O`ZBEKISTON RESPUBLIKASI OLIY VA O`RTA MAXSUS TA`LIM VAZIRLIGI

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Ko`p o`zgaruvchili funksiyalar

(o`quv qo`llanma)

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1-§. Asosiy tushuncha va teoremalar

R^m fazoda ketma-ketlikka va uning limiti.

Ushbu

$$R^m = \underbrace{R \quad R \quad ... \quad R}_{m-ta} = \quad (x_1, x_2, ..., x_m) : x_k \quad R, \;\; k = \overline{1, m}$$

to`plamga $m$ **o`lchovli Evklid fazosi** deyiladi.

Ixtiyoriy $x = (x_1, x_2, ..., x_m) : x_k \quad R^m$ va $y = (\mathrm{y}_1, \mathrm{y}_2, ..., \mathrm{y}_\mathrm{n}) \quad \mathrm{R}^\mathrm{m}$ nuqtalarni olaylik. Quyidagi

$$\rho(x, y) = \sqrt{(y_1 - x_1)^2 + ... + (y_m - x_m)^2} = \sqrt{\sum_{r=1}^{m} (y_r - x_r)^2} \qquad \textbf{(1)}$$

miqdor x va y nuqtalar orasidagi **masofa** deb ataladi.

U quyidagi xossalarga ega

1) $\rho(x, y) \quad 0$ va $(\rho(x, y) = 0 \quad\quad x = y)$;

2) $\rho(x, y) = \rho(y, x)$;

3) $\rho(x, z) \quad \rho(x, y) + \rho(y, z)$ (uchburchak tengsizligi).

Natural sonlar to`plami $N$ va $R^m$ fazo berilgan bo'lib, $f$ har bir $n \quad N$ va $R^m$ fazoning biror $x^{(n)} = (x_1^{(n)}, ..., x_m^{(n)})$ nuqtasini mos qo`yuvchi akslantirish bo`lsin:

$$f : N \to R^m \quad yoki \quad n \to x^{(n)}$$

$f : N \to R^m$ akslantirish obrazlaridan tuzilgan

$$x^{(1)}, x^{(2)}, ..., x^{(n)} \qquad (2)$$

to`plam **ketma-ketlik** deb ataladi va u $\{x^{(n)}\}$ kabi belgilanadi.

Demak, (2)-ketma-ketlikning hadlari R^m fazo nuqtalardan iborat.

$\{x^{(n)}\}$ ketma-ketlikning mos koordinatalaridan tuzilgan $\{x_1^{(n)}\}, ..., \{x_m^{(n)}\}$ lar sonli ketma-ketlik bo`lib, $\{x^{(n)}\}$ ketma-ketlikni shu m ta ketma-ketlikning birgalikda qaralishi deb hisoblash mumkin.

Aytaylik R^m fazoda $\{x^{(n)}\}$ ketma-ketlik va $a=(a_1,...,a_m)$ R^m nuqta berilgan bo`lsin.

1-Ta`rif. $\forall\varepsilon>0,\ \exists n_0(\varepsilon)\quad N:\forall n>n_0\qquad \rho(x^{(n)},a)<\varepsilon$ **, bo'lsa** *unda* a *nuqta* $\{x^{(n)}\}$ *ketma-ketlikning* ***limiti*** *deb ataladi va* $\lim\limits_{n\to} x^{(n)}=a$ *kabi belgilanadi.*

Limitga quyidagicha ham ta`rif berish mumkin

2-Ta`rif. *Agar a nuqtaning* $\forall\ \ _\delta(a)=\{x\quad R^m:\rho(x,a)<\varepsilon\}$ $\{x^{(n)}\}$ *atrofi olinganda ham* $\exists n_0(\varepsilon)\quad N:\forall n>n_0\qquad x_n\qquad _\delta(a)$*, unda* a *nuqta* $\{x^{(n)}\}$ *ketma-ketlikning* ***limiti*** *deb ataladi.*

Teorema. R^m *fazoda* $\{x^{(n)}\}=\{(x_1^{(n)},...x_m^{(n)})\}$ *ketma-ketlikning* $a=(a_{1,...,}a_m)\epsilon$ R^m *nuqtaga yaqinlashishi uchun* $n\to$ *da bir yo`la* $x_1^{(n)}\to a_1,...\ x_m^{(n)}\to a_m$ *bo`lishi zarur va yetarli.*

Eslatma: R^m fazodagi ketma-ketlik uchun ham sonli ketma-ketlik uchun o`rinli bo`lgan xossalar o`rinli.

1 – misol. R^2 fazoda ushbu

$$\{x^{(n)}\}=\ \frac{5}{n};\frac{4}{n}$$

ketma – ketlikning limiti *a(0,0)* ekanligini isbot qiling:

◄ $\forall\varepsilon>0$ *uchun* $\rho\left(x^{(n)},a\right)<\varepsilon$ *ekanligini ko`rsatish kerak*

$$\rho\left(x^{(n)};a\right)=\rho\qquad \frac{5}{n};\frac{4}{n}\ ;(0;0)\ =\sqrt{\ \frac{5}{n}-0\ ^2+\ \frac{4}{n}-0\ ^2}\ =$$

$$=\sqrt{\frac{25}{n^2}+\frac{16}{n^2}}=\sqrt{\frac{41}{n^2}}=\frac{\sqrt{41}}{n}\quad \forall n\ >n_0=\ \frac{\sqrt{41}}{\varepsilon}\ +1\quad ni\ \ olsak$$

$$< \frac{\sqrt{41}}{n_0} = \frac{\sqrt{41}}{\frac{\sqrt{41}}{\varepsilon} \quad +1} < \varepsilon$$

bo`ladi.

Demak ,

$$\rho\left(x^{(n)}, a\right) < \varepsilon \ \blacktriangleright$$

2 – misol. R^2 fazoda ushbu

$$\{x^{(n)}\} = \quad \frac{3+2n}{1-2n}; \frac{n^2+1}{2n^2-3}$$

ketma – ketlikning limiti $a \quad -1; \frac{1}{2}$ ekanligini isbot qiling:

$$\blacktriangleleft \qquad \rho\left(x^{(n)}, a\right) = \quad \frac{3+2n}{1-2n}; \frac{n^2+1}{2n^2-3} \ ; \ -1; \frac{1}{2} \quad = \sqrt{\quad \frac{3+2n}{1-2n} + 1 \quad ^2 + \quad \frac{n^2+1}{2n^2-3} - \frac{1}{2} \quad ^2} =$$

$$= \sqrt{\quad \frac{4}{1-2n} \quad ^2 + \quad \frac{5}{2\left(2n^2-3\right)} \quad ^2}$$

$$a \quad 0, b \quad 0 \quad \sqrt{a^2 - b^2} \quad a + b \quad \frac{4}{2n-1} + \frac{5}{2\left(2n^2-3\right)} \quad \frac{4}{n} + \frac{1}{n} =$$

$$= \frac{5}{n} \quad \frac{5}{n_0} = \frac{1}{\frac{5}{\varepsilon} \quad +1} < \varepsilon \qquad n_0 = \quad \frac{5}{\varepsilon} \quad +1 \ \blacktriangleright$$

3 – misol. R^2 fazoda ushbu

$$\left\{x^{(n)}\right\} = \left\{(-1)^n, (-1)^n\right\}$$

ketma – ketlikning limiti mavjud emasligini ko`rsating.

◄Teskarisini faraz qilamiz, yani berilgan ketma – ketlik limitga ega va u *a=(a₁, a₂)* ga teng bo`lsin. Unda limit ta’rifiga ko`ra $\forall\varepsilon > 0$ uchun, jumladan $\varepsilon=1$ uchun shunday $n_0 \quad N$ topiladiki, $\forall n > n_0$ da

$$\rho\left(\left(-1,-1\right),\left(a_1,a_2\right)\right) < \ \varepsilon,$$

$$\rho\left(\left(1,1\right),\left(a_1,a_2\right)\right) < \ \varepsilon$$

bo`ladi.

Agar $\rho\left(\left(-1,-1\right),\left(1,1\right)\right) = 2\sqrt{2}$ **e**kanligini e`tiborga olsak, yuqoridagi munosabatlardan

$$2\sqrt{2} = \rho\left(\left(-1,-1\right),\left(1,1\right)\right) \quad \rho\left(\left(-1,-1\right),\left(a_1,a_2\right)\right) + \\ + \rho\left(\left(a_1,a_2\right),\left(1,1\right)\right) < \ \varepsilon + \varepsilon = 2\varepsilon = 2,$$

$$2\sqrt{2} < 2$$

ziddiyatga kelamiz. Bunga sabab berilgan ketma – ketlik limitga ega deb qarashdadir. Demak, berilgan ketma – ketlik limitga ega emas►

Endi ketma – ketlik limitini hisoblashga misollar keltiramiz:

4 – misol.

$$\{x^{(n)}\} = \frac{\log_5\left(n^2+1\right)}{n}; \frac{n-\lg n}{\log_2\left(4^n+1\right)}$$

Ketma – ketlikning limitini toping.

