot y_0}{b^2} = 1 \qquad (40)$$

tenglama orqali aniqlanadi.

Giperbolaning asimptotalari

$$y = \pm\frac{b}{a}x \qquad (41)$$

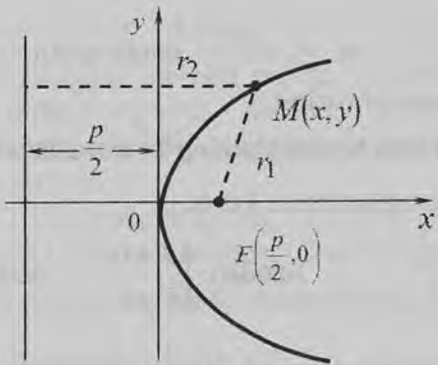
tenglama orqali aniqlanadi.

**Parabola.** Direktrisa deb ataluvchi berilgan to'g'ri chiziq va unda yotmagan fokus deb ataluvchi nuqtadan teng uzoqlikdagi tekislikdagi barcha nuqtalar to'plami **parabola** deyiladi (8-shakl).

Agar $x = -\dfrac{p}{2}$ parabola direktrisasi, $F\left(\dfrac{p}{2}; 0\right)$ parabola fokusi bo'lsa, u holda parabolaning kanonik tenglamasi

$$y^2 = 2px \qquad (42)$$

(42) orqali aniqlanadi ($p > 0$).




**8-shakl**

Parabolaning fokal radiusi

$$r = x + \frac{p}{2} \tag{43}$$

formula bilan aniqlanadi.

Parabolaning $M_0(x_0, y_0)$ nuqtasidan o'tkazilgan urinma tenglamasi

$$yy_0 = p(x + x_0) \tag{44}$$

(44) orqali aniqlanadi.

Parabolaning qutb koordinatalar sistemasidagi tenglamasi

$$\rho = \frac{p}{1 - \cos\varphi} \tag{45}$$

(45) formula orqali aniqlanadi.

**361.** Berilgan aylananing markazi va radiusini toping:

$$2x^2 + 2y^2 - 8x + 5y - 4 = 0.$$

**Yechish.** Tenglamani quyidagicha yozib olamiz:

$$(x^2 - 4x) + (y^2 + \frac{5}{2}y) = 2.$$

Qavslar ichidagi ifodalarni to'la kvadratga keltiramiz:

$$(x^2 - 4x + 4) - 4 + (y^2 + \frac{5}{2}y + \frac{25}{16}) - \frac{25}{16} = 2 \text{ yoki } (x-2)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{121}{16}.$$

Demak, aylana markazi $M_0(2; -\frac{5}{4})$, radiusi $R = \frac{11}{4}$ ekan.

**362.** $M(\frac{5}{2},\frac{\sqrt{6}}{4})$ va $N(-2;\frac{\sqrt{15}}{5})$ nuqtalardan o'tuvchi ellipsning kanonik tenglamasini tuzing.

**Yechish.** M va N nuqtalarning koordinatalari $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ - ellips tenglamasini qanoatlantirishi kerak.

$$\begin{cases} \frac{25}{4a^2}+\frac{3}{8b^2}=1 \\ \frac{4}{a^2}+\frac{3}{5b^2}=1, \end{cases}$$ bundan $\begin{cases} a^2=10 \\ b^2=1 \end{cases}$ natijaga kelamiz.

Demak, ellips tenglamasi: $\frac{x^2}{10}+\frac{y^2}{1}=1$ ko'rinishda ekan.

**363.** Giperbolaning $\frac{x^2}{9}-\frac{y^2}{16}=1$ tenglamasi berilgan. Bu giperbola elementlarini aniqlang.

**Yechish.** Giperbola tenglamasiga asosan $a=3$, $b=4$ fokuslar esa absissalar o'qida yotadi.

$c=\sqrt{a^2+b^2}=\sqrt{9+16}=5$ giperbola ekssentrisiteti: $\varepsilon=\frac{c}{a}=\frac{5}{3}$.

Fokusi va uchlarining koordinatalari quyidagicha:

$$F(5;0); F_1(-5;0), A(3;0), A_1(-3;0), B(0;4), B_1(0;-4).$$

Giperbola asimptotalarining tenglamalari : $y=\frac{4}{3}x,\ y=-\frac{4}{3}x$.

Giperbola ixtiyoriy M(x,y) nuqtasining fokal radiuslari:

$r_1=-3+\frac{5}{3}x,\ r_2=3+\frac{5}{3}x$. Direktrisalarining tenglamalari: $x=\frac{9}{5},\ x=-\frac{9}{5}$;

**364.** Quyidagi shartlarni qanoatlantiruvchi aylana tenglamalarini tuzing.

1) Markazi koordinatalar boshida radiusi $R=3$ ga teng;

2) Markazi $C(2; -3)$ nuqtada radiusi $R=7$ ga teng;

3) Markazi $C$ (6; -8) nuqtada va koordinatalar boshidan o'tuvchi aylana;

4) Markazi $C(1; -1)$ nuqtada va $5x-12y+9=0$ to'g'ri chiziqqa urunuvchi aylana;

5) $A(1; 1)$, $B(1; -1)$ va $C(2; 0)$ nuqtalardan o'tuvchi aylana;

**365.** Qutb koordinatalar sistemasida aylana tenglamasini berilgan. Aylana markazi va radiusini toping.

1) $\rho=4\cos\theta$; 2) $\rho=3\sin\theta$; 3) $\rho=-2\cos\theta$; 4) $\rho=-5\cos\theta$;

5) $\rho = 6\cos(\frac{\pi}{3} - \theta)$; 6) $\rho = 8\sin(\theta - \frac{\pi}{3})$; 7) $\rho = 8\sin(\frac{\pi}{3} - \theta)$.

**366.** Agar quyidagilar ma'lum bo'lsa, markazi koordinatalar boshida, koordinatalar o'qlariga nisbatan simmetrik bo'lgan, fokuslari Ox o'qida yotuvchi ellips tenglamasini tuzing.

1) Yarim o'qlari 5 va 2;

2) Katta o'qi 10, fokuslar orasidagi masofa 8;

3) Kichik o'qi 24, fokuslar orasidagi masofa 10;

4) Fokuslar orasidagi masofa 6, eksentrisiteti $\varepsilon = \frac{3}{5}$;

5) Katta o'qi 20, eksentrisiteti $\varepsilon = \frac{3}{5}$.

**367.** $x$-$y$-5 = 0 to'g'ri chiziqqa urunuvchi fokuslari $F_1$ (-3; 0) va $F_2$ (3; 0) bo'lgan ellips tenglamasini tuzing.

**368.** *Ox* va *Oy* o'qlari bo'yicha mos ravishda $q_1$ va $q_2$ koeffitsiyentlar bilan siqilish natijasida $\frac{x^2}{25} + \frac{y^2}{9} = 1$ ellips $x^2+y^2=16$ aylanaga o'tsa, $q_1$ va $q_2$ ni toping.

**369.** Agar quyidagilar ma'lum bo'lsa, markazi koordinatalar boshida, koordinatalar o'qlariga nisbatan simmetrik bo'lgan, fokuslari Ox o'qida yotuvchi giperbola tenglamalarini tuzing.

1) Haqiqiy o'qi 10, mavhum o'qi 8;

2) Fokuslar orasidagi masofa 10 va mavhum o'qi 8;

3) Fokuslar orasidagi masofa 6 va eksentritsiteti $\varepsilon = \frac{3}{2}$;

4) Haqiqiy o'qi 16 va eksentrisiteti $\varepsilon = \frac{5}{4}$;

5) Asimptotalar tenglamasi $y = \pm\frac{4}{3}x$, fokuslar orasidagi masofa 20;

6) Direktrisalar orasidagi masofa 22, fokuslar orasidagi masofa 26;

7) Direktrisalar orasidagi masofa $\frac{8}{3}$ va eksentritsiteti $\varepsilon = \frac{3}{2}$.

**370.** $5x - 6y - 16 = 0$, $13x - 10y - 48 = 0$ to'g'ri chiziqlarga urunuvchi o'qlari Ox va Oy o'qlarida yotuvchi giperbola tenglamasini tuzing.

**371.** O'qlari Ox va Oy o'qlarida yotuvchi, $A(\sqrt{6}; 3)$ nuqtadan o'tuvchi va

$9x+2y-15 = 0$ to'g'ri chiziqqa urunuvchi giperbola tenglamasini tuzing.

**372.** Parabola uchi koordinata boshida bo'lib, quyidagilar ma'lum bo'lsa, parabola tenglamasini tuzing.

1) $p = 3$, shoxlari Ox o'qi musbat yo'nalishi bo'yicha yo'nalgan va Ox o'qiga nisbatan simmetrik;

2) $p = 0.5$, shoxlari Ox o'qi manfiy yo'nalishi bo'yicha yo'nalgan va Ox o'qiga nisbatan simmetrik;

3) $p = \frac{1}{4}$, shoxlari Oy o'qi musbat yo'nalishi bo'yicha yo'nalgan va Oy o'qiga nisbatan simmetrik;

4) $p = 3$, shoxlari Oy o'qi manfiy yo'nalishi bo'yicha yo'nalgan va Oy o'qiga nisbatan simmetrik;

**373*.** P(-3; 12) nuqtadan $y^2=10x$ parabolaga urinma o'tkazilgan. P nuqtadan urinish nuqtasigacha masofani toping.

**374.** Quyidagi parabola uchining koordinatasini va fokusning koordinatalarini toping.

1) $y = \frac{1}{4}x^2 + x + 2$;  2) $y = 4x^2 - 8x + 7$;  3) $y = -\frac{1}{6}x^2 + 2x - 7$.

## 16 §. Ikkinchi tartibli chiziqlarning turlari. Ikkinchi tartibli chiziqlarni kanonik ko'rinishga keltirish.

Quyidagi ikkinchi tartibli chiziqning berilgan bo'lsin:

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0. \qquad (46)$$

Quyidagicha belgilash kiritaylik:

$$\delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix}, \qquad \Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}. \qquad (47)$$

Quyidagi sxema orqali (46) tenglamaning turini aniqlash mumkin:

| $\delta$ | $\Delta \neq 0$ | $\Delta = 0$ |
|---|---|---|
| $\delta > 0$ | Ellips | Nuqta |
| $\delta < 0$ | Giperbola | Kesishuvchi to'g'ri chiziqlar jufti |
| $\delta = 0$ | Parabola | Parallel to'g'ri chiziqlar jufti |

Bizga biror (46) ko'rinishidagi tenglama berilgan bo'lsin.

$$x = x'\cos\alpha - y'\sin\alpha,\ y = x'\sin\alpha + y'\cos\alpha \qquad (48)$$

$\alpha$ burchakni shunday tanlash mumkinki, (48) almashtirish yordamida (46) tenglamani quyidagicha yozish mumkin

$$A'x'^2 + C'y'^2 + 2D'x' + 2E'y' + F' = 0. \qquad (49)$$

**375.** Quyidagi chiziqni turni aniqlang va kanonik ko'rinishga keltiring:

$$5x^2 + 4xy + 8y^2 + 8x + 14y + 5 = 0.$$

**Yechish.**

**1.** Avval chiziq turini aniqlaymiz. Berilgan tenglamani (46) tenglama bilan taqqoslab $A=5, B=2, C=8, D=4, E=7, F=5$ ekanini topamiz. (47) formuladan

$$\delta = \begin{vmatrix} 5 & 2 \\ 2 & 8 \end{vmatrix} = 36 > 0,\ \Delta = \begin{vmatrix} 5 & 2 & 4 \\ 2 & 8 & 7 \\ 4 & 7 & 5 \end{vmatrix} = 200 + 56 + 56 - 128 - 245 - 20 = -81 \neq 0.$$

Demak, berilgan chiziq ellips ekan.

**2.** Berilgan tenglamani kanonik ko'rinishga keltiramiz. (47) almashtirishdan foydalanib berilgan tenglamani quyidagicha yozamiz:

$$5(x'\cos\alpha - y'\sin\alpha)^2 + 4(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha) + 8(x'\sin\alpha + y'\cos\alpha)^2 +$$

$$+8(x'\cos\alpha - y'\sin\alpha) + 14(x'\sin\alpha + y'\cos\alpha) + 5 = 0$$

yoki

$$(5\cos^2\alpha + 4\sin\alpha\cos\alpha + 8\sin^2\alpha)x'^2 + (5\sin^2\alpha - 4\sin\alpha\cos\alpha + 8\cos^2\alpha)y'^2 +$$

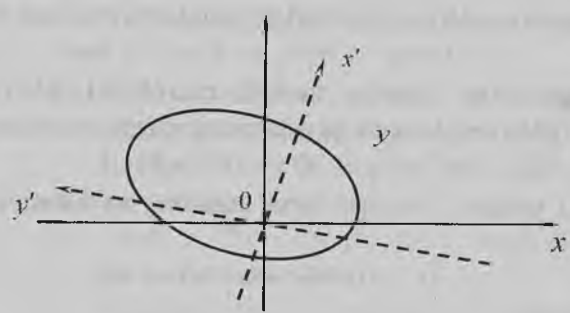
$$+\left[6\sin\alpha\cos\alpha + 4(\cos^2\alpha - \sin^2\alpha)\right]x'y' + (8\cos\alpha + 14\sin\alpha)x' +$$

$$+(14\cos\alpha - 8\sin\alpha)y' + 5 = 0$$

$\alpha$ ni $x'y'$ oldidagi koeffitsiyentni nol bo'ladigan qilib tanlaymiz. Ya'ni,

$4(\cos^2\alpha - \sin^2\alpha) + 6\sin\alpha\cos\alpha = 0$ yoki $2tg^2\alpha - 3tg\alpha - 2 = 0$. Bundan $tg\alpha_1 = 2$, $tg\alpha_2 = -\frac{1}{2}$ ni topamiz.

Shuni ta'kidlash kerakki, $tg\alpha$ ning qiymatlari o'zaro perpendikulyar bo'lgan ikki yo'nalishni aniqladi. $tg\alpha = 2$ holni qarasak, $tg\alpha = -\frac{1}{2}$ holda $x'$ va $y'$ lar o'rni almashadi (9-shakl).




**9-shakl.**

$tg\alpha = 2$ bo'lsin, u holda $\sin\alpha = \frac{2}{\sqrt{5}}$, $\cos\alpha = \frac{1}{\sqrt{5}}$ bo'ladi. U holda

$9x'^2 + 4y'^2 + \frac{36}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' + 5 = 0$, yoki $9\left(x'^2 + \frac{4}{\sqrt{5}}x'\right) + 4\left(y'^2 - \frac{1}{2\sqrt{5}}y'\right) = -5$.

Qavslar ichidagi ifodalarni to'la kvadratga keltirib

$9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{2\sqrt{5}}\right)^2 = \frac{33}{5} + \frac{1}{20} - 5$, yoki $9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{2\sqrt{5}}\right)^2 = \frac{9}{4}$

tenglamaga ega bo'lamiz. Demak, $O'\left(-\frac{2}{\sqrt{5}}; \frac{1}{4\sqrt{5}}\right)$, koordinata o'qlarini parallel ko'chirib $x' = x'' - \frac{2}{\sqrt{5}}, y' = y'' + \frac{1}{4\sqrt{5}}$, quyidagi tenglamani hosil qilamiz:

$9x''^2 + 4y''^2 = \frac{9}{4}$, yoki $\frac{x''^2}{1/4} + \frac{y''^2}{9/16} = 1$. Bu esa avval ta'kidlaganimizdek, ellipsning kanonik tenglamasi.

**376*.** Quyidagi 2-tartibli chiziqlarni kanonik ko'rinishga keltiring, chiziq turini aniqlang va grafigi sxemasini chizing.

1) $4x^2 + 9y^2 - 40x + 36y + 100 = 0$; 2) $9x^2 - 16y^2 - 54x - 64y - 127 = 0$;
3) $9x^2 + 4y^2 + 18x - 8y + 49 = 0$; 4) $4x^2 - y^2 + 8x - 2y + 3 = 0$;
5) $y^2 + 8x - 6y + 11 = 0$. 6) $4x^2 + 4y^2 + 8x - 16y - 29 = 0$;

**377.** Quyidagi 2-tartibli chiziqlarni kanonik ko'rinishga keltiring, chiziq turini aniqlang.

1) $3x^2+10xy+3y^2-2x-14y-13=0$;

2) $25x^2-14xy+25y^2+64x-64y-224=0$;

3) $4xy+3y^2+16x+12y-36=0$;

4) $7x^2+6xy-y^2+28x+12y+28=0$;

5) $19x^2+6xy+11y^2+38x+6y+29=0$;

6) $5x^2-2xy+5y^2-4x+20y+20=0$.

## 17 §. Fazoda tekislik tenglamalari

Berilgan $M_1(x_1,y_1,z_1)$ orqali $\bar{n}(A;B;C)$ vektorga perpendikulyar o'tuvchi tekislik

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \quad (50)$$

tenglama orqali ifodalaniladi.

$M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ va $M_3(x_3,y_3,z_3)$ nuqtalardan o'tuvchi tekislik

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0 \quad (51)$$

tenglama orqali ifodalaniladi.

Tekislikning umumiy tenglamasi

$$Ax+By+Cz+D=0 \quad (52)$$

tenglama orqali ifodalaniladi. Bunda $A^2+B^2+C^2>0$.

$A=0$ bo'lsa, $By+Cz+D=0$ OX o'qqa parallel tekislik;

$B=0$ bo'lsa, $Ax+Cz+D=0$ OY o'qqa parallel tekislik;

$C=0$ bo'lsa, $Ax+By+D=0$ OZ o'qqa parallel tekislik;

$D=0$ bo'lsa, $Ax+By+Cz=0$ koordinatalar boshidan o'tuvchi tekislik hosil bo'ladi.

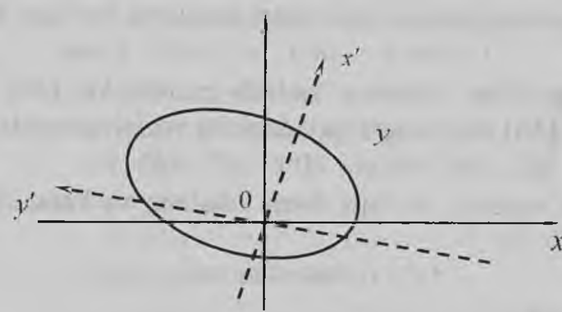
$\Omega_1: A_1x+B_1y+C_1z+D_1=0$, $\Omega_2: A_2x+B_2y+C_2z+D_2=0$ tekisliklar orasidagi burchak

$$\cos\varphi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}} \quad (53)$$

(53) formula orqali topiladi.

Xususan,

$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$ bo'lsa, tekisliklar parallellik bo'ladi;




**9-shakl.**

$tg\alpha = 2$ bo'lsin, u holda $\sin\alpha = \frac{2}{\sqrt{5}}$, $\cos\alpha = \frac{1}{\sqrt{5}}$ bo'ladi. U holda

$$9x'^2 + 4y'^2 + \frac{36}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' + 5 = 0, \text{ yoki } 9\left(x'^2 + \frac{4}{\sqrt{5}}x'\right) + 4\left(y'^2 - \frac{1}{2\sqrt{5}}y'\right) = -5.$$

Qavslar ichidagi ifodalarni to'la kvadratga keltirib

$$9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{2\sqrt{5}}\right)^2 = \frac{33}{5} + \frac{1}{20} - 5, \text{yoki } 9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{2\sqrt{5}}\right)^2 = \frac{9}{4}$$

tenglamaga ega bo'lamiz. Demak, $O'\left(-\frac{2}{\sqrt{5}}; \frac{1}{4\sqrt{5}}\right)$, koordinata o'qlarini parallel ko'chirib $x' = x'' - \frac{2}{\sqrt{5}}, y' = y'' + \frac{1}{4\sqrt{5}}$, quyidagi tenglamani hosil qilamiz:

$9x''^2 + 4y''^2 = \frac{9}{4}$, yoki $\frac{x''^2}{1/4} + \frac{y''^2}{9/16} = 1$. Bu esa avval ta'kidlaganimizdek, ellipsning kanonik tenglamasi.

**376*.** Quyidagi 2-tartibli chiziqlarni kanonik ko'rinishga keltiring, chiziq turini aniqlang va grafigi sxemasini chizing.

1) $4x^2 + 9y^2 - 40x + 36y + 100 = 0$; 2) $9x^2 - 16y^2 - 54x - 64y - 127 = 0$;

3) $9x^2 + 4y^2 + 18x - 8y + 49 = 0$; 4) $4x^2 - y^2 + 8x - 2y + 3 = 0$;

5) $y^2 + 8x - 6y + 11 = 0$. 6) $4x^2 + 4y^2 + 8x - 16y - 29 = 0$;

**377.** Quyidagi 2-tartibli chiziqlarni kanonik ko'rinishga keltiring, chiziq turini aniqlang.

1) $3x^2+10xy+3y^2-2x-14y-13=0$;

2) $25x^2-14xy+25y^2+64x-64y-224=0$;

3) $4xy+3y^2+16x+12y-36=0$;

4) $7x^2+6xy-y^2+28x+12y+28=0$;

5) $19x^2+6xy+11y^2+38x+6y+29=0$;

6) $5x^2-2xy+5y^2-4x+20y+20=0$.

## 17 §. Fazoda tekislik tenglamalari

Berilgan $M_1(x_1,y_1,z_1)$ orqali $\bar{n}(A;B;C)$ vektorga perpendikulyar o'tuvchi tekislik

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \qquad (50)$$

tenglama orqali ifodalaniladi.

$M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ va $M_3(x_3,y_3,z_3)$ nuqtalardan o'tuvchi tekislik

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0 \qquad (51)$$

tenglama orqali ifodalaniladi.

Tekislikning umumiy tenglamasi

$$Ax+By+Cz+D=0 \qquad (52)$$

tenglama orqali ifodalaniladi. Bunda $A^2+B^2+C^2>0$.

$A=0$ bo'lsa, $By+Cz+D=0$ OX o'qqa parallel tekislik;

$B=0$ bo'lsa, $Ax+Cz+D=0$ OY o'qqa parallel tekislik;

$C=0$ bo'lsa, $Ax+By+D=0$ OZ o'qqa parallel tekislik;

$D=0$ bo'lsa, $Ax+By+Cz=0$ koordinatalar boshidan o'tuvchi tekislik hosil bo'ladi.

$\Omega_1: A_1x+B_1y+C_1z+D_1=0$, $\Omega_2: A_2x+B_2y+C_2z+D_2=0$ tekisliklar orasidagi burchak

$$\cos\varphi=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}} \qquad (53)$$

(53) formula orqali topiladi.

Xususan,

$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$ bo'lsa, tekisliklar parallellik bo'ladi;

$A_1A_2+B_1B_2+C_1C_2=0$ bo‘lsa, tekisliklar perpendikulyarlik bo‘ladi;

$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}=\frac{D_1}{D_2}$ bo‘lsa, tekisliklar ustma-ust tushadi.

Koordinata o‘qlarini $A(a,0,0), B(0,b,0), C(0,0,c)$ nuqtalarda kesib o‘tuvchi tekislik

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (54)$$

tenglama bilan aniqlanadi. Bu tenglamani tekislikning kesmalar bo‘yicha tenglamasi deyiladi.

$M_1(x_1,y_1,z_1)$ nuqtadan $\Omega: Ax+By+Cz+D=0$ tekislikkacha bo‘lgan masofa

$$d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}} \qquad (55)$$

(55) formula orqali topiladi.

Tekislikning normal tenglamasi

$$x\cdot Cos\alpha+y\cdot Sin\beta+z\cos\gamma-\rho=0 \qquad (56)$$

(56) formula orqali ifodalaniladi. Bunda $\rho$ koordinata boshidan tekislikka tushirilgan perpendikulyar, $\alpha,\beta,\gamma$ shu perpendikulyar bilan mos ravishda $\vec{i},\vec{j},\vec{k}$ orasidagi burchaklar.

$\Omega_1: A_1x+B_1y+C_1z+D_1=0,\ \Omega_2: A_2x+B_2y+C_2z+D_2=0$ tekisliklar kesishish to‘g‘ri chizig‘ini o‘z ichiga oluvchi tekisliklar dastasi tenglamasi

$$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0 \qquad (57)$$

(57) formula orqali aniqlanadi. Bunda $\lambda$ - ixtiyoriy haqiqiy son.

**378.** M(4;-5;7) nuqta orqali o‘tib, YOZ tekislikka parallel bo‘lgan tekislik tenglamasini tuzing.

**Yechish.** YOZ tekislikka parallel bo‘lgan tekislik tenglamasida $B=C=0$ yoki $Ax+D=0$ bo‘ladi. Oxirgi tenglikni $A$ ga bo‘lib $4+D_1=0$ ni hosil qilamiz. Bundan $x-4=0$ tekislik tenglamasi hosil bo‘ladi.

**379.** $x+y-1=0$ va $2x-y+\sqrt{3}z+3=0$ tekisliklar orasidagi burchakni toping.

**Yechish.** Tekisliklar orasidagi burchak formulasiga ko‘ra,

$$\cos\phi=\frac{1\cdot 2+1\cdot(-1)+0\cdot\sqrt{3}}{\sqrt{1^2+1^2+0^2}\cdot\sqrt{2^2+(-1)^2+\sqrt{3}^2}}=\frac{2-1}{\sqrt{2}\cdot\sqrt{8}}=\frac{1}{4},\quad \phi=\arccos\frac{1}{4}.$$

**380.** M(-1;1;-2) nuqtadan $2x-3y+6z-11=0$ tekislikkacha bo'lgan masofani aniqlang.

**Yechish.** Berilgan nuqtadan tekislikkacha bo'lgan masofa formulasiga asosan,

$$d=\frac{|-2\cdot 1-1\cdot 3-2\cdot 6-11|}{\sqrt{2^2+(-3)^2+6^2}}=\frac{28}{7}=4$$

**381.** $M_1(0;1;3)$ va $M_2(2;4;5)$ nuqtalardan o'tuvchi va OX o'qqa parallel tekislik tenglamasini tuzing.

**382.** OX o'qidan va $M_1(0;-2;3)$ nuqtadan o'tuvchi tekislik tenglamasini tuzing.

**383.** $x+2y+2z-8=0, x+y-6=0$ tekisliklar orasidagi burchakni toping.

**384.** (2;2;-2) nuqtadan o'tuvchi va $x-2y-3z=0$ tekislikka parallel tekislik tenglamasini tuzing.

**385.** $M_1(-1;-2;0)$ va $M_2(1;1;2)$ nuqtadan o'tuvchi hamda $x+2y+2z-4=0$ tekislikka perpendikulyar tekislik tenglamasini tuzing.

**386.** $M_1(1;-1;2)$, $M_2(2;1;2)$ va $M_3(1,1,4)$ nuqtalardan o'tuvchi tekislik tenglamasini tuzing.

**387.** OZ o'qdan o'tib, $2x+y-\sqrt{5}z=0$ tekislik bilan 60° li burchak tashkil etuvchi tekislik tenglamasini tuzing.

**388.** $4x+3y-5z-8=0$ va $4x+3y-5z+12=0$ parallel tekisliklar orasidagi masofani toping.

**389.** (4;3;0) nuqtadan $M_1(1;3;0)$, $M_2(4;-1;2)$ va $M_3(3;0;1)$ nuqtalardan o'tuvchi tekislikkacha bo'lgan masofani toping.

**390*.** $3x-4y-z+5=0$, $4x-3y+z+5=0$ tekisliklar kesishib hosil qilgan o'tmas burchakka bissektrisa bo'luvchi tekislik tenglamasini tuzing.

**391*.** Kubning ikkita yog'i $2x-2y+z-1=0, 2x-2y+z+5=0$ tekisliklarda yotsa, kub hajmini toping.

## 18 §. Fazoda to‘g‘ri chiziq tenglamalari. To‘g‘ri chiziq va tekislikning o‘zaro vaziyati.

Fazoda to‘g‘ri chiziq tenglamasi, $\Omega_1 : A_1x + B_1y + C_1z + D_1 = 0$, $\Omega_2 : A_2x + B_2y + C_2z + D_2 = 0$ tekisliklarning kesishish sifatida

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \qquad (58)$$

orqali ifodalaniladi.

$\vec{s} = m\vec{i} + n\vec{j} + p\vec{k}$ vektor, to‘g‘ri chiziqning yo‘naltiruvchi vektori bo‘lsin.

To‘g‘ri chiziqning vektor tenglamasi,

$$\vec{r} = \vec{r}_1 + t \cdot \vec{s} \qquad (59)$$

(59) orqali ifodalaniladi. Bunda, $\vec{r}, \vec{r}_1$ –radius vektorlar.

Berilgan $M_1(x_1, y_1, z_1)$ nuqtadan o‘tuvchi to‘g‘ri chiziqning kanonik tenglamasi

$$\frac{x - x_1}{m} = \frac{y - y_1}{n} = \frac{z - z_1}{p} \qquad (60)$$

tenglama orqali ifodalaniladi.

To‘g‘ri chiziqning parametrik tenglamasi

$$\begin{cases} x = x_1 + tm \\ y = y_1 + tn \\ z = z_1 + tp \end{cases} \qquad (61)$$

tenglama orqali ifodalaniladi.

Berilgan $M_1(x_1;y_1;z_1)$ va $M_2(x_2;y_2;z_2)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasi esa

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad (62)$$

tenglama orqali ifodalaniladi.

Bizga $l_1 : \frac{x - x_1}{m_1} = \frac{y - y_1}{n_1} = \frac{z - z_1}{p_1}$, $l_2 : \frac{x - x_2}{m_2} = \frac{y - y_2}{n_2} = \frac{z - z_2}{p_2}$ to‘g‘ri chiziqlar berilgan bo‘lsin. Bu to‘g‘ri chiziqlar orasidagi burchak

$$\cos\varphi = \frac{m_1m_2 + n_1n_2 + p_1 \cdot p_2}{\sqrt{m_1^2 + n_1^2 + p_1^2} \cdot \sqrt{m_2^2 + n_2^2 + p_2^2}} \qquad (63)$$

(63) formula orqali topiladi.

$M_0(x_0,y_0,z_0)$ nuqtadan $l_1:\frac{x-x_1}{m_1}=\frac{y-y_1}{n_1}=\frac{z-z_1}{p_1}$ to‘g‘ri chiziqqacha masofani

$$d=\frac{\left|\left[\overline{M_1M_0},\vec{s}\right]\right|}{|\vec{s}|} \tag{64}$$

(64) formula orqali topiladi. Bunda $M_1(x_1,y_1,z_1)\in l_1$. (64) ni quyidagicha yozish mumkin:

$$d=\frac{\sqrt{\begin{vmatrix} y_0-y_1 & z_0-z_1 \\ n_1 & p_1 \end{vmatrix}^2+\begin{vmatrix} z_0-z_1 & x_0-x_1 \\ p_1 & m_1 \end{vmatrix}^2+\begin{vmatrix} x_0-x_1 & y_0-y_1 \\ m_1 & n_1 \end{vmatrix}^2}}{\sqrt{m_1^2+n_1^2+p_1^2}}. \tag{65}$$

Bizga $l:\frac{x-x_1}{m}=\frac{y-y_1}{n}=\frac{z-z_1}{p}$ to‘g‘ri chiziq,

$\Omega: Ax+By+Cz+D=0$ tekislik va $M_0(x_0,y_0,z_0)$ nuqta berilgan bo‘lsin.

To‘g‘ri chiziq va tekislik kesishish nuqtasini topish formulasi quyidagicha:

$$\begin{cases} \frac{x-x_1}{m}=\frac{y-y_1}{n}=\frac{z-z_1}{p} \\ Ax+By+Cz+D=0 \end{cases}. \tag{66}$$

To‘g‘ri chiziqni tekislikda yotish sharti:

$$\begin{cases} Am+Bn+Cp=0 \\ Ax+By+Cz+D=0 \end{cases}. \tag{67}$$

To‘g‘ri chiziq va tekislik orasidagi burchak sinusi:

$$\sin\varphi=\frac{Am+Bn+Cp}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{m^2+n^2+p^2}}. \tag{68}$$

To‘g‘ri chiziq va tekislikni parallellik sharti:

$$Am+Bn+Cp=0. \tag{69}$$

To‘g‘ri chiziq va tekislikni perpendikulyarlik sharti:

$$\frac{A}{m}=\frac{B}{n}=\frac{C}{p}. \tag{70}$$

$M_0$ nuqta va $l$ to‘g‘ri chiziqdan o‘tuvchi tekislik

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ m & n & p \end{vmatrix}=0 \tag{71}$$

tenglama bilan ifodalaniladi.

$l$ to‘g‘ri chiziqni o‘z ichiga oluvchi, $\Omega$ tekislikka perpendikulyar tekislik tenglamasi

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ m_1 & n_1 & p_1 \\ A & B & C \end{vmatrix} = 0 \qquad (72)$$

tenglama bilan ifodalaniladi.

**392.** To‘g‘ri chiziqning quyidagi tenglamasini kanonik shaklga keltiring:

$$\begin{cases} 2x-3y-z-9=0 \\ x-2y+z+3=0 \end{cases}$$

**Yechish. 1-usul.** To‘g‘ri chiziqga tegishli biror nuqtasini aniqlaymiz:

$\begin{cases} z=0 \\ 2x-3y-9=0 \\ x-2y+3=0 \end{cases} \Rightarrow \begin{cases} x=27 \\ y=15 \\ z=0 \end{cases}$. Demak, $M_1(27;15;0)$ ekan.

To‘g‘ri chiziqning yo‘naltiruvchi vektori esa vektor ko‘paytmadan

$$\vec{S} = \left[\vec{n_1}, \vec{n_2}\right] = \begin{vmatrix} i & j & k \\ 2 & -3 & -1 \\ 1 & -2 & 1 \end{vmatrix} = -5i - 3j - k$$

ekani kelib chiqadi.

Bundan, $\frac{x-27}{-5} = \frac{y-15}{-3} = \frac{z}{-1}$ yoki $\frac{x-27}{5} = \frac{y-15}{3} = \frac{z}{1}$ ekanligini olamiz.

**2-usul.** z ni parametr qilib tanlab, x va y ga nisbatan yechamiz: $\begin{cases} x=5z+27 \\ y=3z+15 \end{cases}$.

Bu tenglamalardan z ni topamiz:

$\frac{x-27}{5} = z$, $\frac{y-15}{3} = z$. Bundan $\frac{x-27}{5} = \frac{y-15}{3} = \frac{z}{1}$ ekanligini olamiz.

**393.** M(2,-5,3) nuqtadan o‘tib, Oy o‘qqa parallel bo‘lgan to‘g‘ri chiziqning tenglamasini tuzing.

**Yechish.** $\vec{j}(0,1,0)$ izlanayotgan to‘g‘ri chiziqni yo‘naltiruvchi vektori bo‘ladi, chunki shartga asosan, to‘g‘ri chiziq Oy o‘qqa parallel. Shunga ko‘ra, to‘g‘ri chiziqning paramertik tenglamasi

$$\begin{cases} x = 2 \\ y = t - 5 \\ z = 3 \end{cases}$$ hosil bo‘ladi.

**394.** $A_1(4; -3; 1)$, $A_2(5; -3; 0)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

**Yechish.** (5) formuladan foydalanib,

$$\frac{x-4}{5-4} = \frac{y-(-3)}{-3-(-3)} = \frac{z-1}{0-1} \text{ yoki } \frac{x-4}{1} = \frac{y+3}{0} = \frac{z-1}{-1}$$ ekanini topamiz.

2-kasrning maxraji 0 ekanligini 0 ga bo‘lish emas, yo‘naltiruvchi vektorning ordinatasi deb tushunish kerak. Ushbu noqulaylikdan parametrik tenglamaga o‘tish yordamida qutilish mumkin. Ya’ni,

$$\begin{cases} x = t + 4 \\ y = -3 \\ z = -t + 1 \end{cases}.$$

**395.** $\begin{cases} 2x + y - z - 3 = 0 \\ x + y + z - 1 = 0 \end{cases}$ to‘g‘ri chiziqni koordinata tekisliklari bilan kesishish nuqtalarini toping.

**396*.** $\begin{cases} x - 2y - 3z - 5 = 0 \\ 2x - y - z + 2 = 0 \end{cases}$ to‘g‘ri chiziqni koordinata tekisliklaridagi proyeksiyalarini toping.

**397.** $\begin{cases} 5x - y - 2z - 3 = 0 \\ 3x - 2y - 5 + 2 = 0 \end{cases}$ to‘g‘ri chiziq orqali o‘tib, $x + 19y - 7z - 11 = 0$ tekislikka perpendikulyar tekislik tenglamasini tuzing.

**398.** $M_1$ (2; 0; -3) nuqtadan o‘tuvchi va quyidagilarga parallel bo‘lgan to‘g‘ri chiziqni kanonik tenglamasini tuzing:

1) $\vec{a} = \{2; -3; 5\}$; 2) $\frac{x-1}{5} = \frac{y+2}{2} = \frac{z-1}{-1}$ 3) *Ox* o‘qi;

4) Oy o‘qi; 5) *Oz* o‘qi.

**399.** $M_1(-4; -5; 3)$ nuqta va $\frac{x+1}{3} = \frac{y+1}{-2} = \frac{z-2}{-1}$, $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-1}{-5}$ to‘g‘ri chiziqlar kesishish nuqtasi orqali o‘tuvchi to‘g‘ri chiziq tenglamasini tuzing.

**400.** To‘g‘ri chiziqlar orasidagi o‘tkir burchakni toping:

$$\frac{x-3}{1} = \frac{y+2}{-1} = \frac{z}{\sqrt{2}}; \qquad \frac{x+2}{1} = \frac{y-3}{1} = \frac{z+5}{\sqrt{2}}.$$

**401.** Quyidagi to‘g‘ri chiziqlarni parametrik tenglamasini tuzing:

1) $\begin{cases} 2x+3y-z=0 \\ 3x-5y+2z+1=0 \end{cases}$

2) $\begin{cases} x+2y-z-6=0 \\ 3x-y+z+1=0 \end{cases}$

**402.** To‘g‘ri chiziq va tekislik kesishish nuqtasini toping:

1) $\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z}{6}$, $2x+3y+z-1=0$;

2) $\frac{x+3}{3}=\frac{y-2}{-1}=\frac{z+1}{-5}$, $x-2y+z-15=0$;

3) $\frac{x+2}{-2}=\frac{y-1}{3}=\frac{z-3}{2}$, $x+2y-2z+6=0$.

**403.** Quyidagi ikki to‘g‘ri chiziqlar orasidagi eng qisqa masofani toping:

1) $\frac{x+7}{3}=\frac{y+4}{4}=\frac{z+4}{-2}$; $\frac{x-21}{6}=\frac{y-21}{-4}=\frac{z-2}{-1}$;

2) x=2t -4; y= - t+4; z= -2t- 1, x=- 4t -5; y= -3t+5; z= -5t+5;

3) $\frac{x+5}{3}=\frac{y+5}{2}=\frac{z-1}{-2}$; x=6t+9; y= -2t; z= -t+2.

**404.** Parallel to‘g‘ri chiziqlar orqali o‘tuvchi tekislik tenglamasini tuzing:

$\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{-2}$, $\frac{x-1}{3}=\frac{y-2}{2}=\frac{z+3}{-2}$.

**405.** $M_1$ (-6; 1; -5), $M_2$(7; -2; -1) va $M_1$ (10; -7; 1) nuqtalardan o‘tuvchi tekislikka nisbatan *P(3; -4;-6)* ga simmetrik nuqtaning koordinatasini toping.

**406.** P(1; -1; -2) nuqtadan $\frac{x+3}{3}=\frac{y+2}{2}=\frac{z-8}{-2}$ to‘g‘ri chiziqqacha masofani hisoblang.

**407.** *P(2; 3;-1)* nuqtadan quyidagi to‘g‘ri chiziqlargacha masofani toping:

1) $\frac{x-5}{3}=\frac{y}{2}=\frac{z+25}{-2}$;

2) *x=t+1; y=t+2, z=4t+13*;

3) $\begin{cases} 2x-2y+z+3=0 \\ 2x-2y+2z+17=0 \end{cases}$

**408.** $M_1$ (2; -2; 1) nuqta va x=2t+1; y=-3t+2; z=2t-3 to'g'ri chiziq orqali o'tuvchi tekislik tenglamasini tuzing.

## 19 §. Ikkinchi tartibli sirtlar

$R^3$ fazoda quyidagi tenglamani qaraylik:

$$a_{11}x^2+2a_{12}xy+2a_{13}xz+a_{22}y^2+2a_{23}yz+a_{33}z^2+2b_1x+2b_2y+2b_3z+c=0. \quad (73)$$

(73) tenglama orqali berilgan geometrik shakl **ikkinchi tartibli sirt** deyiladi. Agar (73) tenglama yechimga ega bo'lmasa, **mavhum sirt** deyiladi.

(73) tenglamani muhim xususiy hollarini qarab chiqamiz.

Markazi C (α. β, γ) nuqtada, radiusi $r$ bo'lgan ***sfera***

$$(x-\alpha)^2+(x-\beta)^2+(x-\gamma)^2=r^2 \quad (74)$$

tenglama orqali ifodalaniladi.

***Ellipsoid*** tenglamasi:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad (a,b,c>0). \quad (75)$$

***Bir pallali giperboloid*** tenglamasi:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad (a,b,c>0). \quad (76)$$

***Ikki pallali giperboloid*** tenglamasi:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad (a,b,c>0). \quad (77)$$

***Elliptik paraboloid*** tenglamasi:

$$\frac{x^2}{p}+\frac{y^2}{q}=2z \quad (p,q>0). \quad (78)$$

***Giperbolik paraboloid*** tenglamasi:

$$\frac{x^2}{p}-\frac{y^2}{q}=2z \quad (p,q>0). \quad (79)$$

***Konus sirti*** tenglamasi:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \quad (a,b,c>0). \quad (80)$$

***Nuqta***

$$x^2+y^2+z^2=0. \quad (81)$$

## Silindrik sirtlar

***Elliptik silindr*** tenglamasi:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a,b>0), \qquad (82)$$

***Giperbolik silindr*** tenglamasi:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad (a,b>0), \qquad (83)$$

***Parabolik silindr*** tenglamasi:

$$y^2=2px \quad (p>0). \qquad (84)$$

***Keshuvchi tekisliklar jufti*** tenglamasi:

$$a^2x^2-b^2y^2=0 \quad (a,b>0). \qquad (85)$$

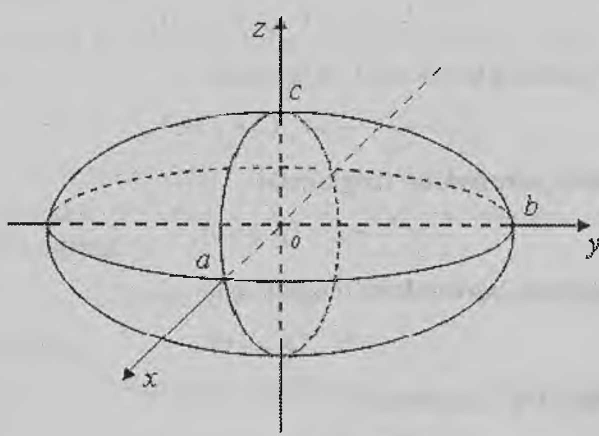
***Parallel*** yoki ***ustma-ust tushuvchi tekisliklar jufti*** tenglamasi:

$$x^2-a^2=0 \quad (a>0), \qquad z^2=0. \qquad (86)$$

***To'g'ri chiziq*** tenglamasi:

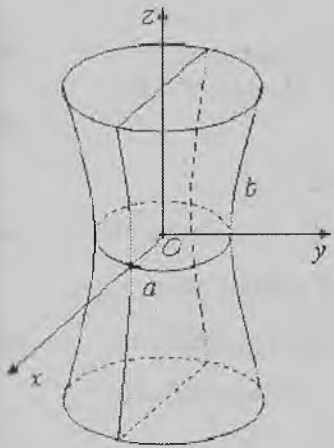
$$x^2+y^2=0. \qquad (87)$$

Quyida bulardan ayrimlarining chizmalari keltirilgan.
Ellipsoid.

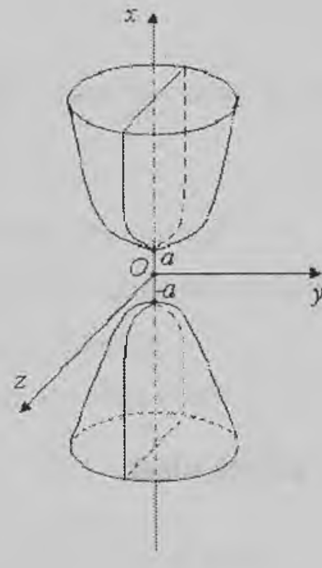



**10-shakl**

Bir va ikki pallali giperboloid.

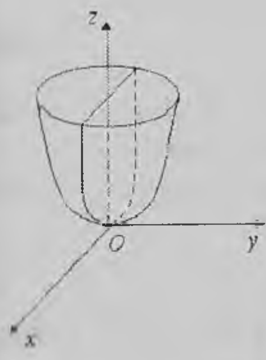



**11-shakl**

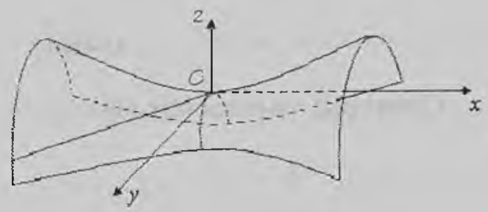



**12-shakl**

Elliptik va giperbolik paraboloid.

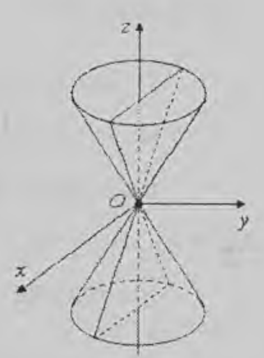



**13-shakl**



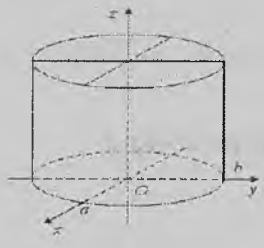

**14-shakl**

Konus.



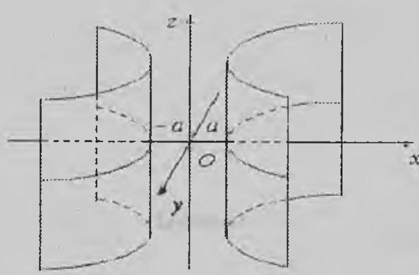

**15-shakl**

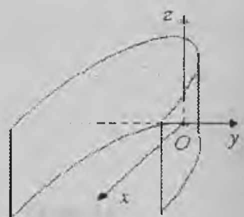
Elliptik silindr.




**16-shakl**

Giperbolik va parabolik silindr.




**17-shakl**

**18-shakl**

**409.** Markazi $C(-5;3;2)$ nuqtada va $2x-2y+z-4=0$ tekislikka urinuvchi sfera tenglamasini tuzing.

**Yechish.** Markazdan tekislikkacha masofa, ya'ni radiusni topamiz:

$$R=\frac{|2\cdot(-5)-2\cdot3+2-4|}{\sqrt{4+4+1}}=6. \qquad (2)$$

tenglamadan $(x+5)^2+(y-3)^2+(z-2)^2=36$ ekanini topamiz.

**410.** $y-2=0$ tekislik bilan $\frac{x^2}{16}+\frac{y^2}{8}+\frac{z^2}{9}=1$ ellipsoid kesilishidan hosil bo'lgan ellipsni uchlari koordinatalarini toping.

**Yechish.** Tenglamada $y=2$ ekanini inobatga olsak, $\frac{x^2}{16}+\frac{z^2}{9}=\frac{1}{2}$ yoki $\frac{x^2}{8}+\frac{z^2}{4.5}=1$ hosil bo'ladi. Bundan $a=\sqrt{8}$, $b=\sqrt{4.5}$, $A_1(-\sqrt{8};2;0)$, $A_2(\sqrt{8};2;0)$ $B_1(0;2;-\sqrt{4.5})$, $B_2(0;2;\sqrt{4.5})$ bo'ladi.

**411.** Quyidagi shartlar asosida sfera tenglamasini tuzing:

1) Markazi C(0; 0; 0), radiusi $r$=9;

2) Markazi *C (5;* -3; 7), radiusi $r$=2;

3) Markazi *C*(4; -4; -2) va koordinatalar boshidan o'tadigan sfera;

4) Markazi *C*(3; -2; 1) va *A* (2; -1; -3) nuqtadan o'tadigan sfera;

5) *A*(2; -3; 5) va *B*(4; 1; -3) diametrial qarama-qarshi nuqtalar bo'ladigan sfera;

6*) Markazi koordinatalar boshi va $16x-15y-12z+75=0$ tekislikga urunuvchi sfera;

7) $M_1(3;\ 1;\ -3)$, $M_2(-2;\ 4;\ 1)$ va $M_3$ (-5; 0; 0) nuqtalar orqali o'tuvchi va $2x+y-2z+3=0$ tekislikka urunuvchi sfera ;

8) $M_1(1;\ -2;\ -1)$, $M_2(-5;\ 10;\ -1)$, $M_3(4;\ 1;\ 11)$, $M_4(-8;\ -2;\ 2)$ nuqtalar orqali o'tuvchi sfera.

**412.** x - 2 = 0 tekislik bilan $\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$ ellipsoid kesilganda hosil bo'lagan ellips yarim o'qlari va uchlari koordinatalarini toping.

**413.** $z+1=0$ tekislik bilan $\frac{x^2}{32}+\frac{y^2}{18}-\frac{z^2}{2}=1$ bir pallali giperboloid kesilganda hosil bo'lgan giperbolaning yarim o'qlari va uchlari koordinatalarini toping.

**414.** $y+6=0$ tekislik bilan $\frac{x^2}{5}+\frac{y^2}{4}=6z$ giperbolik paraboloid kesilganda hosil bo'lgan parabolaning uchini va p parametrini toping.

**415*.** $y^2+z^2=x$ elliptik paraboloid va $x+2y-z=0$ tekislikning kesishishidan hosil bo'lgan kesimning koordinata tekisliklaridagi proyeksiyalarini tenglamasini tuzing.

**416*.** Yasovchilari $x+y-2z-5=0$ tekislikka perpendikulyar bo'lgan silindr, $x^2+y^2+z^2=1$ sferaga tashqi chizilgan. Silindr tenglamasini tuzing.

**417.** Quyidagi to'g'ri chiziq va sirt kesishish nuqtalarini toping:

1) $\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1$, $\frac{x-3}{3}=\frac{y-4}{-6}=\frac{z+2}{4}$;

2) $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{4}=1$, $\frac{x}{4}=\frac{y}{-3}=\frac{z+2}{4}$;

3) $\frac{x^2}{5}+\frac{y^2}{3}=z$, $\frac{x+1}{2}=\frac{y-2}{-1}=\frac{z+2}{-4}$; 4) $\frac{x^2}{9}-\frac{y^2}{4}=z$, $\frac{x}{3}=\frac{y-2}{-2}=\frac{z+1}{2}$.

**418.** $(x-2)^2+(y-1)^2+z^2=25$, $x^2+y^2+z^2=25$ ikkala sferaga ham tashqi chizilgan silindr tenglamasini tuzing.

**419.** Uchi (3;-1;-2) nuqtada bo'lgan, yasovchilari $\begin{cases} x^2+y^2-z^2=1 \\ x-y+z=0 \end{cases}$ tenglama orqali berilgan konus tenglamasini tuzing.

**420.** Uchi koordinatalar boshida, yasovchilari $(x+2)^2+(y-1)^2+(z-3)^2=9$ sferaga urinuvchi konus tenglamasini tuzing.

**421.** $4x^2+16y^2+8z^2=1$ ellipsoidga $x-2y+2z+17=0$ tekislikka parallel qilib urinma o'tkazilgan. Berilgan tekislik va urinma orasidagi masofani toping va urinma tenglamasini tuzing.

# IV BOB. MATEMATIK ANALIZGA KIRISH

## 20 §. To‘plamlar va ular ustida amallar. Haqiqiy sonlar to‘plami. Matematik belgilar.

**To‘plam tushunchasi.** To‘plam matematikaning boshlang‘ich, ayni paytda muhim tushunchalaridan biri. Uni ixtiyoriy tabiatli narsalarning (predmetlarning) ma’lum belgilar bo‘yicha birlashmasi (majmuasi) sifatida tushuniladi. Masalan, javondagi kitoblar to‘plami, bir nuqtadan o‘tuvchi to‘g‘ri chiziqlar to‘plami, $x^2-5x+6=0$ tenglamaning ildizlari to‘plami deyilishi mumkin. To‘plamni tashkil etgan narsalar uning elementlari deyiladi. Matematikada to‘plamlar bosh harflar bilan, ularning elementlari esa kichik harflar bilan belgilanadi. Masalan, $A,B,C$ - to‘plamlar, $a,b,c$ - ularning elementlari.

Ba’zan to‘plamlar ularning elementlarini ko‘rsatish bilan yoziladi:

$$A=\{2,4,6,8,10,12\},\ N=\{1,2,3,...,n,...\},Z=\{...,-2,-1,0,1,2,...\}.$$

Agar $a$ biror $A$ to‘plamning elementi bo‘lsa, $a\in A$ kabi yoziladi va «$a$ element $A$ to‘plamga tegishli» deb o‘qiladi. Agar $a$ shu to‘plamga tegishli bo‘lmasa, uni $a\notin A$ kabi yoziladi va «$a$ element $A$ to‘plamga tegishli emas» deb o‘qiladi. Masalan, yuqoridagi $A$ to‘plamda $10\in A$, $15\notin A$.

Agar $A$ chekli sondagi elementlardan tashkil topgan bo‘lsa, u chekli to‘plam, aks holda cheksiz to‘plam deyiladi. Masalan, $A=\{2,4,6,8,10,12\}$ chekli to‘plam, bir nuqtadan o‘tuvchi barcha to‘g‘ri chiziqlar to‘plami esa cheksiz to‘plam bo‘ladi.

$A$ to‘plamning elementlari orasida biror xususiyatga (bu xususiyatni $P$ bilan belgilaymiz) ega bo‘ladiganlari bo‘lishi mumkin. Bunday xususiyatli elementlardan tuzilgan to‘plam quyidagicha

$$\{x\in A\,|\,P\}\ \text{(yoki ba’zan } \{P\,|\,x\in A\})$$

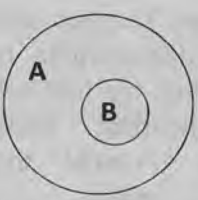
belgilanadi.

Masalan, ratsional sonlar: $Q=\left\{\frac{m}{n}\middle| m\in Z, \text{n}\in N\right\}$.

**1-ta'rif.** $A$ va $B$ to'plamlari berilgan bo'lib, $B$ to'plamning barcha elementlari $A$ to'plamga tegishli bo'lsa, $B$ to'plam $A$ ning qismi (qismiy to'plami) deyiladi va

$$B\subset A \text{ (yoki } A\supset B\text{)}$$

kabi yoziladi (1-shakl).




**1-shakl.**

Yuqoridagi misollarda, $A\subset N\subset Z$ munosabat o'rinli.

Agar $A$ to'plam elementlari orasida $P$ xususiyatli elementlar bo'lmasa, u holda

$$\{x\in A\,|\,P\}$$

bitta ham elementga ega bo'lmagan to'plam bo'lib, uni **bo'sh to'plam** deyiladi. Bo'sh to'plam $\varnothing$ kabi belgilanadi. Masalan, $x^2+x+1=0$ tenglamaning haqiqiy ildizlaridan iborat $A$ bo'sh to'plam bo'ladi:

$$\varnothing=\{x\in R\,|\,x^2+x+1=0\}.$$

Har qanday A to'plam uchun $A\subset A$, $\varnothing\subset A$ deb qarash mumkin.

Odatda, $A$ to'plamning barcha qismiy to'plamlaridan iborat to'plam $P(A)$ kabi belgilanadi. Masalan, $A=\{a,b,c\}$ to'plam uchun

$$P(A)=\{\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\varnothing\}$$

bo'ladi.

Chekli, $n$ ta elementli to'plamning barcha qism to'plamlari soni $2^n$ ta bo'ladi.

**2-ta'rif.** $A$ va $B$ to'plamlari berilgan bo'lib, $A\subset B$, $B\subset A$ bo'lsa, $A$ va $B$ bir biriga teng to'plamlar deyiladi va $A=B$ kabi yoziladi.

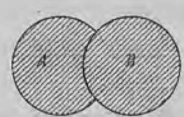
Demak, $A=B$ tenglik $A$ va $B$ to'plamlarning bir xil elementlardan tashkil topganligini bildiradi.

**To'plamlar ustida amallar.** Ikki $A$ va $B$ to'plamlar berilgan bo'lsin.

**3-ta'rif.** $A$ va $B$ to'plamlarning barcha elementlaridan tashkil topgan $E$ to'plam $A$ va $B$ to'plamlar yig'indisi (birlashmasi) deyiladi $A\cup B$ kabi belgilanadi (2-shakl): $E=A\cup B$.

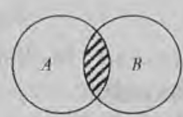
Demak, bu holda $a\in A\cup B$ dan $a\in A$, yoki $a\in B$, yoki bir vaqtda $a\in A$, $a\in B$ bo'lishi kelib chiqadi.

$A\cup B$




**2-shakl.**

$A\cap B$




**3-shakl.**

**422.** A={1; 2; 5; 8} va B={2; 4; 8; 10} to'plamlarning birlashmasini toping.

**Yechish**. Ta'rifga ko'ra, $A\cup B$={1; 2; 4; 5; 8; 10} bo'ladi.

**4-ta'rif.** $A$ va $B$ to'plamlarning barcha umumiy elementlaridan tashkil topgan $F$ to'plam $A$ va $B$ to'plamlar ko'paytmasi (kesishmasi) deyiladi va $A\cap B$ kabi belgilanadi (3-shakl): $F=A\cap B$.

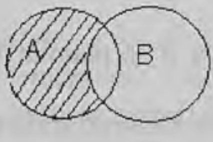
Demak, bu holda $a\in A\cap B$ dan bir vaqtda $a\in A$, $a\in B$ bo'lishi kelib chiqadi.

**423.** A={1; 2; 5; 8} va B={2; 4; 8; 10} to'plamlarning kesishmasini toping.

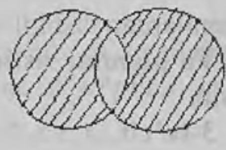
**Yechish.** Ta'rifga ko'ra, $A\cap B$ ={ 2; 8 } bo'ladi.

**5-ta'rif.** $A$ to'plamning $B$ to'plamga tegishli bo'lmagan barcha elementlaridan tashkil topgan G to'plam $A$ to'plamdan $B$ to'plamning ayirmasi deyiladi va $A\setminus B$ kabi belgilanadi (4-shakl): $G=A\setminus B$.

$A \setminus B$ $A \Delta B$



4-shakl



5-shakl

Demak, $a \in A \setminus B$ dan $a \in A$, $a \notin B$ bo‘lishi kelib chiqadi.

**6-ta’rif.** $A$ to‘plamning $B$ ga tegishli bo‘lmagan barcha elementlaridan va B to‘plamning $A$ ga tegishli bo‘lmagan barcha elementlaridan tuzilgan to‘plam $A$ va $B$ to‘plamlarning simmetrik ayirmasi deyiladi va $A\Delta B$ kabi belgilanadi (5-shakl):

$$A\Delta B = (A \setminus B) \cup (B \setminus A).$$

Demak, $a \in A\Delta B$ bo‘lishidan $a \in A$, $a \notin B$ yoki $a \in B$, $a \notin A$ bo‘lishi kelib chiqadi.

**424**. A={1; 3; 5; 7; 9} va B={4; 6; 7; 8; 9} to‘plamlar berilgan. $A \setminus B$, $B \setminus A$, $A\Delta B$ to‘plamlarni toping.

**Yechish.** Yuqoridagi ta’riflardan, $A \setminus B=\{1; 3; 5\}$, $B \setminus A=\{4; 6; 8\}$ ekani kelib chiqadi. Bundan $A\Delta B = \{1;3;5\} \cup \{4;6;8\} = \{1;3;4;5;6;8\}$ ekani kelib chiqadi.

**7-ta’rif.** Aytaylik, $a \in A$, $a \in B$ bo‘lsin. Barcha tartiblangan *(a,b)* ko‘rinishidagi juftliklardan tuzilgan to‘plam $A$ va $B$ to‘plamlarning dekart ko‘paytmasi deyiladi va $A \times B$ kabi belgilanadi. Demak,

$$A \times B = \{(a,b) | a \in A, b \in B\}.$$

Xususan, $A = B$ bo‘lganda $A \times A = A^2$ deb qaraladi.

**425.** A={4; 5; 7} va B={-1; 2; 3; 4} to‘plamlar berilgan bo‘lsin. U holda A va B (B va A) to‘plamlarning dekart ko‘paytmasini toping.

**Yechish.** Ta’rifga ko‘ra

$A \times B = \{(4;-1),(4;2),(4;3),(4;4),(5;-1),(5;2),(5;3),(5;4),(7;-1),(7;2),(7;3),(7;4)\}$

$B \times A = \{(-1;4),(-1;5),(-1;7),(2;4),(2;5),(2;7),(3;4),(3;5),(3;7),(4;4),(4;5),$
$(4;7)\}$ bo‘ladi.

**8-ta’rif.** Aytaylik, $S$ va $A$ to‘plamlar berilgan bo‘lib, $A \subset S$ bo‘lsin. Ushbu

$$S \setminus A$$

to'plam $A$ to'plamni $S$ ga to'ldiruvchi to'plam deyiladi va $CA$ yoki $C_S A$ kabi belgilanadi:

$$CA = S \setminus A.$$

Masalan, $C_Z N = \{0,-1,-2,-3,...\}$.

To'plamlar ustida bajariladigan amallarning ba'zi xossalarini keltiramiz.

$A, B$ va $D$ to'plamlari berilgan bo'lsin.

1) $A \subset B, B \subset D$ bo'lsa, $A \subset D$ bo'ladi;

2) $A \cup A = A, A \cap A = A$ bo'ladi;

3) $A \subset B$ bo'lsa, $A \cup B = B, A \cap B = A$ bo'ladi;

4) $A \cup B = B \cup A, A \cap B = B \cap A$ bo'ladi;

5) $(A \cup B) \cup D = A \cup (B \cup D), (A \cap B) \cap D = A \cap (B \cap D)$;

6) $A \subset S$ bo'lsa, $A \cap CA = \varnothing$;

7) $C(A \cup B) = CA \cap CB, C(A \cap B) = CA \cup CB$, bunda $A \subset S, B \subset S$.

**426.** Ushbu $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ tenglikni isbotlang.

**Yechish.** $a \in (A \setminus B) \bigcup (B \setminus A)$ bo'lsin.

U holda $a \in (A \setminus B)$: $a \in A$, $a \notin B$

yoki $a \in (B \setminus A)$: $a \in B$, $a \notin A$ bo'ladi.

Bundan esa $a \in (A \cup B), a \notin (A \cap B)$

bo'lib, $a \in (A \cup B) \setminus (A \cap B)$ bo'lishi kelib chiqadi.

Demak, $(A \setminus B) \cup (B \setminus A) \subset (A \cup B) \setminus (A \cap B)$

Aytaylik, $a \in (A \bigcup B) \setminus (A \bigcap B)$ bo'lsin. U holda

$$a \in (A \bigcup B): \ a \in A \ yoki \ a \in B,$$

$$a \notin (A \bigcap B): a \notin A, a \notin B \ yoki \ a \in A, a \notin B \ \ yoki \ \ a \notin A, a \in B$$

bo'ladi. Bundan esa $a \in A \setminus B$ yoki $a \in B \setminus A$ bo'lib, $a \in (A \setminus B) \bigcup (B \setminus A)$ bo'lishi kelib chiqadi. Demak, $(A \cup B) \setminus (A \cap B) \subset (A \setminus B) \cup (B \setminus A)$.

Bu munosabatlardan talab qilingan tenglikning o'rinli bo'lishi topiladi.

To'plamlar ustida bajariladigan amallarni bayon etishda to'plamlarning qanday tabiatli elementlardan tuzilganligiga e'tibor qilinmadi.

Aslida, keltirilgan amallar biror universal to'plam deb ataluvchi to'plamning qismiy to'plamlari ustida bajariladi deb qaraladi. Masalan, natural sonlar to‘plamlari ustida amallar bajariladigan bo‘lsa, universal to‘plam sifatida barcha natural sonlardan iborat $N$ to‘plamni olish mumkin.

**Matematik belgilar**. Matematikada tez-tez uchraydigan so‘z va so‘z birikmalari o‘rnida maxsus belgilar ishlatiladi. Ulardan muhimlarini keltiramiz:

1) «agar ... bo‘lsa, u holda ... bo‘ladi» iborasi «$\Rightarrow$» belgi orqali yoziladi;

2) ikki iboraning ekvivalentligi ushbu «$\Leftrightarrow$» belgi orqali yoziladi;

3) «har qanday», «ixtiyoriy», «barchasi uchun» so‘zlari o‘rniga «$\forall$» belgi ishlatiladi;

4) «mavjudki», «topiladiki» so‘zlari o‘rniga «$\exists$» mavjudlik belgisi ishlatiladi.

**Haqiqiy sonlar va ularning to‘plami.** Cheksiz davriy bo‘lmagan o‘nli kasr bilan ifodalanadigan son **irratsional son** deyiladi. Uni $\frac{m}{n}$ $(m \in Z,\ n \in N)$ ko‘rinishida yozib bo‘lmaydi. Ratsional va irratsional sonlar umumiy nom bilan haqiqiy son deyiladi. Barcha haqiqiy sonlar to‘plami $\boldsymbol{R}$ harfi bilan belgilanadi.

Aytaylik, $E = \{x\}$ biror haqiqiy sonlar to‘plami bo‘lsin: $\boldsymbol{E} \subset \boldsymbol{R}$. Agar

$$\exists M \in \boldsymbol{R}, \forall x \in \boldsymbol{E} : x \le M$$
$$(\exists m \in \boldsymbol{R}, \forall x \in \boldsymbol{E} : x \ge m)$$

bo‘lsa, $\boldsymbol{E}$ to‘plam **yuqoridan (quyidan)** chegaralangan deyiladi. Agar $\boldsymbol{E}$ to‘plam ham yuqoridan, ham quyidan chegaralangan bo‘lsa, $\boldsymbol{E}$ **chegaralangan to‘plam** deyiladi.

Agar

**1)** $\forall x \in E \ :\ x \le \alpha$ ;

**2)** $\forall \varepsilon > 0,\ \exists x_0 \in E \ :\ x_0 > \alpha - \varepsilon$

bo‘lsa,

$$\alpha = \sup E = \sup\{x\}$$

son $E$ to‘plamning **aniq yuqori** chegarasi deyiladi.

Agar

1) $\forall x \in E \ : \ x \geq \beta$;

2) $\forall \varepsilon > 0, \exists x_0 \in E \ : \ x_0 < \beta + \varepsilon$

bo'lsa,

$$\beta = \inf E = \inf\{x\}$$

son $E$ to'plamning **aniq quyi** chegarasi deyiladi. Quyidagicha to'plamlarni qaraylik. Ushbu to'plamlar

$\{x \in R | a \leq x \leq b\} = [a,b]$ - **segment** yoki **kesma;**

$\{x \in R | a < x < b\} = (a,b)$ - **interval;**

$\{x \in R | a \leq x < b\} = [a,b)$ - **yarim interval;**

$\{x \in R | a < x \leq b\} = (a,b]$ - **yarim interval**

deb ataladi.

**427**. Ushbu $E = \left\{x = \dfrac{n^2}{n^2+4} : n \in N\right\}$

to'plamning aniq yuqori hamda aniq quyi chegarasi topilsin.

**Yechish.** Ravshanki, $\forall n \in N$ uchun $0 < \dfrac{n^2}{n^2+4} < 1$

bo'ladi. Demak, berilgan to'plam chegaralangan. Oxirgi munosabatdan $\forall x \in E$ uchun $x = \dfrac{n^2}{n^2+4} \leq 1$ bo'lishi kelib chiqadi.

$\forall \varepsilon > 0$ sonni $(0 < \varepsilon < 1)$ olib, $E$ to'plamda, uning

$$x_0 = \frac{n^2}{n^2+4}, \quad n > \sqrt{\frac{4(1-\varepsilon)}{\varepsilon}}$$

elementi qaralsa, uning uchun $\dfrac{n^2}{n^2+4} > 1 - \varepsilon$ tengsizlik bajariladi.

Chunki

$$\frac{n^2}{n^2+4} > 1 - \varepsilon \Rightarrow n^2 > n^2 + 4 - n^2\varepsilon - 4\varepsilon \Rightarrow n^2\varepsilon > 4(1-\varepsilon) \Rightarrow$$

$$\Rightarrow n^2 > \frac{4(1-\varepsilon)}{\varepsilon} \Rightarrow n > \sqrt{\frac{4(1-\varepsilon)}{\varepsilon}}$$

munosabatlardan topamiz: $SupE = Sup\left\{x = \dfrac{n^2}{n^2+4} : n \in N\right\} = 1$.

Xuddi shunga o'xshash $\inf E = \inf\left\{x = \dfrac{n^2}{n^2+4} : n \in N\right\} = 0$ bo'ladi.

**428.** Quyidagi to'plamlarni elementlari orqali yozing.

1) $A=\{x|x\in N, x<6\}$;

2) $A=\{x|x\in Z, -7\le x<2\}$;

3) $A=\{x|x\in N, x<30, x-\text{tub son}\}$;

4) $A=\{x|x\in N, 48 \text{ ning bo‘luvchisi}\}$;

5) $A=\{x|x\in Z,\ 2x^2-5x+2=0\}$;

6) $A=\{x|x\in Q,\ 2x^2-5x+2=0\}$;

7) $A=\{x|x\in R,\ x^2-x-6\le 0\}$.

**429.** O'quv markazida 100 ta talabadan, 70 tasi ingliz tilini, 45 tasi fransuz tilini, 23 tasi har ikki tilni biladi. Nechta talaba na ingliz tilini, na fransuz tilini biladi?

**430.** Agar $A=\{2,3,5,8,13\}$, $B=\{5,9,13,17\}$ bo'lsa, quyidagi to'plamlarni toping: 1) $A\cup B$ 2) $A\cap B$ 3) $A\setminus B$ 4) $B\setminus A$ 5) $A\Delta B$ 6) $A\times B$ 7) $B\times A$ 8) $B\times B$

**431.** Agar $C=\{1,3,5,7,9\}$, $D=\{3,6,9,12\}$ bo'lsa, quyidagi to'plamlarni toping: 1) $C\cup D$ 2) $C\cap D$ 3) $C\setminus D$ 4) $D\setminus C$ 5) $C\Delta D$ 6) $C\times D$ 7) $D\times C$ 8) $D\times D$

**432.** Agar $A=[1,4]$, $B=[2,5]$ bo'lsa, quyidagi to'plamlarni toping: 1) $A\cup B$ 2) $A\cap B$ 3) $A\setminus B$ 4) $B\setminus A$ 5) $A\Delta B$ 6) $A\times B$ 7) $B\times A$ 8) $B\times B$

**433.** Agar $C=[3,5)$, $D=(4,7]$ bo'lsa, quyidagi to'plamlarni toping: 1) $C\cup D$ 2) $C\cap D$ 3) $C\setminus D$ 4) $D\setminus C$ 5) $C\Delta D$ 6) $C\times D$ 7) $D\times C$ 8) $D\times D$

**434.** $A=\{a,b,c,d\}$ to'plamning barcha qism to'plamlarini tuzing.

**435.** To'plamning aniq yuqori va aniq quyi chegaralarini toping.

1) $A=\{x|x\in R, |x+2|<5\}$;

2) $A=\{x|x\in R, |x-2|<2\}$;

3) $A=\{x|x\in R, 2\le|x+1|<5\}$.

**436*.** Quyidagilarni isbotlang:

1) $A\cap(A\cup B)=A$;

2) $A\setminus(B\setminus C)=(A\setminus B)\cup(A\cap C)$;

3) $n(A\cup B)=n(A)+n(B)-n(A\cap B)$, bunda $n(A)$ – $A$ to'plamning elementlari soni.

## 21 §. Sonli ketma-ketliklar. Ketma-ketlik limiti. Yaqinlashuvchi ketma-ketlik xossalari.

**Sonlar ketma-ketligi tushunchasi.** Har bir natural $n$ songa biror haqiqiy $x_n$ sonini mos qo'yuvchi

$$f: n \to x_n, \quad (n=1,2,3,...) \tag{1}$$

akslantirishni qaraymiz.

**1-ta'rif.** 1- akslantirishning akslaridan iborat ushbu

$$x_1,\ x_2,\ x_3,\ ...,x_n,... \tag{2}$$

to'plam **sonlar ketma-ketligi** deyiladi. Uni $\{x_n\}$ yoki $x_n$ kabi belgilanadi.

$x_n\ (n=1,2,3,...)$ sonlar (2) **ketma-ketlikning hadlari** deyiladi. Masalan,

1) $x_n=\frac{1}{n}:\ 1, \frac{1}{2}, \frac{1}{3},..., \frac{1}{n},...,$

2) $x_n=(-1)^n:\ -1,\ 1,\ -1,\ ...,\ (-1)^n,...$

3) $x_n=\sqrt[n]{n}:\ 1,\ \sqrt{2},\ \sqrt[3]{3},\ ...,\sqrt[n]{n},\ ...$

4) $x_n=1:\ 1,1,1,...,1,...$

5) $0{,}3;\ 0{,}33;\ 0{,}333;...;\ 0,\underbrace{333...3}_{n\ \text{та}};\ ...$ .

lar sonlar ketma-ketliklaridir.

Biror $\{x_n\}$ ketma-ketlik berilgan bo'lsin.

**2-ta'rif.** Agar shunday o'zgarmas $M$ soni mavjud bo'lsaki, ixtiyoriy $x_n\ (n=1,2,3,...)$ uchun $x_n \le M$ tengsizlik bajarilsa (ya'ni $\exists M,\ \forall n \in N:\ x_n \le M$ bo'lsa), $\{x_n\}$ ketma-ketlik **yuqoridan chegaralangan** deyiladi.

**3-ta'rif.** Agar shunday o'zgarmas $m$ soni mavjud bo'lsaki, ixtiyoriy $x_n\ (n=1,2,3,...)$ uchun $x_n \ge m$ tengsizlik bajarilsa (ya'ni, $\exists m,\ \forall n \in N: x_n \ge m$ bo'lsa), $\{x_n\}$ ketma-ketlik **quyidan chegaralangan** deyiladi.

**4-ta'rif.** Agar $\{x_n\}$ ketma-ketlik ham yuqoridan, ham quyidan chegaralangan bo'lsa (ya'ni $\exists\, m, M,\ \forall\, n\in N:\ m\le x_n \le M$ bo'lsa), $\{x_n\}$ ketma-ketlik **chegaralangan** deyiladi.

**437.** Ushbu $x_n=\dfrac{n}{4+n^2}$ $(n=1,2,3,...)$ ketma-ketlikning chegaralanganligi isbotlansin.

**Yechish.** Ravshanki, $\forall n\in N$ uchun $x_n=\dfrac{n}{4+n^2}>0$ bo'ladi. Demak, qaralayotgan ketma-ketlik quyidan chegaralangan. Ma'lumki, $0\le(n-2)^2=n^2-4n+4$ bo'lib, undan $4n\le 4+n^2$ ya'ni,

$$\frac{n}{4+n^2}\le\frac{1}{4}$$

bo'lishi kelib chiqadi. Bu esa berilgan ketma-ketlikning yuqoridan chegaralanganligini bildiradi. Demak, ketma-ketlik chegaralangan

**5-ta'rif.** Agar $\{x_n\}$ ketma-ketlik uchun

$$\forall M\in R,\ \exists n_0\in N:\ x_{n_0}>M$$

bo'lsa, **ketma-ketlik yuqoridan chegaralanmagan** deyiladi.

**Sonlar ketma-ketligining limiti.** Aytaylik, $a\in R$ son hamda ixtiyoriy musbat $\varepsilon$ son berilgan bo'lsin.

**6-ta'rif.** Ushbu

$$U_\varepsilon(a)=\{x\in R|\ a-\varepsilon<x<a+\varepsilon\}=(a-\varepsilon,\ a+\varepsilon)$$

to'plam $a$ **nuqtaning** $\varepsilon$ **– atrofi** deyiladi.

Faraz qilaylik $\{x_n\}$ ketma-ketlik va $a\in R$ soni berilgan bo'lsin.

**7-ta'rif.** Agar ixtiyoriy $\varepsilon>0$ son olinganda ham shunday $n_0$ natural soni mavjud bo'lsaki, $n>n_0$ tengsizlikni qanoatlantiruvchi barcha natural sonlar uchun

$$|x_n-a|<\varepsilon \qquad (3)$$

tengsizlik bajarilsa, (ya'ni $\forall\varepsilon>0,\ \exists n_0\in N,\ \forall n>n_0:\ |x_n-a|<\varepsilon$ bo'lsa), $a$ son $\{x_n\}$ **ketma-ketlikning limiti** deyiladi va $a=\lim\limits_{n\to\infty}x_n$ yoki $n\to\infty$ da $x_n\to a$ kabi belgilanadi.

Yuqorida keltirilgan ta'riflardan ko'rinadiki $\varepsilon$ ixtiyoriy musbat son bo'lib, natural $n_0$ soni esa $\varepsilon$ ga va qaralayotgan ketma-ketlikka bog'liq ravishda topiladi.

**438.** Ushbu $x_n = \frac{1}{n}$ $(n = 1,2,3,....)$ ketma-ketlikning limiti 0 ga teng bo‘lishi isbotlansin.

**Yechish.** Ravshanki, $\left|\frac{1}{n} - 0\right| = \frac{1}{n}$ bo‘lib, $\frac{1}{n} < \varepsilon$ $(\varepsilon > 0)$ tengsizlik barcha $n > \frac{1}{\varepsilon}$ bo‘lganda o‘rinli. Bu holda $n_0 = \left[\frac{1}{\varepsilon}\right] + 1$ deyilsa, ($[a] - a$ sonidan katta bo‘lmagan uning butun qismi), unda $\forall n > n_0$ uchun $\left|\frac{1}{n} - 0\right| < \varepsilon$ bo‘ladi. Ta’rifga binoan $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

**439.** Ushbu $x_n = \frac{n}{n+1}$ $(n = 1,2,3,...)$ ketma-ketlikning limiti 1 ga teng bo‘lishi isbotlansin.

**Yechish.** Ixtiyoriy $\varepsilon > 0$ son olamiz. So‘ng ushbu $|x_n - 1| < \varepsilon$ tengsizlikni qaraymiz. Ravshanki, $|x_n - 1| = \left|\frac{n}{n+1} - 1\right| = \frac{n}{n+1}$ Unda yuqoridagi tengsizlik $\frac{n}{n+1} < \varepsilon$ ko‘rinishga keladi. Keyingi tengsizlikdan $n > \frac{1}{\varepsilon} - 1$ bo‘lishi kelib chiqadi. Demak, limit ta’rifidagi $n_0 \in N$ sifatida $n_0 = \left[\frac{1}{\varepsilon} - 1\right] + 1$ olinsa ($\varepsilon > 0$ ga ko‘ra $n_0 \in N$ topilib), $\forall n > n_0$ uchun $|x_n - 1| < \varepsilon$ bo‘ladi. Bu esa $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$ bo‘lishini bildiradi.

**440.** Ushbu $x_n = (-1)^n$ $(n = 1,2,3,...)$ ketma-ketlikning limiti mavjud emasligi isbotlansin.

**Yechish.** Teskarisini faraz qilaylik. Bu ketma-ketlik $a$ limitga ega bo‘lsin. Unda ta’rifga binoan,

$$\forall \varepsilon > 0,\ \exists n_0 \in N,\ \forall n > n_0 :\ |(-1)^n - a| < \varepsilon$$

bo‘ladi. Ravshanki, $n$ juft bo‘lganda $|1 - a| < \varepsilon$, $n$ toq bo‘lganda $|(-1) - a| < \varepsilon$, ya’ni $|1 + a| < \varepsilon$ bo‘ladi. Bu tengsizliklardan foydalanib topamiz:

$$2 = |(1 - a) + (1 + a)| \le |1 - a| + |1 - a| < 2\varepsilon .$$

Bu tengsizlik $\varepsilon > 1$ bo‘lgandagina o‘rinli. Bunday vaziyat $\varepsilon > 0$ sonining ixtiyoriy bo‘lishiga zid. Demak, ketma-ketlik limitga ega emas.

**8-ta'rif.** Agar $\{x_n\}$ ketma-ketlik chekli limitga ega bo'lsa, u **yaqinlashuvchi ketma-ketlik** deyiladi.

**Yaqinlashuvchi ketma-ketlikning xossalari**

**1-teorema.** Agar $\{x_n\}$ ketma-ketlik limitga ega bo'lsa, u yagona bo'ladi.

**2-teorema.** $\{x_n\}$ ketma-ketlik yaqinlashuvchi bo'lsa, u chegaralangan bo'ladi.

**3-teorema.** Agar $\{x_n\}$ ketma-ketlik yaqinlashuvchi va $\lim\limits_{n\to\infty} x_n = a$ bo'lib, $a > p$ $(a < q)$ bo'lsa, u holda shunday $n_0 \in N$ topiladiki, $\forall n > n_0$ bo'lganda $x_n > p$ $(x_n < q)$ bo'ladi.

**4-teorema.** Agar $\{x_n\}$ va $\{z_n\}$ ketma-ketlik yaqinlashuvchi bo'lib,

1) $\lim\limits_{n\to\infty} x_n = a,\quad \lim\limits_{n\to\infty} z_n = a$

2) $\forall n \in N$ uchun $x_n \le y_n \le z_n$

bo'lsa, u holda $\{y_n\}$ ketma-ketlik yaqinlashuvchi va $\lim\limits_{n\to\infty} y_n = a$ bo'ladi.

**5-teorema.** Aytaylik, $\{x_n\}$ va $\{y_n\}$ ketma-ketliklari berilgan bo'lib,

$$\lim_{n\to\infty} x_n = a,\quad \lim_{n\to\infty} y_n = b,\quad (a \in R,\quad b \in R)$$

bo'lsin. U holda $n \to \infty$ da $(c \cdot x_n) \to c \cdot a$;

$$x_n + y_n \to a + b;\qquad x_n \cdot y_n \to ab;\qquad \frac{x_n}{y_n} \to \frac{a}{b}\ (b \ne 0),$$ ya'ni

a) $\forall c \in R$ да $\lim\limits_{n\to\infty}(c \cdot x_n) = c \cdot \lim\limits_{n\to\infty} x_n$;

b) $\lim\limits_{n\to\infty}(x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$;

c) $\lim\limits_{n\to\infty}(x_n \cdot y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$;

d) $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n},\quad (b \ne 0)$

bo'ladi.

**Monoton ketma-ketlik tushunchasi.**

Aytaylik, $\{x_n\}$:

$$x_1, x_2, \ldots, x_n, \ldots \tag{4}$$

ketma-ketlik berilgan bo'lsin.

**9-ta'rif.** Agar (4) ketma-ketlikda $\forall n \in N$ uchun $x_n \le x_{n+1}$ tengsizlik bajarilsa, $\{x_n\}$ **o'suvchi ketma-ketlik** deyiladi. Agar (4) ketma-ketlikda $\forall n \in N$ uchun $x_n < x_{n+1}$ tengsizlik bajarilsa, $\{x_n\}$ **qat'iy o'suvchi ketma-ketlik** deyiladi.

**10-ta'rif.** Agar (4) ketma-ketlikda $\forall n \in N$ uchun $x_n \ge x_{n+1}$ tengsizlik bajarilsa, $\{x_n\}$ **kamayuvchi ketma-ketlik** deyiladi. Agar (4) ketma-ketlikda $\forall n \in N$ uchun $x_n > x_{n+1}$ tengsizlik bajarilsa, $\{x_n\}$ **qat'iy kamayuvchi ketma-ketlik** deyiladi.

**441.** Ushbu $x_n = \frac{n+1}{n}$: $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots$ ketma-ketlik qat'iy kamayuvchi ketma-ketlik bo'lishini ko'rsating.

**Yechish.** Haqiqatdan ham, berilgan ketma-ketlik uchun

$$x_n = \frac{n+1}{n}, \quad x_{n+1} = \frac{n+2}{n+1}$$

bo'lib, $\forall n \in N$ uchun $x_{n+1} - x_n = \frac{n+2}{n+1} - \frac{n+1}{n} = \frac{-1}{n(n+1)} < 0$

bo'ladi. Unda $x_{n+1} < x_n$ bo'lishi kelib chiqadi. Bu esa qaralayotgan ketma-ketlikning qat'iy kamayuvchi bo'lishini bildiradi.

O'suvchi hamda kamayuvchi ketma-ketliklar umumiy nom bilan monoton ketma-ketliklar deyiladi.

**442.** Ushbu $x_n = \frac{n^2}{n^2+1}$ $(n = 1,2,3,\ldots)$ ketma-ketlikning qat'iy o'suvchi ekanligi isbotlansin.

**Yechish.** Bu ketma-ketlikning $n$–hamda $(n+1)$–hadlari uchun

$$x_n = \frac{n^2}{n^2+1} = 1 - \frac{1}{n^2+1}, \quad x_{n+1} = \frac{(n+1)^2}{(n+1)^2+1} = 1 - \frac{1}{(n+1)^2+1}$$

bo'ladi. Ravshanki, $\frac{1}{(n+1)^2} < \frac{1}{n^2}$. Shu tengsizlikni e'tiborga olib, topamiz:

$x_{n+1} = 1 - \frac{1}{(n+1)^2+1} > 1 - \frac{1}{n^2+1} = x_n$. Demak, $\forall n \in N$ uchun $x_n < x_{n+1}$.

Bu esa qaralayotgan ketma-ketlikning qat'iy o'suvchi bo'lishini bildiradi.

**6-teorema.** Agar $\{x_n\}$ ketma-ketlik

1) o'suvchi,

2) yuqoridan chegaralangan bo‘lsa, u chekli limitga ega bo‘ladi.

**7-teorema.** Agar $\{x_n\}$ ketma-ketlik

1) kamayuvchi,

2) quyidan chegaralangan bo‘lsa, u chekli limitga ega bo‘ladi.

**8-teorema.** $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \quad (5)$$

Ushbu limit 2-ajoyib limit deb ataladi. Bu $e$ soni irratsional son bo‘lib, $e = 2{,}718281828459045\ldots$ bo‘ladi.

**443.** Quyidagi limitni toping: $\lim\limits_{n\to\infty}\left(\frac{2n+1}{2n}\right)^n$.

**Yechish.** 2-ajoyib limitdan foydalanib quyidagini topamiz:

$$\lim_{n\to\infty}\left(\frac{2n+1}{2n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n\cdot\frac{1}{2}},$$ $2n = k$ deb belgilash kiritsak,

$$\lim_{n\to\infty}\left(\frac{2n+1}{2n}\right)^n = \left(\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k\right)^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$$ bo‘ladi.

**Ketma-ketlikning quyi hamda yuqori limitlari.** $\{x_n\}$ ketma-ketlik berilgan bo‘lsin. Bu ketma-ketlikning qismiy ketma-ketligining limiti $\{x_n\}$ ning qismiy limiti deyiladi.

**11-ta’rif.** $\{x_n\}$ ketma-ketlik qismiy limitlarining eng kattasi berilgan **ketma-ketlikning yuqori limiti** deyiladi va $\overline{\lim\limits_{n\to\infty}} x_n$ kabi belgilanadi.

$\{x_n\}$ ketma-ketlik qismiy limitlarining eng kichigi berilgan **ketma-ketlikning quyi limiti** deyiladi va $\underline{\lim\limits_{n\to\infty}} x_n$ kabi belgilanadi.

Masalan, ushbu $\{x_n\}: 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$ ketma-ketlikning yuqori limiti $\overline{\lim\limits_{n\to\infty}} x_n = 3$, quyi limiti esa $\underline{\lim\limits_{n\to\infty}} x_n = 1$ bo‘ladi.

Umuman, $\{x_n\}$ ketma-ketlikning quyi hamda yuqori limitlari quyidagicha ham kiritilishi mumkin.

**9-teorema.** $\{x_n\}$ ketma-ketlik $C$ limitga ega bo‘lishi uchun

$$\underline{\lim_{n\to\infty}} x_n = \overline{\lim_{n\to\infty}} x_n = C$$

bo‘lishi zarur va yetarlidir.

**444.** Quyidagi ketma-ketliklarning dastlabki 5 ta hadini yozing:

1) $x_n = 3^{n-1}$; 2) $x_n = (-1)^n + 1$;

3) $x_n = \frac{n+1}{n}$; 4) $x_n = \cos\frac{\pi n}{2}$;

5) $x_n = n^2 - 2n + 3$; 6) $x_n = \frac{1}{n!}$;

7) $x_n = \sum_{k=1}^{n} \frac{1}{2^k}$; 8) $x_n = \sum_{k=1}^{n} (-1)^k$;

9) $x_1 = 0,\ x_2 = 1,\ x_{n+2} = x_n + x_{n+1}$ -Fibonachchi ketma-ketligi.

**445.** Quyidagi ketma-ketliklarning umumiy hadi formulasini yozing.

1) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$; 2) $1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \frac{1}{125}, \ldots$;

3) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \ldots$; 4) $2, 1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}, 1\frac{1}{5}, \ldots$;

5) $-1, 1, -1, 1, \ldots$; 6) $2, 4, 2, 4, 2, 4, \ldots$;

7) $-2, 5, -10, 17, -26, \ldots$; 8) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots$;

9) $\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots$; 10) $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$ .

**446.** Quyidagi ketma-ketliklardan qaysilari quyidan chegaralangan, yuqoridan chegaralangan, chegaralangan, o'suvchi, kamayuvchi bo'lishini aniqlang.

1) $x_n = n$; 2) $x_n = -\frac{n^2}{n+1}$;

3) $x_n = \left(-\frac{1}{2}\right)^n$; 4) $x_n = \cos\frac{\pi n}{2}$;

5) $x_n = -n^3 + 2n$; 6) $x_n = \frac{n^2+1}{n^2-1}$;

7) $x_n = (-2)^n$; 8) $x_n = 2^{-n}$;

9) $x_n = n^{(-1)^n}$; 10) $x_n = \frac{n^2}{n!}$.

**447.** Quyidagi $x_n$ ketma-ketliklarning limiti $a$ ekani ta'rif yordamida ko'rsatilsin.

1) $x_n = \frac{n+1}{n},\ a = 1$; 2) $x_n = \frac{3n+1}{n-5},\ a = 3$;

3) $x_n=\frac{2n-2}{5n+2},\ a=\frac{2}{5}$; 4) $x_n=\frac{3^{n+1}-1}{3^n},\ a=3$.

**448.** Quyidagi ketma – ketliklarning limitlarini toping.

1) $\lim\limits_{n\to\infty}\frac{n-3}{6n+1}$; 2) $\lim\limits_{n\to\infty}\frac{n^3-2n+5}{2n^2-2n^3+7}$;

3) $\lim\limits_{n\to\infty}\frac{n^2-n+3}{n^2-2n^3+1}$; 4) $\lim\limits_{n\to\infty}\frac{n^3-100n+5}{100n^2-2n+1}$;

5) $\lim\limits_{n\to\infty}\frac{(n+5)^4}{1-5n^4}$; 6) $\lim\limits_{n\to\infty}\left(\frac{5}{n-3}-\frac{2}{3n-1}\right)$;

7) $\lim\limits_{n\to\infty}\left(\sqrt{9n^2+n}-3n\right)$; 8) $\lim\limits_{n\to\infty}\left(\sqrt[3]{n^3-4n^2}-n\right)$;

9) $\lim\limits_{n\to\infty}\frac{\sqrt{2n}}{\sqrt{3n+\sqrt{3n+\sqrt{3n}}}}$; 10) $\lim\limits_{n\to\infty}\frac{10n^3-\sqrt{n^3+2}}{\sqrt{4n^6+3}-n}$;

11) $\lim\limits_{n\to\infty}\frac{n^3}{1^2+2^2+...+n^2}$; 12) $\lim\limits_{n\to\infty}\left(\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+...+\frac{1}{(2n-1)\cdot(2n+1)}\right)$;

13) $\lim\limits_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n$; 14) $\lim\limits_{n\to\infty}\left(\frac{2n+3}{2n+1}\right)^{n+1}$;

15) $\lim\limits_{n\to\infty}\left(\frac{n-1}{n+3}\right)^{n+2}$; 16) $\lim\limits_{n\to\infty}\left(\frac{n^2-1}{n^2}\right)^{n^4}$;

17) $\lim\limits_{n\to\infty}\frac{7^n-1}{7^n+1}$; 18**) $\lim\limits_{n\to\infty}\pi\sqrt{\pi\sqrt[3]{\pi\sqrt[4]{\pi...\sqrt[n]{\pi}}}}$.

## 22 §. Funksiya tushunchasi. Elementar funksiyalar sinfi.

### Funksiya ta'rifi, berilish usullari

Aytaylik, $X\subset R, Y\subset R$ to'plamlar berilgan bo'lib, $x$ va $y$ o'zgaruvchilar mos ravishda shu to'plamlarda o'zgarsin: $x\in X$, $y\in Y$.

**1-ta'rif.** Agar $X$ to'plamdagi har bir $x$ songa biror $f$ qoidaga ko'ra $Y$ to'plamdan bitta $y$ son mos qo'yilgan bo'lsa, $X$ to'plamda **funksiya berilgan (aniqlangan)** deyiladi va $f:x\to y$ yoki $y=f(x)$ kabi belgilanadi. Bunda $X$ - **funksiyaning aniqlanish to'plami (sohasi)**, $Y$ - **funksiyaning o'zgarish to'plami (sohasi) deyiladi.**

$x$ – **erkli o'zgaruvchi** yoki **funksiya argumenti**, y esa **erksiz o'zgaruvchi** yoki **funksiyaning qiymati** deyiladi.

**449.** $X=(-\infty,+\infty)$, $Y=(0,+\infty)$ bo'lib, $f$ qoida $f: x \to y = x^2+1$ bo'lsin. Bu holda har bir $x \in X$ ga bitta $x^2+1 \in Y$ mos qo'yilib,

$$y = x^2 + 1$$

funksiyaga ega bo'lamiz.

**450.** Har bir ratsional songa 1 ni, har bir irratsional songa 0 ni mos qo'yish natijasida funksiya hosil bo'ladi. Odatda, bu **Dirixle funksiyasi** deyilib, u $D(x)$ kabi belgilanadi:

$$D(x) = \begin{cases} 1, & agar \;\; x \;\; ratsional\, son\, bo'lsa, \\ 0, & agar \;\; x \;\; irratsional\, son\, bo'lsa. \end{cases}$$

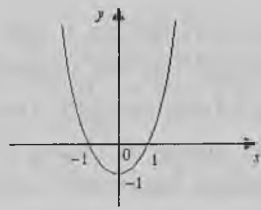
Shunday qilib, $y = f(x)$ funksiya uchta: $X$ to'plam, $Y$ to'plam va har bir $x \in X$ ga bitta $y \in Y$ ni mos qo'yuvchi $f$ qoidaning berilishi bilan aniqlanar ekan.

Tekislikda dekart koordinatalar sistemasini olamiz. Tekislikdagi $(x, f(x))$ nuqtalardan iborat ushbu $\{(x,f(x))\} = \{(x,f(x)) \mid x \in X, f(x) \in Y\}$ to'plam $y = f(x)$ **funksiyaning grafigi** deyiladi.

Masalan,

$$y = x^2 - 1 \quad (x \in X = [-2,2])$$

funksiyaning grafigi 6-shaklda tasvirlangan.




**6-shakl.**

Funksiya ta'rifidagi $f$ qoida turlicha bo'lishi mumkin.