◄ Ketma – ketlikning koordinatalaridan tashkil topgan ketma – ketlik sonli ketma – ketlik bo`lib, ular quyidagiga teng:

$$x_1^{(n)} = \frac{\log_5\left(n^2+1\right)}{n}, \qquad x_2^{(n)} = \frac{n-\lg n}{\log_2\left(4^n+1\right)}$$

bularni limitini topamiz:

$$a_1 = \lim_{n\to} x_1^{(n)} = \lim_{n\to} \frac{\log_5\left(n^2+1\right)}{n} = \text{—}, \textit{Lopital qoidasidan foydalanamiz} =$$

$$= \lim_{n\to} \frac{2n}{\left(n^2+1\right)\ln 5} = 0;$$

$$a_2 = \lim_{n\to} x_2^{(n)} = \lim_{n\to} \frac{n - \lg n}{\log_2\left(4^n+1\right)} = \frac{n - \log n}{\log_2\left(4^n+1\right)} < \frac{n}{\log_2 4^n} = \lim_{n\to} \frac{n}{n \lg_2 4} = \frac{1}{2};$$

Ketma – ketlikning limiti haqidagi teoremadan

$$\lim_{n\to} x^{(n)} = a, \qquad a \;\; 0; \frac{1}{2} \quad ►$$

Amaliy mashg’ulot uchun misol va masalalar.

I. *R^2* fazoda quyidagi ketma-ketliklarning limiti $a(a \quad R^2)$ ekanligi ta`rif yordamida isbotlansin.

1.1 $x^{(n)} = \frac{13-n^2}{1+2n^2}, \frac{2n-1}{2-3n} \;; a \;\; -\frac{1}{2}; -\frac{2}{3}$.

1.2 $x^{(n)} = \frac{3n^2+2}{4n^2-1}, \frac{2n^3}{n^3-2} \;; a \;\; \frac{3}{4}; 2$.

1.3 $x^{(n)} = \frac{1-2n^2}{n^2+3}, \frac{3n^2}{2-n^2} \;; a(-2;-3).$

1.4 $x^{(n)} = \frac{4+2n}{1-3n}, \frac{5n+15}{6-n} \;; a \;\; -\frac{2}{3}; -5$.

1.5 $x^{(n)} = \frac{4n^2+1}{3n^2+2}, \frac{4-n^3}{3+2n^3} \; ; a \; \frac{4}{3}; -\frac{1}{2} \; .$

1.6 $x^{(n)} = \frac{1-2n^2}{2+4n^2}, -\frac{5n}{n+1} \; ; a \; -\frac{1}{2}; -5 \; .$

1.7 $x^{(n)} = \frac{1}{n}; \frac{2}{n}\cos n\pi \; ; a(0,0).$

1.8 $x^{(n)} = \frac{3n-2}{2n-1}, \frac{4n-1}{2n+1} \; ; a \; \frac{3}{2}; 2 \; .$

1.9 $x^{(n)} = \frac{2n}{3n+1}, \frac{1+n}{1-2n} \; ; a \; \frac{2}{3}; -\frac{1}{2} \; .$

1.10 $x^{(n)} = \frac{\cos n}{n}, \frac{n-1}{n^2+1} \; ; a(0,0).$

II. R^2 fazoda quyidagi ketma-ketliklarning limiti topilsin.

2.1 $x^{(n)} = \frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n-1}{n^2}; \; \frac{2n^2+2}{2n^2+1}\;^{n^2} \; .$

2.2 $x^{(n)} = \frac{(2n+1)! + (2n+2)!}{(2n+3)!}; \; \frac{n-1}{n+3}\;^{n+2} \; .$

2.3 $x^{(n)} = \frac{2^{n+1}+3^{n+1}}{2^n+3^n}; \; \frac{n^2-1}{n^2}\;^{n^4} \; .$

2.4 $x^{(n)} = \frac{1+2+\ldots+n}{\sqrt{9n^4+1)}}; \; \frac{2n+3}{2n+1}\;^{n+1} \; .$

2.5 $x^{(n)} = \frac{1+4+7+\ldots+(3n-2)}{\sqrt{5n^4+n+1)}}; \; \frac{n+1}{n-1}\;^{n} \; .$

2.6 $x^{(n)} = \frac{(n+4)!-(n+2)!}{(n+3)!};\ \frac{n+3}{n+5}^{\,n+4}$.

2.7 $x^{(n)} = \frac{\sqrt[3]{n^3+5}-\sqrt{3n^4+2}}{1+3+5+\ldots+(2n-1)};\ \frac{n^3+1}{n^3-1}^{\,2n-n^3}$.

2.8 $x^{(n)} = \frac{3}{4}+\frac{5}{16}+\frac{9}{64}+\ldots+\frac{1+2^n}{4^n}, n\left(\sqrt[3]{5+8n^3}-2n\right)$.

2.9 $x^{(n)} = \frac{1}{1\ 2}+\frac{1}{2\ 3}+\ldots+\frac{1}{n(n+1)}; \sqrt{2}\ \sqrt[4]{2}\ \sqrt[8]{2}\ \ldots\ \sqrt[2n]{2}$.

2.10 $x^{(n)} = \frac{\sqrt[3]{n^2}\sin n!}{n+1}; \frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$.

Mustaqil mashg'ulot uchun misol va masalalar

I. R^2 fazoda quyidagi ketma-ketliklarning limiti $a(a \quad R^2)$ **ekanligi ta`rif yordamida isbotlansin.**

M1.1 $x^{(n)} = \frac{1}{n^2}; \frac{5}{n}\ ; a(0{,}0)$.

M1.2 $x^{(n)} = \frac{2}{n}; \frac{n}{n+1}\ ; a(0{,}1)$.

M1.3 $x^{(n)} = \frac{2-n^2}{3n^2+2}, \frac{5n-1}{2-10n}\ ; a\ -\frac{1}{3}; -\frac{1}{2}$.

M1.4 $x^{(n)} = \frac{5+n^2}{2-n^2}, \frac{2n+1}{4-2n}\ ; a(-1;-1)$.

M1.5 $x^{(n)} = \frac{-2n^2}{3+5n^2}, \frac{1-n^2}{1+n^2}\ ; a\ -\frac{2}{5}; -1$.

M1.6 $x^{(n)} = \quad \frac{1+5n}{2+15n}, \frac{1-n}{5+2n} \quad ; a \quad \frac{1}{3}; -\frac{1}{2} \quad .$

M1.7 $x^{(n)} = \quad \frac{4}{n}; \frac{4}{n}\cos n\pi \quad ; a(0,0).$

M1.8 $x^{(n)} = \quad \frac{2n}{5n+1}, \frac{2n+1}{4n-1} \quad ; a \quad \frac{2}{5}; \frac{1}{2} \quad .$

M1.9 $x^{(n)} = \quad \frac{2}{n^2}, \frac{6}{n} \quad ; a(0;0).$

M1.10 $x^{(n)} = \quad \frac{n^3+1}{n^3-1}, \frac{2n^2+2}{3n^2-2} \quad ; a \quad 1; \frac{2}{3} \quad .$

M1.11 $x^{(n)} = \quad \frac{\cos n}{n}, \frac{n-1}{2n^2+1} \quad ; a(0;0).$

M1.12 $x^{(n)} = \quad \frac{5+2n}{1-n}, \frac{3n+11}{6-n} \quad ; a(-2;-3).$

M1.13 $x^{(n)} = \quad \frac{1+10n}{1-10n}, \frac{2n+1}{3-n^2} \quad ; a(-1;0).$

M1.14 $x^{(n)} = \quad \frac{5}{n}, \frac{n}{n+5} \quad ; a(0;1).$

M1.15 $x^{(n)} = \quad \frac{5+6n}{1-6n}, \frac{n+2}{n^3+1} \quad ; a(-1;0).$

M1.16 $x^{(n)} = \quad \frac{3-n^2}{1-n^3}, \frac{1-n^3}{2+n^3} \quad ; a(0;-1).$