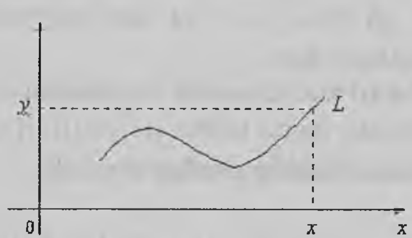
a) Ko'pincha $x$ va $y$ o'zgaruvchilar orasidagi bog'lanish formulalar yordamida ifodalanadi. Bu **funksiyaning analitik usulda berilishi** deyiladi. Masalan, $y = \sqrt{1-x^2}$.

b) Ba'zi hollarda $x \in X$, $y \in Y$ o'zgaruvchilar orasidagi bog'lanish jadval orqali bo'lishi mumkin. Masalan, kun davomida havo haroratini kuzatganimizda, $t_1$ vaqtda havo harorati $T_1$, $t_2$ vaqtda havo harorati $T_2$ va h.k. bo'lsin. Natijada quyidagi jadval hosil bo'ladi.

| $t$ – vaqt | $t_1$ | $t_2$ | $t_3$ | ... | $t_n$ |
|---|---|---|---|---|---|
| $T$ – harorat | $T_1$ | $T_2$ | $T_3$ | ... | $T_n$ |

Bu jadval $t$ vaqt bilan havo harorati $T$ orasidagi bog'lanishni ifodalaydi, bunda $t$-argument, $T$ esa $t$ ning funksiyasi bo'ladi.

c) $x$ va $y$ o'zgaruvchilar orasidagi bog'lanish tekislikda biror egri chiziq orqali ham ifodalanishi mumkin (7-shakl).




**7-shakl.**

Masalan, 7-shaklda tasvirlangan $L$ egri chiziq berilgan bo'lsin. Aytaylik, $[a,b]$ segmentdagi har bir nuqtadan o'tkazilgan perpendikulyar $L$ chiziqni faqat bitta nuqtada kessin. $\forall x \in [a,b]$ nuqtadan perpendikulyar chiqarib, uning $L$ chiziq bilan kesishish nuqtasini topamiz. Olingan $x$ nuqtaga kesishish nuqtasining ordinatasi $y$ ni mos qo'yamiz. Natijada har bir $x \in [a,b]$ ga bitta $y$ mos qo'yilib, funksiya hosil bo'ladi. Bunda $x$ bilan $y$ orasidagi bog'lanishni berilgan $L$ egri chiziq bajaradi.

**Funksiyaning chegaralanganligi.** $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lsin.

**2-ta'rif.** Agar shunday o'zgarmas $M$ soni topilsaki, $\forall x \in X$ uchun $f(x) \leq M$ tengsizlik bajarilsa, $f(x)$ **funksiya** $X$ to'plamda **yuqoridan chegaralangan** deyiladi. Agar shunday o'zgarmas $m$

son topilsaki, $\forall x \in X$ uchun $f(x) \geq m$ tengsizlik bajarilsa, $f(x)$ funksiya $X$ to'plamda **quyidan chegaralangan** deyiladi.

**3-ta'rif.** Agar $f(x)$ funksiya $X$ to'plamda ham yuqoridan, ham quyidan chegaralangan bo'lsa, $f(x)$ funksiya $X$ to'plamda **chegaralangan** deyiladi.

**451.** Ushbu $f(x)=\frac{1+x^2}{1+x^4}$ funksiyani chegaralanganligini ko'rsating.

**Yechish.** Ravshanki, $\forall x \in R$ da $f(x)=\frac{1+x^2}{1+x^4}>0$. Demak, berilgan funksiya R da quyidan chegaralangan. Ayni paytda, $f(x)$ funksiya uchun

$$f(x)=\frac{1}{1+x^4}+\frac{x^2}{1+x^4}\leq 1+\frac{x^2}{1+x^4}$$

bo'ladi. Endi

$$0\leq(x^2-1)^2=x^4-2x^2+1 \quad \Rightarrow 2x^2\leq x^4+1 \Rightarrow \frac{x^2}{x^4+1}\leq\frac{1}{2}$$

bo'lishini e'tiborga olib, topamiz: $f(x)\leq 1+\frac{1}{2}=\frac{3}{2}$.

Bu esa $f(x)$ funksiyaning yuqoridan chegaralanganligini bildiradi. Demak, berilgan funksiya $R$ da chegaralangan.

**4-ta'rif.** Agar har qanday $M>0$ son olinganda ham shunday $x_0 \in X$ nuqta topilsaki, $f(x_0)>M$ tengsizlik bajarilsa, $f(x)$ funksiya $X$ to'plamda **yuqoridan chegaralanmagan** deyiladi.

**Davriy funksiyalar. Juft va toq funksiyalar.** $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lsin.

**5-ta'rif.** Agar shunday o'zgarmas $T\ (T\neq 0)$ son mavjud bo'lsaki, $\forall x \in X$ uchun

1) $x-T\in X$, $x+T\in X$

2) $f(x+T)=f(x)$

bo'lsa, $f(x)$ **davriy funksiya** deyiladi, $T$ son esa $f(x)$ **funksiyaning davri** deyiladi.

Masalan, $f(x)=\sin x$, $f(x)=\cos x$ funksiyalar davriy funksiyalar bo'lib, ularning davri $2\pi$ ga, $f(x)=tgx$, $f(x)=ctgx$ funksiyalarning davri esa $\pi$ ga teng.

Aytaylik, $\forall x \in X$ $(X \subset R)$ uchun $-x \in X$ bo'lsin.

**6-ta'rif.** Agar $\forall x \in X$ uchun $f(-x)=f(x)$ tenglik bajarilsa, $f(x)$ **juft funksiya** deyiladi. Agar $\forall x \in X$ uchun $f(-x)=-f(x)$ tenglik bajarilsa, $f(x)$ **toq funksiya** deyiladi.

Masalan, $f(x)=x^2+1$ juft funksiya, $f(x)=x^3+x$ esa toq funksiya bo'ladi. Ushbu $f(x)=x^2-x$ funksiya juft ham emas, toq ham emas.

**Monoton funksiyalar.** Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lsin.

**7-ta'rif.** Agar $\forall x_1, x_2 \in X$ uchun $x_1 < x_2$ bo'lganda $f(x_1) \le f(x_2)$ tengsizlik bajarilsa, $f(x)$ **funksiya** $X$ **to'plamda o'suvchi** deyiladi.

**8-ta'rif.** Agar $\forall x_1, x_2 \in X$ uchun $x_1 < x_2$ bo'lganda $f(x_1) \ge f(x_2)$ tengsizlik bajarilsa, $f(x)$ **funksiya** $X$ **to'plamda kamayuvchi** deyiladi.

O'suvchi hamda kamayuvchi funksiyalar umumiy nom bilan **monoton funksiyalar** deyiladi.

**452.** Ushbu $f(x)=\frac{x}{1+x^2}$ funksiyaning $X=[1,+\infty)$ to'plamda kamayuvchi ekanligi isbotlansin.

**Yechish.** $[1,+\infty)$ da ixtiyoriy $x_1$ va $x_2$ nuqtalarni olib, $x_1 < x_2$ bo'lsin deylik. Unda

$$f(x_1)-f(x_2)=\frac{x_1}{1+x_1^2}-\frac{x_2}{1+x_2^2}=\frac{x_1+x_1x_2^2-x_2-x_2x_1^2}{(1+x_1^2)(1+x_2^2)}=$$

$$=\frac{x_1-x_2+x_1\cdot x_2(x_2-x_1)}{(1+x_1^2)(1+x_2^2)}=\frac{(x_1-x_2)(1-x_1\cdot x_2)}{(1+x_1^2)(1+x_2^2)}$$

bo'ladi. Keyingi tenglikda $x_1-x_2<0$, $1-x_1\cdot x_2<0$ bo'lishini e'tiborga olib, $f(x_1)-f(x_2)>0$ ya'ni, $f(x_1)>f(x_2)$ ekanini topamiz. Demak, $x_1<x_2 \Rightarrow f(x_1)>f(x_2)$.

**Teskari funksiya. Murakkab funksiyalar.** $y=f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, bu funksiyaning qiymatlaridan iborat to'plam

$$Y_f=\{f(x) \mid x \in X\}$$

bo'lsin.



Faraz qilaylik, biror qoidaga ko'ra $Y_f$, to'plamdan olingan har bir $y$ ga $X$ to'plamdagi bitta $x$ mos qo'yilgan bo'lsin. Bunday moslik natijasida funksiya hosil bo'ladi. Odatda, bu funksiya $y=f(x)$ ga nisbatan **teskari funksiya** deyiladi va $x=f^{-1}(y)$ kabi belgilanadi.

Masalan, $y=\frac{1}{2}x+1$ funksiyaga nisbatan teskari funksiya $x=2y-1$ bo'ladi.

Yuqorida aytilganlardan $y=f(x)$ da $x$ argument, $y$ esa $x$ ning funksiyasi, teskari $x=f^{-1}(y)$ funksiyada $y$ argument, $x$ esa $y$ ning funksiyasi bo'lishi ko'rinadi.

Aytaylik, $Y_f$ to'plamda $u=F(y)$ funksiya berilgan bo'lsin. Natijada $X$ to'plamdan olingan har bir $x$ ga $Y_f$ to'plamda bitta $y$:

$$f: x \to y \quad (y=f(x)),$$

va $Y_f$ to'plamdagi bunday $y$ songa bitta $u$:

$$F: y \to u \quad (u=F(y))$$

son mos qo'yiladi. Demak, $X$ to'plamdan olingan har bir $x$ songa bitta $u$ son mos qo'yilib, yangi funksiya hosil bo'ladi: $u=F(f(x))$. Odatda bunday funksiya **murakkab funksiya** deyiladi.

**Elementar funksiyalar.** Elementar funksiyalar kitobxonga o'rta maktab matematika kursidan ma'lum. Biz quyida elementar funksiyalar haqidagi asosiy ma'lumotlarni bayon etamiz.

**1. Butun ratsional funksiyalar.**

Ushbu

$$y=a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n$$

ko'rinishdagi funksiya butun ratsional funksiya deyiladi. Bunda $a_0, a_1, \ldots, a_n$– o'zgarmas sonlar, $n \in N$. Bu funksiya $R=(-\infty, +\infty)$ da aniqlangan.

Butun ratsional funksiyaning ba'zi xususiy hollari:

**a) Chiziqli funksiya.** Bu funksiya

$$y=ax+b \qquad (a \neq 0)$$

ko'rinishga ega, bunda $a$, $b$ o'zgarmas sonlar.

Chiziqli funksiya $(-\infty, +\infty)$ da aniqlangan $a>0$ bo'lganda o'suvchi, $a<0$ bo'lganda kamayuvchi, grafigi tekislikdagi to'g'ri chiziqdan iborat.

**b) Kvadrat funksiya.** Bu funksiya

$$y = ax^2 + bx + c \qquad (a \neq 0)$$

ko'rinishga ega, bunda $a$, $b$, $c$ – o'zgarmas sonlar.

Kvadrat funksiya $R$ da aniqlangan bo'lib, uning grafigi parabolani ifodalaydi.

**2. Kasr ratsional funksiyalar.** Ushbu

$$y = \frac{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots + b_mx^m}$$

ko'rinishdagi funksiya kasr ratsional funksiya deyiladi. Bunda $a_0, a_1, \ldots, a_n$ va $b_0, b_1, \ldots, b_m$ lar o'zgarmas sonlar $n \in N$, $m \in N$. Bu funksiya

$$X = (-\infty, +\infty) \setminus \{x | b_0 + b_1x + \ldots + b_mx^m = 0\}$$

to'plamda aniqlangan.

Kasr ratsional funksiyaning ba'zi xususiy hollari:

**a) Teskari proportsional bog'lanish.** U

$$y = \frac{a}{x} \qquad (x \neq 0 \quad a = const)$$

ko'rinishga ega. Bu funksiya

$$X = (-\infty, 0) \cup (0, +\infty) = R \setminus \{0\}$$

to'plamda aniqlangan, toq funksiya, $a$ ning ishorasiga qarab funksiya $(-\infty, 0)$ va $(0, +\infty)$ oraliqlarning har birida kamayuvchi yoki o'suvchi bo'ladi.

**b) Kasr chiziqli funksiya.** U ushbu

$$y = \frac{ax + b}{cx + d}$$

ko'rinishga ega. Uning grafigini $y = \frac{a}{x}$ funksiya grafigi yordamida chizish mumkin.

**3. Darajali funksiya.** Ushbu

$$y = x^a, \quad (x \geq 0)$$

ko'rinishdagi funksiya darajali funksiya deyiladi.

Bu funksiyaning aniqlanish to'plami $a$ ga bog'liq. Darajali funksiya $a > 0$, bo'lganda $(0, +\infty)$ da o'suvchi, $a < 0$ bo'lganda kamayuvchi bo'ladi. $y = x^a$ funksiya grafigi tekislikning $(0,0)$ va $(1,1)$ nuqtalaridan o'tadi.

**4. Ko‘rsatkichli funksiya.** Ushbu

$$y = a^x$$

ko‘rinishdagi funksiya ko‘rsatkichli funksiya deyiladi. Bunda $a \in R$, $a > 0$, $a \neq 1$. Ko‘rsatkichli funksiya $(-\infty, +\infty)$ aniqlangan, $\forall x \in R$ da $a^x > 0$; $a > 1$ bo‘lganda o‘suvchi; $0 < a < 1$ bo‘lganda kamayuvchi bo‘ladi.

Xususan, $a = e$ bo‘lsa, matematikada muhim rol o‘ynaydigan $y = e^x$ funksiya hosil bo‘ladi.

Ko‘rsatkichli funksiyaning grafigi $Ox$ o‘qidan yuqorida joylashgan va tekislikning $(0,1)$ nuqtasidan o‘tadi.

**5. Logarifmik funksiya.** Ushbu

$$y = \log_a x$$

ko‘rinishdagi funksiya logarifmik funksiya deyiladi. Bunda $a > 0$, $a \neq 1$.

Logarifmik funksiya $(0, +\infty)$ da aniqlangan, $y = a^x$ funksiyaga nisbatan teskari; $a > 1$ bo‘lganda o‘suvchi, $0 < a < 1$ bo‘lganda kamayuvchi bo‘ladi.

Logarifmik funksiyaning grafigi $Oy$ o‘qining o‘ng tomonida joylashgan va tekislikning $(1,0)$ nuqtasidan o‘tadi.

**6. Trigonometrik funksiyalar.** Ushbu

$$y = \sin x, \quad y = \cos x, \quad y = tgx, \quad y = ctgx,$$

$$\sec x = \frac{1}{\cos x}, \quad \operatorname{cosec} x = \frac{1}{\sin x}$$

funksiyalar trigonometrik funksiyalar deyiladi.

**7. Teskari trigonometrik funksiyalar.**

Ushbu $y = \arcsin x$, $y = \arccos x$, $y = arctgx$, $y = arcctgx$ funksiyalar teskari trigonometrik funksiyalar deyiladi.

**8. Giperbolik funksiyalar.** Ko‘rsatkichli $y = e^x$ funksiya yordamida tuzilgan ushbu

$$shx = \frac{e^x - e^{-x}}{2}, \quad chx = \frac{e^x + e^{-x}}{2}, \quad thx = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad cthx = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

funksiyalar **giperbolik** (mos ravishda **giperbolik sinus, giperbolik kosinus, giperbolik tangens, giperbolik kotangens**) funksiyalar deyiladi.

**453.** Quyidagi funksiyalarning aniqlanish sohasini toping.

1) $y=\ln x+2$;  2) $y=\sqrt{9-x^2}+\frac{1}{x-1}$;

3) $y=\arcsin\sqrt{2x}$ ;  4) $y=\sqrt{\sin x}+\sqrt{16-x^2}$;

5) $y=\arccos\frac{2x}{1+x}$;  6) $y=\arccos\cos x$.

**454.** Quyidagi funksiyalarning qiymatlar sohasini toping.

1) $y=3x^2-12x+13$;  2) $y=4-3\sin 2x$;

3) $y=\pi\,\mathrm{arctgx}$;  4) $y=\sqrt{4-x}+3$;

5) $y=4^{-2x^2-4x-5}$;  6) $y=\ln\cos^2 4x-1$.

**455.** Quyidagi funksiyalarning juft – toqligini aniqlang.

1) $y=\frac{|x|}{x}$;  2) $y=x^6-2x^4$;

3) $y=4x\cos x$;  4) $y=2^x+1$;

5) $y=x^4\sin 3x$;  6) $y=\frac{e^x-e^{-x}}{2}$.

**456.** Quyidagi funksiyalar davriy bo'ladimi? Mavjud bo'lsa, eng kichik musbat davrini toping.

1) $y=\cos\frac{\pi}{4}$;  2) $y=\sin 3x+ctg2x$;

3) $y=\sin 3x\cos 3x$;  4) $y=\cos x^2$;

5) $y=|\sin 2x|$;  6) $y=arctgtgx$.

**457.** Quyidagi berilgan funksiyalar uchun $f(f(x)), g(f(x)), f(g(x))$ murakkab funksiyalarni toping.

1) $f(x)=x^3,\ g(x)=x-1$;  2) $f(x)=|x|,\ g(x)=\cos x$.

**458.** Quyidagi funksiyalardan qaysilarining teskari funksiyalari mavjud? Agar teskari funksiyalari mavjud bo'lsa, teskari funksiyalarini toping.

1) $y=x$;  2) $y=2x^3+5$;  3) $y=2x^2+1$;

4) $y=1+\lg(x+2)$;  5) $y=\frac{2^x}{1+2^x}$;  6) $y=\begin{cases}-x^2, & x<0,\\ 2x, & x\geq 0.\end{cases}$

**459.** Quyidagi funksiyalarni monotonlikka va chegaralanganlikka tekshiring.

1) $y=c,\ c\in R$;  2) $y=\cos^2 x$;  3) $y=\frac{x+3}{x+6}$;

4) $y = tg\sin x$; 5) $y=\sqrt{2x+5}$; 6) $y=\begin{cases}-10, & x<0,\\ x^2, & x\geq 0.\end{cases}$

## 23 §. Funksiyaning limiti. Ajoyib limitlar. Limitga ega bo'lgan funksiyaning xossalari.

**To'plamning limit nuqtasi.** Aytaylik, biror $X\subset R$ to'plam va $x_0\in R$ nuqta berilgan bo'lsin.

**1-ta'rif.** Agar $x_0$ nuqtaning ixtiyoriy

$$U_\varepsilon(x_0)=(x_0-\varepsilon,\ x_0+\varepsilon)\quad (\forall\varepsilon>0)$$

atrofida $X$ to'plamning $x_0$ nuqtadan farqli kamida bitta nuqtasi bo'lsa, ya'ni

$$\forall\varepsilon>0,\ \exists x\in X,\ \ x\neq x_0:\ |x-x_0|<\varepsilon$$

bo'lsa, $x_0$ nuqta $X$ to'plamning **limit nuqtasi** deyiladi.

**460.** Quyidagi to'plamlarning limit nuqtalarini toping:

1. $X=[0,1]$; 2. $X=(0,1)$;

3. $X=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\right\}$; 4. $X=N=\{1,2,3,...\}$.

**Yechish.**

**1.** $X=[0,1]$ to'plamning har bir nuqtasi shu to'plamning limit nuqtasi bo'ladi.

**2.** $X=(0,1)$ to'plamning har bir nuqtasi va $x=0$, $x=1$ nuqtalar shu to'plamning limit nuqtalari bo'ladi.

**3.** $X=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\right\}$ to'plamning limit nuqtasi $x_0=0$ bo'ladi.

**4.** $X=N=\{1,2,3,...\}$ to'plam limit nuqtaga ega emas.

Keltirilgan ta'rif va misollardan ko'rinadiki, to'plamning limit nuqtasi shu to'plamga tegishli bo'lishi ham, bo'lmasligi ham mumkin ekan.

**Funksiya limiti.** Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, $x_0$ nuqta $X$ to'plamning limit nuqtasi bo'lsin.

**2-ta'rif.** Agar $\forall\varepsilon>0$ son olinganda ham shunday $\delta=\delta(\varepsilon)>0$ topilsaki, $0<|x-x_0|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x$ uchun

$$|f(x)-b|<\varepsilon$$

tengsizlik bajarilsa, $b$ soni $f(x)$ **funksiyaning** $x_0$ **nuqtadagi limiti** deyiladi va $\lim\limits_{x\to x_0} f(x)=b$ kabi belgilanadi.

**461.** Ushbu $f(x)=\dfrac{x^2-1}{x-1}$ funksiyaning $x_0=1$ nuqtadagi limiti 2 ga teng ekani ko'rsatilsin.

**Yechish.** $\forall\varepsilon>0$ uchun $\delta=\varepsilon$ deb olsak, u holda $|x-1|<\delta$ $(x\neq 1)$ tengsizlikni qanoatlantiruvchi ixtiyoriy $x$ da

$$\left|\frac{x^2-1}{x-1}-2\right|=|x+1-2|=|x-1|<\delta=\varepsilon$$

bo'ladi. Demak, $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}=2.$

**462.** Quyidagi limitni toping: $\lim\limits_{x\to 10}\dfrac{\sqrt{x-1}-3}{x-10}$.

**Yechish.** $x-10\neq 0$ ekanidan foydalanib

$$\lim_{x\to 10}\frac{\sqrt{x-1}-3}{x-10}=\lim_{x\to 10}\frac{\left(\sqrt{x-1}-3\right)\left(\sqrt{x-1}+3\right)}{(x-10)\left(\sqrt{x-1}+3\right)}=$$

$$=\lim_{x\to 10}\frac{x-10}{(x-10)\left(\sqrt{x-1}+3\right)}=\lim_{x\to 10}\frac{1}{\left(\sqrt{x-1}+3\right)}=\frac{1}{6}$$

ni hosil qilamiz.

**Ajoyib limitlar.** Odatda quyidagi

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad (6)$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e \qquad (7)$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e \qquad (8)$$

(6), (7) va (8) formulalar ajoyib limitlar deyiladi. Xususan, (6) 1-ajoyib limit, (7) va (8) 2-ajoyib limit deb ataladi.

**Funksiyaning o'ng va chap limitlari.**

**3-ta'rif.** Agar $\forall\varepsilon>0, \exists\delta>0, \forall x\in(x_0-\delta, x_0):\ |f(x)-b|<\varepsilon$ bo'lsa, $b$ son $f(x)$ **funksiyaning** $x_0$ **nuqtadagi chap limiti** deyiladi va

$$b=\lim_{x\to x_0-0}f(x)=f(x_0-0)$$

kabi belgilanadi.

**4-ta'rif.** Agar $\forall\varepsilon>0, \exists\delta>0, \forall x\in(x_0,\ x_0+\delta):\ |f(x)-b|<\varepsilon$ bo'lsa, $b$ son $f(x)$ **funksiyaning** $x_0$ **nuqtadagi o'ng limiti** deyiladi va

$$b=\lim_{x\to x_0+0} f(x)=f(x_0+0)$$

kabi belgilanadi.

Masalan,

$$f(x)=\begin{cases} 1, \text{agar } x>0 \text{ bo'lsa}, \\ 0, \text{agar } x=0 \text{ bo'lsa}, \\ -1, \text{agar } x<0 \text{ bo'lsa} \end{cases}$$

funksiyaning 0 nuqtadagi o'ng limiti 1, chap limiti -1 bo'ladi.

**Limitga ega bo'lgan funksiyalarning xossalari.**

Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, $x_0\in R$ nuqta $X$ ning limit nuqtasi bo'lsin.

**1-xossa.** Agar $x\to x_0$ da $f(x)$ funksiya limitga ega bo'lsa, u yagona bo'ladi.

**2-xossa.** Agar $\lim\limits_{x\to x_0} f(x)=b$, ($b$ – chekli son) bo'lsa, u holda $x_0$ nuqtaning shunday $U_\delta(x_0)$ $(\delta>0)$ atrofi topiladiki, bu atrofda $f(x)$ funksiya chegaralangan bo'ladi.

**3-xossa.** Agar $\lim\limits_{x\to x_0} f(x)=b$ bo'lib, $b<p$ bo'lsa, u holda $x_0$ nuqtaning shunday $U_\delta(x_0)$ atrofi topiladiki, bu atrofdagi qiymatlar uchun $f(x)<p$ bo'ladi.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $X\subset R$ to'plamda berilgan bo'lib, $x_0\in R$ nuqta $X$ to'plamning limit nuqtasi bo'lsin.

**4-xossa.** Agar $\lim\limits_{x\to x_0} f(x)=b_1$, $\lim\limits_{x\to x_0} g(x)=b_2$ bo'lib, $\forall x\in X$ da $f(x)\le g(x)$ tengsizlik bajarilsa, u holda $b_1\le b_2$, ya'ni $\lim\limits_{x\to x_0} f(x)\le \lim\limits_{x\to x_0} g(x)$ bo'ladi.

**5-xossa.** Faraz qilaylik, $\lim\limits_{x\to x_0} f(x)=b_1$, $\lim\limits_{x\to x_0} g(x)=b_2$, $(b_1, b_2\in R)$ limitlar mavjud bo'lsin. U holda

a) $\forall c\in R$ da $\lim\limits_{x\to x_0}(c\cdot f(x))=c\cdot \lim\limits_{x\to x_0} f(x)$;

b) $\lim\limits_{x\to x_0}(f(x)+g(x))=\lim\limits_{x\to x_0}f(x)+\lim\limits_{x\to x_0}g(x);$

c) $\lim\limits_{x\to x_0}(f(x)\cdot g(x))=\lim\limits_{x\to x_0}f(x)\cdot\lim\limits_{x\to x_0}g(x);$

d) Agar $b_2\neq 0$ bo‘lsa, $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to x_0}f(x)}{\lim\limits_{x\to x_0}g(x)}$

bo‘ladi.

**463.** Ushbu $\lim\limits_{x\to 1}\dfrac{x+x^2+x^3+...+x^n-n}{x-1}$ limit hisoblansin.

**Yechish.** Bu limitni yuqoridagi xossalardan foydalanib hisoblaymiz:

$$\lim_{x\to 1}\frac{x+x^2+x^3+...+x^n-n}{x-1}=\lim_{x\to 1}\frac{(x-1)+(x^2-1)+(x^3-1)+...+(x^n-1)}{x-1}=$$

$$\lim_{x\to 1}\frac{(x-1)[1+(x+1)+(x^2+x+1)+...+(x^{n-1}+x^{n-2}+x+1)]}{x-1}$$

$$=1+2+3+...+n=\frac{n(n+1)}{2}.$$

**464.** Ushbu $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ limit hisoblansin.

**Yechish.** Ma’lumki, $1-\cos x=2\sin^2\dfrac{x}{2}$. Shuni hisobga olib topamiz:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{1}{2}\cdot\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2=$$

$$=\frac{1}{2}\cdot\left[\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]=\frac{1}{2}.$$

**Funksiya limitining mavjudligi**

**1-teorema.** Agar $f(x)$ funksiya $x_0$ nuqtadagi chap va o‘ng $\lim\limits_{x\to x_0-0}f(x)$, $\lim\limits_{x\to x_0+0}f(x)$ limitlar mavjud va $\lim\limits_{x\to x_0-0}f(x)=\lim\limits_{x\to x_0+0}f(x)$ bo‘lsa, u holda funksiya $x_0$ nuqtada $\lim\limits_{x\to x_0}f(x)$ limitga ega bo‘ladi.

Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $(x_0-\gamma,\ x_0) \subset X$ bo'lsin $(\gamma>0)$.

**2-teorema.** Agar $f(x)$ funksiya $X$ to'plamda o'suvchi bo'lib, u yuqoridan chegaralangan bo'lsa, funksiya $x_0$ nuqtada $\lim\limits_{x\to x_0-0} f(x)$ limitga ega bo'ladi.

**3-teorema.** Agar $f(x)$ funksiya $X$ to'plamda kamayuvchi bo'lib, u quyidan chegaralangan bo'lsa, funksiya $x_0$ nuqtada $\lim\limits_{x\to x_0+0} f(x)$ limitga ega bo'ladi.

**Cheksiz katta va cheksiz kichik funksiyalar.** Aytaylik, $\alpha(x)$ hamda $\beta(x)$ funksiya lap $X \subset R$ to'plamda berilgan bo'lib, $x_0 \in R$ nuqta $X$ to'plamning limit nuqtasi bo'lsin.

**5-ta'rif.** Agar $\lim\limits_{x\to x_0} \alpha(x)=0$ bo'lsa, $\alpha(x)$ funksiya $x \to x_0$ da **cheksiz kichik funksiya** deyiladi.

Masalan, $x\to 0$ da $\alpha(x)=\sin x$ funksiya cheksiz kichik funksiya bo'ladi.

**6-ta'rif.** Agar $\lim\limits_{x\to x_0} \beta(x)=\infty$ bo'lsa, $\beta(x)$ funksiya $x \to x_0$ da **cheksiz katta funksiya** deyiladi.

Masalan, $x\to 0$ da $\beta(x)=\dfrac{1}{x}$ funksiya cheksiz katta funksiya bo'ladi.

**7-ta'rif.** Agar $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\beta(x)}=1$ bo'lsa, $\alpha(x)$ va $\beta(x)$ funksiyalar $x\to x_0$ da **ekvivalent funksiyalar** deyiladi va $\alpha(x)\sim\beta(x)$, $x\to x_0$ kabi belgilanadi.

Ma'lumki, $x\to 0$ da

$$\sin x\sim x,\ tgx\sim x,\ \arcsin x\sim x, arctgx\sim x,\ \log_a(1+x)\sim\frac{x}{\ln a},\ \ln(1+x)\sim x,$$

$$a^x-1\sim x\ln a,\ e^x-1\sim x,\ (1+x)^m-1\sim mx$$

munosabatlar o'rinli bo'ladi.

**465.** $\lim\limits_{x\to 0}\dfrac{\ln(1+6x\arcsin x)\sin 5x}{(e^x-1)tgx^2}$ ni hisoblang.

**Yechish.** $x\to 0$ da $6x\arcsin x\to 0$ ekanini inobatga olib, ekvivalent munosabatlardan quyidagini hosil qilamiz:

$\ln(1+6x\arcsin x)\sim 6x\arcsin x\sim 6x^2,\ \sin 5x\sim 5x,$ .

$e^x-1\sim x,\ tgx^2\sim x^2$

$$\lim_{x\to 0}\frac{\ln(1+6x\arcsin x)\sin 5x}{(e^x-1)tgx^2}=\lim_{x\to 0}\frac{30x^2}{x^2}=30$$

**Cheksiz katta va cheksiz kichik funksiyalarning xossalari**

1) Chekli sondagi cheksiz kichik funksiyalar yig'indisi cheksiz kichik funksiya bo'ladi;

2) Chegaralangan funksiyaning cheksiz kichik funksiya bilan ko'paytmasi cheksiz kichik funksiya bo'ladi;

3) Agar $\alpha(x)$ $(\alpha(x)\neq 0)$ cheksiz kichik funksiya bo'lsa, $\dfrac{1}{\alpha(x)}$ cheksiz katta funksiya bo'ladi.

4) Agap $\beta(x)$ cheksiz katta funksiya bo'lsa, $\dfrac{1}{\beta(x)}$ cheksiz kichik funksiya bo'ladi.

**466.** Ta'rif yordamida quyidagi tengliklarni isbotlang.

1) $\lim\limits_{x\to -1}(3x+2)=-1$; 2) $\lim\limits_{x\to 1}(2-x)=1$;

3) $\lim\limits_{x\to 3}\dfrac{1}{x}=\dfrac{1}{3}$; 4) $\lim\limits_{x\to 2}x^2=4$.

**467.** Quyidagi limitlarni hisoblang.

1) $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{x^2-9}$; 2) $\lim\limits_{x\to 1}\dfrac{1-x^2}{1-x^3}$; 3) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\cos 2x}{\sin x-\cos x}$;

4) $\lim\limits_{x\to 3}\left(\dfrac{1}{x-3}-\dfrac{6}{x^2-9}\right)$; 5) $\lim\limits_{x\to 1}\dfrac{1-x^2}{1-\sqrt{x}}$; 6) $\lim\limits_{x\to -1}\dfrac{x+1}{1-\sqrt{1+x+x^2}}$;

7) $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$; 8) $\lim\limits_{x\to\infty}\dfrac{x-2}{x^2-3x+2}$; 9) $\lim\limits_{x\to -5}\dfrac{\sqrt{3x+17}-\sqrt{2x+12}}{x^2+8x+15}$;

10) $\lim\limits_{x\to\infty}\left(\dfrac{x+8}{x-2}\right)^x$; 11) $\lim\limits_{x\to 0}(1-4x)^{\frac{1-x}{x}}$; 12) $\lim\limits_{x\to 0}\sqrt[2x]{1+3x}$;

13) $\lim\limits_{x\to 0}\dfrac{\sin x}{tg9x}$; 14) $\lim\limits_{x\to 0}\dfrac{\sin^2 2x}{\arcsin^2 3x}$; 15) $\lim\limits_{x\to 0}\dfrac{\sin 3x-\sin x}{\ln(x+1)}$;

16) $\lim\limits_{x\to 0}\dfrac{1-\cos 5x}{1-\cos 3x}$; 17) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{5^x-1}$; 18) $\lim\limits_{x\to\infty}\dfrac{\cos 2x}{x}$;

19) $\lim\limits_{x\to 1}\dfrac{\ln x^2}{x^4-1}$; 20) $\lim\limits_{x\to 0}\dfrac{x\arcsin 3x}{5\sin^2 x}$; 21*) $\lim\limits_{x\to 1}\dfrac{\sin(e^{x-1}-1)}{\ln x}$;

22) $\lim\limits_{x\to 2\pm 0}[x]$;  23) $\lim\limits_{x\to 3\pm 0}\frac{1}{x+2^{\frac{1}{x-3}}}$;  24) $\lim\limits_{x\to \frac{\pi}{4}\pm 0} 3^{tg2x}$.

## 24 §. Funksiyaning uzluksizligi. Uzilish turlari.

**Funksiyaning uzluksizligi ta'riflari.** Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, $x_0\in X$ nuqta $X$ to'plamning limit nuqtasi bo'lsin.

**1-ta'rif.** Agar

$$\lim_{x\to x_0} f(x)=f(x_0) \qquad (9)$$

bo'lsa, $f(x)$ **funksiya** $x_0$ **nuqtada uzluksiz** deyiladi.

Demak, $f(x)$ funksiya ning $x_0$ nuqtada uzluksizligi ushbu

1) $\lim\limits_{x\to x_0} f(x)=b$ ning mavjudligi,

2) $b=f(x_0)$ bo'lishi

shartlarining bajarilishi bilan ifodalanadi. (9) ni quyidagicha ham yozish mumkin:

$$\lim_{x\to x_0} f(x)=f\left(\lim_{x\to x_0} x\right). \qquad (10)$$

**468.** Ushbu $f(x)=x^4+x^2+1$ funksiya $\forall x_0\in R$ nuqtada uzluksiz bo'lishini ko'rsating.

**Yechish.** $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0}(x^4+x^2+1)=x_0^4+x_0^2+1=f(x_0)$ ekanligidan ta'rifga ko'ra berilgan funksiya uzluksiz bo'ladi.

**469**. Ushbu $f(x)=(\text{sign}x)^2=\begin{cases}1, & agar\ \ x\neq 0\ \ bo'lsa,\\ 0, & agar\ \ x=0\ \ bo'lsa\end{cases}$

funksiyani uzluksizlikka tekshiring.

**Yechish.** Ravshanki, $\forall x_0\in R$ nuqtada $\lim\limits_{x\to x_0} f(x)=1$ bo'ladi. Demak, qaralayotgan funksiya $\forall x_0\in R,\ x_0\neq 0$ nuqtada uzluksiz bo'ladi. Ammo $f(0)=0$ bo'lganligi sababli $\lim\limits_{x\to 0} f(x)\neq f(0)$ bo'ladi. Demak, $f(x)$ funksiya $x_0=0$ nuqtada uzluksiz bo'lmaydi.

Funksiya uzluksizligini quyidagicha ham ta'riflash mumkin.

**2-ta'rif.** Agar $\forall\varepsilon>0$ son olinganda ham shunday $\delta=\delta(\varepsilon)>0$ son topilsaki, $\forall x\in X\cap U_\delta(x_0)$ uchun $|f(x)-f(x_0)|<\varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiya $x_0$ nuqtada uzluksiz deyiladi.

Odatda, $x-x_0$ ayirma **argument orttirmasi**, $f(x)-f(x_0)$ esa **funksiya orttirmasi** deyilib, ular mos ravishda $\Delta x$ va $\Delta f$ kabi belgilanadi:

$$\Delta x = x - x_0, \qquad \Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

Unda funksiya uzluksizligining 1-ta'rifidagi (9) munosabat ushbu

$$\lim_{\Delta x \to 0} \Delta f = 0 \qquad (11)$$

ko'rinishga keladi. Demak, (11) munosabatni funksiyaning $x_0$ nuqtada uzluksizligi ta'rifi sifatida qarash mumkin.

Aytaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $x_0 \in X$ nuqta $X$ to'plamning o'ng (chap) limit nuqtasi bo'lsin.

**3-ta'rif.** Agar $\lim\limits_{x \to x_0+0} f(x) = f(x_0)$ $\quad$ $(\lim\limits_{x \to x_0-0} f(x) = f(x_0))$

bo'lsa, $f(x)$ **funksiya** $x_0$ **nuqtada o'ngdan (chapdan) uzluksiz** deyiladi.

Demak, $f(x)$ funksiya $x_0$ nuqtada o'ngdan (chapdan) uzluksiz bo'lganda funksiya ning o'ng (chap) limiti uning $x_0$ nuqtadagi qiymatiga teng bo'ladi:

$$f(x_0+0) = f(x_0) \qquad (f(x_0-0) = f(x_0)).$$

Keltirilgan ta'riflardan, $f(x)$ funksiya $x_0$ nuqtada ham o'ngdan, ham chapdan bir vaqtda uzluksiz bo'lsa, funksiya shu nuqtada uzluksiz bo'lishini topamiz.

Umuman, $f(x)$ funksiyaning $x_0$ nuqtada uzluksiz bo'lishi, $\forall \varepsilon > 0$ berilganda ham unga ko'ra shunday $\delta = \delta(\varepsilon) > 0$ topilib,

$$\forall x \in U_\delta(x_0) \subset X \quad \Rightarrow \quad f(x) \in U_\varepsilon(f(x_0))$$

bo'lishini bildiradi.

**4-ta'rif.** Agar $f(x)$ funksiya $X \subset R$ to'plamning har bir nuqtasida uzluksiz bo'lsa, $f(x)$ funksiya $X$ to'plamda uzluksiz deyiladi.

**5-ta'rif.** $X \subset R$ to'plamda uzluksiz bo'lgan funksiyalardan iborat to'plam uzluksiz funksiyalar to'plami deyiladi va $C(X)$ kabi belgilanadi.

Masalan, $f(x) \in C[a,b]$ bo'lishi, $f(x)$ funksiya ning $[a,b]$ segmentining har bir nuqtasida uzluksiz, ya'ni $f(x)$ funksiya $(a,b)$ intervalning har bir nuqtasida uzluksiz, $a$ nuqtada o'ngdan, $b$ nuqtada esa chapdan uzluksiz bo'lishini bildiradi.

## Uzluksiz funksiyalar ustida amallar

**1-teorema.** $f(x)$ va $g(x)$ funksiyalari $X \subset R$ to'plamda berilgan bo'lib, $x_0 \in X$ nuqtada uzluksiz bo'lsin. U holda

a) $\forall c \in R$ da $c \cdot f(x)$ funksiya $x_0$ nuqtada uzluksiz bo'ladi;

b) $f(x)+g(x)$ funksiya $x_0$ nuqtada uzluksiz bo'ladi;

c) $f(x)\cdot g(x)$ funksiya $x_0$ nuqtada uzluksiz bo'ladi;

d) $\dfrac{f(x)}{g(x)}$ $(g(x)\neq 0)$ funksiya $x_0$ nuqtada uzluksiz bo'ladi.

**470.** $f(x)=x,\ x\in R$ bo'lsa, u holda $f(x)\in C(R)$ bo'lishini ko'rsating.

**Yechish.** Haqiqatan ham, $\forall \varepsilon>0$ ga ko'ra $\delta=\varepsilon$ deyilsa, u holda

$$\forall x,\ |x-x_0|<\delta: \quad |f(x)-f(x_0)|=|x-x_0|<\delta=\varepsilon$$

bo'ladi.

**471.** $f(x)=a_0x^m+a_1x^{m-1}+...+a_{m-1}x+a_m;\ m\in N,\ a_0,a_1,...,a_m\in R$ bo'lsin. U holda $f(x)\in C(R)$ bo'lishini ko'rsating.

**Yechish.** Bu tasdiqning isboti 470-misol hamda 1-teoremadan kelib chiqadi.

**Funksiyaning uzilishi.** Aytaylik, $f(x)$ funksiya $(a,b)$ da $(-\infty\leq a<b\leq+\infty)$ berilgan bo'lib, $x_0\in(a,b)$ bo'lsin.

Ma'lumki, $f(x)$ funksiya ning $x_0$ nuqtadagi o'ng va chap limitlari

$$f(x_0+0),\qquad f(x_0-0) \tag{12}$$

mavjud bo'lib,

$$f(x_0-0)=f(x_0)=f(x_0+0) \tag{13}$$

tenglik o'rinli bo'lsa, u holda $f(x)$ funksiya $x_0$ nuqtada uzluksiz bo'lar edi.

Agar $f(x)$ funksiya $x_0$ nuqtada uzluksiz bo'lmasa, unda $x_0$ nuqta $f(x)$ funksiya ning **uzilish nuqtasi** deyiladi.

**6-ta'rif.** Agar (12) limitlar mavjud va chekli bo'lib, (13) tengliklarning birortasi o'rinli bo'lmasa, $x_0$ nuqta $f(x)$ funksiyaning **birinchi tur uzilish nuqtasi** deyiladi.

Bunda

$$f(x_0+0)-f(x_0-0)$$

ayirma **funksiyaning** $x_0$ **nuqtadagi sakrashi** deyiladi.

**472.** $f(x)=[x]$ funksiya $x=p$ $(p\in Z)$ nuqtada birinchi tur uzilishga ega bo'lishini ko'rsating.

**Yechish.** Ma'lumki, $f(p+0)=p$, $f(p_0-0)=p-1$ bo'lib, $f(p+0)\neq f(p_0-0)$ bo'ladi. Demak, berilgan funksiya $x=p$ $(p\in Z)$ nuqtalarda 1-tur uzilish hosil qilar ekan.

Agar hech bo'lmaganda (12) limitlarning birortasi mavjud bo'lmasa yoki cheksiz bo'lsa, $x_0$ nuqta $f(x)$ funksiyaning **ikkinchi tur uzilish nuqtasi** deyiladi.

**473.** Ushbu $f(x)=\begin{cases}\sin\frac{1}{x}, & agar\ x\neq 0\ bo'lsa,\\ 0, & agar\ x=0\ bo'lsa\end{cases}$

funksiya $x=0$ nuqtada ikkinchi tur uzilishga ega bo'lishini ko'rsating

**Yechish.** Bu funksiyaning $x=0$ nuqtadagi o'ng va chap limitlari mavjud emas. Demak, berilgan funksiya $x=0$ nuqtada 2-tur uzilish hosil qilar ekan.

**2-teorema.** Agar $y=f(x)$ funksiya $x_0\in X$ nuqtada, $u=F(y)$ funksiya esa $y_0\in Y_f$ nuqtada $(y_0=f(x_0))$ uzluksiz bo'lsa, $F(f(x))$ funksiya $x_0$ nuqtada uzluksiz bo'ladi.

**3-teorema.** $[a,b]\subset R$ da monoton bo'lgan $f(x)$ funksiya shu $[a,b]$ning istalgan nuqtasida yoki uzluksiz bo'ladi, yoki birinchi tur uzilishga ega bo'ladi.

Shuni ta'kidlash joizki, barcha elementar funksiyalar o'zining aniqlanish sohalarida uzluksiz bo'ladi.

**Nuqtada uzluksiz bo'lgan funksiyaning xossalari.**

Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to'plamda berilgan bo'lib, $x_0\in X$ bo'lsin.

1. Agar $f(x)$ funksiya $x_0\in X$ nuqtada uzluksiz bo'lsa, $f(x)$ funksiya $x_0$ nuqtaning biror $U_\delta(x_0)$ atrofida chegaralangan bo'ladi.

2. Agar $f(x)$ funksiya $x_0\in X$ nuqtada uzluksiz bo'lib, $f(x_0)\neq 0$ bo'lsa, $f(x)$ funksiyaning biror $U_\delta(x_0)$ dagi ishorasi $f(x_0)$ ning ishorasi kabi bo'ladi.

**474.** Ta'rif bo'yicha berilgan funksiyalarni $\forall x_0 \in R$ da uzluksizligini ko'rsating.

1) $f(x)=C$ 2) $f(x)=x^3$ 3) $f(x)=\sin x$

**475.** $f(x)=\begin{cases} x^2+1, & x\geq 0; \\ 1, & x<0; \end{cases}$ funksiyani $\forall x_0 \in R$ nuqtada uzluksizligini ko'rsating va grafigini chizing.

**476.** $f(x)=\begin{cases} x^2+1, & x\geq 0; \\ 0, & x<0; \end{cases}$ funksiyani $x_0=0$ nuqtada uzluksiz emasligini ko'rsating. grafigini chizing va uzilish turini aniqlang.

**477.** Berilgan funksiyalarning uzilish nuqtalarini toping, uzilish turini aniqlang va grafigini chizing.

1) $f(x)=-\dfrac{6}{x}$; 2) $f(x)=tgx$;

3) $f(x)=\dfrac{4}{4-x^2}$; 4) $f(x)=arctg\dfrac{a}{x-a}$;

5) $f(x)=\dfrac{1}{1+2^{\frac{1}{x}}}$; 6) $f(x)=\begin{cases} \cos x, & -\dfrac{\pi}{2}\leq x\leq \dfrac{\pi}{4}; \\ x^2-\dfrac{\pi^2}{16}, & \dfrac{\pi}{4}\leq x\leq \pi; \end{cases}$.

# V BOB. BIR O'ZGARUVCHILI FUNKSIYANING DIFFERENSIAL HISOBI

## 25 §. Funksiyaning hosilasi. Hosila topish qoidalari. Hosilaning geometrik va mexanik ma'nolari.

**Funksiya hosilasining ta'rifi. Misollar.** Faraz qilaylik, $f(x)$ funksiya $(a,b)\subset R$ da berilgan bo'lib, $x_0 \in (a,b)$, $x_0+\Delta x \in (a,b)$ bo'lsin.

Ushbu $\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)$ ayirma $f(x)$ funksiyaning $x_0$ nuqtadagi **orttirmasi** deyiladi.

**1-ta'rif.** Agar ushbu

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

limit mavjud va chekli bo'lsa, shu limitga $f(x)$ **funksiyaning** $x_0$ **nuqtadagi hosilasi** deyiladi va $\frac{df(x_0)}{dx}$, yoki $f'(x_0)$, yoki $(f(x))'_{x_0}$ kabi belgilanadi. Demak,

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}. \qquad (1)$$

Agar $x_0+\Delta x=x$ deyilsa, unda $\Delta x=x-x_0$ va $\Delta x\to 0$ da $x\to x_0$ bo'lib, (1) munosabat quyidagi

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \qquad (2)$$

ko'rinishga keladi.

**478.** Funksiyaning berilgan nuqtadagi hosilasini toping: $f(x)=x$, $x_0\in R$.

**Yechish.** Bu funksiya uchun

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{x-x_0}{x-x_0}=1$$

bo'lib, $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=1$

bo'ladi. Demak, $f'(x)=(x)'=1$.

**479.** Funksiyaning hosilasini toping: $f(x)=|x|$, $x\in R$.

**Yechish.**

Agar $x>0$ bo'lsa, u holda $f(x)=x$ bo'lib, $f'(x)=1$ bo'ladi.

Agar $x<0$ bo'lsa, u holda $f(x)=-x$ bo'lib, $f'(x)=-1$ bo'ladi.

Agar $x_0=0$ bo'lsa, u holda $\frac{f(x)-0}{x-0}=\frac{|x|}{x}$ bo'lib, $x\to 0$ da bu nisbatlarning limiti mavjud bo'lmaydi. Demak, berilgan funksiya $x_0=0$ nuqtada hosilaga ega bo'lmaydi.

**480.** Funksiyaning hosilasini toping: $f(x)=x|x|$, $x\in R$, $x_0\in R$.

**Yechish.**

a) $x_0>0, x>0$, $x\neq x_0$ uchun

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{x|x|-x_0|x_0|}{x-x_0}=\frac{x^2-x_0^2}{x-x_0}=x+x_0$$

bo'lib, $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=2x_0=2|x_0|$ bo'ladi.

b) $x_0<0, x<0$, $x\neq x_0$ uchun $\frac{f(x)-f(x_0)}{x-x_0}=\frac{-x^2+x_0^2}{x-x_0}=-x-x_0$

bo'lib, $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=-2x_0=2|x_0|$ bo'ladi.

d) $x_0=0$, $x\neq x_0$ uchun $\frac{f(x)-f(x_0)}{x-0}=\frac{x|x|}{x}=|x|$ bo'lib,

$$\lim_{x\to x_0}\frac{f(x)-f(0)}{x-0}=0$$

bo'ladi. Demak, $\forall x\in R$ da $f'(x)=(x|x|)'=2|x|$.

**481.** Funksiyaning $x_0=0$ nuqtadagi hosilasini toping:

$$f(x)=\begin{cases} x\cdot\sin\frac{1}{x}, & agar\ \ x\neq 0\ \ bo'lsa, \\ 0, & agar\ \ x=0\ \ bo'lsa. \end{cases}$$

**Yechish.** $\frac{f(x)-f(x_0)}{x-x_0}=\frac{x\cdot\sin\frac{1}{x}-0}{x-0}=\sin\frac{1}{x}$

bo'lib, uning $x\to 0$ dagi limiti mavjud emas. Demak, berilgan funksiya $x_0=0$ nuqtada hosilaga ega emas.

**Funksiyaning o'ng va chap hosilalari.** Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $(x_0-\delta,\ x_0) \subset X\ \ (\delta>0)$ bo'lsin.

**2-ta'rif.** Agar ushbu $\lim\limits_{x\to x_0-0}\dfrac{f(x)-f(x_0)}{x-x_0}$ limit mavjud bo'lsa, bu limit $f(x)$ **funksiyaning** $x_0$ **nuqtadagi chap hosilasi** deyiladi va $f'(x_0-0)$ kabi belgilanadi: $f'(x_0-0)=\lim\limits_{x\to x_0-0}\dfrac{f(x)-f(x_0)}{x-x_0}$.

Aytaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $(x_0,\ x_0+\delta) \subset X\ \ (\delta>0)$ bo'lsin.

**3-ta'rif.** Agar ushbu $\lim\limits_{x\to x_0+0}\dfrac{f(x)-f(x_0)}{x-x_0}$ limit mavjud bo'lsa, bu limit $f(x)$ **funksiyaning** $x_0$ **nuqtadagi o'ng hosilasi** deyiladi va $f'(x_0+0)$ kabi belgilanadi:

$$f'(x_0+0)=\lim_{x\to x_0+0}\frac{f(x)-f(x_0)}{x-x_0}.$$

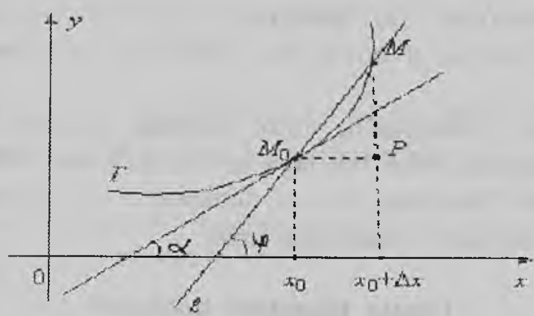
Masalan, $f(x)=|x|$ funksiyaning $x_0=0$ nuqtadagi o'ng hosilasi $f'(+0)=1$, chap hosilasi $f'(-0)=-1$ bo'ladi.

Yuqorida keltirilgan ta'riflardan quyidagi xulosalar kelib chiqadi:

1. Agar $f(x)$ funksiya $x_0$ nuqtada $f'(x_0)$ hosilaga ega bo'lsa, u holda bu funksiya $x_0$ nuqtada o'ng $f'(x_0+0)$ hamda chap $f'(x_0-0)$ hosilalarga ega va $f'(x_0-0)=f'(x_0)=f'(x_0+0)$ tengliklar o'rinli bo'ladi.

2. Agar $f(x)$ funksiya $x_0$ nuqtada o'ng $f'(x_0+0)$ hamda chap $f'(x_0-0)$ hosilalarga ega bo'lib, $f'(x_0-0)=f'(x_0+0)$ bo'lsa, u holda $f(x)$ funksiya $x_0$ nuqtada $f'(x_0)$ hosilaga ega va $f'(x_0-0)=f'(x_0)=f'(x_0+0)$ tengliklar o'rinli bo'ladi.

**Hosilaning geometrik hamda mexanik ma'nolari.** Faraz qilaylik, $f(x)$ funksiya $(a,b)$ da berilgan bo'lib, $x_0\in(a,b)$ nuqtada $f'(x_0)$ hosilaga ega bo'lsin. Bu $f(x)$ funksiyaning grafigi 1-shaklda tasvirlangan $\Gamma$ egri chiziqni ifodalasin:




**1-shakl.**

Bu $\Gamma$ chiziqda $M_0(x_0,y_0)$, $M(x,y)$ nuqtalarni olib, ular orqali o'tuvchi $l$ kesuvchini qaraymiz.

$M_0(x_0,f(x_0))\in\Gamma$, $M(x,f(x))\in\Gamma$, $M\to M_0$ da $l$ kesuvchi limit holati $\Gamma$ chiziqqa $M_0$ nuqtada **o'tkazilgan urinma** deyiladi.

Funksiyaning $x_0$ nuqtadagi $f'(x_0)$ hosilasi urinmaning burchak koeffitsentini ifodalaydi:

$$f'(x_0)=tg\alpha\,.$$

Urinmaning tenglamasi

$$y=f(x_0)+f'(x_0)(x-x_0)$$

ko'rinishda bo'ladi.

Normal tenglamasi

$$y=f(x_0)-\frac{1}{f'(x_0)}(x-x_0)$$

ko'rinishda bo'ladi.

Harakatdagi $P$ nuqtaning $t$ vaqtdagi oniy tezligi $v(t)$, o'tilgan $s(t)$ yo'lning hosilasidan iborat bo'ladi:

$$v(t)=s'(t).$$

Harakatdagi $P$ nuqtaning $t$ vaqtdagi oniy tezlanishi $a(t)$, $v(t)$ tezlikning hosilasidan iborat bo'ladi:

$$a(t)=v'(t).$$

**Hosilaga ega bo'lgan funksiyaning uzluksizligi.** Faraz qilaylik, $f(x)$ funksiya $(a,b)\subset R$ da berilgan bo'lsin.

**Teorema.** Agar $f(x)$ funksiya $x_0 \in (a, b)$ nuqtada chekli $f'(x_0)$ hosilaga ega bo'lsa, u holda $f(x)$ funksiya $x_0$ nuqtada uzluksiz bo'ladi.

***Eslatma.*** Funksiyaning biror nuqtada uzluksiz bo'lishidan uning shu nuqtada chekli hosilaga ega bo'lishi har doim ham kelib chiqavermaydi. Masalan, $f(x)=|x|$ funksiya $x=0$ nuqtada uzluksiz, ammo u shu nuqtada hosilaga ega emas.

## Hosila hisoblash qoidalari

**Ikki funksiya yig'indisi, ayirmasi, ko'paytmasi va nisbatining hosilasi.** Aytaylik, $f(x)$ va $g(x)$ funksiyalari $(a,b)\subset R$ da berilgan bo'lib, $x_0 \in (a,b)$ nuqtada $f'(x_0)$ va $g'(x_0)$ hosilalarga ega bo'lsin. U holda $f(x)\pm g(x), f(x)\cdot g(x), \dfrac{f(x)}{g(x)}$ $(g(x_0)\neq 0)$ funksiya $x_0$ nuqtada hosilaga ega bo'lib ular uchun quyidagi formulalar o'rinli:

$$(f(x)\pm g(x))'_{x_0} = f'(x_0)\pm g'(x_0), \qquad (3)$$

$$(f(x)\cdot g(x))'_{x_0} = f'(x_0)\cdot g(x_0)+f(x_0)\cdot g'(x_0), \qquad (4)$$

$$\left(\frac{f(x)}{g(x)}\right)'_{x_0} = \frac{f'(x_0)\cdot g(x_0)-f(x_0)\cdot g'(x_0)}{g^2(x_0)}. \qquad (5)$$

(4) formuladan

$$(c\cdot f(x))' = c\cdot f'(x) \qquad (6)$$

hosil bo'ladi. Bunda $c$ –o'zgarmas son.

**482.** Funksiyaning hosilasini toping: $f(x)=\dfrac{x-1}{x^2+1}$.

**Yechish.** (5) formuladan foydalanib hisoblaymiz:

$$f'(x)=\frac{(x-1)'(x^2+1)-(x-1)(x^2+1)'}{(x^2+1)^2}=\frac{x^2+1-2x(x-1)}{(x^2+1)^2}=\frac{1+2x-x^2}{(x^2+1)^2}.$$

**483.** Hosila ta'rifidan foydalanib, berilgan funksiyalarning hosilalarini toping:

1) $f(x)=2x^2+7x-3$; 2) $f(x)=x^3+5x^2-2$; 3) $f(x)=\dfrac{x-1}{x+1}$;

4) $f(x)=\sqrt{x}$; 5) $f(x)=\sqrt[3]{x^2}$.

**484.** $f(x)=\frac{7}{x^3}$ funksiya berilgan. $f'(-2)=f'(2)$ ekanligini ko'rsating.

**485.** $f(x)=|x^2-5x+6|$ funksiya $x=2, x=3$ nuqtalarida hosilaga ega emasligini ko'rsating. Shu nuqtalarda o'ng va chap hosilalarni toping.

**486.** Funksiyaning berilgan nuqtadagi hosilasini toping.

1) $y=(4x+1)(3x-1)$ $y'(1)=?$

2) $y=(x-1)(x-2)(x-3)...(x-11)$ $y'(1)=?$

3) $y=\frac{3x+1}{3x-1}$ $y'(1)=?$

4) $y=\frac{\sqrt{x+1}}{x-1}$ $y'(0)=?$

**487.** Qanday nuqtalarda $y=\frac{x}{x+1}$ funksiya grafigiga o'tkazilgan urinma OX o'qning musbat yo'nalishi bilan $45^0$ li burchak tashkil etadi? Shu nuqtada funksiya grafigiga o'tkazilgan urinma va normal tenglamasini tuzing.

**488.** $y=\frac{1}{x}$ va $y=x^2$ funksiyalar kesishish nuqtasidan o'tkazilgan urinmalarining burchak koeffitsiyentlarini toping va chiziqlar orasidagi burchakni toping.

**489.** $S(t)=1+16t-t^2$ qonun bilan harakatlanayotgan moddiy nuqtaning t=1 s dagi oniy tezlik va tezlanishini toping . Moddiy nuqta qachon to'xtaydi?

## 26 §. Elementar funksiyalarning hosilalari. Murakkab, oshkormas, teskari va paramertik usulda berilgan funksiyaning hosilalari. Logarifmik differensiallash.

Elementar funksiyalarning hosilalari hosila ta'rifidan va hosila olish qoidalaridan foydalanib aniqlangan.

**490.** $(\log_a x)'=\frac{1}{x\ln a}$ , ekani ko'rsatilsin. Bunda $a>0$, $a\neq 1$, $x>0$.

**Yechish.** $f(x)=\log_a x$ funksiya uchun

$$\frac{\Delta f(x)}{\Delta x}=\frac{\log_a(x+\Delta x)-\log_a(x)}{\Delta x}=\frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}.$$

$f(x)=\log_a x$ aniqlanish sohasida uzluksiz va $\lim\limits_{\Delta x\to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}=e$ ekanidan foydalanib, $(\log_a x)'=\frac{1}{x}\log_a e=\frac{1}{x\ln a}$ ni topamiz. Xususan, $(\ln x)'=\frac{1}{x}$ bo'ladi.

Shu kabi usullar bilan quyidagi hosilalar jadvalini hosil qilamiz.

| | | | | | |
|---|---|---|---|---|---|
| 1 | $(c)'=0$ | 2 | $x'=1$ | 3 | $(x^\alpha)'=\alpha x^{\alpha-1}$ |
| 4 | $(a^x)'=a^x\ln a$ | 5 | $(e^x)'=e^x$ | 6 | $(\log_a x)'=\frac{1}{x\ln a}$ |
| 7 | $(\ln x)'=\frac{1}{x}$ | 8 | $(\sin x)'=\cos x$ | 9 | $(\cos x)'=-\sin x$ |
| 10 | $(tgx)'=\frac{1}{\cos^2 x}$ | 11 | $(ctgx)'=-\frac{1}{\sin^2 x}$ | 12 | $(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}$ |
| 13 | $(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}$ | 14 | $(\mathrm{arctg}\,x)'=\frac{1}{1+x^2}$ | 15 | $(\mathrm{arcctg}\,x)'=-\frac{1}{1+x^2}$ |
| 16 | $(shx)'=chx$ | 17 | $(chx)'=shx$ | 18 | $(thx)'=\frac{1}{ch^2x}$ |
| 19 | $(cthx)'=-\frac{1}{sh^2x}$ | | | | |

Bu jadvalda $shx, chx, thx, cthx$ funksiyalar mos ravishda giperbolik sinus, giperbolik kosinus, giperbolik tangens va giperbolik kotangens funksiyalar bo'lib, ular quyidagicha aniqlangan:

$$shx=\frac{e^x-e^{-x}}{2},\ chx=\frac{e^x+e^{-x}}{2},\ thx=\frac{shx}{chx}=\frac{e^x-e^{-x}}{e^x+e^{-x}},\ cthx=\frac{chx}{shx}=\frac{e^x+e^{-x}}{e^x-e^{-x}}.$$

**Murakkab funksiyaning hosilasi.** Faraz qilaylik, $y=f(x)$ funksiya $X\subset R$ to'plamda, $g(y)$ funksiya $\{f(x)\,|\,x\in X\}$ to'plamda berilgan bo'lib, $x_0\in X$ nuqtada $f'(x_0)$ hosilaga, $y_0\in\{f(x)\,|\,x\in X\}$ nuqtada $(y_0=f(x_0))$ $g'(y_0)$ hosilaga ega bo'lsin. U holda $g(f(x))$ murakkab funksiya $x_0$ nuqtada hosilaga ega bo'lib,

$$\left(g(f(x))\right)'_{x_0} = g'(f(x_0))\cdot f'(x_0) \qquad (7)$$

bo'ladi.

**491-misol.** $y=\ln\left(x+\sqrt{1+x^2}\right)$ funksiyaning hosilasini toping.

**Yechish.** (7) formula bilan taqqoslab $g(x)=\ln x,\ f(x)=x+\sqrt{1+x^2}$ ekanligidan,

$g'(x)=\dfrac{1}{x},\ f'(x)=1+\dfrac{x}{\sqrt{1+x^2}}=\dfrac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$ ni topamiz.

$g'(f(x))=\dfrac{1}{f(x)}=\dfrac{1}{x+\sqrt{1+x^2}}$ dan va (7) formuladan foydalanib

$y'=(g(f(x)))'=\dfrac{1}{x+\sqrt{1+x^2}}\cdot\dfrac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\dfrac{1}{\sqrt{1+x^2}}$ ekanini topamiz.

**Teskari funksiyaning hosilasi.** Aytaylik, $y=f(x)$ funksiya $(a,b)$ da berilgan, uzluksiz va qat'iy o'suvchi (qat'iy kamayuvchi) bo'lib, $x_0\in(a,b)$ nuqtada $f'(x_0)$ $(f'(x_0)\neq 0)$ hosilaga ega bo'lsin. U holda $x=f^{-1}(y)$ funksiya $y_0$ $(y_0=f(x_0))$ nuqtada hosilaga ega va

$$\left[f^{-1}(y)\right]'_{x_0}=\frac{1}{f'(x_0)} \qquad (8)$$

bo'ladi.

**492.** $(arctgx)'=\dfrac{1}{1+x^2}$ ekani ko'rsatilsin.

**Yechish.** Teskari funksiya hosilasini hisoblash formulasiga asosan $(y=arctgx,\ x=tgy)$

$$y'=(arctgx)'=\frac{1}{(tgy)'}=\cos^2 y=\frac{1}{1+tg^2y}=\frac{1}{1+x^2}$$

bo'ladi.

**Oshkormas funksiyaning hosilasi.** Bizga biror $F(x,y)=0$ oshkormas funksiya berilgan bo'lib, $y=y(x)$ funksiyaning $y'=y'(x)$ hosilasini topish talab etilsin. Buning uchun $y=y(x)$ ekanligini inobatga olgan holda tenglamaning har ikkala tomonidan $x$ bo'yicha hosila olib, so'ngra hosil qilingan tenglamani $y'=y'(x)$ ga nisbatan yechish kerak.

**493.** Oshkormas funksiyaning hosilasini toping: $x^2+y^2=4$.

**Yechish.** Tenglikni ikkala tomonidan $x$ bo‘yicha hosila olamiz: $2x+2yy'=0$.

Bundan $y'$ ni topamiz: $y'=-\frac{x}{y}$.

**Parametrik usulda berilgan funksiyaning hosilasi.** Agar $y=y(x)$ funksiyada o‘zgaruvchilar $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$ parametrik usulda berilgan bo‘lsa, u holda

$$y'=\frac{dy}{dx}=\frac{y'(t)}{x'(t)} \tag{9}$$

bo‘ladi.

**494.** $\begin{cases} x=4\cos t \\ y=3\sin t \end{cases}$ bo‘lsa, $\frac{dy}{dx}$ ni toping.

**Yechish.** (9) formuladan $\frac{dy}{dx}=\frac{(4\cos t)'}{(3\sin t)'}=\frac{-4\sin t}{3\cos t}=-\frac{4}{3}tgt$ ekani kelib chiqadi.

**Logarifmik differensiallash.**

**495.** $y=[u(x)]^{v(x)}$ $(u(x)>0)$ funksiya uchun, $u'(x)$ va $v'(x)$ lar mavjud bo‘lsin. U holda

$$\left([u(x)]^{v(x)}\right)'=[u(x)]^{v(x)}\cdot\left[v'(x)\ln u(x)+\frac{v(x)}{u(x)}u'(x)\right]$$

ekanligini ko‘rsating.

**Yechish.** Ushbu $y=[u(x)]^{v(x)}$ ni logarifmlab,

$\ln y=v(x)\ln u(x)$,

so‘ng murakkab funksiyaning hosilasini hisoblash qoidasidan foydalanib topamiz:

$$\frac{1}{y}y'=v'(x)\cdot\ln u(x)+v(x)\cdot\frac{1}{u(x)}\cdot u'(x),$$

$$y'=y\left[v'(x)\cdot\ln u(x)+v(x)\cdot\frac{v(x)}{u(x)}\cdot u'(x)\right]=$$

$$=[u(x)]^{v(x)}\cdot\left[v'(x)\ln u(x)+\frac{v(x)}{u(x)}u'(x)\right].$$

Demak,

$$\left(u^v\right)=u^v\cdot\ln u\cdot v'+v\cdot u^{v-1}\cdot u'. \tag{10}$$

tenglik o'rinli ekan.

**496.** Ushbu $f(x)=x^x$, $\quad g(x)=x^{x^x}$ funksiyalarning hosilalari topilsin.

**Yechish.** (10) formuladan foydalanib topamiz:

$$f'(x)=\left(x^x\right)'=x^x\cdot\ln x+x\cdot x^{x-1}=x^x(\ln x+1),$$

$$g'(x)=\left(x^{x^x}\right)'=\left(x^{f(x)}\right)'=x^{f(x)}\cdot\ln x\cdot f'(x)+f(x)\cdot x^{f(x)-1}=$$

$$=x^{x^x}\cdot\ln x\cdot(x^x(\ln x+1))+x^{x^x}\cdot x^{x^x-1}=$$

$$=x^{x^x+x-1}(x^x\ln x(\ln x+1)+1).$$

**497.** Quyidagi funksiyalarning hosilalarni jadval va hosila olish qoidalari yordamida toping.

1) $y=\dfrac{1}{x^2}$; 2) $y=\sqrt[3]{x}$; 3) $y=5\sin x+3\cos x$;

4) $y=5(tgx-x)$; 5) $y=\dfrac{1}{e^x+1}$; 6) $y=2^x\arcsin x$;

7) $y=\log_x 2$; 8) $y=sh^2x-ch^2x+2chx$; 9) $y=\dfrac{tgx}{arctgx}$.

**498.** Quyidagi murakkab funksiyalarning hosilalarini toping.

1) $y=\left(2x^3+5\right)^4$; 2) $y=tg^6x$; 3) $y=\cos^2 x$;

4) $y=tg^2\ln x$; 5) $y=\sin^3\dfrac{x}{3}$; 6) $y=\ln tg\dfrac{x}{2}$;

7) $y=\ln\left(\sqrt{2\sin x+1}+\sqrt{2\sin x-1}\right)$; 8) $y=\dfrac{1}{2}tg^2\sqrt{x}+\ln\cos\sqrt{x}$.

**499.** Quyidagi funksiyalarning hosilalarini logarifmik differensiallash yordamida toping.

1) $y=x^{x^2}$; 2*) $y=(\sin x)^{tgx}$;

3*) $y=\dfrac{(2x-1)^3\sqrt{3x+2}}{(5x+4)^2\sqrt[3]{1-x}}$; 4**) $y=\dfrac{(x-1)(x-2)(x-3)(x-4)(x-5)}{(x+1)(x+2)(x+3)(x+4)(x+5)}$.

**500.** Quyidagi oshkormas funksiyalarning hosilalarini toping.

1) $x^3+y^3-3xy=0$; 2) $y^x-x^y=0$;

3) $x^{y^2}+y^2\ln x-4=0$; 4) $x^2\sin y+y^3\cos x-2x-3y+1=0$.

**501.** Quyidagi paramertik usulda berilgan funksiyalarning hosilalarini toping.

1) $\begin{cases} x = cht \\ y = sht \end{cases}$;  2) $\begin{cases} x = t^2 + t + 1 \\ y = \frac{4}{3}t^3 + 2t^2 + t \end{cases}$;

3) $\begin{cases} x = e^{-t}\sin t \\ y = e^{t}\cos t \end{cases}$;  4) $\begin{cases} y = 5\sin t \\ x = 5\cos t \end{cases}$.

**502.** Quyidagi funksiyaga teskari bo'lgan funksiyaning berilgan nuqtadagi hosilasini toping.

1) $y = 2x - \frac{1}{2}\cos x,\ y_0 = -\frac{1}{2}$;  2) $x = shy,\ x_0 = \sqrt{3}$.

## 27 §. Funksiyaning differensiallanuvchiligi. Funksiyaning differensiali. Yuqori tartibli hosila va differensiallar.

**Funksiya differensiali tushunchasi.** Faraz qilaylik, $f(x)$ funksiya $(a, b)$ da berilgan bo'lib, $x_0 \in (a,b)$, $x_0 + \Delta x \in (a,b)$ bo'lsin.

Ma'lumki, $\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0)$ ayirma $f(x)$ funksiyaning $x_0$ nuqtadagi orttirmasi deyiladi.

**1-ta'rif.** Agar $\Delta f(x_0)$ ni ushbu

$$\Delta f(x_0) = A \cdot \Delta x + \alpha \Delta x \tag{11}$$

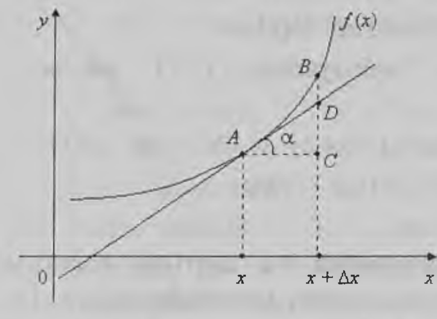
ko'rinishda ifodalash mumkin bo'lsa, $f(x)$ funksiya $x_0$ nuqtada **differensiallanuvchi** deyiladi, bunda $A = const$, $\Delta x \to 0$, da $\alpha \to 0$.

**Teorema.** $f(x)$ funksiya $x \in (a, b)$ nuqtada differensiallanuvchi bo'lishi uchun uning shu nuqtada chekli $f'(x)$ hosilaga ega bo'lishi zarur va yetarli.

**2-ta'rif.** Funksiya orttirmasidagi $f'(x_0) \cdot \Delta x$ ifoda $f(x)$ funksiyaning $x_0$ nuqtadagi **differensiali** deyiladi va $df(x_0)$ kabi belgilanadi:

$$df(x_0) = f'(x_0) \cdot \Delta x .$$

Aytaylik, $x \in (a, b)$ nuqtada differensiallanuvchi $f(x)$ funksiyaning grafigi 2-shakl tasvirlangan egri chiziqni ifodalasin:




**2-shakl.**

Keltirilgan chizmadan ko‘rinadiki,

$$\frac{DC}{AC} = tg\alpha$$

bo‘lib, $DC = tg\alpha \cdot AC = f'(x) \cdot \Delta x$ bo‘ladi.

Demak, $f(x)$ funksiyaning $x$ nuqtadagi differensiali funksiya grafigiga $(x, f(x))$ nuqtada o‘tkazilgan urinma orttirmasi $DC$ ni ifodalar ekan.

Faraz qilaylik, $f(x) = x,\ x \in R$ bo‘lsin. Bu funksiya differensiallanuvchi bo‘lib, $df(x) = (x)' \cdot \Delta x = \Delta x$, ya’ni $dx = \Delta x$ bo‘ladi. Demak, $(a, b)$ da differensiallanuvchi $f(x)$ funksiyaning differensialini

$$df(x) = f'(x) \cdot dx \qquad (12)$$

ko‘rinishda ifodalash mumkin.

Masalan, $d(\sin x) = \cos x dx$ .

**Funksiya differensialining sodda qoidalari.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalari $(a, b)$ da berilgan bo‘lib, $x \in (a, b)$ nuqtada differensiallanuvchi bo‘lsin. U holda $x \in (a, b)$ da

1) $d(c \cdot f(x)) = cdf(x), \quad c = const;$

2) $d(f(x) + g(x)) = df(x) + dg(x);$

3) $d(f(x)g(x)) = g(x)df(x) + f(x)dg(x);$

4) $d\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)df(x) - f(x)dg(x)}{g^2(x)}, \quad (g(x) \neq 0).$

bo‘ladi.

**503.** Ta'rifdan foydalanib, ushbu $f(x)=x-3x^2$ funksiyaning $x_0=2$ nuqtadagi differensiali topilsin.

**Yechish.** Bu funksiyaning $x_0=2$ nuqtadagi orttirmasini topamiz:

$$\Delta f(2)=f(2+\Delta x)-f(2)=2+\Delta x-3(2+\Delta x)^2-2+12=$$
$$=-11\cdot\Delta x-3\Delta x^2=-11\cdot\Delta x+(-3\Delta x)\cdot\Delta x.$$

Demak, $d f(2)=-11\cdot dx$.

**Funksiya differensiali va taqribiy formulalar.** Funksiya differensiali yordamida taqribiy formulalar yuzaga keladi.

Aytaylik, $f(x)$ funksiya $(a,b)$ da berilgan bo'lib, $x_0\in(a,b)$ nuqtada chekli $f'(x_0)$ hosilaga $(f'(x_0)\neq 0)$ ega bo'lsin. U holda $\Delta x\to 0$ da

$$\Delta f(x_0)=f'(x_0)\cdot\Delta x+o(\Delta x)$$

bo'ladi. Bundan,

$$\Delta f(x_0)\approx df(x_0),$$

ya'ni

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\cdot\Delta x \qquad (13)$$

taqribiy formula hosil bo'ladi

**504.** Ushbu $\sin 29^0$ miqdor taqribiy hisoblansin.

**Yechish.** Agar $f(x)=\sin x,\ x_0=30^0$ deyilsa, unda (13) formulaga ko'ra

$$\sin 29^0\approx\sin 30^0+\cos 30^0\cdot(29^0-30^0)\cdot\frac{2\pi}{360^0}=0{,}5-\frac{\sqrt{3}}{2}\cdot\frac{2\pi}{360^0}\approx 0{,}4848$$

bo'ladi.

**Funksiyaning yuqori tartibli hosilalari.** Faraz qilaylik, $f(x)$ funksiya $(a,b)$ da berilgan bo'lib, $\forall x\in(a,b)$ da $f'(x)$ hosilaga ega bo'lsin. Bu $f'(x)$ funksiyani $g(x)$ orqali belgilaymiz:

$$g(x)=f'(x)\qquad (x\in(a,b)).$$

**3-ta'rif.** Agar $x_0\in(a,b)$ nuqtada $g(x)$ funksiya $g'(x_0)$ hosilaga ega bo'lsa, bu hosila $f(x)$ funksiyaning $x_0$ nuqtadagi ikkinchi tartibli hosilasi deyiladi va $f''(x_0)$ yoki $\dfrac{d^2f(x_0)}{dx^2}$ kabi belgilanadi.

Xuddi shunga o'xshash, $f(x)$ ning 3-tartibli $f'''(x)$, 4-tartibli $f^{IV}(x)$ va hokazo, tartibli hosilalari ta'riflanadi.

Umuman, $f(x)$ funksiyaning $n$–tartibli hosilasi $f^{(n)}(x)$ ning hosilasi $f(x)$ funksiyaning $(n+1)$–tartibli hosilasi deyiladi:

$$f^{(n+1)}(x)=\left(f^{(n)}(x)\right)'. \qquad (14)$$

Odatda, $f(x)$ funksiyaning $f''(x)$, $f'''(x)$, ... hosilalari uning yuqori tartibli hosilalari deyiladi. Shuni ta'kidlash lozimki, $f(x)$ funksiyaning $x\in(a,b)$ da $n$–tartibli hosilasining mavjudligi bu funksiyaning shu nuqta atrofida 1–, 2–, ..., $(n-1)$–tartibli hosilalari mavjudligini taqoza etadi. Ammo bu hosilalarning mavjudligidan $n$–tartibli hosila mavjudligi, umuman aytganda, kelib chiqavermaydi.

Masalan, $f(x)=\frac{x|x|}{2}$ funksiyaning hosilasi $f'(x)=|x|$ bo'lib, bu funksiya $x=0$ nuqtada hosilaga ega emas, ya'ni berilgan funksiyaning $x=0$ da birinchi tartibli hosilasi mavjud, ikkinchi tartibli hosilasi esa mavjud emas.

**505.** $f(x)=a^x$, $a>0$, $x\in R$, funksiyaning $n$–tartibli hosilasini toping.

**Yechish.** Bu funksiya uchun $(a^x)'=a^x\ln a$,

$$(a^x)''=(a^x\ln a)'=a^x(\ln a)^2,$$

Umuman $(a^x)^{(n)}=a^x(\ln a)^n$ bo'ladi.

**506.** $f(x)=\sin x$ funksiyaning $n$–tartibli hosilasini toping.

**Yechish.** Bu funksiya uchun

$$(\sin x)'=\cos x=\sin\left(x+\frac{\pi}{2}\right), (\sin x)''=(\cos x)'=-\sin x=\sin\left(x+2\frac{\pi}{2}\right).$$

Umuman, $(\sin x)^{(n)}=\sin\left(x+n\frac{\pi}{2}\right)$ bo'ladi.

**507.** $f(x)=x^{\alpha}$ $x>0$, $\alpha\in R$ funksiyaning $n$–tartibli hosilasini toping.

**Yechish.** Bu funksiya uchun

$$(x^{\alpha})'=\alpha x^{\alpha-1}, (x^{\alpha})''=(\alpha x^{\alpha-1})'=\alpha(\alpha-1)x^{\alpha-2},$$

umuman, $(x^{\alpha})^{(n)}=\alpha(\alpha-1)(\alpha-2)...(\alpha-n+1)x^{\alpha-n}$ bo'ladi.

Xususan, $f(x)=\frac{1}{x}$, $(x>0)$ funksiya uchun $\left(\frac{1}{x}\right)^{(n)}=\frac{(-1)^n n!}{x^{n+1}}$

bo'lib, undan $(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$ bo'lishini topamiz.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $(a,b)$ da berilgan bo'lib, $\forall x \in (a, b)$ da $f^{(n)}(x)$ va $g^{(n)}(x)$ hosilalarga ega bo'lsin. U holda:

1) $(c \cdot f(x))^{(n)} = c \cdot f^{(n)}(x), \quad c = const$;

2) $(f(x) \pm g(x))^{(n)} = f^{(n)}(x) \pm g^{(n)}(x)$;

3) $$(f(x) \cdot g(x))^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)}(x) \cdot g^{(n-k)}(x) \tag{15}$$

$$\left( C_n^k = \frac{n(n-1)...(n-k+1)}{k!} \right), \quad f^{(0)}(x) = f(x)$$

bo'ladi.

(15) formula **Leybnits formulasi** deyiladi.

**508.** Ushbu $y = x^2 \cos 2x$ funksiyaning $n-$tartibli hosilasi topilsin.

**Yechish.** Leybnits formulasida $f(x) = \cos 2x$, $g(x) = x^2$ deb olamiz. Unda bu formulaga ko'ra, ayni paytda $g(x) = x^2$ funksiya uchun $k > 2$ bo'lganda $g^{(k)}(x) = (x^2)^{(k)} = 0, \quad (k > 2)$ bo'lishini e'tiborga olib topamiz:

$$(x^2 \cos 2x)^{(n)} = C_n^0 x^2 (\cos 2x)^{(n)} + C_n^1 (x^2)' \cdot (\cos 2x)^{(n-1)} + C_n^2 (x^2)'' (\cos 2x)^{(n-2)}$$

Ravshanki, $(\cos 2x)^{(n)} = 2^n \cos\left(2x + n \cdot \frac{\pi}{2}\right)$,

$$(\cos 2x)^{(n-1)} = 2^{n-1} \cos\left(2x + (n-1)\frac{\pi}{2}\right) = 2^{n-1} \sin\left(2x + n\frac{\pi}{2}\right),$$

$$(\cos 2x)^{(n-2)} = 2^{n-2} \cos\left(2x + (n-2)\frac{\pi}{2}\right) = -2^{n-2} \cos\left(2x + n\frac{\pi}{2}\right).$$

Demak, $(x^2 \cos 2x)^{(n)} = 2^n \left(x^2 - \frac{n(n-1)}{4}\right) \cos\left(2x + n\frac{\pi}{2}\right) + 2^n nx \sin\left(2x + n\frac{\pi}{2}\right)$.

**Funksiyaning yuqori tartibli differensiallari.**

Faraz qilaylik, $f(x)$ funksiya $(a, b)$ da berilgan bo'lib, $\forall x \in (a, b)$ nuqtada $f''(x)$ hosilaga ega bo'lsin. Ravshanki, $f(x)$ funksiyaning differensiali

$$df(x) = f'(x)dx$$

bo'lib, bunda $dx = \Delta x$ funksiya argumentning ixtiyoriy orttirmasi.

**4-ta'rif.** $f(x)$ funksiyaning $x \in (a, b)$ nuqtadagi differensiali $df(x)$ ning differensiali $f(x)$ funksiyaning $x \in (a, b)$ nuqtadagi ikkinchi tartibli differensiali deyiladi va $d^2 f(x)$ kabi belgilanadi: $d^2 f(x) = d(df(x))$.

Xuddi shunga o'xshash, $f(x)$ funksiyaning uchinchi $d^3 f(x)$, to'rtinchi $d^4 f(x)$ va h.k. tartibdagi differensiallari ta'riflanadi.

Umuman, $f(x)$ funksiyaning $n-$tartibli differensiali $d^n f(x)$ ning differensiali $f(x)$ funksiyaning $(n+1)-$tartibli differensiali deyiladi:

$$d^{n+1} f(x) = d(d^n f(x)).$$

**509.** Ushbu $f(x) = xe^{-x}$ funksiyaning ikkinchi tartibli differensiali topilsin.

**Yechish.** Berilgan funksiyaning ikkinchi tartibli differensialini ta'rifiga ko'ra topamiz:

$$d^2 f(x) = d(df(x)) = d(d(xe^{-x})) = d(xde^{-x} + e^{-x}dx) = d(-xe^{-x}dx + e^{-x}dx) =$$

$$= -d(xe^{-x})dx + (de^{-x})dx = -(xde^{-x} + e^{-x}dx)dx - e^{-x}(dx)^2 =$$

$$= x \cdot e^{-x}(dx)^2 - e^{-x}(dx)^2 - e^{-x}(dx)^2 = (x-2)e^{-x}(dx)^2.$$

Differensiallash qoidasidan foydalanib topamiz:

$$d^2 f(x) = d(df(x)) = d(f'(x)dx) = dx \cdot d(f'(x)) = dx \cdot f''(x)dx = f''(x)(dx)^2,$$

$$d^3 f(x) = d(d^2 f(x)) = f'''(x)(dx)^3,$$

.............................................

$$d^n f(x) = f^{(n)}(x)(dx)^n \qquad (16)$$

Masalan, yuqorida keltirilgan misol uchun

$$d^2(xe^{-x}) = (xe^{-x})''(dx)^2 = (e^{-x} - xe^{-x})'(dx)^2 =$$

$$= (-e^{-x} - e^{-x} - xe^{-x})(dx)^2 = (x-2)e^{-x}(dx)^2$$

bo'ladi.

Aytaylik, $f(x)$ va $g(x)$ funksiyalar $(a, b)$ da berilgan bo'lib, $x \in (a, b)$ nuqtada $n-$tartibli differensiallarga ega bo'lsin. U holda:

1) $d^n(c \cdot f(x)) = c \cdot d^n f(x), \quad c = const;$

2) $d^n(f(x) \pm g(x)) = d^n f(x) \pm d^n g(x);$

3) $d^n(f(x) \cdot g(x)) = d^n f(x) \cdot g(x) + C_n^1 d^{n-1} f(x) \cdot dg(x) +$

$+ \ldots + C_n^k d^{n-k} f(x) \cdot d^k g(x) + \ldots + f(x) \cdot d^n g(x)$

bo'ladi.

**Differensial shaklining invariantligi.** Aytaylik, $y=f(x)$ funksiya $(a,b)$ da differensiallanuvchi bo‘lib, $x$ o‘zgaruvchi o‘z navbatida biror $t$ o‘zgaruvchining $[\alpha,\beta]$ da differensiallanuvchi funksiyasi bo‘lsin:

$$x=\varphi(t) \quad (t\in[\alpha,\beta],\ x=\varphi(t)\in[a,b]).$$

Natijada

$$y=f(x)=f(\varphi(t))$$

bo‘ladi. Bu funksiyaning differensiali

$$dy=(f(\varphi(t)))'dt=f'(\varphi(t))\cdot\varphi'(t)dt=f'(\varphi(t))\cdot d\varphi(t)=f'(x)dx$$

bo‘lib, u (12) ko‘rinishga ega bo‘ladi. Shunday qilib, $y=f(x)$ funksiyada $x$ o‘zgaruvchi erkli bo‘lgan holda ham, u biror $t$ o‘zgaruvchiga bog‘liq bo‘lgan holda ham $y=f(x)$ funksiya differensialining ko‘rinishi bir xil bo‘ladi. Odatda bu xususiyat differensial shaklining **invariantligi** deyiladi.

$y=f(\varphi(t))$ funksiyaning ikkinchi tartibli differensiali quyidagicha bo‘ladi:

$$d^2y=d(df)=d(f'(x)dx)=df'(x)\cdot dx+f'(x)\cdot d(dx)=$$
$$=f''(x)\cdot(dx)^2+f'(x)d^2x.$$

Bu munosabatni (16) munosabat bilan solishtirib ikkinchi tartibli differensiallarda differensial shaklining invariantligi xossasi o‘rinli emasligini topamiz.

**510.** Funksiyaning differensialini ta’rif yordamida toping.

1) $f(x)=x^2+3x-2$; 2) $f(x)=x^3+3x$; 3) $f(x)=3x-2$.

**511.** Funksiyaning differensialini toping.

1) $f(x)=8\sqrt{x}$; 2) $f(x)=\ln x$; 3) $f(x)=\sin x$;

4) $f(x)=e^{-x^2}$; 5) $f(x)=x\ln x-x+1$; 6) $f(x)=tg^4x$.

**512.** Quyidagilarni taqribiy qiymatini hisoblang.

1) $f(x)=x^3+3x^2-2x+4,\ f(1.98)\approx?$; 2) $f(x)=\cos x, f(63^0)\approx?$;

3) $f(x)=\sqrt{\dfrac{x+4}{x-1}},\ f(5.05)\approx?$; 4) $f(x)=\sqrt{x^2+2x+12},\ f(3.96)\approx?$;

5) $\sqrt{65}$; 6) $\sqrt[4]{258}$; 7) $\ln 0.98$.

**513.** Funksiyaning berilgan nuqtada 3-tartibli hosilalari va differensiallarini toping.

1) $f(x)=e^{3x+2},\ x_0=0$; 2) $f(x)=\sin 2x,\ x_0=\dfrac{\pi}{4}$;

3) $f(x)=\ln 3x,\ x_0=1$;  4) $f(x)=\sqrt[3]{1+2x},\ x_0=0$.

**514*.** Funksiyaning $x_0=0$ nuqtadagi $n$ -tartibli hosilalarini toping:

1) $f(x)=x^3-2x^2+4x+6$;  2) $f(x)=e^{2x}$;

3) $f(x)=\sqrt[4]{1-x}$;  4) $f(x)=\ln(1+2x)$;

5) $f(x)=x\ln(1+x)$;  6) $f(x)=x^3\sin x$.

# VI BOB. HOSILANING TADBIQLARI

## 28 §. Differensiallanuvchi funksiyalar haqida asosiy teoremalar

Bu teoremalar funksiyalarni tekshirishda muhim rol o'ynaydi.

**1-teorema (Ferma teoremasi).** $f(x)$ funksiya $X \subset R$ to'p lamda berilgan. $x_0 \in X$ nuqtaning atrofi uchun

$$U_\delta(x_0) = (x_0 - \delta,\ x_0 + \delta) \subset X \qquad (\delta > 0)$$

bo'lib, quyidagi shartlar bajarilsin:

1) $\forall x \in U_\delta(x_0)$ da $f(x) \le f(x_0)$ $\quad (f(x) \ge f(x_0))$,

2) $f'(x_0)$ mavjud va chekli bo'lsin.

U holda $f'(x_0) = 0$ bo'ladi.

**2-teorema (Roll teoremasi).** Faraz qilaylik, $f(x)$ funksiya $[a,\ b]$ da berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $f(x) \in C[a,\ b]$,

2) $\forall x \in (a,\ b)$ da $f'(x)$ mavjud va chekli,

3) $f(a) = f(b)$ bo'lsin.

U holda shunday $x_0 \in (a,\ b)$ nuqta topiladiki, $f'(x_0) = 0$ bo'ladi.

**3-teorema (Lagranj teoremasi).** Faraz qilaylik, $f(x)$ funksiya $[a,\ b]$ da berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $f(x) \in C[a,\ b]$,

2) $\forall x \in (a,\ b)$ da $f'(x)$ hosila mavjud va chekli bo'lsin.

U holda shunday $c \in (a,\ b)$ nuqta topiladiki,

$$f(b) - f(a) = f'(c)(b - a)$$

bo'ladi.

**1-natija.** Aytaylik, $f(x)$ funksiya $(a,\ b)$ da $f'(x)$ hosilaga ega bo'lib, $\forall x \in (a,\ b)$ da $f'(x) = 0$ bo'lsin. U holda $\forall x \in (a,\ b)$ da $f(x) = const$ bo'ladi.

**2-natija.** $f(x)$ va $g(x)$ funksiyalari $(a,\ b)$ da $f'(x)$, $g'(x)$ hosilalarga ega bo'lib, $\forall x \in (a,\ b)$ da $f'(x) = g'(x)$ bo'lsin. U holda $\forall x \in (a,\ b)$ da $f(x) = g(x) + const$ bo'ladi.

**4-teorema (Koshi teoremasi).** Aytaylik, $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni bajarsin.

1) $f(x)\in C[a, b]$, $g(x)\in C[a, b]$,

2) $\forall x\in(a, b)$ da $f'(x)$ va $g'(x)$ hosilalar mavjud va chekli;

3) $\forall x\in(a, b)$ da $g'(x)\neq 0$ bo'lsin.

U holda shunday $c\in(a, b)$ nuqta topiladiki,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

bo'ladi.

**515.** $\forall x', x''\in R$ uchun $|\sin x'-\sin x''|\leq|x'-x''|$ tengsizlik isbotlansin.

**Yechish.** Aytaylik, $x'<x''$ bo'lsin. $f(x)=\sin x$ ga $[x', x'']$ da Lagranj teoremasini qo'llaymiz. Unda shunday $c\in(x', x'')$ nuqta topiladiki,

$$|\sin x'-\sin x''| = |\cos c|\cdot(x''-x')$$

bo'ladi. Agar $\forall t\in R$ da $|\cos t|\leq 1$ ekanini e'tiborga olsak, unda yuqoridagi munosabatdan

$$|\sin x'-\sin x''| \leq |x'-x''| \qquad (\forall x', x''\in R)$$

bo'lishi kelib chiqadi.

**516.** Ushbu

$$\frac{a-b}{a}<\ln\frac{a}{b}<\frac{a-b}{b} \qquad (0<b<a)$$

tengsizlik isbotlansin.

**Yechish.** $[b, a]$ segmentda $f(x)=\ln(x)$ funksiyani qaraymiz. Bu funksiya shu segmentda uzluksiz va $(b, a)$ da $f'(x)=\frac{1}{x}$ hosilaga ega. Unda Lagranj teoremasiga ko'ra shunday $c$ $(b<c<a)$ nuqta topiladiki,

$\frac{\ln a-\ln b}{a-b}=\frac{1}{c}$ bo'ladi. Ravshanki, $b<c<a \Rightarrow \frac{1}{a}<\frac{1}{c}<\frac{1}{b}$.

Bulardan $\frac{a-b}{a}<\ln\frac{a}{b}<\frac{a-b}{b}$ bo'lishi kelib chiqadi.

**Lopital qoidalari.** Ma'lum shartlarda funksiya limitini hisoblash qoidalari o'rganilgan edi. Ko'p hollarda bunday shartlar bajarilmaganda, ya'ni

$$x\to x_0 \text{ da } f(x)\to 0,\ g(x)\to 0:\ \frac{f(x)}{g(x)} \text{ ning limiti } \left(\frac{0}{0}\right),$$

$$x\to x_0 \text{ da } f(x)\to +\infty,\ g(x)\to +\infty:\ \frac{f(x)}{g(x)} \text{ ning limiti } \left(\frac{\infty}{\infty}\right),$$

$$x\to x_0 \text{ da } f(x)\to +\infty,\ g(x)\to +\infty:\ f(x)-g(x) \text{ ning limiti } (\infty-\infty),$$

$$x\to x_0 \text{ da } f(x)\to 0,\ g(x)\to 0:\ (f(x))^{g(x)} \text{ ning limiti } \left(0^0\right),$$

$$x\to x_0 \text{ da } f(x)\to 1,\ g(x)\to +\infty:\ (f(x))^{g(x)} \text{ ning limiti } \left(1^\infty\right)$$

$x\to x_0$ da $f(x)\to\infty, g(x)\to 0$: $f(x)\,g(x)$ ni limiti $\infty^0$ni topishda funksiyaning hosilalariga asoslangan qoidaga ko'ra hisoblash qulay bo'ladi. Bunday usul bilan funksiya limitini topish **Lopital qoidalari** deyiladi.

## $\frac{0}{0}$ *va* $\frac{\infty}{\infty}$ ko'rinishidagi hollar

**5-teorema.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $(a, b)$ da berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $\lim\limits_{x\to b-0} f(x)=0,\ \lim\limits_{x\to b-0} g(x)=0;$

2) $\forall x\in(a, b)$ da $f'(x)$ va $g'(x)$ hosilalar mavjud;

3) $\forall x\in(a, b)$ da $g'(x)\neq 0;$

4) Ushbu $\lim\limits_{x\to b-0}\frac{f'(x)}{g'(x)}=\ell,\ (\ell\in R)$ mavjud. U holda $\lim\limits_{x\to b-0}\frac{f(x)}{g(x)}=\ell$ bo'ladi.

**517.** Ushbu

$$\lim_{x\to e}\frac{(\ln x)^\alpha-\left(\frac{x}{e}\right)^\beta}{x-e}=\frac{\alpha-\beta}{e}$$

munosabat isbotlansin.

**Yechish.** $f(x)=(\ln x)^\alpha-\left(\frac{x}{e}\right)^\beta$, $g(x)=x-e$ funksiyalari uchun $(1,\ e)$ da 5-teoremaning barcha shartlari bajariladi:

1) $\lim\limits_{x\to e} f(x)=\lim\limits_{x\to e}\left[\ ((\ln x)^\alpha-\left(\frac{x}{e}\right)^\beta\ )\right]=0,$

$\lim\limits_{x\to e} g(x)=\lim\limits_{x\to e}(x-e)=0;$

2) $f'(x)=\alpha(\ln x)^{\alpha-1}\cdot\frac{1}{x}-\frac{\beta}{e}\left(\frac{x}{e}\right)^{\beta-1}$, $g'(x)=1$;

3) $g'(x)=1\neq 0$;

4) $\lim\limits_{x\to e}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to e}\frac{\alpha(\ln x)^{\alpha-1}\cdot\frac{1}{x}-\frac{\beta}{e}\cdot\left(\frac{x}{e}\right)^{\beta-1}}{1}=\frac{\alpha-\beta}{e}$.