M1.17 $x^{(n)} = \quad \sqrt[n]{3}; \frac{\log_3 n}{n} \quad ; a(1;0).$

M1.18 $x^{(n)} = \quad \frac{2^n}{n!}; \frac{n}{3^n} \quad ; a(0;0).$

M1.19 $x^{(n)} = \; \frac{1}{n}; \frac{\sin n}{n} \; ; a(0;0).$

M1.20 $x^{(n)} = \; \sqrt[n]{n}; \frac{n^3}{3^n} \; ; a(1;0).$

II. R^2 fazoda quyidagi ketma-ketliklarning limiti topilsin.

M2.1 $x^{(n)} = \; \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + ... + \frac{2n-1}{2^n}; \; \frac{n^3-1}{n^3} \; ^{n^4} \; .$

M2.2 $x^{(n)} = \; \frac{5^n}{n!}; \; \frac{n+2}{n-2} \; ^{n} \; .$

M2.3 $x^{(n)} = \; \frac{1}{2\;\;5} + \frac{1}{5\;\;8} + \frac{1}{8\;\;11} + ... + \frac{1}{n(n+3)}; \frac{1}{2} \;\; \frac{3}{4} \;\;\;\; \frac{2n-1}{2n} \; .$

M2.4 $x^{(n)} = \;\; 1 - \frac{1}{2^2} \;\;\; 1 - \frac{1}{3^2} \;\;\;\; 1 - \frac{1}{n^2} \; ; \sqrt{n+1} - \sqrt{n} \; .$

M2.5 $x^{(n)} = \;\; 1 - \frac{1}{3} \;\;\; 1 - \frac{1}{6} \;\;\;\; 1 - \frac{1}{\frac{n(n+1)}{2}} \; ; \frac{\log_5 n}{n} \; .$

M2.6 $x^{(n)} = \; \frac{2^3-1}{2^3+1} \; \frac{3^3-1}{3^3+1} \; ... \; \frac{n^3-1}{n^3+1}; \; \frac{n-2}{n+4} \; ^{n+3} \; .$

M2.7 $x^{(n)} = \;\; \frac{n^2-1}{2n+1} \; ^{6} ; \frac{(n+1)^{20}+3}{n^{21}-5} \; .$

M2.8 $x^{(n)} = \; \frac{2^{n+2}+3^{n+3}}{2^n+3^n}; \frac{5\;\;2^n - 3\;\;5^{n+1}}{100\;\;2^n + 5^{n+1}} \; .$

M2.9 $x^{(n)} = \left(\sqrt[4n]{16}; \sqrt[4n]{0,5}\right).$

M2.10 $x^{(n)} = \dfrac{\sqrt[n]{8}-1}{\sqrt[n]{2}-1}; 0{,}5^{\frac{2n+5}{2n+1}}$.

M2.11 $x^{(n)} = \left(1+13^n\right)^{\frac{1}{n+1}}; a^{\frac{1}{n+p}}$, *a>0, p>0.*

M2.12 $x^{(n)} = \left(\sqrt[n]{n^3}; \sqrt[n^2]{n}\right).$

M2.13 $x^{(n)} = \left(\sqrt[n]{3n-2}; \sqrt[n]{n^3+3n}\right).$

M2.14 $x^{(n)} = \dfrac{\log_2\left(n^2+1\right)}{n}; \dfrac{n-\log_3 n}{\log_3\left(9^n+1\right)}$.

M2.15 $x^{(n)} = \dfrac{(-2)^n}{(n+2)!}; \dfrac{1}{(0{,}5)^n \quad n!}$.

M2.16 $x^{(n)} = \dfrac{1}{1\ 2\ 3} + \dfrac{1}{2\ 3\ 4} + \ldots + \dfrac{1}{n(n+1)(n+2)}; \sqrt[n]{4n+1}$.

M2.17 $x^{(n)} = \dfrac{3^n+2^n}{6^n}; \ \dfrac{0{,}5n}{n+1}^{n^2}$.

M2.18 $x^{(n)} = \dfrac{5^n+2^n}{10^n}; \ \dfrac{9n+1}{8n-3}^{\frac{n}{2}} \ \dfrac{8}{9}^{\frac{2n}{3}}$

M2.19 $x^{(n)} = \dfrac{n \ \ 3^n+1}{n!+1}; \dfrac{4^n+n^2 2^n-1}{n^4+(n!)^2}$.

M2.20 $x^{(n)} = \dfrac{\sqrt[n]{6n-n}}{\sqrt[n]{0{,}1}}; \dfrac{\sqrt[n]{n^3+3n}}{\sqrt[n^2]{0{,}01}}$.

2-§. Ko`p o`zgaruvchili funksiya va uning limiti

Ko`p o`zgaruvchili funksiya, funksiyaning aniqlanish sohasi va qiymatlar to`plami, ko`p o`zgaruvchili murakkab funksiya ta`riflari bir o`zgaruvchili funksiyadagi mos ta`riflar kabi kiritiladi.

2.1-misol. Quyidagi

$$u = 2x \arcsin \frac{x}{y^2} + 3y^2 \arcsin(1-y)$$

Funksiyaning aniqlanish sohasi topilsin va chizmada tasvirlansin.

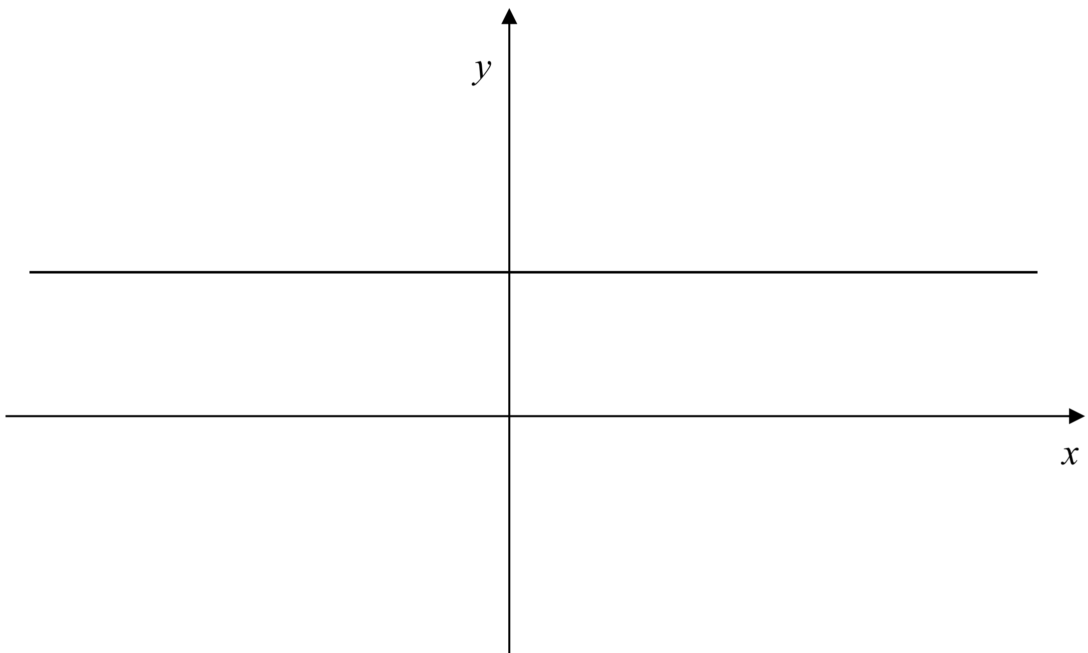
◀Arksinus funksiyaning aniqlanish sohasidan foydalanamiz.

$$D(u) = \begin{cases} y \quad 0, \\ \left|\frac{x}{y^2}\right| \quad 1, \\ |1-y| \quad 1. \end{cases} \qquad \begin{cases} y \quad 0, \\ -1 \quad \frac{x}{y^2} \quad 1 \\ -1 \quad 1-y \quad 1 \end{cases} \qquad \begin{cases} y \quad 0, \\ -y^2 \quad x \quad y^2 \\ 0 < y \quad 2 \end{cases}$$

yoki

$$D(u) = \{(x,y) \quad R^2 : 0 < y \quad 2, \ -y^2 \quad x \quad y^2\}$$

bo’lib, bu soha 2.1-chizmada tasvirlangan▶



2.1-chizma.

2-misol. Quyidagi

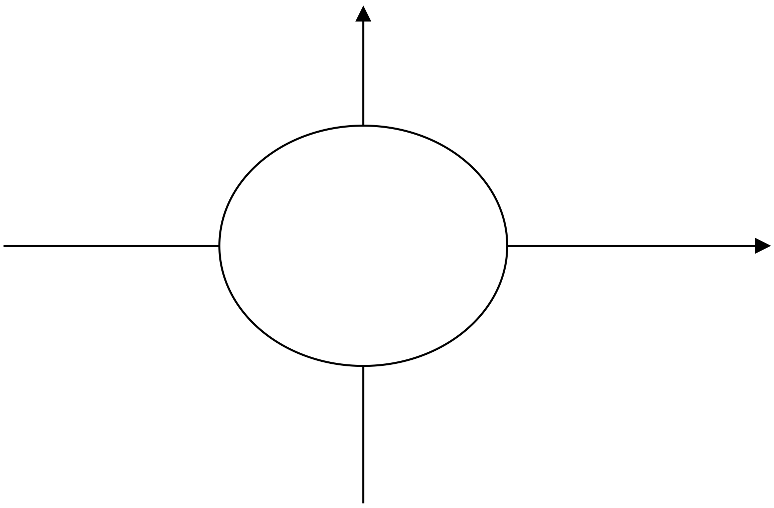
$$u = \frac{\sqrt{4x - y^2}}{\ln(1 - x^2 - y^2)}$$

funksiyaning aniqlanish sohasi topilsin va chizmada tasvirlansin.

◀Aniqlanish sohasini topish uchun kasr ratsional funksiya, logarifmik funksiya va kvadrat ildiz ostidagi funksiya xossalaridan foydalanamiz:

$$D(u) = \begin{matrix} \ln(1 - x^2 - y^2) \quad 0, \\ 1 - x^2 - y^2 \quad 0, \\ 4x - y^2 \quad 0. \end{matrix} \qquad \begin{matrix} (x,y) \quad 0, \\ x^2 + y^2 < 1, \\ x \quad \frac{y^2}{4} \end{matrix} \qquad (x,y) \quad R^2 : (x,y) \quad (0,0), \;\; x^2 + y^2 < 1, \;\; x \quad \frac{y^2}{4}$$

bo'lib, bu soha 2.2-chizmada tasvirlangan▶



2.2-chizma.

$M \quad R^m$ to`plam berilgan bo`lib, a nuqta M to`plamning limit nuqtasi va $y= f(x)=f(x_1,...,x_m)$ funksiya $M$ to`plamda aniqlangan bo`lsin.

1-Ta`rif (Koshi ta`rifi) . *Agar* $\forall\varepsilon >0$ *uchun* $\exists\delta = \delta(\varepsilon, a) >0$ *son topilsaki, ushbu* $0<\rho(x,a)<\delta$ *tengsizlikni qanoatlantiruvchi* $\forall x \quad M$ *uchun*

$$\left| f(x) - b \right| < \varepsilon$$

tengsizlik bajarilsa, unda b son $f(x)$ *funksiyaning a nuqtadagi limiti (yoki* ***karrali limiti*** *) deyiladi va*

$$\lim_{x \to a} f(x) = b$$

yoki

$$\lim_{\substack{x_1 \to a_1 \\ \dots \\ x_m \to a_m}} f(x_1, x_2, ..., x_m) = b$$

kabi belgilanadi.

2- Ta`rif (Geyne ta`rifi). *Agar M to'plamning nuqtalaridan tuzilgan, a ga intiluvchi har qanday* $\{x^{(n)}\}$ $\left(x^{(n)} \quad a, \quad n = 1,2,...\right)$ *ketma – ketlik olinganda ham mos* $\left\{f(x^{(n)})\right\}$ *ketma – ketlik hamma vaqt yagona b (chekli yoki cheksiz) limitga intilsa, b son f(x) funksiyaning a nuqtadagi limiti deb ataladi.*

Agar $a=$ yoki $b=$ bo`lsa, unda ham shu kabi ta`riflarni berish mumkin. Ko`p o`zgaruvchili funksiyalar uchun boshqa formadagi limit tushunchasini ham kiritish mumkin. Masalan, bunda avval bir o`zgaruvchi bo`yicha limitga o`tilib, qolgan $m$-$1$ ta o`zgaruvchini fiksirlangan (tayin) deb faraz qilinadi. Keyin, qolgan m-1 ta o`zgaruvchining biri bo`yicha limitga o`tilib, $m$-$2$ ta o`zgaruvchini fiksirlangan deb faraz qilinadi. Bu jarayonni m marta qaytarish natijasida hosil qilingan limitga $f(x_1, ..., x_m)$ funksiyaning **takroriy limiti** deyiladi. m o`zgaruvchili funksiyaning jami **m!** ta takroriy limitini qarash mumkin. Masalan, ikki o`zgaruvchili $f(x_1,x_2)$ funksiya uchun 2 ta $\lim\limits_{x_1 \to a_1} \lim\limits_{x_2 \to a_2} f(x_1,x_2)$ va $\lim\limits_{x_2 \to a_2} \lim\limits_{x_1 \to a_1} f(x_1,x_2)$ takroriy limitni ko`rish mumkin.

5 - misol. Ushbu

$$f(x, y) = \begin{cases} x + y\sin\frac{1}{x}, & x \quad 0, \\ 0, & x = 0 \end{cases}$$

funksiyaning (0,0) nuqtadagi takroriy va karrali limitlari hisoblansin.

◀ $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0}\lim\limits_{y\to 0} \quad x + y\lim\frac{1}{x} \quad = \lim\limits_{x\to 0} x = 0$ mavjud.

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = \lim\limits_{y\to 0} \quad y\lim\limits_{x\to 0}\sin\frac{1}{x}$ mavjud emas, lekin $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$

mavjud va u nolga teng.

Darhaqiqat ,

$$0 \quad |f(x,y) - 0| = \left|x + y\sin\frac{1}{x}\right| \quad |x| + |y| \quad (x \quad 0) \quad \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 .$$ ▶

6 – misol. Ushbu

$$f(x,y) = \frac{x - y}{x + y}$$

funksiyaning $x \to 0, \; y \to 0$ dagi takroriy va karrali limitlarni toping.

◀ $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0}\lim\limits_{y\to 0}\frac{x - y}{x + y} = \lim\limits_{x\to 0}\frac{x}{x} = 1$,

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}\lim_{x\to 0}\frac{x - y}{x + y} = \lim_{y\to 0}\frac{-y}{y} = -1.$$

Karrali limitini topishda Geyne ta`rifidan foydalanamiz:

$n \to$ da ikkita (0,0) nuqtaga intiluvchi

$$\{x_n, y_n\} = \frac{1}{n}, \frac{1}{n} \quad , \quad \{x'_n, y'_n\} = \frac{2}{n}, \frac{1}{n}$$

ketma – ketliklarda funksiya limiti har xil sonlarga intiluvchiligini ko`rsatamiz.

$$f(x_n, y_n) = \frac{\frac{1}{n} - \frac{1}{n}}{\frac{1}{n} + \frac{1}{n}} = 0, \quad f(x_n, y_n) \to 0$$

$$f(x_n, y_n) = \frac{\frac{2}{n} - \frac{1}{n}}{\frac{2}{n} + \frac{1}{n}} = \frac{\frac{1}{n}}{\frac{3}{n}}, \quad f(x_n, y_n) \to \frac{1}{3}$$

Demak, $n \to$ $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ mavjud emas ►

7 – misol. Ushbu

$$\lim_{\substack{x \to + \\ y \to +}} \left(x^2 + y^2\right) e^{-(x+y)}$$

limitni hisoblang.

◄Elementar tengsizliklardan foydalanamiz. (x>0, y>0).

$$0 < \left(x^2 + y^2\right) e^{-(x+y)} = \frac{x^2}{e^{x+y}} + \frac{y^2}{e^{x+y}} < \frac{x^2}{e^x} + \frac{y^2}{e^y}.$$

$$0 \quad \lim_{\substack{x \to + \\ y \to +}} \left(x^2 + y^2\right) e^{-(x+y)} \quad \lim_{\substack{x \to + \\ y \to +}} \frac{x^2}{e^x} + \frac{y^2}{e^y} \quad = 0.$$

Demak,

$$\lim_{\substack{x \to + \\ y \to +}} \left(x^2 + y^2\right) e^{-(x+y)} = 0 \; ►$$

8 – misol. **Ushbu**

$$\lim_{\substack{x\to \\ y\to}} \frac{x^2+y^2}{x^4+y^4}$$

limitni hisoblang $(x \quad 0; \quad y \quad 0)$.