Demak,

$$\lim_{x\to e}\frac{f(x)}{g(x)}=\lim_{x\to e}\frac{(\ln x)^{\alpha}-\left(\frac{x}{e}\right)^{\beta}}{x-e}=\frac{\alpha-\beta}{e}.$$

**6-teorema.** Aytaylik, $f(x)$ va $g(x)$ funksiyalar $(a,+\infty)$ da berilgan bo‘lib, quyidagi shartlarni bajarsin:

1) $\lim\limits_{x\to+\infty}f(x)=0$, $\lim\limits_{x\to+\infty}g(x)=0$;

2) $\forall x\in(a,+\infty)$ da $f'(x)$, $g'(x)$ hosilalar mavjud;

3) $\forall x\in(a,+\infty)$ da $g'(x)\neq 0$;

4) Ushbu $\lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}=\ell$ mavjud $(\ell\in R)$. U holda $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\ell$ bo‘ladi.

**518.** Ushbu $\lim\limits_{x\to+\infty}\frac{e^{-\frac{1}{x^2}}-1}{2\operatorname{arctg}x^2-\pi}$ limitni hisoblang.

**Yechish.** Agar $f(x)=e^{-\frac{1}{x^2}}-1$, $g(x)=2\operatorname{arctg}x^2-\pi$ deyilsa, ular uchun 6-teoremaning barcha shartlari bajariladi, jumladan,

$$f'(x)=-\frac{2}{x^3}e^{\frac{1}{x^2}},\qquad g'(x)=\frac{4x}{1+x^4}\text{ bo‘lib,}$$

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{-\frac{2}{x^3}e^{-\frac{1}{x^2}}}{\frac{4x}{1+x^4}}=-\lim_{x\to+\infty}\frac{1+x^4}{2x^4}=-\frac{1}{2}$$

bo‘ladi. 6-teoremaga ko‘ra

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{e^{-\frac{1}{x^2}}-1}{2\operatorname{arctg}x^2-\pi}=-\frac{1}{2}$$

bo‘ladi.

**7-teorema.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $(a, b)$ da berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $\lim\limits_{x\to b-0} f(x)=\infty, \quad \lim\limits_{x\to b-0} g(x)=\infty;$

2) $\forall x\in(a,b)$ da $f'(x), g'(x)$ hosilalar mavjud;

3) $\forall x\in(a,b)$ da $g'(x)\neq 0;$

4) Ushbu $\lim\limits_{x\to b-0}\dfrac{f'(x)}{g'(x)}=\ell, \ (\ell\in R)$ mavjud. U holda

$$\lim_{x\to b-0}\frac{f(x)}{g(x)}=\ell$$

bo'ladi.

**8-teorema.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $(a, +\infty)$ da berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $\lim\limits_{x\to+\infty} f(x)=\infty, \quad \lim\limits_{x\to+\infty} g(x)=\infty;$

2) $\forall x\in(a,+\infty)$ da $f'(x), g'(x)$ hosilalar mavjud;

3) $\forall x\in(a,+\infty)$ da $g'(x)\neq 0;$

4) Ushbu $\lim\limits_{x\to+\infty}\dfrac{f'(x)}{g'(x)}=\ell, \ (\ell\in R)$ mavjud. U holda $\lim\limits_{x\to+\infty}\dfrac{f(x)}{g(x)}=\ell$ bo'ladi.

### $0\cdot\infty,\ \infty-\infty,\ 1^{\infty},\ 0^0$ ko'rinishidagi hollar

Bu ko'rinishdagi aniqmasliklar $\dfrac{0}{0}, \dfrac{\infty}{\infty}$ hollarga keltirilib, so'ng yuqoridagi teoremalar qo'llaniladi.

**519*.** Ushbu $\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$ limit hisoblansin.

**Yechish.** Avvalo, $y=\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$ deb olamiz. Ravshanki, $x\to0$ da

$$f(x)=\frac{\sin x}{x}\to1, \qquad g(x)=\frac{1}{x^2}\to+\infty.$$

Sodda hisoblashlar yordamida topamiz:

$$\lim_{x\to0}\ln y=\lim_{x\to0}\frac{\ln\frac{\sin x}{x}}{x^2}=\lim_{x\to0}\frac{\left(\ln\frac{\sin x}{x}\right)'}{\left(x^2\right)'}=$$

$$=\lim_{x\to 0}\frac{\frac{x}{\sin x}\cdot\frac{x\cos x-\sin x}{x^2}}{2x}=$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}=\frac{1}{2}\lim_{x\to 0}\frac{(x\cos x-\sin x)'}{(x^3)'}=$$

$$=-\frac{1}{2}\lim_{x\to 0}\frac{x\sin x}{3x^2}=-\frac{1}{6}.$$

Demak, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}=e^{-\frac{1}{6}}$

**520.** $f(x)=\sqrt[3]{x^2}-1$ funksiya uchun x=0 nuqta va (-1;1) oraliqda Ferma teoremasining shartlarini tekshiring.

**521.** $f(x)=x^2-4x+5$ funksiya uchun x=2 nuqta va (1;3) oraliqda Ferma teoremasining shartlarini tekshiring.

**522.** $f(x)=\sin x$ funksiya uchun $[0;2\pi]$ kesmada Roll teoremasini qo'llash mumkinmi?

**523.** $f(x)=x(x-1)(x-2)(x-3)$ ko'phad hosilasining ildizlari haqiqiy va $(0;1),(1;2).(2;3)$ oraliqda yotishini isbotlang.

**524.** $f(x)=x^3$ egri chiziqda shunday nuqta topingki, bu nuqtadan unga o'tkazilgan urinma $A(-1;-1)$ va $B(2;8)$ nuqtalarni tutashtiruvchi vatarga parallel bo'lsin.

**525.** Lagranj teoremasidan foydalanib tengsizliklarni isbotlang.

1) $|arctgb-arctga|\le|b-a|$; 2) $e^x>1+x \quad (x>0)$.

**526.** $f(x)=x^2$ va $g(x)=x^3$ funksiyalar uchun a) [-1;1]; b) [1;2] kesmalarda Koshi teoremasi o'rinli bo'ladimi?

**527.** Quyidagi limitlarni Lopital qoidasidan foydalanib yeching.

1) $\lim_{x\to 3}\frac{x-3}{x^4-81}$; 2) $\lim_{x\to 0}\frac{e^x-e^{2x}}{\sin x}$; 3) $\lim_{x\to 0}\frac{1-\cos x}{1-\cos 4x}$;

4) $\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3}$; 5) $\lim_{x\to 0}\frac{x-arctgx}{2x^3}$; 6) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$;

7) $\lim_{x\to 0}\left(ctgx-\frac{1}{x}\right)$; 8*) $\lim_{x\to 0}\left(\frac{1}{x}\right)^{tgx}$; 9) $\lim_{x\to 1}x^{\frac{2}{1-x}}$.

## 29 §. Funksiyaning monotonligi, ekstremumlari, grafigini qavariq va botiqligi

### Funksiyaning to'la tekshirish

**Funksiyaning monotonligi.** Faraz qilaylik, $f(x)$ funksiya $(a, b)$ da berilgan bo'lsin.

Ma'lumki, $\forall x_1, x_2 \in (a, b)$, uchun

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \quad (f(x_1) < f(x_2))$$

bo'lsa, $f(x)$ funksiya $(a, b)$ da o'suvchi (qat'iy o'suvchi), $\forall x_1, x_2 \in (a, b)$ uchun $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ $(f(x_1) > f(x_2))$ bo'lsa, $f(x)$ funksiya $(a, b)$ da kamayuvchi (qat'iy kamayuvchi) deyiladi.

**1-teorema.** Aytaylik, $f(x)$ funksiya $(a, b)$ da berilgan bo'lib, $\forall x \in (a, b)$ da $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiyaning $(a, b)$ da o'suvchi bo'lishi uchun $\forall x \in (a, b)$ da $f'(x) \geq 0$ bo'lishi zarur va yetarli.

**2-teorema.** Faraz qilaylik, $f(x)$ funksiya $(a, b)$ da berilgan bo'lib, $\forall x \in (a,b)$ da $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiya $(a, b)$ da kamayuvchi bo'lishi uchun $\forall x \in (a,b)$ da $f'(x) \leq 0$ bo'lishi zarur va yetarli.

Demak, $(a, b)$ da

$f'(x) \geq 0 \Rightarrow f(x)$ o'suvchi $\Rightarrow f'(x) \geq 0$,

$f'(x) \leq 0 \Rightarrow f(x)$ kamayuvchi $\Rightarrow f'(x) \leq 0$,

$f'(x) > 0 \Rightarrow f(x)$ qat'iy o'suvchi $\Rightarrow f'(x) \geq 0$,

$f'(x) < 0 \Rightarrow f(x)$ qat'iy kamayuvchi $\Rightarrow f'(x) \leq 0$

bo'ladi.

**528.** Ushbu $f(x) = \frac{x^2}{2^x}$ funksiyaning o'suvchi, kamayuvchi bo'lish oraliqlari topilsin.

**Yechish.** Ravshanki, $f'(x) = x \cdot 2^{-x}(2 - x\ln 2)$ bo'ladi. Ushbu $f'(x) > 0$, $x \cdot 2^{-x}(2 - x\ln 2) > 0$ tengsizlik $x \in \left(0, \frac{2}{\ln 2}\right)$ da o'rinli bo'ladi.

Demak, $f(x)$ funksiya $x \in \left(0, \frac{2}{\ln 2}\right)$ da o'suvchi, $(-\infty, 0) \cup (\frac{2}{\ln 2}, +\infty)$ da kamayuvchi bo'ladi.

**Funksiyaning ekstremumlari.** Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib. $x_0 \in X$ bo'lsin.

**1-ta'rif.** Agar shunday $\delta > 0$ son topilsaki,

$$\forall x \in U_\delta(x_0) = (x_0 - \delta,\ x_0 + \delta) \subset X \text{ nuqtalarda}$$

$$f(x) \le f(x_0) \qquad \left(f(x) \ge f(x_0)\right)$$

tengsizlik bajarilsa, $f(x)$ funksiya $x_0$ nuqtada maksimumga (minimumga) erishadi deyiladi, $x_0$ nuqtaga esa $f(x)$ funksiyaning maksimum (minimum) nuqtasi deyiladi.

**2-ta'rif.** Agar shunday $\delta > 0$ son topilsaki, $\forall x \in U_\delta(x_0) \setminus \{x_0\}$ $(U_\delta(x_0) \subset X)$ nuqtalarda

$$f(x) < f(x_0) \qquad \left(f(x) > f(x_0)\right)$$

tengsizlik bajarilsa, $f(x)$ funksiya $x_0$ nuqtada qat'iy maksimumga (qat'iy minimumga) erishadi deyiladi.

Funksiyaning maksimum hamda minimumi umumiy nom bilan uning ekstremumlari, maksimum hamda minimum nuqtalari esa uning **ekstremum** nuqtalari deyiladi.

**3-teorema.** Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, $x_0 \in X$ nuqtada ekstremumga erishsin.

Agar $f(x)$ funksiya $x_0$ nuqtada $f'(x_0)$ hosilaga ega bo'lsa, u holda $f'(x_0) = 0$ bo'ladi.

**3-ta'rif.** Funksiya hosilasini nolga aylantiradigan nuqta uning **statsionar** (kritik) nuqtasi deyiladi.

**Eslatma.** Agar $f(x)$ funksiya biror nuqtada ekstremumga erishsa, u shu nuqtada hosilaga ega bo'lishi shart emas. Masalan, $f(x) = |x|$ funksiya $x_0 = 0$ nuqtada minimumga erishadi, biroq u shu nuqtada hosilaga ega emas.

Demak, $f(x)$ funksiyaning ekstremum nuqtalari uning statsionar hamda hosilasi mavjud bo'lmagan nuqtalari bo'lishi mumkin.

**4-ta'rif.** Agar shunday $\delta > 0$ son topilsaki,

$$\forall x \in (x_0 - \delta,\ x_0) \ da \ g(x) > 0 \ yoki \ \forall x \in (x_0 - \delta,\ x_0) \ da \ g(x) < 0$$

bo'lsa, $g(x)$ funksiya $x_0$ nuqtaning chap tomonida ishora saqlaydi deyiladi.

Agar shunday $\delta > 0$ son topilsaki,

$$\forall x \in (x_0,\ x_0 + \delta) \ da \ g(x) > 0 \ yoki \ \forall x \in (x_0,\ x_0 + \delta) \ da \ g(x) < 0$$

bo'lsa, $g(x)$ funksiya $x_0$ nuqtaning o'ng tomonida ishora saqlaydi deyiladi.

**4-teorema.** Aytaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $\exists\delta > 0,\ \forall x \in U_\delta(x_0) \subset X$ да $f'(x)$ hosila mavjud;

2) $f'(x_0) = 0$;

3) $f'(x)$ hosila $x_0$ nuqtaning o'ng va chap tomonlarida ishora saqlasin.

Agar $f'(x)$ hosila $x_0$ nuqtani o'tishda ishorasini o'zgartirsa, $f(x)$ funksiya $x_0$ nuqtada ekstremumga erishadi.

Agar $f'(x)$ hosila $x_0$ nuqtani o'tishda ishorasini o'zgartirmasa, $f(x)$ funksiya $x_0$ nuqtada ekstremumga erishmaydi.

**5-teorema.** $f(x)$ funksiya $X \subset R$ to'plamda berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $f(x) \in C(X)$;

2) $\exists\delta > 0,\ \forall x \in U_\delta(x_0) \setminus \{x_0\}$ да $f'(x)$ hosila mavjud va chekli;

3) $f'(x)$ hosila $x_0$ nuqtaning o'ng va chap tomonlarida ishora saqlansin.

Agar $f'(x)$ hosila $x_0$ nuqtani o'tishda ishorasini o'zgartirsa, $f(x)$ funksiya $x_0$ nuqtada ekstremumga erishadi.

Agar $f'(x)$ hosila $x_0$ nuqtani o'tishda ishorasini o'zgartirmasa, $f(x)$ funksiya $x_0$ nuqtada ekstremumga erishmaydi.

**6-teorema.** Faraz qilaylik, $f(x)$ funksiya $X \subset R$ to'plamda berilgan va $m \in N$, $m \geq 2$, $x_0 \in X$ bo'lib, quyidagi shartlarni bajarsin:

1) $\exists\delta > 0,\ \forall x \in U_\delta(x_0) \subset X$ *da* $f^{(m-1)}(x)$ hosila mavjud;

2) $f^{(m)}(x_0)$ hosila mavjud;

3) $f'(x_0) = f''(x_0) = \ldots = f^{(m-1)}(x_0) = 0,\quad f^{(m)}(x_0) \neq 0.$

U holda $m = 2k,\ k \in N$ bo'lganda $f(x)$ funksiya $x_0$ nuqtada ekstremumga erishib, $f^{(m)}(x_0) < 0$ bo'lganda $x_0$ nuqtada maksimumga, $f^{(m)}(x_0) > 0$ da minimumga erishadi.

Agar $m = 2k+1,\ k \in N$ bo'lsa, $f(x)$ funksiya $x_0$ nuqtada ekstremumga erishmaydi.

Xususan, agar $x_0$ nuqta $f(x)$ funksiyaning statsionar nuqtasi bo‘lib, $f(x)$ funksiya $x_0$ nuqtada chekli $f''(x_0)\neq 0$ hosilaga ega bo‘lsa, shu nuqtada $f(x)$ funksiya $f''(x_0)<0$ bo‘lganda maksimumga, $f''(x_0)>0$ minimumga ega bo‘ladi.

**529.** Ushbu $f(x)=2\sqrt[3]{x^5}-5\sqrt[3]{x^2}+1$ funksiya ekstremumga tekshirilsin.

**Yechish.** Bu funksiya $R=(-\infty;+\infty)$ aniqlangan bo‘lib, u shu to‘plamda uzluksiz. Uning hosilasini topamiz:

$$f'(x)=2\cdot\frac{5}{3}\cdot x^{\frac{2}{3}}-5\cdot\frac{2}{3}\cdot x^{-\frac{1}{3}}=\frac{10(x-1)}{3\sqrt[3]{x}}$$

Ravshanki, funksiyaning hosilasi $x_1=1$ nuqtada nolga alanadi: $f'(1)=0$; $x_2=0$ nuqtada esa funksiyaning hosilasi mavjud emas.

Hosila ifodasidan ko‘rinadiki, $x=1$ nuqtaning chap tomonidagi nuqtalarda $f'(x)<0$ o‘ng tomonidagi nuqtalarda $f'(x)>0$ bo‘ladi. Demak, berilgan funksiya $x=1$ nuqtada minimumga erishadi va $\min f(x)=f(1)=-2$ bo‘ladi.

Yana hosila ifodasidan ko‘rinadiki, $x=0$ nuqtaning chap tomonidagi nuqtalarda $f'(x)>0$, o‘ng tomonidagi nuqtalarda $f'(x)<0$ bo‘ladi.

Demak, $f(x)$ funksiya $x=0$ nuqtada maksimumga erishadi va $\max f(x)=f(0)=1$ bo‘ladi.

**Funksiyaning qavariqligi va botiqligi.** Faraz qilaylik, $f(x)$ funksiya $(a,b)$ da berilgan bo‘lib, $x_1,x_2\in(a,b)$ uchun $x_1<x_2$ bo‘lsin.

$f(x)$ funksiya grafigining $(x_1, f(x_1))$, $(x_2, f(x_2))$ nuqtalaridan o‘tuvchi to‘g‘ri chiziqni $y=l(x)$ desak, u quyidagicha

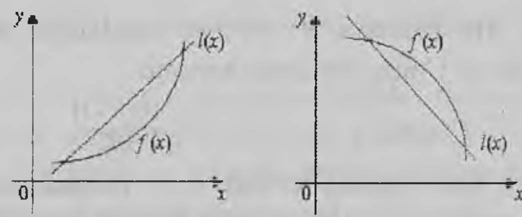
$$l(x)=\frac{x_2-x}{x_2-x_1}f(x_1)+\frac{x-x_1}{x_2-x_1}f(x_2)$$

bo‘ladi.

**5-ta’rif.** Agar har qanday oraliq $(x_1,x_2)\subset(a,b)$ da joylashgan $\forall x\in(x_1,x_2)$ uchun $f(x)\leq l(x)$ $(f(x)<l(x))$ bo‘lsa, $f(x)$ funksiya $(a,b)$ da botiq (qat’iy botiq) funksiya deyiladi.

**6-ta'rif.** Agar har qanday oraliq $(x_1, x_2) \subset (a, b)$ da joylashgan $\forall x \in (x_1, x_2)$ uchun $f(x) \ge l(x)$ $(f(x) > l(x))$ bo'lsa, $f(x)$ funksiya $(a, b)$ da qavariq (qat'iy qavariq) funksiya deyiladi.

Botiq hamda qavariq funksiyalarning grafiklari 1-shaklda tasvirlangan:




**1-shakl.**

Aytaylik, $\alpha_1 \ge 0$, $\alpha_2 \ge 0$, $\alpha_1 + \alpha_1 = 1$ bo'lib, $\forall x_1, x_2 \in (a, b)$ bo'lsin. Funktsiyaning botiqligi hamda qavariqligini quyidagicha ta'riflash ham mumkin.

**7-ta'rif.** Agar $f(\alpha_1 x_1 + \alpha_2 x_2) \le \alpha_1 f(x_1) + \alpha_2 f(x_2)$ bo'lsa, $f(x)$ funksiya $(a, b)$ da botiq deyiladi.

**8-ta'rif.** Agar $f(\alpha_1 x_1 + \alpha_2 x_2) \ge \alpha_1 f(x_1) + \alpha_2 f(x_2)$ bo'lsa, $f(x)$ funksiya $(a, b)$ da qavariq deyiladi.

**530.** Ushbu $f(x) = x^2$ funksiya $R$ da qat'iy botiq funksiya bo'lishini isbotlang.

**Yechish.** 7-ta'rifdan foydalanib topamiz:

$$f(\alpha_1 x_1 + \alpha_2 x_2) = (\alpha_1 x_1 + \alpha_2 x_2)^2 = (\alpha_1 x_1)^2 + 2\alpha_1 \alpha_2 x_1 x_2 + (\alpha_2 x_2)^2 <$$

$$< \alpha_1^2 x_1^2 + \alpha_1 \alpha_2 (x_1 + x_2)^2 + \alpha_2^2 x_2^2 = \alpha_1 x_1^2 (\alpha_1 + \alpha_2) + \alpha_2 x_2^2 (\alpha_1 + \alpha_2) =$$

$$= \alpha_1 x_1^2 + \alpha_2 x_2^2 = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

**7-teorema.** $f(x)$ funksiya $(a, b)$ intervalda botiq (qavariq) bo'lishi uchun $(a, b)$da $f''(x) \ge 0$ $(f''(x) \le 0)$ bo'lishi zarur va yetarli.

**531.** Ushbu $f(x) = \ln x$ $(x > 0)$ funksiya qavariq bo'lishini ko'rsating.

**Yechish.** Bu funksiya uchun $f''(x) = -\frac{1}{x^2} < 0$

bo‘ladi. Berilgan $f(x)=\ln x$ funksiya $(0,+\infty)$ da qat'iy qavariq bo‘ladi.

**Funksiyaning egilish nuqtalari.** Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to‘plamda berilgan bo‘lib, $x_0\in X$, $(x_0-\delta,\ x_0+\delta)\subset X$, $\delta>0$ bo‘lsin.

**9-ta’rif.** Agar $f(x)$ funksiya $(x_0-\delta,\ x_0)$ da botiq (qavariq), $(x_0,\ x_0+\delta)$ da qavariq (botiq) bo‘lsa, $x_0$ nuqta $f(x)$ funksiyaning **egilish nuqtasi** deyiladi.

Aytaylik, $f(x)$ funksiya $(x_0-\delta,\ x_0+\delta)$ da $f''(x)$ hosilaga ega bo‘lsin. Agar $\forall x\in(x_0-\delta, x_0)$ da $f''(x)\ge 0$ $(f''(x)\le 0)$,

$\forall x\in(x_0, x_0+\delta,)$ da $f''(x)\le 0$ $(f''(x)\ge 0)$,

bo‘lsa, $f'(x)$ funksiya $x_0$ nuqtada ekstremumga erishadi va demak, $f''(x_0)=0$ bo‘ladi. Demak, $f(x)$ funksiya egilish nuqtasida $f''(x)=0$ bo‘ladi.

**532.** Ushbu $f(x)=x^3$ funksiya $x_0=0$ nuqtada egilishini ko‘rsating.

**Yechish.** Bu funksiya uchun $f''(x)=6x$ bo‘lib, $\forall x\in(-\delta,0)$ da $f''(x)<0$, $\forall x\in(0,\delta,)$ da $f''(x)>0$ $(\delta>0)$ bo‘ladi.

**Funksiya grafigining asimptotalari.** Faraz qilaylik, $f(x)$ funksiya $X\subset R$ to‘plamda berilgan bo‘lib, $x_0$ nuqta $X$ to‘plamning limit nuqtasi bo‘lsin.

**10-ta’rif.** Agar ushbu $\lim\limits_{x\to x_0+0} f(x)$, $\lim\limits_{x\to x_0-0} f(x)$

limitlardan biri yoki ikkalasi xam cheksiz bo‘lsa, $x=x_0$ to‘g‘ri chiziq $f(x)$ funksiya grafigining vertikal asimptotasi deyiladi.

Masalan, $f(x)=\dfrac{1}{x}$ funksiya grafigi uchun $x=0$ to‘g‘ri chiziq vertikal asimptota bo‘ladi.

Aytaylik, $f(x)$ funksiya $(x_0,+\infty)$ da aniqlangan bo‘lsin.

**11-ta’rif.** Agar shunday $k$ va $b$ sonlari topilsaki, $f(x)=kx+b+\alpha(x)$ $(x\to\infty$ da $\alpha(x)\to 0)$

bo‘lsa, $y=kx+b$ to‘g‘ri chiziq $f(x)$ funksiya grafigining og‘ma asimptotasi deyiladi.

**8-teorema.** $f(x)$ funksiya grafigi $y=kx+b$ og'ma asimptotaga ega bo'lishi uchun

$$\lim_{x\to+\infty}\frac{f(x)}{x}=k,\quad \lim_{x\to+\infty}(f(x)-kx)=b$$

bo'lishi zarur va yetarli.

**533.** $f(x)=\dfrac{x^3}{(x-1)^2}$ funksiyaning og'ma asimptotasi topilsin.

**Yechish.** Bu funksiya uchun $k=\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to+\infty}\dfrac{x^2}{(x-1)^2}=1;$

$$b=\lim_{x\to+\infty}(f(x)-kx)=\lim_{x\to+\infty}\left(\frac{x^3}{(x-1)^2}-x\right)=2$$

bo'ladi. Demak, $y=x+2$ to'g'ri chiziq berilgan funksiya grafigining og'ma asimptotasi bo'ladi.

**Funksiyaning to'la tekshirish sxemasi**

1) Funksiyaning aniqlanish sohasi topiladi;
2) Koordinata o'qlarini kesuvchi nuqtalar topiladi;
3) Funksiyaning juft-toqligiga tekshiriladi;
4) Uzluksizlikka tekshiriladi, uzilish nuqtalar topiladi va turi aniqlanadi;
5) Funksiyaning monotonlik intervali topiladi, lokal maksimum va lokal mimimumlari hisoblanadi;
6) Funksiyaning qavariq- botiqlik intervali topiladi;
7) Funksiyaning asimptotalar va qiymatlar sohasi topiladi;
8) Funksiyaning grafigi yasaladi.

**534.** Funksiyani to'liq tekshiring va grafigini yasang:

$$y=\frac{x^2}{2(x-1)}.$$

**Yechish.** 1) Funksiyaning aniqlanish sohasi maxraji nolga aylanadigan nuqtalardan boshqa barcha nuqtalar to'plami: $D(y)=(-\infty;1)\cup(1;+\infty)$.

2) $x=0$ da quyidagini hosil qilamiz: $y(0)=\dfrac{0^2}{2(0-1)}=0$,

$x=0$ ***nuqta koordinata o' qlarini kesib o' tuvchi yagona nuqta.***

3) Funksiyaning juft-toqligiga tekshiramiz.

$$y(-x)=\frac{(-x)^2}{2(-x-1)}=-\frac{x^2}{2(x+1)}$$

$$y(-x)\neq y(x),\ y(-x)\neq -y(x).$$

Demak, funksiya juft ham, toq ham emas;

4) $x=1$ nuqta uzilish nuqta. Shu nuqtada o'ng va chap limitini hisoblaymiz:

$$\lim_{x\to 1-0}\frac{x^2}{2(x-1)}=-\infty,\ \lim_{x\to 1+0}\frac{x^2}{2(x-1)}=+\infty.$$

$x=1$ nuqta 2-tur uzilish nuqta ekan.

5) Funksiyaning monotonlik intervalini hisoblash uchun funksiyaning 1- tartibli hosilasini hisoblaymiz va son o'qini quyidagi intervallarga bo'lamiz:

$$y'=\frac{2x(x-1)-x^2}{2(x-1)^2}=\frac{x(x-2)}{2(x-1)^2},\ (-\infty;0)\cup(0;1)\cup(1;2)\cup(2;+\infty).$$

Bu intervallarda funksiya hosilasining ishorasini aniqlaymiz.

$$y'(-1)=\frac{1}{4}>0,\ y'(0.5)=-1.5<0,\ y'(1.5)=-1.5<0,\ y'(3)=\frac{3}{8}>0.$$


$(-\infty;0)\cup(2;+\infty)$ intervalda funksiya o'sadi, $(0;1)\cup(1;2)$ intervalda funksiya kamayadi. $x=0$ nuqta lokal maksimum, $y_{\max}(0)=\frac{0^2}{2(0-1)}=0$,

$y_{\min}(2)=\frac{2^2}{2(2-1)}=2$ bo'ladi.

6) Funksiyaning qavariq- botiqlik intervalini topamiz. Buning uchun 2- tartibli hosilani hisoblaymiz:

$$y''=\frac{(2x-2)(x-1)^2-2x(x-2)(x-1)}{2(x-1)^4}=\frac{1}{(x-1)^3}.$$

Funksiya $(-\infty;1)$ intevalda qavariq, $(1;+\infty)$ intervalda botiq bo'ladi.

7) $x=1$ to'g'ri chiziq funksiyaning vertikal asimptota bo'ladi. O'g'ma asimptota quyidagicha ko'rinishda bo'ladi: $y=kx+b$,

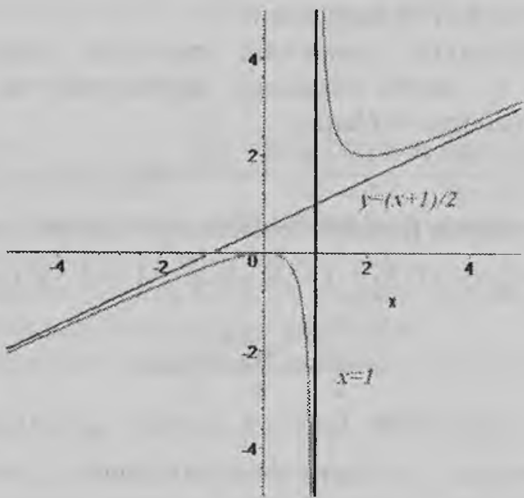
bunda $k=\lim\limits_{x\to\pm\infty}\frac{f(x)}{x}, b=\lim\limits_{x\to\pm\infty}(f(x)-kx)$.

$$k=\lim_{x\to\pm\infty}\frac{x^2}{2x(x-1)}=\frac{1}{2},\quad b=\lim_{x\to\pm\infty}\left(\frac{x^2}{2(x-1)}-\frac{1}{2}x\right)=\lim_{x\to\pm\infty}\left(\frac{x^2-x^2+x}{2(x-1)}\right)=\frac{1}{2}.$$

Demak, o'g'ma asimptota $y=0,5(x+1)$.

Funksiyaning qiymatlar sohasi $E(y)=(-\infty;0)\cup(2;+\infty)$ ekan.

8) Yuqoridagi natijalarga tayangan holda funksiyaning grafini yasaymiz. Avvalo vertikal, og'ma asimptotlarni yasaymiz, keyin bir necha qiymatlarni topamiz va grafik yasaymiz.




**2-shakl**

**535**. Quyidagi funksiyalarning o'shish va kamayish oralig'ini toping:

1. $y=x^2-6x+8$; 2. $y=x^3-9x^2-21x+1$; 3. $y=x^3+3x^2+3x+1$;
4. $y=x^4-2x^2+5$; 5. $y=xe^{-x}$; 6. $y=e^{-x^2}$;
7. $y=2^x+4^{-x}$; 8. $y=2x-e^{2x}$; 9. $y=x\ln x$.

**536**. Quyidagi funksiyalarning ekstremumlarini toping:

1. $y=x^2-4x+5$; 2. $y=2x^3-6x^2-18x+1$; 3. $y=3x^4-4x^3+3$;
4. $y=x\sqrt{1-x^2}$; 5. $y=x^2\sqrt[3]{7-x}$; 6. $y=e^x\sin x$;
7. $y=x-\ln(1+x)$; 8. $y=\dfrac{x}{x^2+x+1}$; 9. $y=\ln x-2arctgx$.

**537.** Quyidagi funksiyalarning berilgan kesmadagi eng katta va eng kichik qiymatlarini toping:

1. $y=x^3-9x^2+15x+1,\ [-2;6]$; 2. $y=4x^4-2x^2+2,\ [0;2]$;

3. $y=\frac{x^3+2x^2}{x-2},\ [-1;1]$; 4. $y=x-2\sqrt{x},\ [0;4]$;

5. $y=2x-tgx,\ [0;\frac{\pi}{3}]$; 6. $y=\ln 2x-x^2+x,\ [0.5;2]$.

**538.** Quyidagi funksiyalarning qavariq-botiqlik oralig'larini toping. Egilish nuqtalarini aniqlang:

1. $y=x^3-3x^2+4x-1$; 2. $y=x^4-6x^2+x$; 3. $y=\frac{2x^2+4}{x^2-4}$;

4. $y=\frac{x^2+2x+4}{x+2}$; 5. $y=(x+1)e^{-x}$; 6. $y=x^2\ln x$.

**539.** Quyidagi funksiyalarning grafiklari asimptotalarini toping:

1. $y=\frac{x+2}{x-1}$; 2. $y=\frac{2x^2+x+1}{x-2}$; 3. $y=\frac{2x^2}{\sqrt{x^2-4}}$;

4. $y=\sqrt{x^2-x}$; 5. $y=xe^x$; 6. $y=\frac{\ln x}{x}$.

**540*.** Quyidagi funksiyalarni to'la tekshiring va grafigini yasang:

1. $y=3x^5-5x^3$; 2. $y=\frac{x^2+1}{x^2-1}$; 3. $y=\frac{(x+1)(x+8)}{x}$;

4. $y=\frac{x^2-1}{x}$; 5. $y=\frac{4x}{x^2+4}$; 6. $y=\frac{1}{x^2+1}$;

7. $y=\frac{2x-1}{(x-1)^2}$; 8. $y=\frac{x^2}{x-1}$; 9. $y=\frac{1}{9}x(x-4)^3$;

10. $y=\frac{x^3}{x^2-1}$; 11. $y=\frac{1}{1-x^2}$; 12. $y=\frac{2x^2}{x^2+1}$;

13. $y=\frac{x}{x^2-1}$; 14. $y=\frac{(x+1)^2}{x-2}$; 15. $y=\frac{x-1}{x^2-2x}$;

16. $y=\frac{x^3+4}{x^2}$; 17. $y=\frac{x^2-1}{x^2+1}$; 18. $y=\frac{6x^2-x^4}{9}$;

19. $y=\frac{x^3}{3-x^2}$; 20. $y=xe^x$; 21. $y=e^{2x-x^2}$;

22. $y=\frac{\ln x}{\sqrt{x}}$; 23. $y=\frac{e^{x-1}}{x}$; 24. $y=\ln\frac{x-2}{x+1}$;

25. $y=x^2e^{-x}$; 26. $y=\frac{\ln x}{x}$; 27. $y=x\ln^2 x$;

28. $y=x^3e^{-x}$; 29. $y=\frac{x}{\ln x}$; 30. $y=|x\ln|x||$.

## 30 §. Teylor va Makloren formulalari

**Ko'phad uchun Teylor formulasi:**

$$P(x)=P(x_0)+\frac{P'(x_0)}{1!}(x-x_0)+\frac{P''(x_0)}{2!}(x-x_0)^2+...+\frac{P^{(n)}(x_0)}{n!}(x-x_0)^n \qquad (1)$$

bo'ladi.

**Ixtiyoriy funksiyaning Teylor formulasi va uning qoldiq hadlari.** Faraz qilaylik, $f(x)$ funksiya $(a,b)$ da berilgan bo'lib, $x_0\in(a,b)$ bo'lsin. Bu funksiya $x_0$ nuqtaning

$$\cup_\delta(x_0)=(x_0-\delta,\ x_0+\delta)\subset(a,b) \quad \delta>0$$

atrofida $f'(x), f''(x),...,f^{(n)}(x), f^{(n+1)}(x)$ hosilalarga ega bo'lsin.

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x) \qquad (2)$$

Bu (2) formula $f(x)$ funksiyaning Teylor formulasi deyiladi. (2) formuladagi $R_n(x)$ esa Teylor formulasining qoldiq hadi deyiladi.

Endi qoldiq had $R_n(x)$ ni aniqlaymiz.

**a) Koshi ko'rinishidagi qoldiq hadli Teylor formulasi:**

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+$$
$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{n!}(x-x_0)^{n+1}(1-\theta)^n \qquad (3)$$

bo'ladi. Bunda bunda $c=x_0+\theta(x-x_0)$ $(0<\theta<1)$.

**b) Lagranj ko'rinishidagi qoldiq hadli Teylor formulasi:**

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{n!}(x-x_0)^{n+1},$$
$$(c=x_0+\theta(x-x_0),\quad 0<\theta<1) \qquad (4)$$

formula hosil bo'lib, uni $f(x)$ funksiyaning Lagranj ko'rinishidagi qoldiq hadli Teylor formulasi deyiladi.

**d) Peano ko'rinishidagi qoldiq hadli Teylor formulasi:**

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+$$
$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n), \qquad (x\to x_0) \qquad (5)$$

bo'ladi.

**Ba'zi funksiyalarning Teylor formulalari.** $f(x)$ funksiyaning Peano ko'rinishidagi qoldiq hadli Teylor formulasini olamiz:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+$$

$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n), \qquad (x\to x_0)$$

Bu tenglikda $x_0=0$ deb, ushbu

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n), \quad (x\to 0) \qquad (6)$$

formulaga kelamiz. (6) formula $f(x)$ funksiyaning Makloren formulasi deyiladi.

1) $f(x)=e^x$ bo'lsin. Bu funksiya uchun $f(0)=1, \quad f^{(n)}(0)=1$ bo'lib,

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+o(x^n), \quad x\to 0$$

bo'ladi.

2) $f(x)=(1+x)^\alpha, \ \alpha\in R$ bo'lsin. Bu funksiya uchun

$f(0)=1, \quad f^{(n)}(0)=\alpha(\alpha-1)\ldots(\alpha-n+1)$

bo'lib,

$$(1+x)^\alpha=\sum_{k=0}^{n}\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}x^k+o(x^n), \quad x\to 0$$

bo'ladi. Xususan, $\frac{1}{1-x}=\sum_{k=0}^{n}x^k+o(x^n), \quad x\to 0$

$$\frac{1}{1+x}=\sum_{k=0}^{n}(-1)^k x^k+o(x^n), \quad x\to 0$$

bo'ladi.

3) $f(x)=\ln(1+x)$ bo'lsin. Bu funksiya uchun

$f(0)=0, \quad f^{(k)}(0)=(-1)^{k-1}(k-1)!$

bo'lib,

$$\ln(1+x)=\sum_{k=1}^{n}\frac{(-1)^k x^k}{k}+o(x^n), \quad x\to 0$$

bo'ladi.

Shuningdek, $\ln(1-x)=-\sum_{k=1}^{n}\frac{x^k}{k}+o(x^n), \quad x\to 0$ bo'ladi.

4) $f(x)=\sin x$ bo‘lsin. Bu funksiya uchun $f(0)=0$, $f^{(2k+1)}(0)=(-1)^k$ bo‘lib,

$$\sin x=\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}+o(x^{2n+2}),\quad x\to 0$$

bo‘ladi.

5) $f(x)=\cos x$ bo‘lsin. Bu funksiya uchun $f(0)=1$, $f^{(2k)}(0)=(-1)^k$ bo‘lib,

$\cos x=\sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}+o(x^{2n+1}),\quad x\to 0$ bo‘ladi.

**541.** Ushbu $f(x)=\dfrac{1}{3x+2}$

funksiyaning Teylor (Makloren) formulasi yozilsin.

**Yechish.** Bu funksiyani quyidagicha $f(x)=\dfrac{1}{3x+2}=\dfrac{1}{2\left(1+\frac{3}{2}x\right)}$

yozib, so‘ng $\dfrac{1}{1+x}=\sum_{k=0}^{n}(-1)^k x^k+o(x^n),\quad x\to 0$

bo‘lishidan foydalanib topamiz:

$$\frac{1}{3x+2}=\sum_{k=0}^{n}(-1)^k\frac{3^k}{2^{k+1}}x^k+o(x^n),\quad x\to 0.$$

**542.** Teylor formulasidan foydalanib $y=\sqrt[3]{x}$ funksiyani $(x-1)^5$ hadigacha yoying.

**543.** Makloren formulasidan foydalanib $y=a^x$ funksiyani $x^3$ hadigacha yoying.

**544.** Makloren formulasidan foydalanib $\sqrt[3]{29}$ ni $10^{-3}$ aniqlikda hisoblang.

**545.** Makloren formulasidan foydalanib $\sqrt{e}$ ni $10^{-1}$ aniqlikda hisoblang.

**546.** Makloren formulasidan foydalanib quyidagi limitlarni hisoblang.

1) $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}$; 2) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$; 3) $\lim\limits_{x\to 0}\dfrac{\ln\cos x+x^2}{\sin x\,tgx}$.

# VII BOB. ANIQMAS VA ANIQ INTEGRAL

## 31 §. Boshlang'ich funksiya va aniqmas integral tushunchasi. Integrallash usullari.

**Boshlang'ich funksiya tushunchasi.** Faraz qilaylik, $f(x)$ va $F(x)$ funksiyalari $(a,b)\subset R$ intervalda berilgan bo'lib, $F(x)$ funksiya shu $(a,b)\subset R$ da differensiallanuvchi bo'lsin.

**1-ta'rif.** Agar $(a,b)$ intervalda $F'(x)=f(x)$ $(x\in(a,b))$ bo'lsa, $(a,b)$ da $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi deyiladi.

Masalan, $f(x)=\dfrac{1}{x}$ funksiyaning $(0,+\infty)$ da boshlang'ich funksiyasi $F(x)=\ln x$ bo'ladi, chunki $(0,+\infty)$ da $F'(x)=(\ln x)'=\dfrac{1}{x}=f(x)$.

Aytaylik, $f(x)$ va $F(x)$ funksiyalari $[a,b]$ segmentda berilgan bo'lib, $F(x)$ funksiya shu $[a,b]$ da differentsiallanuvchi bo'lsin.

**2-ta'rif.** Agar $(a,b)$ intervalda $F'(x)=f(x)$ $(x\in(a,b))$ bo'lib, $a$ va $b$ nuqtalarda esa $F'(a+0)=f(a)$, $F'(b-0)=f(b)$ tengliklar o'rinli bo'lsa, $[a,b]$ segmentda $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi deyiladi.

**1-teorema.** Agar $(a,b)$ intervalda $F(x)$ va $\Phi(x)$ funksiyalarning har biri $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, u holda $F(x)$ va $\Phi(x)$ funksiyalar $(a,b)$ da bir-biridan o'zgarmas songa farq qiladi: $F(x)-\Phi(x)=C$

$\Phi(x)-F(x)=C.$ $(C=const)$

**Natija.** Agar $(a,b)$ da $F(x)$ funksiya $f(x)$ ning biror boshlang'ich funksiyasi bo'lsa, u holda $f(x)$ funksiyaning $(a,b)$ dagi ixtiyoriy boshlang'ich funksiyasi $\Phi(x)$ uchun $\Phi(x)=F(x)+C.$ $(C=const)$ bo'ladi.

***Eslatma.*** $(a,b)$ da berilgan har qanday funksiya ham boshlang'ich funksiyaga ega bo'lavermaydi.

**547.** $(-1,1)$ intervalda berilgan funksiyaning boshlang'ich funksiyaga ega emasligini ko'rsating:

$$f(x)=\begin{cases} -1, \ agar \ -1<x<0 \ bo'lsa, \\ 0, \ agar \ x=0 \ bo'lsa, \\ 1, \ agar \ 0<x<1 \ bo'lsa \end{cases}.$$

**Yechish.** Teskarisini faraz qilaylik, ya'ni berilgan funksiya $(-1,1)$ da boshlang'ich funksiya $F(x)$ ga ega bo'lsin: $F'(x)=f(x)$ $(x\in(-1,1))$. Ravshanki, $F'(0)=f(0)=0$ bo'ladi. Bu $F(x)$ funksiyaga $[0,x]$ segmentda $(0<x<1)$ Lagranj teoremasini qo'llab topamiz: $F(x)-F(0)=F'(c)\cdot x=f(c)\cdot x=x$ $(c\in(0,x))$. Keyingi tenglikdan

$$\frac{F(x)-F(0)}{x}=1, \quad \lim_{x\to+0}\frac{F(x)-F(0)}{x}=1$$

bo'lib, $F'(+0)=1$ bo'lishi kelib chiqadi. Bu esa $F'(0)=f(0)=0$ munosabatga ziddir. Demak, qaralayotgan $f(x)$ funksiya $(-1,1)$ da boshlang'ich funksiyaga ega bo'lmaydi.

**2-teorema.** Agar $f(x)\in C(a,b)$ bo'lsa, u holda $f(x)$ funksiya $(a,b)$ da boshlang'ich funksiyaga ega bo'ladi.

**Funksiyaning aniqmas integrali. Integralning xossalari.** Aytaylik, $(a,b)$ da $f(x)$ funksiya berilgan bo'lib, $F(x)$ funksiya uning biror boshlang'ich funksiyasi bo'lsin: $F'(x)=f(x)$ $(x\in(a,b))$.

**3-ta'rif.** Ushbu $F(x)+C$ $(x\in(a,b))$ ifoda $f(x)$ funksiyaning **aniqmas integrali** deyiladi va $\int f(x)dx$ kabi belgilanadi. Bunda $\int$ - integral belgisi, $f(x)$ integral ostidagi funksiya, $f(x)dx$ integral ostidagi ifoda deyiladi.

Demak, $\int f(x)dx=F(x)+C$ $(C=const)$

**548.** Ushbu $\int x^3dx$ aniqmas integral topilsin.

**Yechish.** Aniqmas integral ta'rifiga ko'ra, shunday $F(x)$ funksiya topilishi kerakki, $F'(x)=x^3$ bo'lsin. Agar $F(x)=\frac{1}{4}x^4$ deyilsa, ravshanki, $F'(x)=x^3$ bo'ladi. Demak, $\int x^3dx=\frac{1}{4}x^4+C$ $(C=const)$.

Endi aniqmas integralning xossalarini keltiramiz. Bundan buyon aniqmas integral haqida gap borganda uni qaralayotgan oraliqda mavjud deb, ya'ni integral ostidagi funksiya qaralayotgan oraliqda boshlang‘ich funksiyaga ega deb qaraymiz va oraliqni ko‘rsatib o‘tirmaymiz.

Aniqmas integralning xossalari.

1) $d(\int f(x)dx) = f(x)dx$;

2) $\int dF(x) = F(x) + C \quad (C = const)$;

3) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$;

4) $\int kf(x)dx = k\int f(x)dx$, bunda $k$ o‘zgarmas son va $k \neq 0$.

**549.** Ushbu aniqmas integralni toping: $I = \int (\frac{5}{1+x^2} - 3\sin x)dx$.

**Yechish.** Aniq integralning 3) va 4) xossalaridan foydalansak, unda $\int (\frac{5}{1+x^2} - 3\sin x)dx = 5\int \frac{1}{1+x^2}dx - 3\int \sin xdx$ bo‘lishi kelib chiqadi.

Endi $(-\cos x)' = \sin x$, $(\text{arctg}x)' = \frac{1}{1+x^2}$ bo‘lishini e’tiborga olib topamiz:

$5\int \frac{1}{1+x^2}dx - 3\int \sin xdx = 5\text{arctg}x + 3\cos x + C$.

Demak, $I = 5\text{arctg}x + 3\cos x + C$.

## Asosiy aniqmas integrallar jadvali

Elementar funksiyalarning hosilalari jadvali hamda aniqmas integral ta’rifidan foydalanib, sodda funksiyalarning aniqmas integrallari topiladi. Ularni jamlab, jadval ko‘rinishiga keltiramiz:

1) $\int 0 \cdot dx = C, \quad C = const.$

2) $\int 1 \cdot dx = x + C.$

3) $\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad (\alpha \neq -1).$

4) $\int \frac{dx}{x} = \ln|x| + C, \quad (x \neq 0).$

5) $\int a^x dx = \frac{a^x}{\ln a} + C, \quad (a > 0,\ a \neq 1).$

$\int e^x dx = e^x + C.$

6) $\int \sin xdx = -\cos x + C.$

7) $\int \cos xdx = \sin x + C.$

$$8)\ \int \frac{dx}{\cos^2 x} = \mathrm{tg}x + C, \qquad (x \neq \frac{\pi}{2} + \pi n,\ n \in Z).$$

$$9)\ \int \frac{dx}{\sin^2 x} = -\mathrm{ctg}x + C, \qquad (x \neq \pi n,\ n \in Z).$$

$$10)\ \int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C, \\ -\arccos x + C. \end{cases} \qquad (-1 < x < 1).$$

$$11)\ \int \frac{dx}{\sqrt{1+x^2}} = \begin{cases} \mathrm{arctg}x + C, \\ -\mathrm{arcctg}x + C. \end{cases}$$

$$12)\ \int shxdx = chx + C.$$

$$13)\ \int chxdx = shx + C.$$

$$14)\ \int \frac{dx}{sh^2 x} = -cthx + C\ .$$

$$15)\ \int \frac{dx}{ch^2 x} = thx + C\ .$$

## Aniqmas integralni integrallash usullari. O‘zgaruvchini almashtirib integrallash usuli.

Faraz qilaylik, $f(x)$ funksiyaning aniqmas integrali

$$\int f(x)dx \qquad (1)$$

berilgan bo‘lib, uni hisoblash talab etilsin.