◀ $0 \quad \frac{x^2+y^2}{x^4+y^4} = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} \quad \frac{x^2}{x^4} + \frac{y^2}{y^4} = \frac{1}{x^2} + \frac{1}{y^2}$

$$\lim_{\substack{x\to+ \\ y\to+}} \frac{1}{x^2} + \frac{1}{y^2} \quad = 0$$ dan $$\lim_{\substack{x\to \\ y\to}} \frac{x^2+y^2}{x^4+y^4} = 0$$ ▶

Ayrim hollarda $x = a + r\cos\varphi, \quad y = b + r\sin\varphi$ almashtirish

$$\lim_{\substack{x\to 0 \\ y\to 0}} f(x,y)$$

karrali limitni topishni yengillashtiradi. Bunda

$$f(x,y) = f(a + r\cos\varphi, \ b + r\sin\varphi) = F(r,\varphi)$$

bo`lib,

$$\lim_{\substack{x\to a \\ y\to b}} f(x,y) = c \qquad \lim_{r\to 0} F(r,\varphi) = c$$

9-misol. **Ushbu**

$$\lim_{\substack{x\to 0 \\ y\to 0}} \frac{\sqrt[3]{x^4y^3}}{x^2+y^2}$$

limitni hisoblang.

◀ $x = a + r\cos\varphi, \quad y = b + r\sin\varphi$ almashtirishdan foydalamiz, bunda

$a=0; \quad b=0$ va $x \to 0 \ \ y \to 0 \ \ da \ \ r \to 0;$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt[3]{x^4 y^3}}{x^2+y^2} = \lim_{r\to 0} \frac{\sqrt[3]{(r\cos\varphi)^4 \ (r\sin\varphi)^3}}{r^2\cos^2\varphi + r^2\sin^2\varphi} =$$

$$= \lim_{r\to 0} \frac{r^2\sqrt[3]{r} \ \sqrt[3]{\cos^4\varphi \sin^3\varphi}}{r^2} = \lim_{r\to 0} \sqrt[3]{r} \ \sqrt[3]{\cos^4\varphi \sin^3\varphi} = 0,$$

chunki

$$\sqrt[3]{\cos^4\varphi \sin^3\varphi}$$

chegaralangan ►

10-misol. Agar $f(x,y) = \dfrac{x^y}{1+x^y}$ bo`lsa, ushbu takroriy limitlarni hisoblang.

$$\lim_{x\to +} \ \lim_{y\to +0} f(x,y) \qquad \text{va} \qquad \lim_{y\to +0}\left(\lim_{x\to +} f(x,y)\right).$$

◄ x ni o`zgarmas desak $y > 0$ da $x^y$ $y$ – ning funksiyasi sifatida uzliksiz bo`ladi, shu sababli

$$\lim_{y\to +0} x^y = 1$$

bo`ladi;

y ning o`zgarmas ($y > 0$) qiymatida, x ning barcha $x > 0$ qiymatida x^y x ning funksiyasi sifatida uzluksizligidan.

$$\lim_{y\to +} x^y = +$$

bo'ladi

$$\lim_{x\to +} \ \lim_{y\to +0} \frac{x^y}{1+x^y} = \lim_{x\to +} \frac{1}{1+1} = \lim_{x\to +} \frac{1}{2} = \frac{1}{2},$$

$$\lim_{y\to+0}\ \lim_{x\to+}\frac{x^y}{1+x^y}=\lim_{y\to+0}\ \lim_{x\to+}\frac{1}{\frac{1}{x^y}+1}=\lim_{x\to+0}1=1\ ►$$

Tabiiy savol tug`iladi: Qanday shartlar bajarilganda karrali va takroriy limitlar bir – biriga teng bo`ladi?

Bu savolga quyidagi teoremalar javob beradi.

Aytaylik, $f(x,y)$ funksiya

$$M=\{(x,y)\ \ R^2:|x-x_0|\ a_1,|y-y_0|\ a_2\}$$

to`plamda aniqlangan bo`lsin.

1- Teorema. Agar

1) $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=b$ mavjud ,

2) Har bir tayinlangan x da $\lim\limits_{y\to y_0} f(x,y)=\varphi(x)$ mavjud bo`lsa, u holda $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)$ takroriy limit mavjud bo`lib,

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=b$$

bo`ladi.

2-Teorema. Agar

1) $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=b$ mavjud ,

2) Har bir tayinlangan y da $\lim\limits_{x\to x_0} f(x,y)=\varphi(y)$ mavjud bo`lsa, u holda $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$ takroriy limit mavjud bo'lib,

$$\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)=b$$

bo`ladi.

Natija. Agar bir vaqtning o`zida 1 va 2-teoremalarning shartlari bajarilsa, unda

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$$

bo`ladi.

Amaliy mashg'ulot uchn misol va masalalar

I. Quyidagi funksiyalarning aniqlanish sohalari topilsin va chizmada ko`rsatilsin.

1.1. $u=\arccos\dfrac{x}{x+y}.$ **1.2.** *u=ln(xyz).*

1.3 *u=ln(-x-y).* **1.4**. $u=\arcsin\dfrac{y}{x}$.

1.5. $u=\sqrt{\sin(x^2+y^2)}.$ **1.6.** $u=\sqrt{1-(x^2+y^2)^2}.$

1.7. $u=\sqrt{\dfrac{x^2+y^2-x}{2x-x^2-y^2}}.$ **1.8.** $u=\sqrt{(x^2+y^2-1)\ (4-x^2-y^2)}.$

1.9. $u=\sqrt{1-x^2}+\sqrt{y^2-1}.$ **1 .10.** $u=\sqrt{\dfrac{x^2+2x+y^2}{x^2-2x+y^2}}.$

II. Karrali limitlar hisoblansin.

2.1. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin xy}{\sqrt{x^2+y^2}}$ **.** **2.2.** $\lim\limits_{\substack{x\to\\ y\to}}\dfrac{x+y}{x^2-xy+y^2}$ **.**

2.3. $\lim\limits_{\substack{x\to\\ y\to 3}}\left(1+\dfrac{1}{x}\right)^{\frac{x^2}{x+y}}$ **2.4.** $\lim\limits_{\substack{x\to +\\ y\to +}}\left(\dfrac{xy}{x^2+y^2}\right)^{x^2}$ **.**

2.5. $\lim\limits_{\substack{x\to\\ y\to}}\dfrac{x^2+y^2}{x^4+y^4}$ **2.6.** $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left(x^2+y^2\right)^{x^2y^2}$ **.**

2.7. $\lim\limits_{\substack{x\to+ \\ y\to+0}} \left(1+\dfrac{2}{x}\right)^{\frac{x^2}{x+y}}$

2.8. $\lim\limits_{\substack{x\to \\ y\to}} \left(x^2+y^2\right)e^{-(x+v)}$

2.9. $\lim\limits_{\substack{x\to1 \\ y\to0}} \dfrac{\ln(x+e^y)}{\sqrt{x^2+y^2}}$

2.10. $\lim\limits_{\substack{x\to0 \\ y\to0}} \dfrac{\sin(x^4y^2)}{(x^2+y^2)}$

III. $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)$ **va** $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$

takroriy limitlar hisoblansin.

3.1. $f(x,y)=\sin\dfrac{\pi x}{2x+y};\, x_0= \quad ,\, y_0= \quad .$

3.2. $f(x,y)=\dfrac{x^2+xy+y^2}{x^2-xy+y^2};\, x_0=y_0=0.$

3.3. $f(x,y)=\log_x(x+y);\, x_0=1,\, y_0=0$

3.4. $f(x,y)=\dfrac{\sin(x+y)}{2x+3y};\, x_0=y_0=0$

3.5. $f(x,y)=\dfrac{\cos x-\cos y}{x^2+y^2};\, x_0=y_0=0$ **.**

3.6. $f(x,y)=\dfrac{\sin|x|-\sin|y|}{\sqrt{x^2+y^2}};\, x_0=y_0=0.$

3.7. $f(x,y)=\dfrac{\sin 3x+tg2y}{6x+3y};\, x_0=y_0=0.$

3.8. $f(x,y)=\dfrac{x^2+y^2}{x^2+y^4};\, x_0=y_0=0.$

3.9. $f(x,y)=\sin\dfrac{x^y}{1+x^v};\, x_0=+ \quad ,\, y_0=+0.$

3.10. $f(x,y)=\frac{1}{xy}tg\frac{xy}{1+xy}; x_0=0, y_0=$.

Mustaqil mashg'ulot uchun misol va masalalar

I. Quyidagi funksiyalarning aniqlanish sohalari topilsin va chizmada ko'rsatilsin

M3.1 $u=\arccos\frac{z}{\sqrt{x^2+y^2}}$. **M3.2.** $u=xy+\sqrt{\ln\frac{9}{x^2+y^2}}+\sqrt{x^2+y^2-9}$.

M3.3. $u=\frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$. **M3.4.** $u=\ln\left(\frac{x^2}{9}-\frac{y^2}{4}-1\right)$.

M3. 5. $u=arctg\frac{x-y}{1+x^2y^2}$. **M3.6.** $u=1+\sqrt{-(x-y)^2}$.

M3. 7. $u=\sqrt{x-\sqrt{y}}$. **M3.8.** $u=\sqrt{y\sin x}$.

M3.9. $u=\sqrt{x\cos y}$. **M3.10.** $u=\arccos\frac{x^2+y^2}{9}$.

M3. 11. $u=arcsin\frac{x}{y^2}+\arcsin(1-y)$. **M3.12.** $u=arcctg\frac{x-y}{1+x^2y^2}$.

M3.13. $u=5+x+y+\sqrt{-x+y}$. **M3.14.** $u=2arccos\frac{y}{x^2}$.

M3.15. $u=x^2y^2+\frac{1}{x-y}+\sqrt{\ln\frac{4}{x^2+y^2}}$. **M3.16** $u=\sqrt{x+\sqrt{y}}-\sqrt{-x+\sqrt{y}}$.

M3.17. $u=\sqrt{y\sin x}+\sqrt{y\cos x}$. **M 3.18.** $u=arccos\frac{x}{y^2}+arcsin\frac{y}{x^2}$

M3.19. $u=\arcsin(1+x)+\arccos(1+y)$). **M3.20.** $u=\arccos\frac{x^2+y^2}{4}+\arcsin\frac{x^2+y^2}{16}$.

II. Karrali limitlar hisoblansin.

M4.1. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{e^{\frac{1}{x^4+y^4}}}{x^4+y^4}$.

M4.2. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(1+x^2y^2\right)^{\frac{1}{x^2+y^2}}$.

M4.3. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(1+x^2+y^2\right)^{\frac{1}{x^2+y^2}}$

M4.4. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\sqrt[3]{x^4y^2}}{x^2+y^2}$

M4.5. $\lim\limits_{\substack{x\to \\ y\to }} \left(x^2+y^2\right)\sin\dfrac{1}{x^2+y^2}$.

4.6. $\lim\limits_{\substack{x\to \\ y\to }} \left(x^2+y^2\right)\ln\ 1+\sin\dfrac{1}{x^2+y^2}$.

M4.7. $\lim\limits_{\substack{x\to \\ y\to }} (x+y)e^{-(x^2+v^2)}$.

M4. 8. $\lim\limits_{\substack{x\to \\ y\to }} \dfrac{x^2+y^2}{|x|^3+|y|^3}$.

M4.9. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(x^2+y^2\right)^{|x|}$

M4.10. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1-cos\left(x^2y^2\right)}{(x^2+y^2)^2}$.

M4.11. $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{\ln^2(x+y)}{\sqrt{x^2+y^2-2x+1}}$.

M4.12. $\lim\limits_{\substack{x\to 0\\ y\to 3}} \dfrac{\sin xy}{x}$

M4.13. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1-cosxy}{x^2y^2}$.

M4.14. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(x^2y^2\right)e^{\frac{1}{xy^2}}$

M4.15. $\lim\limits_{\substack{n\to \\ m\to }} \dfrac{\ln^2 n-\ln^2 m}{\ln n^2+\ln^2 m}$.

M4.16. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{tgxy}{xy(1+tg^2xy)}$

M4.17. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1-cosx^4y^2}{(x^2+y^2)^2}$.

M4.18. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(1+x^4y^4\right)^{\frac{1}{x^4+y^4}}$

M4.19. $\lim\limits_{\substack{x\to \\ y\to }} \left(5+x^4+y^4\right)\ln\ 1+\sin\dfrac{1}{5+x^4+y^4}$

M4.20. $\lim\limits_{\substack{x\to \\ y\to}} \dfrac{tg\dfrac{1}{x^2+y^2}}{\arcsin\dfrac{1}{x^2+y^2}}$.

III. $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)$ **va** $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$

takroriy limitlar hisoblansin.

M5. 1. $f(x,y) = \begin{matrix} \dfrac{x^3+y^3}{x^4+y^2}, (x,y) \quad (0,0) \\ 0, (x,y)=(0,0). x_0 = y_0 = 0 \end{matrix}$

M5.2. $f(x,y) = \begin{matrix} \left(1+\dfrac{1}{x+y}\right)^{x+y}, x+y \quad 0 \\ 1, x+y=0. x_0 = y_0 = \end{matrix}$

M5.3. $f(x,y) = \begin{matrix} \dfrac{x-y+x^2+y^2}{x+y}, x \quad -y \\ 0, x=-y, x_0 = y_0 = 0 \end{matrix}$

M5.4. $f(x,y) = \dfrac{\sin x - \sin y}{x+y}; x_0 = y_0 = 0$

M5.5. $f(x,y) = \dfrac{x^2\sin\dfrac{1}{x} + y}{x+y}; x_0 = y_0 = 0$.

M5.6. $f(x,y) = \begin{matrix} \dfrac{x^2-y^2}{|x|-|y|}, |x| \quad |y| \\ 0, |x|=|y|, x_0 = y_0 = 0 \end{matrix}$

M5.7. $f(x,y) = \dfrac{\ln\left(x+e^y\right)}{\sqrt{x^2+y^2}}; x_0 = 1, y_0 = 0$.

M5.8. $f(x,y)=\dfrac{5-\sqrt{25-xy}}{xy};x_0=y_0=0$.

M5.9. $f(x,y)=\dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2};x_0=y_0=0$.

M5.10. $f(x,y)=\dfrac{\ln(x+y)}{y};x_0=1,y_0=0$.

M5.11. $f(x,y)=\sin\dfrac{\pi(x+y)}{2x+3y};x_0=y_0=$.

M5.12. $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2};x_0=y_0=0$.

M5.13. $f(x,y)=\log_x(x+y);x_0=1,y_0=0$.

M5.14. $f(x,y)=\dfrac{\sin x+\sin y}{x+y};x_0=y_0=0$.

M5.15. $f(x,y)=\dfrac{x-y}{x+y};x_0=y_0=0$.

M5.16 $f(x,y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2};x_0=y_0=0$.

M5.17. $f(x,y)=(x+y)\sin\dfrac{1}{x}\ \sin\dfrac{1}{y};x_0=y_0=0$.

M5.18. $f(x,y)=\dfrac{2xy}{x^2+y^2};x_0=y_0=0$.

M5.19. $f(x,y)=\begin{matrix}\dfrac{x^3+y^3}{x^4+y^2},(x,y)\quad(0,0)\\ 0,(x,y)=(0,0).x_0=y_0=0\end{matrix}$

M5.20. $f(x,y)=\left(1+\frac{1}{x+y}\right)^{x+y},\ x+y\quad 0$

$1,\ x+y=0.\ x_0\ =y_0\ =$

IV. Funksiya limitiga doir qo'shimcha mashqlar.

1. $f(x,y)=\frac{x^2}{|x|+|y|}$ funksiya O(0,0) nuqtada cheksiz kichik bo`lishi isbotlansin.

2. $f(x,y)=\sin(x+y)\ \ln\left(x^2+y^2\right)$ funksiya O(0,0) nuqtada cheksiz kichik bo`lishi isbotlansin.

3. $f(x,y)=\frac{x^2y}{x^4+y^2}$ funksiya quyidagi xossalarga ega ekanligi isbotlansin.

a) *M(x,y)* nuqta O(0,0) nuqtaga shu O(0,0) nuqtadan o`tuvchi $\forall$ to`gri chiziq bo`ylab intilganda ham funksiya limiti 0 ga teng.

b) O(0,0) nuqtada funksiya limiti mavjud emas.