Ko‘pincha, o‘zgaruvchi $x$ ni ma’lum qoidaga ko‘ra boshqa o‘zgaruvchiga almashtirish natijasida berilgan integral sodda integralga keladi va uni hisoblash oson bo‘ladi.

Aytaylik, (1) integraldagi o‘zgaruvchi $x$ yangi o‘zgaruvchi $t$ bilan ushbu

$$t = \varphi(x)$$

munosabatda bo‘lib, quyidagi shartlar bajarilsin:

1) $\varphi(x)$ funksiya differensiallanuvchi bo‘lsin;

2) $g(t)$ funksiya boshlang‘ich funksiya $G(t)$ ga ega, ya`ni

$$G'(t) = g(t), \qquad \int g(t)dt = G(t) + C; \qquad (2)$$

3) $f(x)$ funksiya quyidagicha

$$f(x) = g(\varphi(x)) \cdot \varphi'((x) \qquad (3)$$

ifodalansin. U holda

$$\int f(x)dx = \int g(\varphi(x))\varphi'(x)dx = G(\varphi(x)) + C \qquad (4)$$

bo'ladi. Shu yo'l bilan (1) integralni hisoblash o'zgaruvchini almashtirib integrallash usuli deyiladi.

Bu usulda, o'zgaruvchini juda ko'p munosabat bilan almashtirish imkoniyati bo'lgan holda ular orasidan qaralayotgan integralni sodda, hisoblash uchun qulay holga keltiradiganini tanlab olish muhimdir.

**550.** Berilgan aniqmas integralni toping: $I = \int \frac{dx}{e^x + e^{-x}}$.

**Yechish.** Avvalo berilgan integralni quyidagicha

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x} + 1}$$

yozib olamiz. Bu integralni o'zgaruvchini almashtirish usulidan foydalanib hisoblaymiz:

$$I = \int \frac{e^x dx}{e^{2x} + 1} = \left| \begin{matrix} e^x = t \\ e^x dx = dt \end{matrix} \right| = \int \frac{dt}{1 + t^2} = arctgt + C = arctge^x + C$$

**551.** Berilgan aniqmas integralni toping:

$$I = \int \frac{dx}{\sqrt{x^2 + a}} \quad (a \neq 0,\ a \in R).$$

**Yechish.** Integralda o'zgaruvchini quyidagicha almashtiramiz: $x + \sqrt{x^2 + a} = t$. Unda $dt = d(x + \sqrt{x^2 + a}) = (1 + \frac{x}{\sqrt{x^2 + a}})dx =$ $= \frac{\sqrt{x^2 + a} + x}{\sqrt{x^2 + a}} dx = \frac{t}{\sqrt{x^2 + a}} dx$ bo'lib, undan $\frac{dx}{\sqrt{x^2 + a}} = \frac{dt}{t}$ bo'lishi kelib chiqadi. Natijada $I = \int \frac{dt}{t} = \ln|t| + C = \ln\left|x + \sqrt{x^2 + a}\right| + C$ bo'lishini topamiz.

## Bo'laklab integrallash usuli

Faraz qilaylik, $u(x)$ va $v(x)$ funksiyalar uzluksiz $u'(x)$, $v'(x)$ hosilalarga ega bo'lsin. U holda

$$\int u(x) \cdot dv(x) = u(x) \cdot v(x) - \int v(x) du(x) \qquad (5)$$

ham yozish mumkin. (5) formula bo'laklab integrallash formulasi deyiladi. Uning yordamida $\int u(x)\cdot v'(x)dx$ integralni hisoblash $\int u'(x)\cdot v(x)dx$ integralni hisoblashga keltiriladi.

**552.** Ushbu aniqmas integralni toping: $\int x\cos x dx$.

**Yechish.** Bo'laklab integrallash formulasidan foydalanib topamiz:

$$\int x\cos x dx = \left| \begin{matrix} u = x, & du = dx \\ \cos x dx = dv & v = \sin x \end{matrix} \right| = x\sin x - \int \sin x dx =$$

$$= x\sin x + \cos x + C.$$

**553.** Ushbu aniqmas integralni toping:

$$I_n = \int \frac{dx}{(x^2+a^2)^n} \quad (n\in N,\ a\in R,\ a\neq 0).$$

**Yechish.** Bu integralda $u = \frac{1}{(x^2+a^2)^n}$, $dv = dx$ deb olsak, u holda $du = -\frac{2nxdx}{(x^2+a^2)^{n+1}}$, $v = x$ bo'ladi. (5) formuladan foydalanib topamiz:

$$I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx =$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\left[\int \frac{dx}{(x^2+a^2)^n} - a^2\int \frac{dx}{(x^2+a^2)^{n+1}}\right].$$

Natijada $I_n = \frac{x}{(x^2+a^2)^n} + 2n\cdot I_n - 2na^2\cdot I_{n+1}$ bo'ladi. Bu tenglikdan

$$I_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n}\frac{1}{a^2}\cdot I_n \qquad (6)$$

bo'lishi kelib chiqadi. Odatda, (6) munosabat rekkurent formula deyiladi.

Ravshanki, $n=1$ bo'lganda

$$I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a}arctg\frac{x}{a} + C$$

bo'ladi. $n=2$ da,

$$I_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\frac{x}{x^2+a^2} + \frac{1}{2a^2}\cdot J_1 = \frac{1}{2a^2}\frac{x}{(x^2+a^2)} + \frac{1}{2a^3}\mathrm{arctg}\frac{x}{a} + C$$

bo'ladi.

**554.** Aniqmas integralning xossalari va aniqmas integrallar jadvali yordamida quyidagilarni integrallang:

1. $\int\left(x^4-2x^3-6x^2+8x+7\right)dx$; 2. $\int\frac{(x-1)(x^3-1)}{x^2}dx$;

3. $\int\frac{x^4}{x^2+1}dx$; 4. $\int\left(\frac{3}{\sqrt{x}}-\frac{x\sqrt{x}}{3}\right)dx$;

5. $\int e^x\left(5+\frac{3e^{-x}}{x^4}\right)dx$; 6. $\int 2^x3^{2x}5^{3x}dx$;

7. $\int\cos^2\frac{x}{2}dx$; 8. $\int tg^2xdx$;

9. $\int\frac{1+\cos^2 x}{1+\cos 2x}dx$; 10. $\int\frac{\cos 2x}{4\cos^2 x\sin^2 x}dx$.

**555.** O'zgaruvchini almashtirib integrallash usuli yordamida quyidagilarni integrallang:

1. $\int(5-2x)^6dx$; 2. $\int x^3\left(1-2x^4\right)^3dx$;

3. $\int\frac{1}{\sqrt{2-7x}}dx$; 4. $\int x\sqrt{1-x}dx$;

5. $\int\frac{1}{1+\sqrt[3]{x+1}}dx$; 6. $\int\frac{2-\sqrt{x+1}}{2+\sqrt[3]{x+1}}dx$;

7. $\int x^2e^{x^3-1}dx$; 8. $\int\frac{1}{x\ln x\ln(\ln x)}dx$;

9. $\int\frac{dx}{tg2x}$; 10*. $\int\frac{arctg\sqrt{x}}{(1+x)\sqrt{x}}dx$.

**556.** Bo'laklab integrallash usuli yordamida quyidagilarni integrallang:

1. $\int xe^xdx$; 2. $\int e^{\sqrt{x}}dx$;

3. $\int x^3e^{x^2}dx$; 4. $\int\left(e^x+x\right)^3dx$;

5. $\int e^{2x}\sin 5xdx$; 6. $\int xe^x\sin xdx$;

7. $\int\ln(x+2)dx$; 8. $\int\log_2(1-2x)dx$;

9. $\int x^2\sin 7xdx$; 10*. $\int\cos\sqrt{x}dx$.

## 32 §. Kasr-ratsional funksiyalarni integrallash

**Sodda kasr-ratsional funksiyalarni integrallash**

Ushbu

$$\frac{A}{(x-a)^m}\ (x\neq a)\ ,\ \frac{Bx+C}{(x^2+px+q)^m}$$

ko'rinishdagi funksiyalar sodda kasr-ratsional funksiyalar deyiladi, bunda $m\in N$; $A,B,C,a,p,q$ – haqiqiy sonlar bo'lib, $x^2+px+q$ kvadrat uchhad haqiqiy ildizga ega emas, ya'ni $q-\frac{p^2}{4}>0$.

$m=1$ bo'lganda sodda kasrlarning integrallari

$$\int\frac{A}{x-a}dx\ ,\ \int\frac{Bx+C}{x^2+px+q}dx$$

lar quyidagicha hisoblanadi:

$$\int\frac{A}{x-a}dx=A\int\frac{d(x-a)}{x-a}=A\ln|x-a|+C\ ;$$

$$\int\frac{Bx+C}{x^2+px+q}dx=\int\frac{Bx+C}{(x+\frac{p}{2})^2+(q-\frac{p^2}{4})}dx=$$

$$=\left|\begin{matrix} x+\frac{p}{2}=t, & x=t-\frac{p}{2} \\ dx=dt, & q-\frac{p^2}{4}=a^2\end{matrix}\right|=$$

$$=B\int\frac{tdt}{t^2+a^2}+(C-\frac{Bp}{2})\int\frac{dt}{t^2+a^2}=$$

$$=\frac{B}{2}\ln(t^2+a^2)+(C-\frac{Bp}{2})\frac{1}{a}\text{arctg}\frac{t}{a}+C_1=$$

$$=\frac{B}{2}\ln(x^2+px+q)+\frac{2C-Bp}{2\sqrt{q-\frac{p^2}{4}}}\text{arctg}\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C_1.$$

**557.** Integrallang: $\int\frac{x+3}{x^2-8x+25}dx$.

**Yechish.** Maxrajdagi kvadrat uch haddan to'la kvadrat ajratamiz: $x^2-8x+25=(x-4)^2+9$ hamda $x-4=t,\ dx=dt$ almashtirish kiritib, quyidagini hosil qilamiz:

$$\int\frac{x+3}{x^2-8x+25}dx=\int\frac{t+7}{t^2+9}dt=\frac{1}{2}\int\frac{2t}{t^2+9}dt+7\int\frac{1}{t^2+3^2}dt=$$

$$=\frac{1}{2}\ln|t^2+9|+\frac{7}{3}\operatorname{arctg}\frac{t}{3}+C=\frac{1}{2}\ln|x^2-8x+25|+\frac{7}{3}\operatorname{arctg}\frac{x-4}{3}+C$$

Aytaylik, $m\in N, m>1$ bo'lsin. Bu holda sodda kasrlarning integrallari

$$\int\frac{A}{(x-a)^m}dx,\quad \int\frac{Bx+C}{(x^2+px+q)^m}dx$$

lar quyidagicha hisoblanadi:

$$\int\frac{A}{(x-a)^m}dx=A\int(x-a)^{-m}d(x-a)=-\frac{A}{(m-1)(x-a)^{m-1}}+C,$$

$$\int\frac{Bx+C}{(x^2+px+q)^m}dx=\left|\begin{matrix} x+\frac{p}{2}=t, & x=t-\frac{p}{2} \\ dx=dt, & q-\frac{p^2}{4}=a^2\end{matrix}\right|=$$

$$=\frac{B}{2}\int\frac{2tdt}{(t^2+a^2)^m}+(C-\frac{p}{2}B)\int\frac{dt}{(t^2+a^2)^m}=$$

$$=-\frac{B}{2}\frac{1}{(m-1)(t^2+a^2)^{m-1}}+(C-\frac{p}{2}B)\int\frac{dt}{(t^2+a^2)^m}.$$

Keyingi munosabatdagi $\int\frac{dt}{(t^2+a^2)^m}$ integral avvalgi mavzudagi, (6) rekkurent formula yordamida topiladi.

**Kasr ratsional funksiyalarni sodda kasrlarga yoyib integrallash**

Bizga $\frac{P_n(x)}{Q_m(x)}$, (n < m) kasr ratsional funksiya berilgan bo'lsin. Bunda, $P_n(x), Q_m(x)$ mos ravishda $n,m$ darajali ko'phadlar. Agar $n\geq m$ bo'lsa, u holda $\frac{P_n(x)}{Q_m(x)}=P_{n-m}(x)+\frac{P_s(x)}{Q_m(x)}$, (s < m) ko'rinishida yozib olishimiz mumkin. Ko'phadning integrallash sodda bo'lgani uchun to'g'ri kasr – ratsional funksiyani integrallash masalasiga kelamiz.

Faraz qilaylik

$$Q_m(x)=(x-a_1)^{\alpha_1}(x-a_2)^{\alpha_2}\cdot\ldots\cdot(x-a_l)^{\alpha_l}\cdot$$

$$\cdot(x^2+p_1x+q_1)^{\beta_1}(x^2+p_2x+q_2)^{\beta_2}\cdot\ldots\cdot(x^2+p_rx+q_r)^{\beta_r}$$

bo'lsin. Bunda $\alpha_1+\alpha_2+\ldots+\alpha_l+2(\beta_1+\beta_2+\ldots+\beta_r)=m$ va

$D = p_i^2 - 4q_i < 0, i = \overline{1,r}$. U holda quyidagicha

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_{11}}{x-a_1} + \frac{A_{12}}{(x-a_1)^2} + \ldots + \frac{A_{1\alpha_1}}{(x-a_1)^{\alpha_1}} + \frac{A_{21}}{x-a_2} + \frac{A_{22}}{(x-a_2)^2} + \ldots + \frac{A_{2\alpha_2}}{(x-a_2)^{\alpha_2}} + \ldots +$$

$$+ \frac{A_{l1}}{x-a_l} + \frac{A_{l2}}{(x-a_l)^2} + \ldots + \frac{A_{l\alpha_l}}{(x-a_l)^{\alpha_l}} + \frac{M_{11}x + N_{11}}{x^2 + p_1x + q_1} + \frac{M_{12}x + N_{12}}{(x^2 + p_1x + q_1)^2} + \ldots +$$

$$+ \frac{M_{1\beta_1}x + N_{1\beta_1}}{(x^2 + p_1x + q_1)^{\beta_1}} + \frac{M_{21}x + N_{21}}{x^2 + p_2x + q_2} + \frac{M_{22}x + N_{22}}{(x^2 + p_2x + q_2)^2} + \ldots + \frac{M_{2\beta_2}x + N_{2\beta_2}}{(x^2 + p_2x + q_2)^{\beta_2}} + \ldots +$$

$$+ \frac{M_{r1}x + N_{r1}}{x^2 + p_rx + q_r} + \frac{M_{r2}x + N_{r2}}{(x^2 + p_rx + q_r)^2} + \ldots + \frac{M_{r\beta_r}x + N_{r\beta_r}}{(x^2 + p_rx + q_r)^{\beta_r}}$$

ko‘rinishda sodda kasrlarga yoyish mumkin bo‘ladi. Demak, berilgan kasr-ratsional funksiyani integrallash, bir nechta sodda kasr ratsional funksiyalarni integrallash masalasiga keltirib yechilar ekan.

**558.** Ushbu $\dfrac{3x^2+8}{x^3+4x^2+4x}$ to‘g‘ri kasrni sodda kasrlarga yoying.

**Yechish.** Bu kasrning maxraji

$$x^3 + 4x^2 + 4x = x(x^2 + 4x + 4) = x(x+2)^2$$

bo‘lgani uchun $\dfrac{3x^2+8}{x^3+4x^2+4x} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$ bo‘ladi. Uni $\dfrac{3x^2+8}{x^3+4x^2+4x} = \dfrac{A(x+2)^2 + x(x+2)B + Cx}{x(x+2)^2}$ ko‘rinishda yozib, ushbu

$$3x^2 + 8 = A(x+2)^2 + Bx(x+2) + Cx =$$
$$= (A+B)x^2 + (4A + 2B + C)x + 4A$$

tenglikka kelamiz. Ikki ko‘phadning tengligidan foydalanib, ushbu

$$\begin{cases} A + B = 3 \\ 4A + 2B + C = 0 \\ 4A = 8 \end{cases}$$

sistemani hosil qilamiz va uni yechib $A = 2$, $B = 1$, $C = -10$ bo‘lishini topamiz. Demak, $\dfrac{3x^2+8}{x^3+4x^2+4x} = \dfrac{2}{x} + \dfrac{1}{x+2} + \dfrac{-10}{(x+2)^2}$.

**559.** Ushbu kasr-ratsional funksiyani integrallang:

$$\int \frac{3x^2+8}{x^3+4x^2+4x}dx$$

**Yechish.** Integral ostidagi ratsional funksiyani sodda kasrlarga yoyamiz (avvalgi misolga qarang):

$$\frac{3x^2+8}{x^3+4x^2+4x}=\frac{2}{x}+\frac{1}{x+2}-\frac{10}{(x+2)^2}.$$

Demak, $\int\frac{3x^2+8}{x^3+4x^2+4x}dx=2\int\frac{dx}{x}+\int\frac{dx}{x+2}-10\int\frac{dx}{(x+2)^2}=$

$$=2\ln|x|+\ln|x+2|+\frac{10}{x+2}+C.$$

**560.** Ushbu kasr-ratsional funksiyani integrallang:

$$\int\frac{x^6+2x^4+2x^2-1}{x(x^2+1)^2}dx$$

**Yechish.** Integral ostidagi funksiya kasr-ratsional funksiya bo‘lib, u noto‘g‘ri kasrdir. Bu kasrning surati $x^6+2x^4+2x^2-1$ ko‘phadni maxraji $x(x^2+1)^2$ ko‘phadga bo‘lib, uning butun qismini ajratamiz:

$$\begin{array}{r|l} x^6+2x^4+2x^2-1 & x^5+2x^3+x \\ \underline{-\;x^6+2x^4+x^2\qquad} & \overline{\quad x \quad} \\ x^2-1 & \end{array}$$

Demak, $\frac{x^6+2x^4+2x^2-1}{x(x^2+1)^2}=x+\frac{x^2-1}{x(x^2+1)^2}.$

Endi $\frac{x^2-1}{x(x^2+1)^2}$ to‘g‘ri kasrni sodda kasrlarga yoyamiz:

$$\frac{x^2-1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2},$$

$$x^2-1=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x=$$

$$=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A.$$

Keyingi tenglikdan $A=-1,\ B=1,\quad C=0,\quad D=2,\quad E=0$

bo‘lishini topamiz. Demak, $\frac{x^2-1}{x(x^2+1)^2}=\frac{-1}{x}+\frac{x}{x^2+1}+\frac{2x}{(x^2+1)^2}.$

Natijada, $\frac{x^6+2x^4+2x^2-1}{x(x^2+1)^2}=x-\frac{1}{x}+\frac{x}{x^2+1}+\frac{2x}{(x^2+1)^2}$

bo‘lib, $\int\frac{x^6+2x^4+2x^2-1}{x(x^2+1)^2}=\int xdx-\int\frac{dx}{x}+\int\frac{x}{x^2+1}dx+$

$$+\int\frac{2x}{(x^2+1)^2}dx=\frac{x^2}{2}-\ln|x|+\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}+\int\frac{d(x^2+1)}{(x^2+1)^2}=$$

$$=\frac{x^2}{2}-\ln|x|+\frac{1}{2}\ln(x^2+1)-\frac{1}{x^2+1}+C$$

bo'ladi.

**561.** Quyidagi sodda kasr-ratsional funksiyalarni integrallang.

1) $\int\frac{dx}{x+7}$; 2) $\int\frac{3}{1-2x}dx$; 3) $\int\frac{dx}{x^2+121}$;

4) $\int\frac{dx}{3x^2+11}$; 5) $\int\frac{dx}{(x-5)^2}$; 6) $\int\frac{5dx}{(3-2x)^7}$;

7) $\int\frac{dx}{(x^2+4)^2}$; 8) $\int\frac{dx}{2x^2-3x+4}$; 9) $\int\frac{dx}{x^2-x+1.5}$;

10) $\int\frac{x+6}{x^2+2x+5}dx$; 11) $\int\frac{x+5}{2x^2+2x+3}dx$; 12) $\int\frac{1-3x}{x^2-4x+8}dx$;

13) $\int\frac{2x+1}{(x^2+2x+5)^2}dx$; 14) $\int\frac{dx}{(x^2+6)^3}$; 15*) $\int\frac{dx}{(x^2-6x+10)^3}$.

**562.** Quyidagi kasr-ratsional funksiyalarni sodda kasrlarga yoyib integrallang.

1) $\int\frac{dx}{27-3x^2}$; 2) $\int\frac{1}{x^2+7x}dx$; 3) $\int\frac{dx}{x^2-x-6}$;

4) $\int\frac{2x+7}{2-x-x^2}dx$; 5) $\int\frac{1-2x}{(x-2)(1-x)}dx$; 6) $\int\frac{dx}{x^3-64}$;

7) $\int\frac{dx}{1-x^4}$; 8) $\int\frac{dx}{x^3-3x+2}$; 9*) $\int\frac{dx}{x^4+2x^3-13x^2-14x+24}$;

10) $\int\frac{4-x}{x^3-2x^2+x-2}dx$; 11) $\int\frac{x^3+x^2+x+3}{x^2+x+1}dx$; 12) $\int\frac{11x-9}{(x+1)(x+3)^2}dx$;

13) $\int\frac{x+1}{(x^2+x+1)(x^2+1)}dx$; 14) $\int\frac{x^2dx}{x^6+2x^3+3}$; 15*) $\int\frac{3x^4+4}{2x^2(x^2+1)^3}dx$.

## 33 §. Trigonometrik funksiyalarni integrallash

Ushbu

$$\int R(\sin x,\cos x)\,dx \tag{7}$$

integralni qaraymiz. Bunda $R(\sin x,\cos x)$ funksiya $\sin x$ va $\cos x$ ga nisbatan kasr-ratsional funksiya.

Bu integralda $t=tg\frac{x}{2}$ almashtirishni bajaramiz. Unda

$$\sin x=\frac{2\text{tg}\frac{x}{2}}{1+\text{tg}^2\frac{x}{2}}=\frac{2t}{1+t^2},\quad \cos x=\frac{1-\text{tg}^2\frac{x}{2}}{1+\text{tg}^2\frac{x}{2}}=\frac{1-t^2}{1+t^2},$$

$$x=2\,\text{arctg}\,t,\quad dx=\frac{2dt}{1+t^2}$$

bo'lib, $\int R(\sin x,\cos x)\,dx=2\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{1}{1+t^2}dt$ bo'ladi.

Ravshanki, $R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{1}{1+t^2}$ ifoda $t$ ning ratsional funksiyasidir.

Demak, (7) integralni hisoblash $t=\text{tg}\frac{x}{2}$ almashtirish bilan ratsional funksiyani integrallashga keladi.

**563.** Trigonometrik funksiyani integrallang: $\int\frac{dx}{1+\sin x}$.

**Yechish.** Bu integralda $t=\text{tg}\frac{x}{2}$ almashtirish bajarib topamiz:

$$\int\frac{dx}{1+\sin x}=\int\frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}}=2\int\frac{dt}{(1+t)^2}=-\frac{2}{1+t}=-\frac{2}{1+\text{tg}\frac{x}{2}}+C.$$

Ayrim hollarda $t=\cos x$, $t=\sin x$, $t=\text{tg}x$ almashtirishlar qulay bo'ladi.

Aytaylik, $\int R(\sin x,\cos x)dx$ integralni topish talab etilsin.

1) $R(-\sin x,\cos x)=-R(\sin x,\cos x)$ bo'lsa, $t=\cos x$

2) $R(\sin x,-\cos x)=-R(\sin x,\cos x)$ bo'lsa, $t=\sin x$

3) $R(-\sin x,-\cos x)=R(\sin x,\cos x)$ bo'lsa, $t=tgx$

almashtirishlar bajarib oson integrallash mumkin.

**564.** Trigonometrik funksiyani integrallang: $\int\sin^3 x\cos^4 x dx$.

**Yechish.** Integral ostidagi funksiya uchun $R(-\sin x,\cos x)=-R(\sin x,\cos x)$ bo'ladi. Shuning uchun $\cos x=t$ deyilsa, unda $-\sin x dx=dt$ bo'lib,

$$\int \sin^3 x\cos^4 x dx=\int (t^2-1)t^4 dt=\frac{t^7}{7}-\frac{t^5}{5}+C=\frac{1}{7}\cos^7 x-\frac{1}{5}\cos^5 x+C$$

bo'ladi.

Agar integral

$$\int \sin mx\cos nx\, dx, \int \cos mx\cos nx\, dx, \int \sin mx\sin nx\, dx$$

ko'rinishda bo'lsa ,

a) $\sin x\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ ,

b) $\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$,

c) $\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

mos formuladan foydalanib hisoblanadi.

**565.** Trigonometrik funksiyani integrallang: $\int \sin x\sin 3x\sin 5x dx$.

**Yechish.** Yuqoridagi formulalardan foydalanib quyidagicha shakl almashtirishlar bajaramiz:

$$\sin 2x\sin 3x\sin 5x=\sin 2x\cdot\frac{1}{2}\cdot(\cos 4x-\cos 6x)=\frac{1}{2}\sin 2x\cos 4x-\frac{1}{2}\sin 2x\cos 6x=$$

$$=\frac{1}{4}(\sin 6x-\sin 2x-\sin 8x+\sin 4x)$$

Natijada, $\int \sin x\sin 3x\sin 5x dx=\frac{1}{8}\left(\frac{1}{4}\cos 8x+\cos 2x-\frac{1}{3}\cos 6x-\frac{1}{2}\cos 4x\right)+C$

hosil bo'ladi.

Agar integral $\int R(tgx)dx$ yoki $\int R(\text{ctg}x)dx$ ko'rinishda berilgan bo'lsa, mos ravishda $t=tgx$ yoki $t=ctgx$ almashtirishlar yordamida integrallanadi.

Agar integral $\int\frac{1}{\cos^{2n+1}x}dx$ va $\int\frac{1}{\sin^{2n+1}x}dx$ ko'rinishida bo'lsa bu integrallar quyidagi formullalar orqali hisoblanadi:

$$\int\frac{1}{\cos^{2n+1}x}dx=\frac{1}{2n}\cdot\frac{\sin x}{\cos^{2n}x}+\left(1+\frac{1}{2n}\right)\int\frac{1}{\cos^{2n-1}x}dx \qquad (8)$$

$$\int\frac{1}{\sin^{2n+1}x}dx=\frac{1}{2n}\cdot\frac{\cos x}{\sin^{2n}x}+\left(1+\frac{1}{2n}\right)\int\frac{1}{\sin^{2n-1}x}dx\cdot \qquad (9)$$

Bo'laklab integrallash orqali hosil qilinadigan «tartibini pasaytirish» formulasi orqali

$$\int \sin^n x dx=-\frac{1}{n}\cos x\sin^{n-1}x+\frac{n-1}{n}\int \sin^{n-2}x dx \qquad (10)$$

$$\int \cos^n x dx=\frac{1}{n}\sin x\cos^{n-1}x+\frac{n-1}{n}\int \cos^{n-2}x dx \qquad (11)$$

(10) va (11) ko'rinishidagi integrallarni integrallash mumkin.

**566.** Trigonometrik funksiyani integrallang: $\int \sin^6 x dx$.

**Yechish.** (10) formula orqali integrallaymiz:

$$\int \sin^6 x dx = -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\int \sin^4 x dx = -\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}\left(-\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x dx\right) =$$

$$= -\frac{1}{6}\cos x \sin^5 x - \frac{5}{24}\cos x \sin^3 x + \frac{5}{8}\left(-\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right) + C =$$

$$= -\frac{1}{6}\cos x \sin^5 x - \frac{5}{24}\cos x \sin^3 x - \frac{5}{16}\cos x \sin x + \frac{5}{16}x + C.$$

**567.** Trigonometrik funksiyani integrallang.

1) $\int \frac{dx}{17+15\cos x}$; 2) $\int \frac{dx}{5-\cos x}$; 3) $\int \frac{dx}{2-3\sin x}$;

4) $\int \frac{dx}{7-5\sin x+6\cos x}$; 5) $\int \frac{1-\sin x}{1+\sin x} dx$; 6*) $\int \frac{dx}{2-\sqrt{3}\sin x+\cos x}$;

7) $\int \frac{dx}{\sin^5 x \cos x}$; 8) $\int \cos^5 \frac{x}{2} dx$; 9) $\int \frac{\cos^5 3x}{\sin^3 x} dx$;

10) $\int \frac{\cos^3 x dx}{1+\cos^2 x}$; 11) $\int \sin^4 x \cos^2 x dx$; 12) $\int \sin 3x \cos 8x dx$;

13) $\int \sin x \sin \frac{x}{2} \sin \frac{x}{3} dx$; 14) $\int ctg^6 x dx$; 15) $\int tg^4 \frac{x}{6} dx$.

## 34 §. Ba'zi irratsional funksiyalarni integrallash

**$\int R(x, \sqrt[n]{\frac{ax+b}{cx+d}})dx$ ko'rinishidagi integrallarni hisoblash.**

Ushbu

$$\int R(x, \sqrt[n]{\frac{ax+b}{cx+d}})dx \quad , \quad ad-bc \neq 0, \qquad (12)$$

ko'rinishidagi integrallarni qaraymiz. Bu integral o'zgaruvchini almashtirish yordamida ratsional funksiyaning integraliga keladi:

$$\int R(x, \sqrt[n]{\frac{ax+b}{cx+d}})dx = \left| \begin{array}{l} \sqrt[n]{\frac{ax+b}{cx+d}} = t, x = \frac{b-t^n d}{ct^n - a} \\ dx = \frac{(ad-bc)n}{(a-ct^n)^2} t^{n-1} dt \end{array} \right| =$$

$$= \int R\left(\frac{dt^n - b}{a-ct^n}, t\right) \cdot \frac{(ad-bc)nt^{n-1}}{(a-ct^n)^2} dt .$$

(12) integralning umumlashmasi bo'lgan quyidagi:

$$\int R\left(x, \sqrt[n_1]{\left(\frac{ax+b}{cx+d}\right)^{m_1}}, \ldots, \sqrt[n_k]{\left(\frac{ax+b}{cx+d}\right)^{m_k}}\right)dx$$

integralda $\frac{ax+b}{cx+d} = t^{EKUK(n_1,\ldots,n_k)}$ almashtirish ratsional funksiyaning integraliga olib keladi.

**568.** Ushbu $\int \sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{1-x} dx$ integral toping.

**Yechish.** Bu integralda $t = \sqrt{\frac{1+x}{1-x}}$ almashtirishni bajaramiz. Unda

$x = \frac{t^2-1}{t^2+1}$, $dx = \frac{4tdt}{(t^2+1)^2}$ bo'lib, $\int \sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{1-x} dx = 2\int \frac{t^2 dt}{t^2+1}$

bo'ladi. Ravshanki, $\int \frac{t^2 dt}{t^2+1} = t - \text{arctg}t + C$.

Demak, $\int \sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{1-x} dx = 2\sqrt{\frac{1+x}{1-x}} - 2\text{arctg}\sqrt{\frac{1+x}{1-x}} + C$.

$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$ **ko'rinishidagi integrallarni hisoblash.**

Bizdan

$$\int R(x, \sqrt{ax^2+bx+c})dx \qquad (13)$$

ko'rinishidagi integralni integrallash talab etilsin. Bunda integraldagi $a, b, c$ -haqiqiy sonlar bo'lib, $a \neq 0, D = b^2 - 4ac \neq 0$ bo'lsin.

1) $ax^2+bx+c$ **kvadrat uchhad ildizga ega bo'lmasin.** Ma'lumki, bu holda $ax^2+bx+c$ kvadrat uchhadning ishorasi $a$ ning ishorasi bilan bir xil. Shuning uchun $a > 0$ **bo'ladi** va qaralayotgan integral quyidagi almashtirish yordamida ratsional funksiya integraliga keladi.

(13) integralda ushbu

$$t = \sqrt{a}x + \sqrt{ax^2+bx+c} \qquad (\text{yoki } t = -\sqrt{a}x + \sqrt{ax^2+bx+c})$$

almashtirishni bajaramiz. U holda

$$ax^2+bx+c = t^2 - 2\sqrt{a}\,xt + ax^2,$$

$$x = \frac{t^2-c}{2\sqrt{a}t+b}, \quad dx = \frac{2(\sqrt{a}t^2+bt+c\sqrt{a})}{(2\sqrt{a}t+b)^2}dt.$$

$$\sqrt{ax^2+bx+c}=\frac{\sqrt{a}t^2+bt+c\sqrt{a}}{2\sqrt{a}t+b}$$

bo‘ladi. Natijada

$$\int R(x,\sqrt{ax^2+bx+c})dx=$$

$$=\int R\left(\frac{t^2-c}{2\sqrt{a}t+b},\frac{\sqrt{a}t^2+bt+c\sqrt{a}}{2\sqrt{a}t+b}\right)\cdot\frac{2(\sqrt{a}t^2+bt+c\sqrt{a})}{(2\sqrt{a}t+b)^2}dt$$

bo‘ladi.

**569.** Ushbu $\int\frac{dx}{x+\sqrt{x^2+x+1}}$ integral hisoblansin.

**Yechish.** Bu integralda $t=x+\sqrt{x^2+x+1}$ almashtirishni bajaramiz. Natijada $x=\frac{t^2-1}{1+2t}$, $dx=2\frac{t^2+t+1}{(1+2t)^2}dt$

bo‘lib, $\int\frac{dx}{x+\sqrt{x^2+x+1}}=2\int\frac{t^2+t+1}{(1+2t)^2t}dt$ bo‘ladi.

Agar $\frac{2(t^2+t+1)}{t(1+2t)^2}=\frac{2}{t}-\frac{3}{1+2t}-\frac{3}{(1+2t)^2}$

bo‘lishini e’tiborga olsak, unda

$$\int\frac{dx}{x+\sqrt{x^2+x+1}}=\int\left(\frac{2}{t}-\frac{3}{1+2t}-\frac{3}{(1+2t)^2}\right)dt=$$

$$=2\ln|t|-\frac{3}{2}\ln|1+2t|+\frac{3}{2(1+2t)}+C=$$

$$=2\ln\left|x+\sqrt{x^2+x+1}\right|-\frac{3}{2}\ln\left|1+2x+2\sqrt{x^2+x+1}\right|+$$

$$+\frac{3}{2(1+2x+2\sqrt{x^2+x+1})}+C$$

bo‘lishi kelib chiqadi.

**2)** $ax^2+bx+c$ **kvadrat uchhad turli** $x_1$ **va** $x_2$ **haqiqiy ildizga ega bo‘lsin:**

$$ax^2+bx+c=a(x-x_1)\cdot(x-x_2)\,.$$

Bu holda (1) integralda ushbu $t=\frac{1}{x-x_1}\sqrt{ax^2+bx+c}$ almashtirishni bajaramiz. Natijada

$$x=\frac{-ax_2+x_1t^2}{t^2-a}\,,\quad \sqrt{ax^2+bx+c}=\frac{a(x_1-x_2)}{t^2-a}t\,,\quad dx=\frac{2a(x_1-x_2)t}{(t^2-a)^2}dt$$

bo'lib, $\int R(x, \sqrt{ax^2+bx+c})dx =$

$$= \int R\left(\frac{-ax_2 + x_1 t^2}{t^2 - a}, \frac{a(x_1 - x_2)}{t^2 - a} t\right) \cdot \frac{2a(x_1 - x_2)t}{(t^2 - a)^2} dt$$

bo'ladi.

**570.** Ushbu $I = \int \frac{x - \sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}} dx$ integral hisoblansin.

**Yechish.** Ravshanki, $x^2+3x+2=(x+1)\cdot(x+2)$. Shuni e'tiborga olib berilgan integralda $t = \frac{1}{x+1}\sqrt{x^2+3x+2}$ almashtirishni bajaramiz. U holda

$$x = \frac{2-t^2}{t^2-1}, \quad dx = -\frac{2tdt}{(t^2-1)^2} \text{ bo'lib,}$$

$$\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}} dx = \int \frac{-2t^2-4t}{(t-2)\cdot(t-1)\cdot(t+1)^3} dt \text{ bo'ladi.}$$

Endi $\frac{-2t^2-4t}{(t-2)\cdot(t-1)\cdot(t+1)^3} = \frac{\frac{3}{4}}{t-1} - \frac{\frac{16}{27}}{t-2} - \frac{\frac{17}{108}}{t+1} + \frac{\frac{5}{18}}{(t+1)^2} + \frac{\frac{1}{3}}{(t+1)^3}$

bo'lishini e'tiborga olib topamiz:

$$I = \int \frac{-2t^2-4t}{(t-2)\cdot(t-1)\cdot(t+1)^3} dt = \frac{3}{4}\int \frac{dt}{t-1} - \frac{16}{27}\int \frac{dt}{t-2} -$$

$$-\frac{17}{108}\int \frac{dt}{t+1} + \frac{5}{18}\int \frac{dt}{(t+1)^2} + \frac{1}{3}\int \frac{dt}{(t+1)^3} = \frac{3}{4}\ln|t-1| -$$

$$-\frac{16}{27}\ln|t-2| - \frac{17}{108}\ln|t+1| - \frac{5}{18}\cdot\frac{1}{t+1} - \frac{1}{6}\cdot\frac{1}{(t+1)^2} + C.$$

**Binomial funksiyalarni integrallash.**

Quyidagi binomial ko'rinishidagi funksiyaning integralini qaraylik:

$$\int x^m (a+bx^n)^p dx. \qquad (14)$$

Bunda $a \in R$, $b \in R$, $m, n, p$ – ratsional sonlar. Bu integral quyidagi hollarda ratsional funksiyaning integraliga keladi:

**1)** $p$**-butun son.** Bu holda $m$ va $n$ ratsional sonlar maxrajlarining eng kichik umumiy karralisini $\delta$ orqali belgilab, (14) integralda

$$x = t^{\delta}$$

almashtirish bajarilsa, (14) integral ratsional funksiyaning integraliga keladi.

**571.** Ushbu $I=\int\frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx$ integral hisoblansin.

**Yechish.** Bu integralni quyidagicha

$$\int\frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx=\int x^{\frac{1}{2}}(1+x^{\frac{1}{3}})^{-2}dx$$

yozib, bunda $p=-2$ bo'lishini aniqlaymiz. Integralda $x=t^6$ almashtirish bajarib $I=6\int\frac{t^8}{(1+t^2)^2}dt$ bo'lishini topamiz.

Ravshanki, $\frac{t^8}{(1+t^2)^2}=t^4-2t^2+3-\frac{4}{t^2+1}+\frac{1}{(t^2+1)^2}$.

Demak, $\int\frac{t^8}{(1+t^2)^2}dt=\frac{t^5}{5}-\frac{2t^3}{3}+3t-4\operatorname{arctg}t+\frac{1}{2}\cdot\frac{t}{t^2+1}+\frac{1}{2}\operatorname{arctg}t+C$

bo'lib,

$$I=\frac{6}{5}\sqrt[6]{x^5}-4\sqrt{x}+18\sqrt[6]{x}-21\operatorname{arctg}\sqrt[6]{x}+\frac{3\sqrt[6]{x}}{\sqrt[3]{x}+1}+C$$ bo'ladi.

2) $\frac{m+1}{n}$ **- butun son.** Bu holda (14) integralda

$x=t^{\frac{1}{n}}$ almashtirishni bajarib $\int x^m(a+bx^n)^p dx=\frac{1}{n}\int(a+bt)^p\cdot t^q\,dt$ bo'lishini topamiz, bunda $q=\frac{m+1}{n}-1$. So'ng $p$ ning maxrajini $s$ deb $z=(a+bt)^{\frac{1}{s}}$ almashtirishni bajaramiz. Natijada (14) integral ratsional funksiyaning integraliga keladi.

**572.** Ushbu $\int\frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$ integralni hisoblang.

**Yechish.** Bu integralda

$$\int\frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}=\int x(1+x^{\frac{2}{3}})^{-\frac{1}{2}}dx,\ m=1,\ n=\frac{2}{3},\ p=-\frac{1}{2}$$

bo'lib, $\frac{m+1}{n}=3$ bo'ladi. Shuni e'tiborga olib, berilgan integralda, $t=(1+x^{\frac{2}{3}})^{\frac{1}{2}}$

almashtirishni bajaramiz. Unda

$$1+x^{\frac{2}{3}}=t^2\ ,\ x=(t^2-1)^{\frac{3}{2}}\ ,\quad dx=\frac{3}{2}(t^2-1)^{\frac{1}{2}}\cdot 2tdt$$

bo‘lib, $\int x(1+x^{\frac{2}{3}})^{\frac{1}{2}}dx=3\int(t^2-1)^2t^2dt=3\frac{t^7}{7}-6\frac{t^5}{5}+t^3+C,\ \ t=\sqrt{1+x^{\frac{2}{3}}}$ bo‘ladi.

3) $\frac{m+1}{n}+p$ - **butun son.** Ma’lumki, (14) integral $x=t^{\frac{1}{n}}$ almashtirish bilan ushbu

$$\frac{1}{n}\int(a+bt)^p\cdot t^{\frac{m+1}{n}-1}dt=\frac{1}{n}\int(a+bt)^p\cdot t^q dt=\frac{1}{n}\int\left(\frac{a+bt}{t}\right)^p\cdot t^{p+q}dt$$

ko‘rinishga keladi. Agar keyingi integralda $z=\left(\frac{a+bt}{t}\right)^{\frac{1}{s}}$ almashtirish bajarilsa ($s$ soni $p$ ning maxraji), u ratsional funksiyaning integraliga keladi.

**573.** Ushbu $\int\frac{dx}{x^2\sqrt{2+3x^2}}$ integral hisoblansin.

**Yechish.** Ravshanki. $\int\frac{dx}{x^2\sqrt{2+3x^2}}=\int x^{-2}(2+3x^2)^{-\frac{1}{2}}dx$.

Demak, $m=-2, n=2\ , p=-\frac{1}{2}\ ,\ \ \frac{m+1}{n}+p=-1$

bo‘lib, $\frac{m+1}{n}+p$-butun son bo‘ladi. Berilgan integralda $t=\left(\frac{2+3x^2}{x^2}\right)^{\frac{1}{2}}=\sqrt{\frac{2}{x^2}+3}$ almashtirish bajarib,

$$x=\frac{\sqrt{2}}{\sqrt{t^2-3}}\quad ,\quad dx=-\frac{\sqrt{2}tdt}{\sqrt{(t^2-3)^3}}$$

$$\int\frac{dx}{x^2\cdot\sqrt{2+3x^2}}=\int x^{-2}(2+3x^2)^{-\frac{1}{2}}dx=$$

$$=\int(-\frac{dt}{2})=-\frac{t}{2}+C=-\frac{\sqrt{\frac{2}{x^2}+3}}{2}+C$$

bo‘lishini topamiz.

Ba'zan yuqoridagi almashtirishlar ancha qiyinchilik tug'dirishi mumkin. Ba'zi xususiy hollarda quyidagi usul va almashtirishlar irratsional funksiyalarni integrallashga qulaylik tug'diradi.

1) Agar integral $\int \frac{(Mx+N)dx}{(x-d)\sqrt{ax^2+bx+c}}$ ko'rinishda bo'lsa, $x-d=\frac{1}{t}$ almashtirish yordamida hisoblanadi;

2) Agar integral $\int R\left(x,\sqrt{a^2-x^2}\right)dx$ ko'rinishda bo'lsa, $x=a\sin t$ yoki $x=a\cos t$ almashtirish yordamida hisoblanadi;

3) Agar integral $\int R\left(x,\sqrt{x^2-a^2}\right)dx$ ko'rinishda bo'lsa, $x=\frac{a}{\sin t}$ yoki $x=\frac{a}{\cos t}$ almashtirish yordamida hisoblanadi;

4) Agar integral $\int R\left(x,\sqrt{a^2+x^2}\right)dx$ ko'rinishda bo'lsa, $x=atgt$ yoki $x=actgt$ almashtirish yordamida hisoblanadi;

5) $\int \frac{a_0x^m+a_1x^{m-1}+...+a_m}{\sqrt{ax^2+bx+c}}dx$ ko'rinishdagi integraldan ushbu

$$\int \frac{a_0x^m+...+a_m}{\sqrt{ax^2+bx+c}}dx=\left(A_0x^{m-1}+...+A_{m-1}\right)\sqrt{ax^2+bx+c}+A_m\int\frac{dx}{\sqrt{ax^2+bx+c}} \qquad (15)$$

formula asosida algebraik qismini ajratish mumkin. Bunda $A_1, A_2, ..., A_m$ lar noma'lum koeffitsiyentlar bo'lib (15) tenglikni ikki tomonini differensiallab va maxrajdan qutqargandan so'ng chap va o'ng tomonidagi bir xil darajali $x$ lar oldidagi koeffitsiyentlarni tenglashtirib topiladi.

**574.** $\int \frac{dx}{x\sqrt{5x^2-2x+1}}$ integralni hisoblang.

**Yechish.** $x=\frac{1}{t}$ almashtirish bajarib hisoblaymiz:

$$\int\frac{dx}{x\sqrt{5x^2-2x+1}}=\left|\begin{matrix}x=\frac{1}{t}\\ dx=-\frac{dt}{t^2}\end{matrix}\right|=-\int\frac{\frac{dt}{t^2}}{\frac{1}{t}\cdot\sqrt{\frac{5}{t^2}-\frac{2}{t}+1}}=-\int\frac{dt}{\sqrt{t^2-2t+5}}=$$

$$=-\ln\left|t-1+\sqrt{t^2-2t+5}\right|+C=\left|t=\frac{1}{x}\right|=-\ln\left|\frac{1}{x}-1+\sqrt{\frac{1}{x^2}-\frac{2}{x}+5}\right|+C$$

**575.** $\int\sqrt{a^2-x^2}dx$ integralni hisoblang.

**Yechish.** $x = a\sin t$ almashtirish bajarib hisoblaymiz:

$$\int\sqrt{a^2-x^2}dx = \left|\begin{matrix} x = a\sin t \\ dx = a\cos tdt \end{matrix}\right| = \int\sqrt{a^2-a^2\sin^2 t}\cdot a\cos tdt = \int\sqrt{a^2\cos^2 t}\cdot a\cos tdt =$$

$$= a^2\int\cos^2 tdt = a^2\int\frac{1+\cos 2t}{2}dt = \frac{a^2}{2}\int dt + \frac{a^2}{2}\int\cos 2tdt = \frac{a^2}{2}t + \frac{a^2}{4}\sin 2t + C =$$

$$= \left|\begin{matrix} t = \arcsin\frac{x}{a} \\ \sin 2t = 2\sin t\cdot\cos t = \frac{2x}{a}\sqrt{1-\frac{x^2}{a^2}} \end{matrix}\right| = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{4}\cdot\frac{2x}{a}\sqrt{1-\frac{x^2}{a^2}} + C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C.$$

**576.** $\int\frac{x^3+2x^2+3x+4}{\sqrt{x^2+2x+2}}dx$ integralni hisoblang.

**Yechish.** (4) formula yordamida hisoblaymiz:

$$\int\frac{x^3+2x^2+3x+4}{\sqrt{x^2+2x+2}} = \left(A_0x^2 + A_1x + A_2\right)\sqrt{x^2+2x+2} + A_3\int\frac{dx}{\sqrt{x^2+2x+2}}$$

Tenglikning har ikkala tomonini differensiallasak, quyidagi ko‘rinishga keladi:

$$\frac{x^3+2x^2+3x+4}{\sqrt{x^2+2x+2}} =$$

$$= (2A_0x + A_1)\sqrt{x^2+2x+2} + \left(A_0x^2 + A_1x + A_2\right)\cdot\frac{x+1}{\sqrt{x^2+2x+2}} + \frac{A_3}{\sqrt{x^2+2x+2}}$$

bunda,

$$x^3+2x^2+3x+4 = (2A_0x + A_1)(x^2+2x+2) + (A_0x^2 + A_1x + A_2)(x+1) + A_3$$

$$x^3+2x^2+3x+4 = 3A_0x^3 + (5A_0 + 2A_1)x^2 + (4A_0 + 3A_1 + A_2)x + (2A_1 + A_2 + A_3)$$

$$\begin{cases} 3A_0 = 1 \\ 5A_0 + 2A_1 = 2 \\ 4A_0 + 3A_1 + A_2 = 3 \\ 2A_1 + A_2 + A_3 = 4 \end{cases}$$ bu sistemani yechib, $A_0 = \frac{1}{3}, A_1 = \frac{1}{6}, A_2 = \frac{7}{6}, A_3 = \frac{5}{2}$ ni

olamiz. Natijada quyidagiga ega bo‘lamiz :

$$\int\frac{x^3+2x^2+3x+4}{\sqrt{x^2+2x+2}}dx =$$

$$= \left(\frac{1}{3}x^2 + \frac{1}{6}x + \frac{7}{6}\right)\sqrt{x^2+2x+2} + \frac{5}{2}\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C.$$

**577.** Quyidagi irratsional funksiyalarni integrallang.;

1. $\int \frac{dx}{\sqrt{x^2+13x+41}}$; 2. $\int \frac{dx}{\sqrt{x^2-4x+8}}$; 3. $\int \frac{dx}{\sqrt{-3x^2-2x}}$;

4. $\int \frac{dx}{\sqrt{-x^2+2x+8}}$; 5. $\int \frac{3x+13}{\sqrt{1-x^2}}dx$; 6. $\int \frac{\sqrt[3]{2x-5}}{1+\sqrt[3]{2x-5}}dx$;

7. $\int \frac{\sqrt{x-7}}{1+\sqrt[4]{(x-7)^3}}dx$; 8. $\int \sqrt[3]{\frac{x+7}{x+5}} \cdot \frac{1}{(x+5)^3}dx$; 9. $\int \sqrt{2-x^2}dx$;

10. $\int \frac{1}{x\sqrt{3x-x^2}}dx$; 11. $\int \frac{(x-1)^3}{\sqrt{x^2-2x+3}}dx$; 12. $\int \frac{3x-16}{\sqrt{x^2-11x+32}}dx$;

13. $\int \frac{1}{x^3\sqrt[3]{2-x^3}}dx$; 14*. $\int \sqrt[3]{x}\sqrt{5x\sqrt[3]{x}+3}dx$; 15. $\int \frac{1}{1+\sqrt{x^2+1}}dx$.