3-§. Ko`p o`zgaruvchili funksiyaning uzluksizligi va tekis uzluksizligi

$M\quad R^m$ to`plamda $f(x)=f(x_1,\dots,x_m)$ funksiya berilgan bo`lib, $a=(a_1,\dots,a_m)\in M$ nuqta M to`plamning limit nuqtasi bo`lsin. Quyidagi belgilashlarni kiritamiz:

$$\Delta f(a)=f(a_1+\Delta x_1,\dots a_m+\Delta x_m)-f(a_1,\dots a_m)$$

funksiyaning a nuqtadagi **to`liq orttirmasi**;

$$\Delta x_k f(a) = f(a_1,...a_{k-1}, a_k + \Delta x_k, a_{k+1},...a_m) - f(a_1,...a_m)$$

funksiyaning a nuqtadagi $\boldsymbol{x_k}$ **o`zgaruvchi bo`yicha xususiy orttirmasi** *(k=1,m).*

**1-Ta`rif** *Agar* $\lim\limits_{x \to a} f(x) = f(a)$ *yoki* $\lim\limits_{\substack{\Delta x_1 \to 0 \\ \cdots \\ \Delta x_m \to 0}} \Delta f(a) = 0$ *bo`lsa , unda*

f(x) funksiya a ***nuqtada uzluksiz*** *deb ataladi.*

2-Ta`rif** *Agar* $\lim\limits_{\Delta x_k \to 0} \Delta_{x_k} f(a) = 0 \quad \left(k = \overline{1, m}\right)$ *bo`lsa,* $f(x_1,...,x_m)$ *funksiya* $a=(a_1,...,a_m)$ *nuqtada* $\boldsymbol{x_k}$ ***o`zgaruvchi bo`yicha uzluksiz* *deb ataladi. Odatda funksiyaning bunday uzluksizligi uning* ***xususiy uzluksizligi*** *deb ataladi.*

3-Ta`rif** (Geyne ta`rifi). *Agar M to'plamning limit nuqtalaridan tuzilgan a ga (*a M*) intiluvchi har qanday* $\left\{x^{(n)}\right\}$ *ketma – ketlik olinganda ham mos* $\left\{f(x^{(n)})\right\}$ *ketma – ketlik hamma vaqt f(a) ga intilsa, f(x) funksiya* ***a nuqtada uzluksiz* *deb ataladi.*

4-Ta`rif (Koshi ta`rifi). *Agar* $\forall \varepsilon > 0$ *son olinganda ham* $\exists \delta > 0$ *topilsaki,* $\rho(x,a) < \delta$ *tengsizlikni qanoatlantiruvchi barcha x M nuqtalarda*

$$\left|f(x) - f(a)\right| < \varepsilon$$

tengsizlik bajarilsa, unda ***f(x) funksiya a nuqtada uzluksiz*** *deb ataladi.*

5-Ta`rif.** *Agar f(x) funksiya M to`plamning har bir nuqtasida uzluksiz bo`lsa, funksiya shu* ***to`plamda uzluksiz* *deyiladi.*

11-misol. Ushbu

$$f(x,y)=ax+by+c \quad \left(a \quad R, \quad b \quad R, \quad c \quad R\right)$$

funksiyaning R^2 da uzluksiz bo`lishini ko`rsating.

◄ $\forall \varepsilon>0$ sonni olamiz. Unga ko`ra $\delta=\dfrac{\varepsilon}{2d}$ deyilsa, u holda

$$\rho\left((x,y),(x_0,y_0)\right)=\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$$

tengsizlikni qanoatlantiruvchi, $\forall(x_0 y_0) \quad R^2$ nuqtalarda

$$\left|f(x,y)-f(x_0,y_0)\right|=\left|ax+by+c-(ax_0+by+c)\right|=$$
$$=\left|a(x-x_0)+b(y-y_0)\right|$$
$$d=\max\left(|a|,|b|\right),$$
$$|a||x-x_0|+|b||y-y_0| \quad |x-x_0|< \quad \sqrt{(x-x_0)^2+(y-y_0)^2},$$
$$|y-y_0|< \quad \sqrt{(x-x_0)^2+(y-y_0)^2}.$$
$$2d \quad \sqrt{(x-x_0)^2+(y-y_0)^2}<2\delta=\varepsilon.$$

tengsizlik o`rinli bo`ladi. Bu esa Koshi ta`rifiga ko`ra *f(x,y)* funksiyaning $\forall(x_0,y_0)$ nuqtada uzluksiz bo`lishini bildiradi.

12 – misol. Ushbu

$$f(x,y)=\frac{y}{x^2+y^2+5}$$

funksiyaning ixtiyoriy $(x_0, y_0) \quad R^2$ nuqtada uzluksiz bo`lishini ko`rsating.

$\lhd$ (x_0, y_0) nuqtaga $\Delta x,$ Δy orttirmalar berib, funksiyaning to`liq orttirmasini topamiz:

$$\Delta f(x_0, y_0) = f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0, y_0) = \frac{y_0 + \Delta y}{(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 + 5} - \frac{y_0}{x_0^2 + y_0^2 + 5} =$$

$$= \frac{(y_0 + \Delta y)(x_0^2 + y_0^2 + 5) - y_0\left[(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 + 5\right]}{\left[\ (x_0 + \Delta x)^2 + (y_0 + \Delta y)^2 + 5\ \right](x_0^2 + y_0^2 + 5)}$$

bu tengliklardan

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta f(x_0, y_0) = 0$$

bo`lishi kelib chiqadi. 1-tarifdan berilgan funksiya $(x_0, y_0)$ nuqtada uzliksiz bo`ladi. ►

1-Teorema. *Agar $f(x_1, \ldots, x_m)$ funksiya $a \quad M$ nuqtada uzluksiz bo`lsa, funksiya shu nuqtada har bir o`zgaruvchisi bo`yicha ham xususiy uzluksiz bo`ladi.*

Izoh: Teoremaning aksi har doim ham o`rinli bo`lavermaydi.

13 – misol. Ushbu

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, x^2 + y^2 \quad 0, \\ 0, \quad x^2 + y^2 = 0 \end{cases}$$

funksiya (0,0) nuqtada har bir o`zgaruvchi bo`yicha xususiy uzluksiz , lekin shu nuqtada bir yo`la uzluksiz emas, bu nuqtada hatto limitga ega emas.

◄Oldin x o`zgaruvchi bo`yicha uzluksizligini ko`rsatamiz.

Agar $y \quad 0$ va $x \to x_0 \quad 0$ bo`lsa,

$$\lim_{x\to x_0} f(x,y)=\lim_{x\to x_0}\frac{2xy}{x^2+y^2}=\frac{2x_0y}{x_0^2+y^2}=f(x_0,y).$$

Agar $y=0$ va $x\to x_0$ 0 bo'lsa

$$\lim_{x\to x_0} f(x,0)=\lim_{x\to x_0}\frac{2x\ \ 0}{x^2+0}=\lim_{x\to x_0} 0=0=f(x_0,0)$$

Endi $y=0$ va $x\to x_0=0$ desak,

$$\lim_{x\to x_0} f(x,0)=0=f(0,0)$$

bo`ladi.

Demak, $\lim_{x\to x_0} f(x,y)=f(x_0,y)$.

Bu berilgan $f(x,y)$ funksiyaning x o`zgaruvchisi bo`yicha xususiy uzluksiz bo`lishini ko`rsatadi. Xuddi shu usulda funksiyaning y o`zgaruvchisi bo`yicha ham uzluksizligi ko`rsatiladi.

Berilgan funksiya (0,0) nuqtada ikkala o`zgaruvchi bo'yicha bir yo`la uzluksiz emas, chunki $x\to 0,\quad y\to 0$ da

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{2xy}{x^2+y^2}$$

mavjud emas.

Aytaylik, (x,y) nuqta (0,0) nuqtaga tekislikdagi $y=kx$ to`g`ri chiziq bo`yicha intilsin.

$$\lim_{\substack{x\to 0\\ y\to 0\\ (y=kx)}}\frac{2xy}{x^2+y^2}=\lim_{x\to 0}\frac{2x\ \ kx}{x^2+(kx)^2}=\lim_{x\to 0}\frac{2kx^2}{x^2+(kx)^2}=\frac{2k}{1+k^2}$$

bo`ladi. Demak, $(x$,y) nuqta turli to`g`ri chiziqlar bo`yicha (0,0) ga intilganda limitning qiymati turlicha bo`ladi. Bu hol esa qaralayotgan limitning mavjud emasligini bildiradi.