## 35 §. Aniq integral. Aniq integralni integrallash usullari.

$f(x)$ funksiya biror $[a,b] \subset R$ kesmada aniqlangan va shu kesmada chegaralangan bo'lsin. Ushbu

$$a = x_0 < x_1 < x_2 < ..x_{n-1} < x_n = b$$

munosabatda bo'lgan $x_0, x_1, x_2,..x_{n-1}, x_n$ nuqtalar to'plami $[a,b]$ kesmani **bo'laklash** deyiladi va $P = \{x_0, x_1, x_2,..x_{n-1}, x_n\}$ kabi belgilanadi.

Bunda har bir $x_k$ $(k=1,2,...,n)$ nuqta $[a,b]$ kesmaning bo'luvchi nuqtasi, $[x_{k-1}, x_k]$ $(k=1,2,...,n)$ kesma esa $P$ bo'laklashning oralig'i deyiladi.

Quyidagi $\lambda_p = \max\{\Delta x_k\}$, $\Delta x_k = x_k - x_{k-1}$ miqdor $P$ bo'laklashning diametri deyiladi. Har bir bo'laklash orag'idan ixtiyoriy $\xi_k \in [x_k, x_{k+1}]$ nuqtalar tanlaymiz va quyidagi

$$\sum_{k=1}^{n} f(\xi_k)\cdot\Delta x_k = f(\xi_1)\cdot\Delta x_1 + f(\xi_2)\cdot\Delta x_2 + ... + f(\xi_n)\cdot\Delta x_n \qquad (16)$$

**integral yig'indi** yoki **Riman yig'indisi** deb ataluvchi (16) yig'indini qaraylik.

Agar

$$\lim_{\lambda_p \to 0} \sum_{k=1}^{n} f(\xi_k)\cdot\Delta x_k \qquad (17)$$

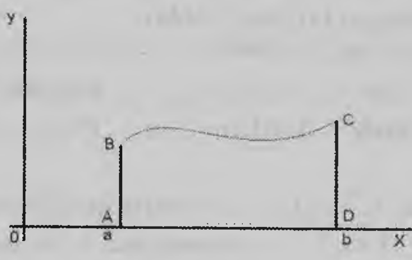
limit mavjud, chekli, $P$ bo'laklash va $\xi_k$ ni tanlanishiga bog'liq bo'lmasa, $f(x)$ funksiya $[a,b]$ kesmada Riman ma'nosida **integrallanuvchi** deyiladi. Limitning qiymati esa $f(x)$ funksiya $[a,b]$ kesmadagi **aniq integrali** deyiladi va $\int_a^b f(x)dx$ kabi belgilanadi. Demak,

$$\int_a^b f(x)dx := \lim_{\lambda_p \to 0} \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k \ . \qquad (18)$$

Bunda $a$ va $b$ aniq inregralning mos ravishda **quyi** va **yuqori** chegaralari deyiladi.

**Aniq integralning geomertik ma'nosi.**

Agar $\forall x \in [a,b],\ f(x) \geq 0$ bo'lsa, aniq integralning qiymati quyidagi 1-shakldagi $x = a, x = b. y = 0, y = f(x)$ chiziqlar bilan chegrelangan egri chiziqli trapetsiyaning yuziga teng.




**1-shakl**

## Aniq integralning asosiy xossalari

Agar integral ostidagi funksiyalarni integrallari mavjud bo'lib, $a < b$ bo'lsa, quyidagi xossalar o'rinli:

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ ;

2. $\int_a^a f(x)dx = 0$ ;

3. $\int_a^b f(x)dx = \int_b^c f(x)dx + \int_c^b f(x)dx$;

4. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;

5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;

6. Agar $\forall x \in (a,b)$, $f(x) \geq 0$ ($f(x) \leq 0$) bo'lsa, $\int_a^b f(x)dx \geq 0$, $\left(\int_a^b f(x)dx \leq 0\right)$ bo'ladi;

7. Agar $\forall x \in (a,b)$, $f(x) \geq g(x)$ bo'lsa, $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ bo'ladi;

8. Agar $\forall x \in [a,b]$, $m \leq f(x) \leq M$ bo'lsa, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$ baho o'rinli bo'ladi;

9. **O'rta qiymat haqidagi teorema.** Agar $f(x) \in C[a,b]$ bo'lsa, shunday $\xi \in [a,b]$ topilib, $\int_a^b f(x)dx = f(\xi)(b-a)$ tenglik o'rinli bo'ladi;

10. Agar $f(x)$ uzluksiz funksiya va $\Phi(x) = \int_a^x f(t)dt$ bo'lsa, u holda $\Phi'(x) = f(x)$ bo'ladi;

11. **Nyuton-Leybnits formulasi.** Agar $\int f(x)dx = F(x)$ bo'lsa, u holda $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$ bo'ladi.

**578.** Ta'rif yordamida quyidagi aniq integralni hisoblang: $\int_0^2 x^2 dx$.

**Yechish.** $f(x) = x^2$, $a = 0$, $b = 2$, berilgan kesmani teng $n$ bo'lakka bo'lsak, u holda $\Delta x_k = \frac{b-a}{n} = \frac{2}{n}$ bo'ladi. $\xi_k = x_k$ deb tanlasak, $x_k = \frac{2k}{n}$, $f(x_k) = \frac{4k^2}{n^2}$, $k = 0,1,...,n$ bo'ladi.

Demak,

$$\int_0^2 x^2dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{4k^2}{n^2}\cdot\frac{2}{n} = \lim_{n\to\infty}\frac{8}{n^3}\sum_{k=1}^{n}k^2 = \lim_{n\to\infty}\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} =$$

$$= \frac{4}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{8}{3}.$$

**579.** Aniq integralni baholang: $\int_5^{13}\frac{\cos x}{\sqrt{1+x^4}}dx$.

**Yechish.** $|\cos x|\le 1,\ x\ge 5\ \left|\frac{1}{\sqrt{1+x^4}}\right| < \frac{1}{x^2} \le 5^{-2}$ tengsizliklardan va 8-xossadan $\left|\int_5^{13}\frac{\cos x}{\sqrt{1+x^4}}dx\right| < 8\cdot 5^{-2} = 0.32$ ekani kelib chiqadi.

**580.** Aniq integralni hisoblang: $\int_3^6 e^{\frac{x}{3}}dx$.

**Yechish.** Nyuton-Leybnits formulasidan

$\int_3^6 e^{\frac{x}{3}}dx = 3e^{\frac{x}{3}}\Big|_3^6 = 3e^2 - 3e = 3e(e-1)$ ekanini topamiz.

**581.** $y = \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$ funksiyaning $(1;4)$ intervaldagi o‘rtacha qiymatini toping.

**Yechish.** O‘rta qiymat haqidagi teoremaga ko‘ra

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{3}\int_1^4\left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)dx = \frac{1}{3}\left(\frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}}\right)\Bigg|_1^4 = \sqrt[3]{4} + \sqrt[3]{2} - \frac{3}{4}$$

ekanini topamiz.

**O‘zgaruvchilarini almashtirish formulasi**

Aytaylik, aniq integralda $x$ o‘zgaruvchi ushbu

$$x = \varphi(t)$$

formula bilan almashtirilgan bo‘lib, bunda $\varphi(t)$ funksiya quyidagi shartlarni bajarsin:

1) $\varphi(t)\in C[\alpha,\beta]$ bo‘lib, $\varphi(t)$ funksiyaning barcha qiymatlari $[a,b]$ ga tegishli;

2) $\varphi(\alpha) = a,\quad \varphi(\beta) = b$;

3) $\varphi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hosilaga ega bo‘lsin. U holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \qquad (19)$$

bo‘ladi.

**582.** Ushbu $\int_0^1 \sqrt{1-x^2}dx$ integral hisoblansin.

**Yechish.** Berilgan integralda $x=\sin t$ almashtirishni bajaramiz. Unda

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cos tdt = \int_0^{\frac{\pi}{2}} \cos^2 tdt =$$

$$= \int_0^{\frac{\pi}{2}} (\frac{1}{2}+\frac{1}{2}\cos 2t)dt = (\frac{1}{2}t+\frac{1}{4}\sin 2t)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

bo‘ladi.

**Bo‘laklab integrallash formulasi**

Aytaylik , $u(x)$ va $v(x)$ funksiyalarning har biri $[a,b]$ segmentda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo‘lsin. U holda

$$\int_a^b u(x)dv(x) = (u(x)\cdot v(x))\Big|_a^b - \int_a^b v(x)du(x) \qquad (20)$$

bo‘ladi.

Agar $f(x)$ - toq funksiya bo‘lsa, u holda $\int_{-a}^{a} f(x)dx = 0$;

Agar $f(x)$ - juft funksiya bo‘lsa, u holda $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$;

Agar $f(x)$, T - davrli davriy funksiya bo‘lsa, u holda $\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$ bo‘ladi.

**583.** Ushbu $\int_1^2 x\ln xdx$ integral hisoblansin.

**Yechish.** Bu intervalda $u(x)=\ln x, dv(x)=x$ deb $du(x)=\frac{1}{x}dx, v(x)=\frac{x^2}{2}$ bo‘lishini topamiz. Unda (20) formulaga ko‘ra:

$\int_1^2 x\ln xdx = (\frac{x^2}{2}\ln x)\Big|_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}dx = 2\ln 2 - \frac{1}{2}\int_1^2 xdx = 2\ln 2 - \frac{3}{4}$ bo‘ladi.

**584*.** Ushbu $\int_0^{12} x(x-2)(x-4)(x-6)(x-8)(x-10)(x-12)dx$ integral hisoblansin.

**Yechish.** O'zgaruvchini almashtirish usulidan foydalanib hisoblaymiz:

$$\int_0^{12} x(x-2)(x-4)(x-6)(x-8)(x-10)(x-12)dx = \left| \begin{array}{l} x-6=t \\ x=t+6, dx=dt \\ x=0, t=-6 \\ x=12, t=6 \end{array} \right| =$$

$$= \int_{-6}^{6} (t+6)(t+4)(t+2)t(t-2)(t-4)(t-6)dt = \int_{-6}^{6} t(t^2-2^2)(t^2-4^2)(t^2-6^2)dt = 0.$$

Oxirgi integralda, chegarasi 0 ga simmetrik bo'lgan oraliqdagi toq funksiyaning integrali 0 ekanidan foydalanildi.

**585.** Aniq integralni ta'rif yordamida hisoblang:

1) $\int_0^1 xdx$; 2) $\int_0^1 x^2dx$; 3) $\int_0^1 e^x dx$.

**586.** Quyidagi aniq integrallarni baholang:

1) $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{x}dx$; 2) $\int_1^2 \sqrt{8+x^3}dx$; 3) $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{5+2\sin x}}dx$.

**587.** Quyidagi funksiyalarning berilgan oraliqdagi o'rtacha qiymatini toping:

1) $f(x)=x^2, [4;5]$; 2) $f(x)=x^3+2x-1, [0;1]$;

3) $f(x)=5-2\sin x+3\cos x, \left[\frac{\pi}{2};\pi\right]$.

**588.** Quyidagi aniq integrallarni hisoblang:

1) $\int_1^6 \left(7-x-\frac{6}{x}\right)dx$; 2) $\int_1^4 \frac{\left(1+\sqrt{x}\right)^3}{\sqrt{x}}dx$; 3) $\int_1^2 \frac{1}{x^2+x}dx$;

4) $\int_0^1 e^{x+e^x}dx$; 5) $\int_0^1 \frac{1}{e^x+e^{-x}}dx$; 6) $\int_1^e \frac{1}{x(1+\ln^2 x)}dx$;

7) $\int_0^1 (x+1)\sqrt{1-x}dx$; 8) $\int_3^6 \frac{\sqrt{x^2-9}}{x}dx$; 9) $\int_0^{\pi} \frac{dx}{2-\sin\frac{x}{2}}$;

10) $\int_{0}^{0.5} xe^{-2x}dx$; 11) $\int_{0}^{2} x^3e^xdx$; 12) $\int_{-\pi}^{\pi} e^{\frac{x}{2}}\cos x dx$;

13) $\int_{2}^{2e^{\frac{\pi}{2}}} \cos\ln\frac{x}{2}dx$; 14) $\int_{0}^{\frac{\pi}{4}} x^2\sin x dx$; 15) $\int_{0}^{1.5} \arcsin\frac{x}{3}dx$.

## 36 §. Aniq integralni taqribiy hisoblash

**To'g'ri to'rtburchaklar formulasi.** $\int_a^b f(x)dx$ aniq integralni taqribiy hisoblash talab etilsin. $[a,b]$ kesmani

$$a = x_0 < x_1 < x_2 < \dots x_{n-1} < x_n = b$$

munosabatda bo'lgan $n$ ta qismga bo'laylik.

$$x_k = a + k\frac{b-a}{n},\quad x_{k+\frac{1}{2}} = \frac{x_k + x_{k+1}}{2} = a + (k+\frac{1}{2})\frac{b-a}{n},\quad x_{k+1} - x_k = \frac{b-a}{n} \text{ va}$$

$y_k = f(x_k),\ k = \overline{0,n}$ bo'lsin. U holda

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{k=1}^{n-1} f\left(\frac{x_k + x_{k+1}}{2}\right) = \frac{b-a}{n}\sum_{k=1}^{n-1} f\left(x_{k+\frac{1}{2}}\right) \qquad (21)$$

(21) formula **to'g'ri to'rtburchaklar** formulasi o'rinli. Bu taqribiy hisoblash xatoligi quyidagi

$R_n = \frac{(b-a)^3}{24n^2} f''(\zeta) \quad (\zeta \in (a,b))$ formula bilan ifodalanadi.

**Trapetsiyalar formulasi.**

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\left[\frac{f(x_0) + f(x_n)}{2} + f(x_1) + f(x_2) + \dots + f(x_{n-1})\right]. \qquad (22)$$

(22) formula trapetsiyalar formulasi deyiladi.

Bu taqribiy formulaning xatoligi $R_n'$, $f(x)$ funksiya $[a,b]$ da uzluksiz $f''(x)$ hosilaga ega bo'lishi shartida ,

$$R_n' = -\frac{(b-a)^3}{12n^2} f''(\zeta) \qquad (\zeta \in (a,b))$$

bo'ladi.

**Simpson formulasi**

$$\int_a^b f(x)dx \approx \frac{b-a}{6n}[f(x_0) + f(x_{2n}) + 4(f(x_1) + f(x_3) + \dots$$

$$\dots + f(x_{2n-1})) + 2(f(x_2) + f(x_4) + \dots + f(x_{2n-2}))]. \qquad (23)$$

(3) formula Simpson formulasi deyiladi.

Bu taqribiy formulaning xatoligi $R_n''$, $f(x)$ funksiya $[a,b]$ da uzluksiz $f^{(iv)}(x)$ hosilaga ega bo‘lishi shartida,

$$R_n''=-\frac{(b-a)^5}{2880n^4}f^{(iv)}(\zeta) \quad (\zeta\in(a,b))$$

bo‘ladi.

**589.** Trapetsiya formulasi orqali aniq integralni taqribiy qiymati hisoblansin: $\int_0^1\frac{dx}{2+x}$, $n=5$.

**Yechish.** $f(x)=\frac{1}{2+x}$, $\frac{b-a}{n}=0.2$.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_k$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
| $y_k$ | $\frac{1}{2}$ | $\frac{1}{2,2}$ | $\frac{1}{2,4}$ | $\frac{1}{2,6}$ | $\frac{1}{2,8}$ | $\frac{1}{3}$ |

$$\int_0^1\frac{dx}{2+x}\approx 0,2\left(\frac{0,5+0,333}{2}+0,4545+0,4167+0.3846+0,3571\right)\approx 0,2\cdot 2,0296\approx 0,4059$$

Endi xatolikni baholaymiz.

$$f'(x)=-\frac{1}{(2+x)^2},\ f''(x)=\frac{2}{(2+x)^3},\ f'''(x)=-\frac{6}{(2+x)^4}.$$

$f''(x)$ funksiya $[0,1]$ da statsionar nuqtalarga ega emas. Demak,

$$\max_{[0,1]}|f''(x)|=\frac{2}{(2+x)^3}\bigg|_{x=0}=\frac{1}{4}\cdot|R_5|\le\frac{1}{12\cdot 25}\cdot\frac{1}{4}\approx 0,00083<0,001.$$

**590.** Aniq integralni 0.01 aniqlikda taqribiy hisoblang.

1) $\int_0^3\frac{dx}{x+2}$ ;  2) $\int_1^2\frac{dx}{x}$ ;

3) $\int_0^{1.2}e^x dx$ ;  4) $\int_0^2 e^{-x^2}dx$.

**591*.** Aniq integralni trapetsiya formulasi yordamida $n=10$ uchun taqribiy hisoblang.

1) $\int_0^1\frac{dx}{x+1}$;  2) $\int_0^1\frac{dx}{x^2+1}$;

$3)\int_0^1 \frac{1}{x^3+1}dx;$ $4)\int_0^1 \frac{1}{x^3+4}dx.$

## 37 §. Xosmas integrallar

Aniq integralning ta'rifida integrallash chegaralari chekli va integral ostidagi funksiya $[a,b]$ oraliqda chegaralangan deb olingan edi. Bu shartlardan hech bo'lmaganda birortasi bajarilmasa, integralning aniq integral ta'rifi o'z ma'nosini yo'qotadi. Biroq nazariy va amaliy mulohazalarga muvofiq aniq integralning ta'rifi bu cheklanishlar bajarilmaydigan hollar uchun ham umumlashtirilishi mumkin. Bunday integrallar bizga tanish bo'lgan aniq integrallarga xos bo'lmagan qisqacha **xosmas integrallar** deb aytiladi.

Xosmas integrallarning ikki asosiy turini qaraymiz.

**1. Uzluksiz funksiyalarning cheksiz oraliq bo'yicha xosmas integrallari.** $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan va uning istalgan qismi $[a,+A]$ da integrallanuvchi, ya'ni istalgan $A>a$ da aniq integral mavjud bo'lsin. Bu holda $\lim\limits_{A\to\infty}\int_a^A f(x)dx=I$ limitga $f(x)$ funksiyaning $[a,\infty)$ oraliqdagi **xosmas integrali** deyiladi va quyidagicha belgilanadi: $I=\int_a^\infty f(x)dx:=\lim\limits_{A\to\infty}\int_a^A f(x)dx$.

limit chekli bo'lsa, **xosmas integral yaqinlashuvchi** deyiladi. Limit mavjud bo'lmasa yoki cheksiz bo'lsa, **xosmas integral uzoqlashuvchi** deyiladi.

$f(x)$ funksiyadan $(-\infty,a]$ oraliq bo'yicha olingan xosmas integral ham xuddi yuqoridagiga o'xshash aniqlanadi:

$$\int_{-\infty}^a f(x)dx=\lim_{A\to-\infty}\int_A^a f(x)dx.$$

$f(x)$ funksiyadan $(-\infty,+\infty)$ oraliq bo'yicha olingan xosmas integral quyidagicha aniqlanadi: $\int_{-\infty}^\infty f(x)dx=\int_{-\infty}^a f(x)dx+\int_a^\infty f(x)dx$, bu yerda $a$ istalgan son. Oxirgi integrallarda o'ng tomondagi ikkala integral ham yaqinlashsa chap tomondagi integral ham yaqinlashuvchi deyiladi. O‘ng tomondagi integrallardan aqalli bittasi uzoqlashsa, chap tomondagi integral ham uzoqlashuvchi bo‘ladi.

**592.** $\int_1^{\infty} \frac{dx}{(1+x^2)}$ integralning yaqinlashishini tekshiring.

**Yechish.** $\int_1^{\infty} \frac{dx}{1+x^2} = arctgx \Big|_1^{\infty} = \lim_{x\to\infty} arctgx - arctg1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. Demak, integral yaqinlashuvchi va $\frac{\pi}{4}$ ga teng.

**2. Chegaralanmagan funksiyalarning chekli oraliq bo‘yicha xosmas integrallari.** $(a,b]$ intervalda uzluksiz va $x=a$ da aniqlanmagan yoki uzilishga ega bo‘lgan $f(x)$ funksiyaning xosmas integrali quyidagicha belgilanib aniqlanadi: $\int_a^b f(x)dx := \lim_{\varepsilon\to 0} \int_{a+\varepsilon}^b f(x)dx$.

Oxirgi limit mavjud bo‘lsa, xosmas integral yaqinlashuvchi aks holda uzoqlashuvchi deyiladi. Bunday integrallarga **2-tur xosmas integral** deyiladi.

Integral ostidagi $f(x)$ funksiya uchun $F(x)$ boshlang‘ich funsiya ma’lum bo‘lsa, Nyuton - Leybnits formulasini qo‘llash mumkin:

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0} F(x)\Big|_{a+\varepsilon}^{b} = \lim_{\varepsilon\to 0}[F(b) - F(a+\varepsilon)]$$

Shunday qilib. $x\to a$ da $F(x)$ boshlang‘ich funksiyaning limiti mavjud bo‘lsa, xosmas integral yaqinlashuvchi, mavjud bo‘lmasa, xosmas integral uzoqlashuvchi bo‘ladi. $[a.b)$ intervalda $x=b$ nuqtada uzilishga ega bo‘lgan $f(x)$ funksiya xosmas integrali ham shunga o‘xshash bo‘ladi, ya’ni

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0} \int_a^{b-\varepsilon} f(x)dx = \lim_{x\to 0} F(x) \Big|_a^{b-\varepsilon} = \lim_{\varepsilon\to 0}[F(b-\varepsilon) - F(a)].$$

$f(x)$ funksiya $[a,b]$ kesmaning biror $x=c$ nuqtasida uzilishga ega bo‘lsa xosmas integral quyidagicha aniqlanadi: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. O‘ng tomondagi integrallardan aqalli bittasi uzoqlashuvchi bo‘lsa, xosmas integral uzoqlashuvchidir. O‘ng tomondagi ikkala integral ham yaqinlashuvchi bo‘lsa, chap tomondagi xosmas integral yaqinlashuvchi bo‘ladi.

**593.** $\int_0^4 \frac{dx}{\sqrt{x}}$ integralning yaqinlashuvchiligini tekshiring.

**Yechish.** $x\to 0$ *da* $f(x)=\frac{1}{\sqrt{x}}\to\infty$ demak,

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{4}\frac{1}{\sqrt{x}}dx=\lim_{\varepsilon\to 0}2\sqrt{x}\Big|_{\varepsilon}^{4}=\lim_{\varepsilon\to 0}\left[2\sqrt{4}-2\sqrt{\varepsilon}\right]=2\cdot 2=4.$$

Demak, $\int_0^4\frac{dx}{\sqrt{x}}$ integral yaqinlashuvchi.

**594.** $\int_{-1}^{8}\frac{dx}{\sqrt[3]{x^2}}$ integralning yaqinlashuvchiligini tekshiring.

**Yechish.** $x\to 0$ *da* $f(x)=\frac{1}{\sqrt[3]{x^2}}\to+\infty$, $x=0$ nuqta $[-1,8]$ kesmaning ichki nuqtasi. 2-tur xosmas integrali formuladan foydalansak, $\int_{-1}^{8}\frac{dx}{\sqrt[3]{x^2}}=\int_{-1}^{8}x^{-\frac{2}{3}}dx=\lim_{\varepsilon\to 0}\left(3\sqrt[3]{x}\Big|_{-1}^{\varepsilon}+3\sqrt[3]{x}\Big|_{\varepsilon}^{8}\right)=0+3+6=9$ bo'ladi. Demak, berilgan xosmas integral yaqinlashuvchi.

**Xosmas integrallarning asosiy xossalari.**

**1) Taqqoslash alomati.** Bizga $\int_a^{+\infty}f(x)\mathrm{dx}$, $\int_a^{+\infty}g(x)\mathrm{dx}$ berilgan bo'lib,

$\int_a^{A}f(x)\mathrm{dx}$, $\int_a^{A}g(x)\mathrm{dx}$ integrallar mavjud bo'lib,

$\forall x\in[a,A]$, $0\le f(x)\le g(x)$ bo'lsa, u holda $\int_a^{+\infty}f(x)\mathrm{dx}$ uzoqlashsa, $\int_a^{+\infty}g(x)\mathrm{dx}$ ham uzoqlashadi, $\int_a^{+\infty}g(x)\mathrm{dx}$ yaqinlashsa, $\int_a^{+\infty}f(x)\mathrm{dx}$ ham yaqinlashadi.

2) Bizga $\int_a^{+\infty}f(x)\mathrm{dx}$, $\int_a^{+\infty}g(x)\mathrm{dx}$ berilgan bo'lib, $\int_a^{A}f(x)\mathrm{dx}$, $\int_a^{A}g(x)\mathrm{dx}$ integrallar mavjud bo'lib, $\forall x\in[a,A]$, $\lim_{x\to+\infty}\frac{f(x)}{g(x)}=l>0$ bo'lsa, u holda $\int_a^{+\infty}f(x)\mathrm{dx}$ va $\int_a^{+\infty}g(x)\mathrm{dx}$ bir vaqtda yaqinlashadi yoki uzoqlashadi.

3) $\int_a^{+\infty}|f(x)|\mathrm{dx}$ yaqinlashsa, u holda $\int_a^{+\infty}f(x)\mathrm{dx}$ ham yaqinlashadi va bu holda $\int_a^{+\infty}f(x)\mathrm{dx}$ absolut uzoqlashuvchi deyiladi.

**595.** Xosmas integrallarni hisoblang yoki uzoqlashuvchi ekanini ko'rsating:

1) $\int\limits_{2}^{+\infty}\frac{dx}{x^2}$;  2) $\int\limits_{-\infty}^{-3}\frac{dx}{x+2}$;  3) $\int\limits_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2}$;

4) $\int\limits_{1}^{+\infty}\frac{dx}{x^\alpha}$;  5) $\int\limits_{e}^{+\infty}\frac{dx}{x\ln^2 x}$;  6*) $\int\limits_{0}^{+\infty}xe^{-x^2}dx$;

7) $\int\limits_{0}^{1}\frac{dx}{\sqrt{x}}$;  8) $\int\limits_{0}^{1}\frac{dx}{\sqrt{x(1-x)}}$;  9) $\int\limits_{0}^{2}\frac{dx}{\sqrt[3]{(x-1)^2}}$;

10) $\int\limits_{0}^{1}\frac{dx}{x^\alpha}$;  11) $\int\limits_{1}^{3}\frac{xdx}{\sqrt{x^2-1}}$;  12*) $\int\limits_{2}^{e+1}\frac{dx}{(x-1)\sqrt[3]{\ln(x-1)}}$.

**596.** Xosmas integrallarni yaqinlashishga tekshiring:

1) $\int\limits_{0}^{e}\frac{dx}{e^x-1}$;  2) $\int\limits_{0}^{1}\frac{e^x dx}{\sqrt{1-\cos x}}$;  3) $\int\limits_{1}^{+\infty}\frac{e^{-x^2}dx}{x^2}$;

4) $\int\limits_{0}^{\frac{2}{\pi}}\sin\frac{1}{x}\cdot\frac{dx}{x^2}$;  5) $\int\limits_{0}^{1}\frac{dx}{tgx-x}$;  6) $\int\limits_{0}^{5}\frac{\cos x dx}{\sqrt{x}}$.

## 38 §. Aniq integralning tatbiqlari

### Aniq integralning geometrik masalalarga tatbiqlari

**Tekis yuzani hisoblash.** $y=f(x)$ funksiya grafigi, $x=a, x=b$ ikkita to'g'ri chiziqlar va $OX$ o'qi bilan chegaralangan egri chiziqli trapetsiyaning yuzi

$$S=\int\limits_{a}^{b}ydx=\int\limits_{a}^{b}f(x)dx \qquad (24)$$

formula bilan hisoblanadi

Umumiy hol, ya'ni $y_1=f_1(x)$, $y_2=f_2(x)$, $f_2(x)\geq f_1(x)$ chiziqlar bilan chegaralangan yuza

$$S_1=\int\limits_{x_1}^{x_2}\left[f_2(x)-f_1(x)\right]dx \qquad (25)$$

aniq integralga teng bo'ladi.

$x=\varphi(y),\ y=c,\ y=d,\ x=0$ chiziqlar bilan chegaralangan yuza

$$S_2=\int_c^d x\,dy=\int_c^d \varphi(y)dy \qquad (26)$$

aniq integral bilan hisoblanadi.

Egri chiziq parametrik $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$, tenglama bilan berilgan bo'lsa, yuza

$$S_3=\int_{t_1}^{t_2} y(t)x'(t)dt \qquad (27)$$

formula bo'yicha hisoblanadi.

Qutb koordinatalar sistemasida berilgan egri chiziqli sektorning yuzi

$$S_4=\frac{1}{2}\int_\alpha^\beta \rho^2(\varphi)d\varphi, \qquad (28)$$

bu yerda $\varphi=\alpha, \varphi=\beta, (\alpha<\beta)$.

**597.** $xy=8, x=1, x=e, y=0$ chiziqlar bilan chegaralangan yuzani hisoblang:

**Yechish.** $y=\frac{8}{x}$ bo'lib, (1) formulaga asosan,

$S=\int_a^b ydx=\int_1^e \frac{8}{x}dx=8\ln|x|\Big|_1^e=8(\ln e-\ln 1)=8$ bo'ladi.

**598.** $y=x^2, y^2=x$ chiziqlar bilan chegaralangan yuzani toping:

**Yechish.** $\begin{cases} y=x^2, \\ y^2=x \end{cases}$ tenglamalar sistemasidan

$x^4=x, x^4-x=0, x_1=0; x_2=1$ kesishish nuqtalarining abtsissalari bo'lib,

bu yuza $S=\int_0^1\left[\sqrt{x}-x^2\right]dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1-\frac{x^3}{3}\Big|_0^1=\left(\frac{2}{3}-0\right)-\left(\frac{1}{3}-0\right)=\frac{1}{3}$ bo'ladi.

**599.** Ellipsning $\begin{cases} x=3\cos t \\ y=2\sin t \end{cases}$ parametrik tenglamasidan foydalanib uning yuzini toping.

**Yechish**. Ellips koordinata o'qlariga nisbatan simmetrikligidan foydalanib, hamda $x=3\cos t$ tenglamada $x=0, x=3$ bo'lganda $t_1=\frac{\pi}{2}$, $t_2=0$ bo'lganligini hisobga olib,

$$S=4\int_{\frac{\pi}{2}}^{0} ydx=-4\int_{0}^{\frac{\pi}{2}} 2\sin t(-3\sin t)dt=24\int_{0}^{\frac{\pi}{2}}\sin^2 tdt=24\int_{0}^{\frac{\pi}{2}}\frac{1-\cos 2t}{2}dt=12\int_{0}^{\frac{\pi}{2}}(1-\cos 2t)dt=$$

$$=12t\Big|_{0}^{\frac{\pi}{2}}-\frac{12}{2}\sin 2t\Big|_{0}^{\frac{\pi}{2}}=12\left(\frac{\pi}{2}-0\right)-6(\sin\pi-\sin 0)=6\pi$$

ni hosil qilamiz.

**Egri chiziq yoyi uzunligini hisoblash.** To‘g‘ri burchakli koordinatlar sistemasida $y=f(x)$ funksiya $[a,b]$kesmada silliq (ya’ni $y'=f'(x)$ hosila mavjud) bo‘lsa, bu egri chiziq yoyining uzunligi

$$l=\int_a^b\sqrt{1+(y')^2}\,dx \qquad (29)$$

formula yordamida hisoblanadi.

Egri chiziq parametrik tenglama $\begin{cases}x=x(t)\\ y=y(t)\end{cases}$ bilan berilgan bo‘lsa, yoy uzunligi

$$l=\int_{t_1}^{t_2}\sqrt{(x')^2+(y')^2}\,dt \qquad (30)$$

aniq integral bilan hisoblanadi.

Silliq egri chiziq qutb koordinatalarida $r=r(\varphi)$, $(\alpha\le\varphi\le\beta)$ tenglama bilan berilgan bo‘lsa, yoy uzunligi

$$l=\int_\alpha^\beta\sqrt{r^2(\varphi)+(r'(\varphi))^2}\,d\varphi \qquad (31)$$

formula bilan hisoblanadi.

**600.** $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ astroida yoyining uzunligini toping.

**Yechish.**Astroida koordinata o‘qlariga nisbatan simmetrik bo‘lganligi uchun 1/4 qism yoy uzunligini topamiz.

Oshkormas funksiya hosilasiga asosan $\frac{2}{3x^{\frac{1}{3}}}+\frac{2}{3y^{\frac{1}{3}}}y'=0$ bundan,

$y'=-\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$. Yoy uzunligi formulasiga asosan,

$$l=4\int_0^a\sqrt{1+(y')^2}dx=4\int_0^a\sqrt{1+\left(\sqrt[3]{y}/\sqrt[3]{x}\right)^2}dx=4\int_0^a\sqrt{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}}dx=$$

$$4\int_0^a\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}}dx=4\sqrt[3]{a}\int_0^a x^{-\frac{1}{3}}dx=4\sqrt[3]{a}\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\Bigg|_0^a=4\frac{3}{2}\sqrt[3]{a}\cdot\left(a^{\frac{2}{3}}-0\right)=6a$$

bo'ladi.

**Aylanma jism hajmini hisoblash.** $y=f(x), x=a, x=b, y=0$ chiziqlar bilan chegaralangan figuraning $OX$ o'qi atrofida aylanishidan hosil bo'lgan jismning hajmi

$$V_x=\pi\int_a^b y^2dx=\pi\int_a^b f^2(x)dx \qquad (32)$$

aniq integral bilan hisoblanadi.

$x=\varphi(y), y=c, y=d, x=0$ chiziqlar bilan chegaralangan figuraning $OY$ o'qi atrofida aylanishidan hosil bo'lgan jismning hajmi

$$V_y=\pi\int_c^d x^2dy=\pi\int_c^d \varphi^2(y)dy \qquad (33)$$

formula bilan hisoblanadi.

$y_1=f_1(x)$ va $y_2=f_2(x)$ $\left(0\le f_2(x)\le f_1(x)\right)$ egri chiziqlar, hamda $x=a, x=b,$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiya $OX$ o'q atrofida aylanishdan hosil bo'lgan jism hajmi

$$V_x=\pi\int_a^b\left(y_1^2-y_2^2\right)dx \qquad (34)$$

formula bilan hisoblanadi.

**601.** $y^2=2x$ parabola, $x=3$ to'g'ri chiziq va $OX$ o'qi bilan chegaralangan figuraning $OX$ o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini hisoblang.

**Yechish.** Masala shartiga ko'ra $x$ 0 dan 3 gacha o'zgaradi.

Demak, $V_x=\pi\int_0^3 y^2dx=\pi\int_0^3 2xdx=\pi x^2\Big|_0^3=\pi(3^2-0^2)=9\pi$.

**602.** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipsning $OY$ o'qi atrofida aylanishidan hosil bo'lgan jism hajmini hisoblang.

**Yechish**. Bunday jismga aylanma ellipsoid deyiladi. Ellips tenglamasidan $x^2 = a^2\left(1-\frac{y^2}{b^2}\right)$ bo'lib, integralning chegaralari $c=-b,\ d=b$ bo'ladi. (10) formulaga asosan,

$$V_y = \pi\int_{-b}^{b} a^2\left(1-\frac{y^2}{b^2}\right)dy = \pi a^2\int_{-b}^{b} dy - \frac{\pi a^2}{b^2}\int_{-b}^{b} y^2 dy = \pi a^2 y\Big|_{-b}^{b} - \pi\frac{a^2}{b^2}\cdot\frac{y^3}{3}\Big|_{-b}^{b} =$$

$$= \pi a^2\left[b-(-b)\right] - \pi\frac{a^2}{3b^2}\left[b^3-(-b)^3\right] = 2\pi a^2 b - \frac{2}{3}\pi a^2 b = \frac{4}{3}\pi a^2 b$$

Demak, $V_y = \frac{4}{3}\pi a^2 b$. $a=b=R$ bo'lsa, shar hosil bo'lib $V_{sh} = \frac{4}{3}\pi R^3$ bo'ladi.

**Aylanma jism sirtining yuzini hisoblash.** $y=f(x),\ a\le x\le b$ egri chiziq $AB$ yoyini $OX$ o'qi atrofida aylanishdan hosil bo'lgan jism sirtining yuzi

$$S_{ox} = 2\pi\int_a^b y\sqrt{1+(y')^2}\,dx \qquad (35)$$

formula bilan hisoblanadi.

$x=x(t),\ y=y(t),\ t\in[t_1,t_2]$ parametrik tenglama bilan berilgan egri chiziqni $OX$ o'qi atrofida aylanishdan hosil bo'lgan jism sirtining yuzi

$$S_{ox} = 2\pi\int_{t_1}^{t_2} y\sqrt{(x')^2+(y')^2}\,dt \qquad (36)$$

formula bilan hisoblanadi.

**603**. $\frac{x}{2}+\frac{y}{3}=1$ to'g'ri chiziqning koordinata o'qlari bilan kesishib hosil qilgan kesmasini $OX$ o'qi atrofida aylanishidan hosil bo'lgan jism yon sirtini yuzasini hisoblang.

**Yechish**. $0\le x\le 2,\ y=3\left(1-\frac{x}{2}\right),\ y'=-\frac{3}{2}$ ekanligini inobatga olib (35) formuladan quyidagini topamiz:

$$S_{ox} = 2\pi\int_0^2 3\left(1-\frac{x}{2}\right)\sqrt{1+\frac{9}{4}}dx = \frac{3\sqrt{13}\pi}{2}\int_0^2 (2-x)dx =$$

$$= \frac{3\sqrt{13}\pi}{2}\left(2x-\frac{x^2}{2}\right)\Big|_0^2 = 3\sqrt{13}\pi.$$

## Aniq integralning mexanika va fizikaga tatbiqlari

**Inersiya momenti.** Mexanikada moddiy nuqta harakati muhim tushunchalardan biri hisoblanadi. Odatda, o‘lchami yetarli darajada kichik va massaga ega bo‘lgan jism moddiy nuqta deb qaraladi.

Aytaylik, tekislikda $m$ massaga ega bo‘lgan $A$ moddiy nuqta berilgan bo‘lib, bu nuqtadan biror $l$ o‘qqacha (yoki $O$ nuqtagacha) bo‘lgan masofa $r$ ga teng bo‘lsin. Ushbu $J = mr^2$ miqdor $A$ moddiy nuqtaning $l$ o‘qqa ($O$ nuqtaga) nisbatan inertsiya momenti deyiladi.

Masalan, $A = A(x, y)$ moddiy nuqtaning koordinata o‘qlariga hamda koordinata boshiga nisbatan inertsiya momentlari mos ravishda

$$J_x = my^2\ ,\ J_y = mx^2\ ,\ J_0 = m\sqrt{x^2 + y^2}$$

bo‘ladi. Massaga ega bo‘lgan $\breve{AB}$ egri chiziqning koordinata o‘qlariga hamda koordinata boshiga nisbatan inertsiya momentlari aniq integrallar yordamida topiladi:

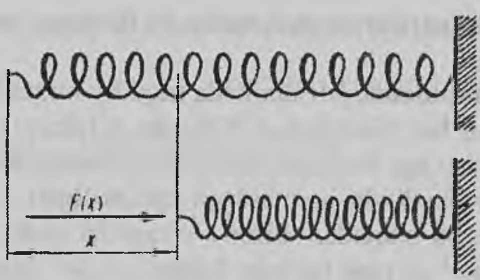
$$\begin{cases} J_x = \int\limits_a^b f^2(x)\sqrt{1 + f'^2(x)}\, dx\ , \\ J_y = \int\limits_a^b x^2\sqrt{1 + f'^2(x)}\, dx\ , \\ J_0 = \int\limits_a^b (x^2 + f^2(x))\sqrt{1 + f'^2(x)}\, dx. \end{cases} \tag{37}$$

**O‘zgaruvchi kuchning bajargan ishi.** Biror jismni $Ox$ o‘qi bo‘ylab, shu o‘q yo‘nalishida bo‘lgan $F = F(x)$ kuch ta’siri ostida $a$ nuqtadan $b$ nuqtaga $(a < b)$ o‘tkazish uchun bajarilgan ishi

$$A = \int\limits_a^b F(x)dx \tag{38}$$

formula bilan ifodalanadi.

**604.** Vintsimon prujinaning bir uchi mustahkamlangan, ikkinchi uchiga esa $F = F(x)$ kuch ta’sir etib, prujina qisilgan (2-shakl)




**2-shakl**

Agar prujinaning qisilishi unga ta'sir etayotgan $F(x)$ kuchga proportsional bo'lsa, prujinani $a$ birlikka qisish uchun $F(x)$ kuchning bajargan ishi topilsin.

**Yechish.** Agar $F(x)$ kuch ta'sirida prujinaning qisilish miqdorini $x$ orqali belgilasak, u holda $F(x)=kx$ bo'ladi, bunda $k$ - proportsionallik koeffitsiyenti (qisilish koeffitsiyenti). (15) formulaga ko'ra bajarilgan ish

$$A=\int_0^a kxdx=\frac{ka^2}{2}$$

bo'ladi.

**605.** Quyidagi chiziqlar bilan chegralangan shakl yuzini hisoblang:

1) $y=2x-x^2,\ y=0$; 2) $y=0, y=2x^2+1,\ x=-1,\ x=1$;

3) $y=x^2-x,\ y=3x$; 4) $y=x^2-2x+2, y=2+4x-x^2$.

**606.** Quyidagi chiziqlar bilan chegralangan shaklni OY o'qi atrofida aylanishidan hosil bo'lgan jism hajmini hisoblang:

1) $y^2=6x, y=0, x=3$; 2) $y=\frac{1}{x}, y=x,\ y=0,\ x=2$;

3) $4y=x^2,\ 8y=x^3$; 4) $xy=4,\ x=1, x=4, y=0$.

**607.** Quyidagi chiziqlarni berilgan oraliqdagi uzunligini hisoblang:

1) $y=\frac{x^2}{4},\ 0\le x\le 2$; 2) $y=\sqrt{2x-x^2}-1,\ \frac{1}{4}\le x\le 1$;

3) $y=\frac{x^2}{2}-\frac{\ln x}{4},\ 1\le x\le 3$; 4) $x=\ln\cos y,\ 0\le y\le\frac{\pi}{3}$.

**608.** Quyidagi chiziqlar bilan chegralangan shaklni OX o'qi atrofida aylanishidan hosil bo'lgan jism sirt yuzasini hisoblang:

1) $y = 2ch\frac{x}{2}, 0 \le x \le 2$; 2) $y = x^3, 0 \le x \le \frac{1}{2}$;

3*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; 4) $x = t - \sin t, y = 1 - \cos t$ bir arkasi.

**609.** Moddiy nuqta $v = 100 + 8t$ tezlik qonuni bilan harakatlanmoqda. Vaqtning $[0;10]$ oralig'ida qancha yo'l bosib o'tadi?

**610.** Agar prujinani 1 sm ga cho'zish uchun 1N kuch talab etilsa, 4 sm ga cho'zish uchun qancha ish bajarish kerak?

# JAVOBLAR

## I BOB

**4.** -7 **5.** 14 **6.** -0,5 **7.** $4\sqrt{ab}$ **8.** 1 **9.** 0 **10.** 0 **11.** 2 **12.** 0;3 **13.** -2 **14.** -3;3
**15.** $\frac{\pi}{2}+\frac{n\pi}{3}$ **16.** (-2;1) **17.** (1;5) **18.** 40 **19.** -24 **20.** -23 **21.** 14 **22.** 1
**23.** $a^2(a-2)$ **24.** $4x$ **25.** 0 **26.** $\Delta=(y-x)(z-x)(z-y)$
**27.** $\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)$ **28.** (2;3) **29.** (-4;1;2) **35.** 8x+15y+12z-19t
**36.** 3a-b+2c+d **37.** -48 **38.** 40 **39.** -30 **40.** 18 **41.** -36 **42.** -40 **43.** -150 **44.** -10
**45.** 5
**46.** -720 **47.** a=-1 **48.** a=-9 **49.** a=5 **50.** a=2 **51.** 900 **52.** 12 **53.** 39520 **54.** $a^2b^2$
**55.** -2(n-2)! **62.** $x_1=-1$ $x_2=3$ $x_3=2$ **63.** $x_1=2$ $x_2=1$ $x_3=2$
**64.** $x_1=-1$ $x_2=3$ $x_3=1$ **65.** $x_1=-7$ $x_2=7$ $x_3=5$ **66.** (1;2;3) **67.** $\Delta=0$, $\Delta_1\neq 0$ **68.** cheksiz ko‘p **69.** (1;1;1) **70.** (2;-1;0)
**71.** (1;2;3) **72.** $x_1=x_2=1$; $x_3=x_4=-1$ **73.** $x_1=1$; $x_2=x_3=2$; $x_4=0$ **74.** x=-3, y=0, z=-0.5 t=2/3 **75.** (-1;0;1) **76.** (2;-1;-3) **77.** (1;-1;2) **78.** (1;0;2) **83.** $C=\begin{pmatrix}-7 & 2 & 4\\ -12 & -1 & -7\\ 5 & 2 & -8\end{pmatrix}$ **84.** $C=\begin{pmatrix}7 & -4 & -17 & 6\\ 4 & -9 & -4 & -1\\ 1 & -1 & -10 & 0\end{pmatrix}$

**85.** $\begin{pmatrix}-1 & 8\\ 0 & -9\end{pmatrix}$ **86.** $\begin{pmatrix}3 & -9 & -13\\ 8 & -5 & 13\end{pmatrix}$ **87.** $\begin{pmatrix}4 & 7 & 11\\ 4 & 2 & -2\\ 3 & 3 & 3\end{pmatrix}$ **88.** $\begin{pmatrix}2 & 0\\ 4 & -1\\ 5 & 3\end{pmatrix}$ **89.** $\begin{pmatrix}1 & -1\\ 0 & 1\\ -1 & 0\end{pmatrix}$ **90.** $\begin{pmatrix}5 & 0 & 4\\ 10 & 10 & 33\\ -11 & -7 & 25\end{pmatrix}$ **91.** $(6\ \ 7)$ **92.** $\begin{pmatrix}22 & 11\\ -19 & -1\end{pmatrix}$

**93.** $\begin{pmatrix}9 & -5 & -9\\ 25 & 15 & -12\end{pmatrix}$ **94.** $\begin{pmatrix}14 & 22\\ 17 & -9\\ 16 & 8\end{pmatrix}$ **95.** $\begin{pmatrix}-6 & -2\\ 13 & -4\\ 21 & 1\end{pmatrix}$ **96.** $\begin{pmatrix}-1 & 15\\ 17 & 13\end{pmatrix}$

**97.** $\begin{pmatrix}11 & -14 & 10\\ 7 & -6 & 7\end{pmatrix}$ **98.** $\begin{pmatrix}6 & 7 & 3\\ 0 & -8 & 6\\ 0 & -6 & 6\end{pmatrix}$ **99.** $(-12)$ **100.** $\begin{pmatrix}2 & 4 & -6\\ -1 & -2 & 3\\ 4 & 8 & -12\end{pmatrix}$

**101.** $\begin{pmatrix}31\\ -14\\ 18\end{pmatrix}$ **102.** (-6 -17 -1). **103.** $\begin{pmatrix}-6 & 1 & 3\\ 6 & 2 & 9\\ -12 & -3 & 14\end{pmatrix}$ **104.** $\begin{pmatrix}8\\ 19\end{pmatrix}$

**105.** $\begin{pmatrix}6 & 10\\ 6 & 5\\ 2 & 3\end{pmatrix}$ **106.** $\begin{pmatrix}-9 & -16 & -3\\ 19 & 21 & 17\end{pmatrix}$ **107.** $\begin{pmatrix}8 & 0 & 7\\ 16 & 10 & 4\\ 13 & 5 & 7\end{pmatrix}$; $\begin{pmatrix}4 & 6 & 6\\ 1 & 7 & 3\\ 8 & 11 & 14\end{pmatrix}$

**108.** $X=\begin{pmatrix}1 & 8 & 14\\ -7 & 1 & 8\\ 8 & 6 & 5\end{pmatrix}$ **109.** $X=\frac{1}{3}\cdot\begin{pmatrix}-30 & -6 & -2 & -25\\ -13 & 13 & -22 & 16\\ -9 & -5 & -3 & -1\end{pmatrix}$

**110.** $A^3=\begin{pmatrix}13 & -14\\ 21 & -22\end{pmatrix}$ **111.** $A^5=\begin{pmatrix}304 & -61\\ 305 & -62\end{pmatrix}$ **112.** $A^{10}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $A^{15}=\begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}$ **113.** $A^{10}=\begin{pmatrix}1 & 10\\ 0 & 1\end{pmatrix}$ **114.** $A=\begin{pmatrix}a^n & na^{n-1}\\ 0 & a^n\end{pmatrix}$ **115.** $\begin{pmatrix}11 & 14\\ 7 & 18\end{pmatrix}$

**116.** $\begin{pmatrix}28 & 15 & 16\\ 19 & 36 & 15\\ 30 & 19 & 28\end{pmatrix}$ **117.** $\begin{pmatrix}9 & 7\\ 2 & 9\end{pmatrix}$ **118.** $\begin{pmatrix}1 & 0 & 10\\ 6 & -3 & 15\\ 34 & 0 & 82\end{pmatrix}$ **119.** $\begin{pmatrix}73 & 25\\ 1 & -11\end{pmatrix}$

**120.** $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ **121.** $f(A)=\begin{pmatrix}2 & 8\\ 4 & 6\end{pmatrix}$ **122.** $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

**123.** $\begin{pmatrix}5 & 1 & 3\\ 8 & 0 & 3\\ -2 & 1 & -2\end{pmatrix}$ **124.** $\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}$ **125.** $\begin{pmatrix}16 & -10\\ 0 & -6\end{pmatrix}$ **126.** $\begin{pmatrix}0 & 6 & -2\\ 0 & -2 & 4\\ 0 & -4 & -2\end{pmatrix}$ **127.**

$\begin{pmatrix}-9 & 0 & 0 & 0\\ 0 & 9 & 0 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & -5\end{pmatrix}$ **128.** $\begin{pmatrix}-18 & 40\\ 0 & 62\end{pmatrix}$ **129.** $\begin{pmatrix}-8 & -6\\ -6 & -20\end{pmatrix}$ **130.** $\begin{pmatrix}3 & 2 & 2\\ 2 & -1 & -2\\ -2 & 0 & -3\end{pmatrix}$ **131.**

$X=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$, B=A*X **132.** $X=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$, B=X*A **133.** $X=\begin{pmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$,

B=X*A **134.** $X=\begin{pmatrix}3 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{pmatrix}$, B=X*A **136.** $\frac{1}{3}\begin{pmatrix}9 & -6\\ -4 & 3\end{pmatrix}$ **137.**

$\frac{1}{2}\begin{pmatrix}2 & -3\\ -4 & 7\end{pmatrix}$ **138.** $\frac{1}{2}\begin{pmatrix}5 & -3\\ -6 & 4\end{pmatrix}$ **139.** $\frac{1}{4}\begin{pmatrix}8 & -4\\ 5 & -3\end{pmatrix}$

**140.** $\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 2 & 3 & 4\end{pmatrix}$ **141.** $\begin{pmatrix}-1 & 1 & 0\\ 1 & -5 & 3\\ 0 & 3 & -2\end{pmatrix}$ **142.** $\frac{1}{3}\begin{pmatrix}1 & -4 & -3\\ 1 & -7 & -3\\ -1 & 10 & 6\end{pmatrix}$

**143.** $\frac{1}{5}\begin{pmatrix}1 & 1 & 0\\ -3 & 12 & 10\\ -1 & 4 & 5\end{pmatrix}$ **144.** $\begin{pmatrix}-2 & 3 & -1\\ -1{,}5 & 2{,}5 & -1\\ 9 & -13 & 5\end{pmatrix}$ **145.** $\begin{pmatrix}-10 & 3 & 8\\ -11 & 3 & 9\\ 14 & -4 & -11\end{pmatrix}$

**146.** $\begin{pmatrix}-2 & 5\\ -1 & 3\end{pmatrix}$ **147.** $\begin{pmatrix}-\frac{3}{5} & \frac{4}{5}\\ -\frac{2}{5} & \frac{1}{5}\end{pmatrix}$ **148.** mavjud emas **149.** $\begin{pmatrix}-\frac{1}{5} & \frac{4}{15}\\ 0 & \frac{1}{3}\end{pmatrix}$

**150.** $\frac{1}{18}\begin{pmatrix}7 & 5 & 1\\ -8 & 2 & 4\\ 3 & -3 & 3\end{pmatrix}$ **151.** $\frac{1}{38}\begin{pmatrix}8 & -2 & 4\\ 9 & -7 & -5\\ 5 & 13 & -7\end{pmatrix}$ **152.** $\frac{1}{60}\begin{pmatrix}12 & 6 & 6\\ -18 & 11 & 1\\ -18 & 1 & 11\end{pmatrix}$

**153.** $\frac{1}{6}\begin{pmatrix}-6 & -2 & 4\\ 0 & -4 & 2\\ 6 & 3 & -3\end{pmatrix}$ **154.** $\frac{1}{5}\begin{pmatrix}1 & 3 & 2\\ -3 & 1 & 1\\ 1 & -2 & 3\end{pmatrix}$ **155.** $\begin{pmatrix}0 & 0 & 0 & 12\\ 0 & 0 & 6 & 0\\ 0 & 4 & 0 & 0\\ 3 & 0 & 0 & 0\end{pmatrix}$

**156.** $\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}$ **157.** $\begin{pmatrix}1 & 3 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & -1 & 2\end{pmatrix}$ **158.** a=-2, b=2, c=4 **159.**a=-1, b=3, c=2

**160.**a=3, b=-1, c=-3 **161.**a=1, b=9, c=-4 **162.** a=-5, b=-2, c=3 **163.** a=5, b=5, c=-3 **164.** a≠-3 **165.** a≠1, a≠4 **166.** a ning bunday qiymati mavjud emas **167.** har qanday a larda **169.**(1, 2, -1) **170.** (1, 0, -1) **171.**(3, 7, -1) **172.**(5, -11, -13) **173.** (1, 1, 1) **174.** (1, 0, -1) **175.** (1;1;1) **176.**(2;-1;0) **177.**(1;2;3) **178.** $x_1 = x_2 = 1;\ x_3 = x_4 = -1$ **181.** 2 **182.** 1 **183.** 2 **184.** 3 **185.** 3 **186.** 2 **187.** 3 **188.** 3. **189.**3 **190.** 2 **191.**r=2, M=$\begin{vmatrix}2 & 0\\ 0 & -1\end{vmatrix}$, **192.**r=2, M=$\begin{vmatrix}1 & 2\\ 2 & 5\end{vmatrix}$

**193.** r=1, M=|2| **194.** r=2, M=$\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}$ **195.** r=1, M=|4|, **196.** r=3, M=$\begin{vmatrix}1 & 3 & 0\\ 2 & 4 & 0\\ 0 & 0 & 2\end{vmatrix}$ **197.** r=3, M=$\begin{vmatrix}4 & 2 & 0\\ 20 & 10 & -40\\ 10 & -30 & 40\end{vmatrix}$ **198.** r=3, M=$\begin{vmatrix}1 & -2 & 0\\ -1 & 3 & 1\\ 2 & -1 & 0\end{vmatrix}$

**199.** r=2, **200.** r=2, **201.**r=3 **202.** r=5 **203.** r=2 **204.** r=3, **205.** r=2, **206.** r=2, **207.** r=3 **208.** $\gamma$=0,5 **209.** $\gamma = 3$ **210.** $\gamma_1 = 0, \gamma_2 = 2$ **211.** har qanday $\gamma$ **212.** $\gamma = \frac{7}{9}$ **213.** r=3 **214.** r=2 **218.** $x_1 = -1, x_2 = 3, x_3 = 2$, **219.** $x_1 = 2, x_2 = 1, x_3 = 2$, **220.** $x_1 = -1, x_2 = 3, x_3 = 1$, **221.** $x_1 = -7, x_2 = 7, x_3 = 5$. **222.**birgalikda **223.** birgalikda emas **224.**birgalikda emas. **225.** a≠-6, **226.** a=2. **227.** a≠5 **228.** a=0 **229.** a≠5 **230.** a=-4 **231.** a=11 **232.** a=18 **233.** a=-2 **234.** birgalikda emas **235.** $\left(\frac{5-7x_3}{5}, \frac{8x_3}{5}, x_3\right)$, $x_3 \in R$, **236.** (1,1,1) **237.** ($11x_3$-4.3-$7x_3$, $x_3$), $x_3 \in R$ **238.** (1,1,1) **239.** birgalikda emas **240.** ($x_1$,($5x_1$-$4x_4$-11)/10), (-7-$3x_4$)/5, $x_4$), $x_1, x_4 \in R$ **241.** birgalikda emas **242.** ($x_1, x_2, (3 - 5x_1 + 25x_2)/9$), ($10x_2$-$2x_1$)/3), $x_1, x_2 \in R$ **243.** ((9-$x_3$-$14x_4$-$x_5$)/7, ((-1+$4x_3$-$x_4$-$3x_5$)/7, $x_3, x_4, x_5$)), где $x_3, x_4, x_5 \in R$

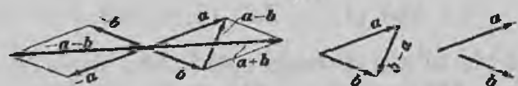
**246.**$(c, -2c, c), c \in R$ **247.** (0,0,0) **248.**$\left(\frac{4c_1-c_2}{3}, c_1, c_2, 0\right), c_1, c_2 \in R$

**249.** $\left(-\frac{1}{4}c, c, \frac{3}{4}c, 0\right)$,c∈ $R$ **250.**$(5c, c, 7c)$,c∈ $R$ **251.** $\left(-\frac{1}{4}c_1, \frac{5}{4}c_1 +\right.$

$c_2, c_1, c_2\big), c_1, c_2 \in R$ **252.** (0,0,0) **253.** $\big(-\frac{4}{5}(c_1 + c_2, \frac{1}{3}(c_2 -$
$5c_1), c_1, c_2\big), c_1, c_2 \in R$ **254.** $x_3 = -\frac{5}{2}x_1 + 5x_2,\ x_4 = \frac{7}{2}x_1 - 7x_2$.
255. $x_2 = x_1 - 13x_4 + x_5,\ x_3 = 5x_4 - x_5$.

## II BOB

256. $|a|=7$. **257.** $z = \pm 3$. **258.** $\overline{AB} = \{-4;-3;-1\}$, $\overline{BA} = \{4;-3;1\}$. **259.** $N(4;1;1)$. **260.** (-1; 2; 3). **261.** $X = \sqrt{2}$, $Y = 1$, $Z = -1$. **262.** $\cos\alpha = \frac{12}{25}$, $\cos\beta = -\frac{3}{5}$, $\cos\gamma = -\frac{16}{25}$. **263.** $\cos\alpha = \frac{3}{13}$, $\cos\beta = \frac{4}{13}$, $\cos\gamma = \frac{12}{13}$. **264.** 1) mumkin; 2) mumkinmas; 3) mumkin . **265.** 60° yoki 120°. **266.** a = $\{1;-1;\sqrt{2}\}$ yoki α = $\{1;-1;-\sqrt{2}\}$. **267.** $M_1(\sqrt{2}, \sqrt{3}, \sqrt{3})$, $M_2(-\sqrt{3}, -\sqrt{3}, -\sqrt{3})$,




**268.** shaklda.

**269.** $|a-b| = 22$. **270.** $|a+b| = 20$. **271.** $|a+b| = |a-b| = 13$. **272.** $|a+b| = \sqrt{129} \approx 11,4$, $|a-b| = 7$. **273.** $|a+b| = \sqrt{19} \approx 4,4$, $|a-b| = 7$. **274.** 1) *a* va *b* vektorlar o`zaro perpendikulyar; 2) *a* va *b* vektorlar orasidagi burchak o'tkir; 3) *a* va *b* vektorlar orasidagi burchak o'tmas. **275.** $|a| = |b|$. **277.** $\alpha = 4$, $\beta = -1$. **278.** $|a+b| = 6$, $|a-b| = 14$. **284.** 1) -62 2) 162; 3) 16; 4) 13; 5) -61; 6) 37; 7) 73. **285.** 1) -62; 2) 162; 3) 373. **286.** Paralelogramm diagonallari kvadrati yig'indisi uning tomonlari yig'indisi kvadratiga teng. **287.** $ab + bc + ca = -\frac{3}{2}$. **288.** $ab + bc + ca = -13$. **289.** $|p| = 10$. **290.** $\alpha = \pm\frac{3}{5}$. **291.** $a = \frac{3}{5}$; $\cos\alpha = \frac{1}{3}$; $\cos\beta = \cos\gamma = \frac{2}{3}$ **292.** $-3$ **293.** $\alpha = \arccos\frac{2}{\sqrt{7}}$. **294.** 1) 22; 2) 6; 3) 7; 4)-200; 5) 129; 6) 41. **295.** 1) -524; 2) 13; 3) 3; **297.** $\alpha = -6$. **298.** 45°. **299.** $\arccos(-\frac{4}{9})$. **301.** $x = \{2;-3;0\}$. **302.** $x = 2i + 3j - 2k$. **305.** $|[ab]| = 15$. **306.** $|[ab]| = 16$. **307.** $ab = \pm 30$. **308.** 1) 24; 2) 60. **309.** 1) 3; 2) 27; 3) 300. **310.** ***a*** va ***b*** kolleniar. **312.** ***a*** va ***b*** perpendikulyar. **315.** 1) {5; 1; 7}; 2) {10; 2; 14}; 3) {20; 4; 28}. **316.** 1) {6; -4; -6}; 2) {-2; 8; 12}. **317.** {2; 11; 7}. **318.**
{-4; 3; 4}. **319.** 15; $\cos\alpha = \frac{2}{3}$,

$\cos\beta = -\frac{2}{15}$, $\cos\gamma = \frac{11}{15}$. **320.** 14 kv.b. **321.** $x = \{7, 5; 1\}$. **322.** 1) o'ng; 2) chap; 3) o'ng; 4) o'ng; 5) vektorlar komplanar; 6) chap. **323.** $abc = 24$. **324.** $abc = \pm 27$, $a$, $b$, $c$ o'ng uchlik bo'lsa "+", chap uchlik bo'lsa "-". **326.** $abc = -7$. **327.** 1) komplanar; 2) komplanar emas; 3) komplanar. 329. $V = 7/6$ . **330.** 3 kub. b.

## III BOB

**336.** $A_y(0; 2)$, $B_y(0; 1)$, $C_y(0; -2)$, $D_y(0, 1)$, $E_y(0; -2)$. **337.** 1) (2; -3); 2) (-3; -2);
3) (-1; 1); 4) (-3; 5); 5) (-4; -6); 6) ($a$; -$b$). **339.** (1; -3), (3; 1), (-5; 7).
**341.** 1) 14 ; 2) 12 ; 3) 26 **342.** 5. **343.** 20 **344.** (5; 0) yoki $(-\frac{1}{3};0)$. **345.** $C(3; \frac{5}{9}\pi)$, $D(5; -\frac{11}{14}\pi)$. **346.** (1; -5). **347.** (3;-4) **348.** $\left(-1-2\sqrt{3}; -\sqrt{3}+2\right)$
**351.** $M_1$, $M_3$ va $M_4$ to'g'ri chiziqda yotadi, $M_2$, $M_5$ va $M_6$ to'g'ri chiziqda yotmaydi. **352.**(6;0), (0;-4) **353.**(4;-2), $45^0$ **354.** 1. $(-3;-1),(2;4),(3;1)$ 2.10 3. $90^0, arctg2, arctg0.5$ **355.** 1) $\varphi = 45°$, 2)$\varphi = 60°$; 3)$\varphi = 90°$. **356.** 4x + 7y -1=0,y -3 = 0, 4x + 3y -5 = 0. **357.** 1) $m=-4, n\neq 2$ yoki $m=4, n\neq -2$; 2) $m=-4, n=2$ yoki $m=4, n=-2$; 3) $m=0$, n - ixtiyoriy. **358.** $\sqrt{10}$,(2;2),(5;1) **359.** (2; 0), (0; -3) yoki (-4; 0), (0; 4).
**364.** 1) $x^2 + y^2 = 9$; 2) $(x-2)^2 + (y+3)^2 = 49$; 3) $(x-6)^2 + (y+8)^2 = 100$; 4) $(x-1)^2 + (y+1)^2 = 4$; 5) $(x-1)^2 + y^2 = 1$; **365.** 1) (2; 0) R = 2; 2) $(\frac{3}{2}; \frac{\pi}{2})$ R = $\frac{3}{2}$; 3) (1; π). R = 1; 4) $(\frac{5}{2}; -\frac{\pi}{2})$ R = $\frac{5}{2}$; 5) (3; 4) R = 3; 6 ) $(4; \frac{5}{6}\pi)$ R= 4;
7) $(4; -\frac{\pi}{6})$ R = 4. **366.** 1) $\frac{x^2}{25}+\frac{y^2}{4}=1$; 2) $\frac{x^2}{25}+\frac{y^2}{9}=1$;3) $\frac{x^2}{169}+\frac{y^2}{144}=1$; 4) $\frac{x^2}{25}+\frac{y^2}{16}=1$; 5) $\frac{x^2}{100}+\frac{y^2}{64}=1$; **367.** $\frac{x^2}{17}+\frac{y^2}{8}=1$ **368.** $q_1 = \frac{4}{3}$ $q_2 = \frac{4}{5}$ **369.** 1) $\frac{x^2}{25}-\frac{y^2}{16}=1$. 2) $\frac{x^2}{9}-\frac{y^2}{16}=1$ 3) $\frac{x^2}{4}-\frac{y^2}{5}=1$. 4) $\frac{x^2}{64}-\frac{y^2}{36}=1$ 5) $\frac{x^2}{36}-\frac{y^2}{64}=1$. 6) $\frac{x^2}{144}-\frac{y^2}{25}=1$ 7) $\frac{x^2}{4}-\frac{y^2}{5}=1$ **370.** $\frac{x^2}{16}-\frac{y^2}{4}=1$
**371.** $\frac{x^2}{5}-\frac{y^2}{45}=1$, $\frac{3x^2}{10}-\frac{4y^2}{45}=1$ **372.** 1) $y^2 = 6x$; 2) $y^2 = -x$. 3) $x^2 = \frac{1}{2}y$; 4) $x^2 = -6y$.
**373.** $d = 13\frac{5}{13}$. **374.** 1) $A(-2; 1)$, $p = 2$; 2) $A(1; 3)$, $p = \frac{1}{8}$; 3) $A(6; -1)$, $p = 3$.

**376.** 1) ellips $\frac{x^2}{9}+\frac{y^2}{4}=1$; $O'(5; —2)$; 2) giperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$; $O'(3; -2)$ 3) mavhum ellips $\frac{x^2}{4}+\frac{y^2}{9}=-1$; 4) giperbola $4x^2-y^2=0$ $O'(-1; -1)$; 5) parabola ; 6) aylana.

**377.** 1) $x'^2-\frac{y'^2}{4}=1$ 2) $\frac{x'^2}{16}+\frac{y'^2}{9}=1$ 3) $\frac{x'^2}{9}-\frac{y'^2}{36}=1$
4) $x'^2-4y'^2=0$ 5) $x'^2+2y'^2=-1$
6) $2x'^2+3y'^2=0$ **381.** $2y-3z+7=0$ **382.** $3y+2z=0$ **383.** $45^0$ **384.** $x-2y-3z=4$

**385.** $2x-2y+z=2$ **386.** $2x-y+z=0$ **387.** $3x-y=0$ va $x+3y=0$ **388.** $2\sqrt{2}$ **389.** $\sqrt{6}$ **390.** $x+y+2z=0$. **391.** 8 **395.** $(2; -1; 0)$; $(1\frac{1}{3};\ 0;\ -\frac{1}{3})$. **396.** $\begin{cases}7x-y+1=0,\\ z=0;\end{cases}$ $\begin{cases}5x-z-1=0,\\ y=0,\end{cases}$, $\begin{cases}5y-7z-12=0,\\ x=0;\end{cases}$ $3y+2z=0$ **397.** $x+19y-7z-11=0$ $45^0$

**398.** 1) $\frac{x-2}{2}=\frac{y}{-3}=\frac{z+3}{5}$; 2) $\frac{x-2}{5}=\frac{y}{2}=\frac{z+3}{-1}$; 3) $\frac{x-2}{1}=\frac{y}{0}=\frac{z+3}{0}$;
4) $\frac{x-2}{0}=\frac{y}{1}=\frac{z+3}{0}$; 5) $\frac{x-2}{0}=\frac{y}{0}=\frac{z+3}{1}$. **399.** $\frac{x+4}{3}=\frac{y+5}{2}=\frac{z-3}{-1}$

**400.** $60°$ **401.** 1) $x=t+1,\ y=-7t,\ z=-19t-2$; **2)** $x=-t+1,\ y=3t+2,\ z=5t-1$ $3x-y=0$ va $x+3y=0$ **402.** 1) $(2; —3; 6)$; 2) parallel 3) to‘g‘ri chiziq tekislikda yotadi. **403.** 1) 13; 2) 3; 3) 7. **404.** $6x-20y-11z+1=0$. **405.** $(1; 2; -2)$ .**406.** $d=7$. **407.** 1) 21; 2) 6; 3) 15. **408.** $4x+6y+5z-1=0$.

**411.** **1)** $x^2+y^2+z^2=81$; 2) $(x-5)^2+(y+3)^2+(z-7)^2=4$;
3) $(x-4)^2+(y+4)^2+(z+2)^2=36$; 4) $(x-3)^2+(y+2)^2+(z-1)^2=18$; 5) $(x-3)^2+(y+1)^2+(z-1)^2=21$; 6) $x^2+y^2+z^2=9$; 7) $(x-1)^2+(y+2)^2+(z-3)^2=49$;
8) $(x+2)^2+(y-4)^2+(z-5)^2=81$.

**412.** 3, $\sqrt{3}$; $(2; 3; 0)$, $(2; -3; 0)$, $(2; 0; \sqrt{3})$, $(2; 0; -\sqrt{3})$. **413.** 4,3; $(4; 0; -1)$, $(-4; 0; -1)$. **414.** 15; $(0; -6; -\frac{3}{2})$.

**415.** Oxy: $\begin{cases}x^2+4xy+5y^2-x=0,\\ z=0;\end{cases}$ Oxz: $\begin{cases}x^2-2xz+5z^2-4x=0,\\ y=0;\end{cases}$ Oyz: $\begin{cases}y^2+z^2+2y-z=0,\\ x=0.\end{cases}$

**416.** $5x^2+5y^2+2z^2-2xy+4xz+4yz-6=0$

**417.** 1) $(3; 4; -2)$ $(2;6; -2)$; 2) $(4; -3; 2)$; 3) kesishmaydi; 4) To‘g‘ri chiziq sirtga tegishli. **418.** $x^2+4y^2+5z^2-4xy-125=0$.

**419.** $3x^2-5y^2+7z^2-6xy+10xz-2yz-4x+4y-4z+4=0$.

**420.** $x^2+4y^2-4z^2+4xy+12xz-6yz=0$.

**421.** x -2y + 2z-1=0, x- 2y + 2z + 1 = 0; $\frac{2}{3}$.

## IV BOB

**428.** 1) $A=\{1,2,3,4,5\}$; 2) $A=\{-7,-6,-5,-4,-3,-2,-1,0,1\}$;
3) $A=\{2,3,5,7,11,13,17,19,23,29\}$; 4) $A=\{1,2,3,4,6,8,12,16,24,48\}$;
5) $A=\{2\}$; 6) $A=\{0.5,2\}$; 7) $A=[-2,3]$. **429.** 8.
**430.** 1) $A\cup B=\{2,3,5,8,9,13,17\}$ 2)$A\cap B=\{5,13\}$ 3)$A\setminus B=\{2,3,8\}$
4)$B\setminus A=\{9,17\}$ 5)$A\Delta B=\{2,3,8,9,17\}$
6)$A\times B=\{(2,5),(2,9),(2,13),(2,17),(3,5),(3,9),(3,13),(3,17),(5,5),(5,9),(5,13),(5,17)$
$(8,5),(8,9),(8,13),(8,17),(13,5),(13,9),(13,13),(13,17)\}$
**432.** 1) $A\cup B=[1,5]$ 2)$A\cap B=[2,4]$ 3)$A\setminus B=[1,2)$
4)$B\setminus A=(4,5]$ 5) $A\Delta B=[1,2)\cup(4,5]$
6)$A\times B$ tekislikdagi uchlari $(1,2),(1,5),(4,5),(4,2)$ nuqtalarda bo'lgan kvadratning nuqtalari **435.** 1) $supA=3,\inf A=-7$ 2) $supA=4,\inf A=0$
3) $supA=4,\inf A=-6$

**444.** 1) $1,3,9,27,81$ 2) $0,2,0,2,0$ 3) $2,\frac{3}{2},\frac{4}{3},\frac{5}{4},\frac{6}{5}$ 4) $0,-1,0,1,0$ 5) $2,3,6,11,18$
6) $1,\frac{1}{2},\frac{1}{6},\frac{1}{24},\frac{1}{120}$ 7) $\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\frac{31}{32}$ 8) $-1,0,-1,0,-1$ 9) $0,1,1,2,3,5,8$
**445.** 1) $x_n=\frac{1}{2n-1}$ 2) $x_n=\frac{1}{n^3}$ 3) $x_n=\frac{2n}{2n+1}$ 4) $x_n=\frac{n+1}{n}$
5) $x_n=(-1)^n$ 6) $x_n=3+(-1)^n$ 7) $x_n=(-1)^n(n^2+1)$ 8) $x_n=\frac{1}{3n-1}$
9) $x_n=\frac{1}{(n+1)!}$ 10) $x_n=n^{(-1)^{n+1}}$ **446.** 1) ↗, quyidan chegaralangan 2) ↘, yuqoridan chegaralangan 3) chegaralangan 4) chegaralangan 5) ↘, yuqoridan chegaralangan 6) ↘, chegaralangan 7) chegaralanmagan,monoton emas 8) ↘, chegaralangan 9) quyidan chegaralangan 10) ikkinchi hadidan boshlab ↘
**448.** 1) $\frac{1}{6}$ 2) $-\frac{1}{2}$ 3) 0 4) $\infty$ 5) $-\frac{1}{5}$ 6) 0 7) $\frac{1}{6}$ 8) $-\frac{4}{3}$ 9) $\sqrt{\frac{2}{3}}$ 10) 5
11) 3 12) $\frac{1}{2}$ 13) $e^2$ 14) $e$ 15) $e^{-4}$ 16) 0 17) 1 18) $\pi^{e-1}$
**453.** 1) $(0,+\infty)$ 2) $[-3,1)\cup(1,3]$ 3) $[0,2]$ 4) $[-4,-\pi]\cup[0,\pi]$ 5) $\left[-\frac{1}{3},1\right]$

6) $(-\infty,+\infty)$ **454.** 1) $[1,+\infty)$ 2) $[1,7]$ 3) $\left(-\frac{\pi^2}{2},\frac{\pi^2}{2}\right)$ 4) $(3,+\infty)$ 5) $\left[-\frac{1}{64},0\right)$

6) $(-\infty,-1]$ **455.** 1) toq 2) juft 3) toq 4) juft yoki toq emas 5) toq 6) toq

**456.** 1) davriy, eng kichik musbat davri mavjud emas 2) davriy, $T=2\pi$

3) davriy, $T=\frac{\pi}{3}$ 4) davriy emas 5) davriy, $T=\frac{\pi}{2}$ 6) davriy, $T=\pi$

**457.** 1) $x^9, x^3-1, (x-1)^3$ 2) $|x|, \cos|x|, |\cos x|$ **458.** 1) $y=x$ 2) $y=\sqrt[3]{\frac{x-5}{2}}$

3) Aniqlanish sohasida teskari funksiyasi mavjud emas.

4) $y=-2+10^{x-1}$ 5) $y=\log_2\frac{x}{1-x}$ 6) $y=\begin{cases}-\sqrt{-x}, & x<0,\\ \frac{x}{2}, & x\geq 0.\end{cases}$

**459.** 1)monoton, chegaralangan 2) chegaralangan 3) kamayuvchi 4) chegaralangan 5) o'suvchi 6) kamaymaydigan funksiya

**467.** 1) $\frac{1}{6}$ 2) $\frac{2}{3}$ 3) $-\sqrt{2}$ 4) $\frac{1}{6}$ 5) 4 6) 2 7) 1.5 8) 0 9) $-\frac{\sqrt{2}}{8}$ 10) $e^{10}$

11) $e^{-4}$ 12) $e^{\frac{3}{2}}$ 13) $\frac{1}{9}$ 14) $\frac{4}{9}$ 15) 2 16) $\frac{25}{9}$ 17) $\frac{2}{\ln 5}$ 18) 0 19) $\frac{1}{2}$ 20) 0.6

21) 1

22) $f(2-0)=1, f(2+0)=2$ 23) $f(3-0)=\frac{1}{3}, f(3+0)=0$

24) $f(\frac{\pi}{4}-0)=+\infty, f(\frac{\pi}{4}+0)=0$

**477.** 1) $x=0$ 2-tur uzilish nuqta 2) $x=\frac{2n-1}{2}\pi,\ n\in Z$ 2-tur uzilish nuqtalar

3) $x=\pm 2$ 2-tur uzilish nuqtalar 4) $x=a$ 1-tur uzilish nuqta,

$f(a-0)=-\frac{\pi}{2}, f(a+0)=\frac{\pi}{2}$ 5) $x=0$ 1-tur uzilish nuqta,

$f(-0)=1, f(+0)=0$ 6) $x=\frac{\pi}{4}$ 1-tur uzilish nuqta,

$f(\frac{\pi}{4}-0)=\frac{\sqrt{2}}{2},\ f(\frac{\pi}{4}+0)=0$

## V BOB

**483.** 1) $4x+7$ 2) $3x^2+10x$ 3) $\frac{2}{(x+1)^2}$ 4) $\frac{1}{2\sqrt{x}}$ 5) $\frac{2}{3\sqrt[3]{x}}$

**485.** $f'(2-0)=-1,\ f'(2+0)=1,\ f'(3-0)=-1,\ f'(3+0)=1$

**486.** 1) 23 2) 10! 3) -1.5 4) -2 **487.** $O(0,0), y=x, y=-x$ **488.** $arctg3$

**489.** 14,-2, 7

**497.** 1) $-\frac{2}{x^3}$ 2) $\frac{1}{\sqrt[3]{x^2}}$ 3) $5\cos x-3\sin x$ 4) $5tg^2x$ 7) $-\frac{1}{x\ln 2\log^2{}_2 x}$ 8) $2shx$

**498.** 1) $24x^2\left(2x^3+5\right)^3$ 3) $-\sin 2x$ 6) $\frac{1}{\sin x}$ 7) $\frac{\cos x}{\sqrt{4\sin^2 x-1}}$ 8) $\frac{tg^3\sqrt{x}}{2\sqrt{x}}$

**499.** 1) $x^{x^2+1}\left(1+2\ln x\right)$ 4) $y=e^y\sum_{k=1}^{5}\frac{2k}{x^2-k^2}$ 23. 1) $y'=\frac{x^2-y}{x-y^2}$ 2)

$y'=\frac{y(y-x\ln y)}{y(y-x\ln y)}$

**501.** 2) $2t+1$ 4) $-\text{ctgt}$ **502.** 1) 5 2) 0.5 **510. 1)** $df(x)=(2x+3)dx$ 2) $df(x)=(3x^2+3)dx$ 3) $df(x)=3dx$ **511. 4)** $df(x)=-2xe^{-x^2}dx$ 5) $df(x)=\ln x dx$ 6) $df(x)=\frac{4tg^3x}{\cos^2 x}dx$ **512.** 1) 19.56 2) 0.4557 3) 1.4948 4) 5.9777 5) 8.0625 6) 4.0078 7) -0.02 **513.** 1) $27e^2$ 2) 0 3) 2 4) $\frac{80}{27}$ **514.** 1) 0, ( $n\ge 0$ ) 2) $2^n$

3) $\frac{(-1)^n(-3)(-7)...(5-4n)}{4^n}$ 4) $(-1)^{n-1}2^n(n-1)!$ 5) $(-1)^{n-1}n!$

## VI BOB

**524.** (1;1) **527.** 1) $\frac{1}{108}$; 2) $-1$; 3) $\frac{1}{16}$; 4) $-\frac{1}{6}$; 8) 1 9) $e^{-2}$

**535.** 1. $(-\infty;3)\searrow$, $(3;\infty)\nearrow$ 2. $(-1;7)\searrow$, $(-\infty;-1)\cup(7;\infty)\nearrow$ 3. $(-\infty;\infty)\nearrow$

6. $(-\infty;0)\nearrow$, $(0;\infty)\searrow$ 9. $(0;\frac{1}{e})\searrow$, $(\frac{1}{e};\infty)\nearrow$

**536.** 1. $x_{min}=2$ 2. $x_{min}=3,\ x_{max}=-1$ 4. $x_{min}=-\frac{1}{\sqrt{2}},\ x_{max}=\frac{1}{\sqrt{2}}$

8. $x_{min}=-1,\ x_{max}=1$ 9.ekstremum mavjud emas. **537.** 1. Eng kichik qiymat: $y(-2)=-73$, eng katta qiymat: $y(1)=8$ 2. Eng kichik qiymat: $y(0.5)=1.75$,

eng katta qiymat: $y(2)=58$ 3. Eng kichik qiymat: $y(1)=-3$, eng katta qiymat: $y(0)=0$ 4. Eng kichik qiymat: $y(1)=-1$, eng katta qiymat: $y(0)=y(4)=0$ 5. Eng kichik qiymat: $y(0)=0$, eng katta qiymat: $y(\frac{\pi}{4})=\frac{\pi}{2}-1$ 6. Eng kichik qiymat: $y(2)=\ln 4-2$, eng katta qiymat: $y(1)=\ln 2$. **538.** 1. $x=1.\ (-\infty;-1)\cap,\ (1;\infty)\cup$

2. $x=\pm 1.\ (-1;1)\cap,\ (-\infty;-1)\text{ va }(1;\infty)\cup$ 5. $x=1,\ (-\infty;-1)\cup,(1;\infty)\cap$

**539.** 1. $x=1$ vertikal asimptota, $y=1$ gorizontal asimptota 2. $x=2$ vertikal asimptota, $y=2x+5$ og'ma asimptota. 3. $x=\pm 2$ vertikal asimptota, $y=\pm 2x$ og'ma asimptota 4. $y=\pm(x-0.5)$ og'ma asimptota 5. $y=0$ gorizontal asimptota 6. $x=0$ vertikal asimptota, $y=0$ gorizontal asimptota. **544.** 3.072 **545.** 1.6487

**546.** 1) $-\frac{1}{6}$ 2) $\frac{1}{2}$ 3) $\frac{1}{2}$

## VII BOB

**554.** 1. $\frac{1}{5}x^5-\frac{1}{2}x^4-2x^3+4x^2+7x+C$ 2. $\frac{1}{3}x^3-\frac{1}{2}x^2-\ln|x|-\frac{1}{x}+C$

3. $\frac{1}{3}x^3-x+arctgx+C$ 4. $6\sqrt{x}-\frac{2}{15}x^2\sqrt{x}+C$ 5. $5e^x-\frac{1}{x^3}+C$

6. $\frac{2250^x}{\ln 2250}+C$ 7. $\frac{1}{2}x+\frac{1}{2}\sin x+C$ 8. $tgx-x+C$ 9. $-ctgx-\frac{1}{2}x+C$

10. $-\frac{1}{4}ctgx-\frac{1}{4}tgx+C$

**555.** 1. $-\frac{1}{14}(5-2x)^7+C$ 2. $-\frac{1}{32}(1-2x^4)^4+C$

3. $-\frac{2}{7}\sqrt{2-7x}+C$ 4. $-\frac{2}{3}(1-x)\sqrt{1-x}-\frac{2}{5}(1-x)^2\sqrt{1-x}+C$

5. $\frac{3}{2}\sqrt[3]{(x+1)^2}-3\sqrt[3]{x+1}+3\ln|1+\sqrt[3]{x+1}|+C$

6.

$-\frac{6}{7}t^7+\frac{12}{5}t^5+3t^4-8t^3-12t^2+48t+24\ln(t^2+2)-\frac{96}{\sqrt{2}}arctg\frac{t}{\sqrt{2}}+C,\ t=\sqrt[6]{x+1}$

7. $\frac{1}{3}e^{x^3+1}+C$ 8. $\ln|\ln|\ln x||+C$ 9. $\frac{1}{2}\ln|\sin 2x|+C$ 10. $\left(arctg\sqrt{x}\right)^2+C$

**556.** 1. $e^x(x-1)+C$ 2. $2e^{\sqrt{x}}(\sqrt{x}-1)+C$

3. $\frac{1}{3}x^3 - x + arctgx + C$ 4. $\frac{1}{3}e^{3x} + \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + 3e^x(x^2 - 2x + 2) + \frac{1}{4}x^4 + C$

5. $\frac{1}{29}e^{2x}(2\sin 5x - 5\cos 5x) + C$ 6. $\frac{1}{2}xe^x(\sin x - \cos x) + \frac{1}{2}e^x\cos x + C$

7. $C - x + (x+2)\ln|x+2|$ 8. $\frac{1}{\ln 2}\big((x-0.5)\ln(x-2x) - x\big) + C$

9. $-\frac{x^2}{7}\cos 7x + \frac{2}{49}x\sin 7x + \frac{2}{243}\cos 7x + C$ 10. $2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + C$

**561.** 1) $\ln|x+7| + C$ 2) $-\frac{3}{2}\ln|1-2x| + C$ 3) $\frac{1}{11}arctg\frac{x}{11} + C$

6) $-\frac{5}{12}\cdot\frac{1}{(3-2x)^6} + C$ 7) $\frac{1}{16}\left(arctg\frac{x}{2} + \frac{2x}{x^2+4}\right) + C$ 8) $\frac{2}{\sqrt{23}}arctg\frac{4x-3}{\sqrt{23}} + C$

11) $\frac{1}{4}\ln\left(2x^2 + 2x + 3\right) + \frac{9}{2\sqrt{5}}arctg\frac{2x+1}{\sqrt{5}} + C$

14) $\frac{1}{96}\left(\frac{x(x^2+10)}{(x^2+6)^2} + \frac{1}{\sqrt{6}}arctg\frac{x}{\sqrt{6}}\right) + C$

15) $\frac{1}{8}\left(\frac{(x-3)(3x^2-18x+32)}{(x^2-6x+10)^2} + 3arctg(x-3)\right) + C$

**562.** 1) $\frac{1}{18}\ln\left|\frac{x+3}{x-3}\right| + C$ 2) $\frac{1}{7}\ln\left|\frac{x}{x+7}\right| + C$ 3) $\frac{1}{5}\ln\left|\frac{x-3}{x+2}\right| + C$

4) $\ln\left|\frac{x+2}{(x-1)^3}\right| + C$ 5) $\ln\left|\frac{(x-2)^2}{x-1}\right| + C$

6) $\frac{1}{96}\ln\left|\frac{(4-x)^2}{x^2+4x+16}\right| - \frac{1}{16\sqrt{3}}arctg\frac{x+2}{2\sqrt{3}} + C$

9) $\frac{1}{30}\ln\left|\frac{x+2}{x-1}\right| + \frac{1}{70}\ln\left|\frac{x-3}{x+4}\right| + C$ 10) $\frac{1}{5}\left(\ln\frac{(x-2)^2}{x^2+1} - 9arctgx\right) + C$

14) $\frac{1}{3\sqrt{2}}\mathrm{arctg}\frac{x^3+1}{\sqrt{2}} + C$ 15) $-\frac{57}{16}arctgx - \frac{57x^4 + 103x^2 + 32}{16x(x^2+1)^2} + C$

**567.** 1) $\frac{1}{4}arctg\left(\frac{1}{4}tg\frac{x}{2}\right) + C$ 2) $\frac{1}{\sqrt{6}}arctg\left(\frac{\sqrt{3}}{\sqrt{2}}tg\frac{x}{2}\right) + C$

3) $\frac{1}{\sqrt{5}}\ln\left|\frac{2tg\frac{x}{2} - 3 - \sqrt{5}}{2tg\frac{x}{2} - 3 + \sqrt{5}}\right| + C$ 6) $\frac{2}{\sqrt{3} - tg\frac{x}{2}} + C$

9) $-\frac{1}{6\sin^2 3x}-\frac{2}{3}\ln|\sin 3x|+\frac{1}{6}\sin^2 3x+C$

13) $\frac{3}{2}\cos\frac{x}{6}-\frac{3}{10}\cos\frac{5x}{6}-\frac{3}{14}\cos\frac{7x}{6}+\frac{3}{22}\cos\frac{11x}{6}+C$

14) $-\frac{1}{5}ctg^5x+\frac{1}{3}ctg^3x-ctgx-x+C$ 15) $2tg^3\frac{x}{6}-6tg\frac{x}{6}+x+C$

**577.** 1. $\ln\left|x+6.5+\sqrt{x^2+13x+41}\right|+C$ 3. $\frac{1}{\sqrt{3}}\arcsin(3x+1)+C$

4. $\arcsin\frac{x-1}{3}+C$ 5. $-3\sqrt{1-x^2}+13\arcsin x+C$

7. $\frac{4}{3}t^3-\frac{4}{3}\ln|t^3+1|+C,\ t=\sqrt[4]{x-7}$ 9. $\frac{1}{2}x\sqrt{2-x^2}+\arcsin\frac{x}{\sqrt{2}}+C$

10. $-\frac{2}{3}\sqrt{\frac{3-x}{x}}+C$ 11. $\frac{1}{3}\left(x^2-2x-3\right)\sqrt{x^2-2x+3}+C$

13. $-\frac{\sqrt[3]{\left(2-x^3\right)^2}}{4x^2}+C$ 14. $\frac{1}{10}\sqrt{\left(5x^{\frac{4}{3}}+3\right)^3}+C$ 15.

$\frac{1-\sqrt{x^2+1}}{x}+\ln\left|x+\sqrt{x^2+1}\right|+C$

**585.** 1) 0.5 2) $\frac{1}{3}$ 3) $e$ **586.** 1) $0<I<1$ 2) $12\le I\le 16$ 3) $\frac{\pi}{2\sqrt{7}}\le I\le\frac{\pi}{2\sqrt{3}}$

**587.** 1) $\frac{1}{3}$ 2) $\frac{1}{4}$ 3) $\frac{5}{\pi}(\pi-2)$ **588.** 1) $17.5-6\ln 6$ 2) 32.5 3) $\ln\frac{4}{3}$ 4) $e^e-e$

5) $arctge-\frac{\pi}{4}$ 6) $\frac{\pi}{4}$ 7) $\frac{14}{15}$ 8) $3\sqrt{3}-\pi$ 9) $\frac{4\pi\sqrt{3}}{9}$ 10) $\frac{e-2}{4e}$ 11) $2(2^e+3)$

12) $\frac{2}{5}\left(e^{-\frac{\pi}{2}}-e^{\frac{\pi}{2}}\right)$ 13) $e^{\frac{\pi}{2}}-1$ 14) 0.089 15) $\frac{\pi}{4}+\frac{3\sqrt{3}}{2}-3$

**590.** 1) 0.916 2) 0.693 3) 2.320 4) 0.882 **591.** 1) 0.69377 2) 0.78458 3) 0.84259 4) 0.23661. **595.** 1) 0.5 2) uzoqlashuvchi 3) $\pi$

4) $\alpha>1,\frac{1}{\alpha-1};\ \alpha\le 1,$ *uzoqlashadi* 5) 1 6) 0.5 7) 2 8) $\pi$ 9) 6

10) $\alpha<1,\frac{1}{1-\alpha};\ \alpha\ge 1,$ *uzoqlashuvchi* 11) $2\sqrt{2}$ 12) 1.5

**596.** 1) uzoqlashadi 2) uzoqlashadi
3) yaqinlashuvchi 4) uzoqlashadi
5) uzoqlashadi 6) absolut yaqinlashuvchi.

**605.** 1) $\frac{4}{3}$ 2) $\frac{10}{3}$ 3) $\frac{32}{3}$ 4) 9 **606.** 1) $27\pi$ 2) $\frac{5\pi}{6}$ 3) $\frac{4}{35}\pi$ 4) $12\pi$

**607.** 1) $\sqrt{2}+\ln(1+\sqrt{2})$ 2) $\arcsin\frac{3}{4}$ 3) $4+\frac{\ln 3}{4}$ 4) $\ln\left(2+\sqrt{3}\right)$

**608.** 1) $\pi\left(e^2-e^{-2}+4\right)$ 2) $\frac{61\pi}{1728}$ 3) $2\pi b\left[b+\frac{a^2}{c^2}\arcsin\frac{c}{a}\right], c=\sqrt{a^2-b^2}$ 4) $\frac{64\pi}{3}$

**609.** 1400 **610.** 0.08 J

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# MUNDARIJA

YUSUPJON PULATOVICH APAKOV,
BAXRIDDINXO'JA ISMOILOVICH JAMALOV,
AKBARXO'JA MAMAJONOVICH TO'XTABAYEV

# OLIY MATEMATIKADAN MISOL VA MASALALAR

**Ikki jildlik**

**1-jild**

**Toshkent – «Donishmand ziyosi» – 2022**

| | |
|---|---|
| Muharrir: | T.Mirzayeva |
| Tex. muharrir: | R.Axmedov |
| Rassom-dizayner: | D.Mulla-Axunov |
| Kompyuterda sahifalovchi: | G.Axmedova |

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Matematikaning xususiy hosilali differensial tenglamalar, yuqori tartibli karrali xarakteristikali tenglamalar va aralash parabolo-giperbolik tenglamalar sohasida ilmiy tadqiqotlar olib boradi. Uning rahbarligida 1 ta PhD himoya qilingan, 1 ta DSc va 2 ta PhD ilmiy izlanishlar olib borishmoqda.

Olimning «Oddiy differensial tenglamalardan misol va masalalar» (Turgunov N., Gafarov I.lar bilan hammualliflikda) o'quv qo'llanmasi, «К теории уравнений третьего порядка с кратными характеристиками» monografiyasi hamda «Oliy matematika» darsligi nashr qilingan. 200 dan ortiq ilmiy maqolalari (shulardan 30 dan ortig'i nufuzli xalqaro nashrlarda) chop etilgan.

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Matematikaning xususiy hosilali differensial tenglamalar, yuqori tartibli aralash parabolo-giperbolik tenglamalar sohasida ilmiy tadqiqotlar olib boradi.

Bir necha ilmiy va ilmiy-ommabop maqolalar muallifi.

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Bir nechta ilmiy va ilmiy-ommabop maqolalar muallifi.

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