2 – usul. Yuqoridagi funksiyani (0,0) nuqtada uzluksiz emasligini, ya`ni (0,0) nuqtada limit mavjud emasligini boshqacha yo`l bilan ko`rsatamiz. (0,0) nuqtaga intiluvchi ikkita ketma – ketlikni qaraylik:

$$\left(x_n, y_n\right) = \frac{1}{n}, \frac{1}{n} \to (0,0),$$

$$\left(x_n^{'}, y_n^{'}\right) = \frac{2}{n}, \frac{1}{n} \to (0,0).$$

Bu ketma – ketliklarda

$$f\left(x_n, y_n\right) = \frac{2\,\frac{1}{n}\,\frac{1}{n}}{\frac{1}{n^2} + \frac{1}{n^2}} = 1 \to 1$$

$$f\left(x_n^{'}, y_n^{'}\right) = \frac{2\,\frac{2}{n}\,\frac{1}{n}}{\frac{4}{n^2} + \frac{1}{n^2}} = \frac{4}{5} \to \frac{4}{5}$$

bo`ladi va berilgan funksiyaning karrali limitning mavjud emasligini va (0,0) nuqtada funksiya qiymati 0 bo`lish sharti bajarilmadi. Demak, $f(x,y)$ funksiya (0,0) nuqtada uzluksiz emas. ►

6-Ta`rif.** *Agar* $\lim\limits_{x\to a} f(x) -$ *mavjud emas yoki* = *, yoki* $\lim\limits_{x\to a} f(x) = b$ $f(a)$ *bo`lsa, u holda f(x) funksiya* ***a nuqtada uzilishga ega* *deyiladi.*

14 – misol.

$$f(x,y) = \frac{x+y}{x^3+y^3}$$

funksiyani uzluksizlikka tekshiring.

◄Kasrning surati va maxraji (x,y) ning funksiyasi sifatida uzluksiz. Funksiya maxraji x^3+y^3 nol bo`lgan nuqtalarda uzilishga ega bo`ladi.

$x^3+y^3=0$ tenglamani *y* ga nisbatan yechib, *y=-x* ni topamiz. Funksiya *y=-x* nuqtada uzilishga ega.

Keling, $x_0 \quad 0, \quad y_0 \quad 0$ va $x_0 + y_0 = 0$. Unda

$$\lim_{\substack{x\to x_0\\ y\to y_0}} \frac{x+y}{x^3+y^3} = \lim_{\substack{x\to x_0\\ y\to y_0}} \frac{1}{x^2-xy+y^2} = \frac{1}{x_0{}^2 - x_0y_0 + y_0{}^2}$$

Demak, $x_0 \quad 0,$ da *y=-x* chiziq nuqtalari bartaraf qilish mumkin bo`lgan uzilish nuqtalari bo`ladi.

Quyidagi

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x+y}{x^3+y^3} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{1}{x^2-xy+y^2} = +$$

natijadan, funksiya (0,0) nuqtada cheksiz uzilishga ega►

7-Ta`rif.** *Agar* $\forall\varepsilon$ *>0 uchun* $\exists\delta=\delta(\varepsilon)$ *>0* $M$ *to`plamning* $\rho(x^{`}, x^{``}) < \delta$ *tengsizlikni qanoatlantiruvchi* $\forall x^{'}$ *va* $x^{''}$ *nuqtalarida* $\left|f(x^{''}) - f(x^{'})\right| < \varepsilon$ *tengsizlik bajarilsa, f(x) funksiya* ***M* to`plamda tekis uzluksiz funksiya *deb ataladi.*

15 – misol. Ushbu

$$f(x,y)=ax+by+c$$

funksiyaning

$$M = \{(x,y) \quad R^2 : |x| \quad + \quad , \ |y| \quad + \quad , \ a \quad R, \ b \quad R, \ c \quad R\}$$

to`plamda tekis uzluksiz bo`lishini ko`rsating.

◄ $(x_1, y_1) \quad M$ va $(x_2, y_2) \quad M$ nuqtalar uchun quyidagiga ega bo`lamiz

$$|f(x_1,y_1)-f(x_2,y_2)| = |ax_1+by_1+c-(ax_2+by_2+c)| =$$
$$= |a(x_1-x_2)+b(y_1-y_2)| \quad |a|\ |x_1-x_2|+|b|\ |y_1-y_2|$$

$\forall\varepsilon>0$ sonni olib, unga ko\`ra olinadigan $\delta>0$ *so*<u>nda</u>

$$|x_1-x_2|< \ \delta,\ |y_1-y_2|< \ \delta, \quad \delta=\frac{\varepsilon}{2d}, \quad d=\max(|a|,\ |b|)$$

shart bajarilganda

$$|f(x_1,y_1)-f(x_2,y_2)| \quad d(|x_1-x_2|+|y_1-y_2|)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

bo\`lib, 7 – ta\`rifdan berilgan funksiyaning M da tekis uzluksizligi kelib chiqadi.►

2-Teorema (Kantor teoremasi). Agar *f(x) funksiya chegaralangan yopiq M* R^m *to\`plamda uzluksiz bo\`lsa, u holda funksiya shu to\`plamda tekis uzluksiz bo\`ladi.*

Amaliy mashg'ulot uchun misol va masalalar

I. Quyidagi funksiyalarni berilgan nuqtada har bir o\`zgaruvchi bo\`yicha xususiy va ikkala o\`zgaruvchi bo\`yicha birgalikda uzluksizlikka tekshiring.

(M6.1-M6.20 m**ustaqil mashg'ulot uchun mashqlar**)

I.1. $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^4+y^4}, x^4+y^4 \quad 0\\ 0, x^4-y^4=0\end{cases}$ 0(0,0) **va** A(1;2)**.**

I.2. $f(x,y)=\begin{cases}\dfrac{x^3y^2}{x^4+y^4}, x^4+y^4 \quad 0\\ 0, x^4+y^4=0\end{cases}$ 0(0,0) **va** A(10^{-4};10^{-5})

I.3. $f(x,y)= \begin{array}{l} \dfrac{x^2+y^2}{x+y}, x+y \quad 0 \\ 0, x+y=0 \end{array}$ 0(0,0) **va** A(-1;-1).

M6.1. $f(x,y)= \begin{array}{l} \dfrac{x^2-y^2}{x^2+y^2}, x^2+y^2 \quad 0 \\ 1, x^2+y^2=0 \end{array}$ 0(0,0) **va** A(0;1).

M6.2. $f(x,y)= \begin{array}{l} \dfrac{\sin x+\sin y}{x+y}, x^2+y^2 \quad 0 \\ 1, x-y=0 \end{array}$ 0(0,0) **va** A $\frac{\pi}{3};-\frac{\pi}{3}$.

M6.3. $f(x,y)= \begin{array}{l} \dfrac{\cos x-\cos y}{x-y}, x-y \quad 0 \\ 0, x-y=0 \end{array}$ 0(0,0) **va** A $\frac{\pi}{4};\frac{\pi}{4}$.

M6.4. $f(x,y)= \begin{array}{l} \dfrac{1}{x^2+y^2}, (x,y) \quad (0,0) \\ 0, (x,y)=(0,0) \end{array}$ 0(0,0) **va** A(1;0).

M6.5. $f(x,y)= \begin{array}{l} \dfrac{x^2 y}{x^4+y^2}, x^2+y^2 \quad 0 \\ 0, x^2+y^2=0 \end{array}$ 0(0,0) **va** A(1;0).

M6.6. $f(x,y)= \begin{array}{l} \dfrac{2xy}{x^2+y^2}, x^2+y^2 \quad 0 \\ 0, x^2+y^2=0 \end{array}$ 0(0,0) **va** A(1;1).

M6.7. $f(x,y)= \begin{array}{l} \dfrac{x^2 y^2}{x^4+y^4}, x^4+y^4 \quad 0 \\ 0, x^4+y^4=0 \end{array}$ 0(0,0) **va** A(1,5;2,5).

M6.8. $f(x,y)=\begin{array}{l}\dfrac{x^3y^2}{x^4+y^4}, x^4+y^4 \quad 0\\ 0, x^4+y^4=0\end{array}$ 0(0,0) **va** A(10^{-2};10^{-3})

M6.9. $f(x,y)=\begin{array}{l}\dfrac{x^2+y^2}{x+y}, x+y \quad 0\\ 0, x+y=0\end{array}$ 0(0,0) **va** A(-2;-2).

M6.10. $f(x,y)=\begin{array}{l}\dfrac{x^2-y^2}{x^2+y^2}, x^2+y^2 \quad 0\\ 1, x^2+y^2=0\end{array}$ 0(0,0) **va** A(0;2).

M6.11. $f(x,y)=\begin{array}{l}\dfrac{\sin x+\sin y}{x+y}, x^2+y^2 \quad 0\\ 1, x-y=0\end{array}$ 0(0,0) **va** A $\frac{\pi}{4};-\frac{\pi}{4}$ **.**

M6.12. $f(x,y)=\begin{array}{l}\dfrac{\cos x-\cos y}{x-y}, x-y \quad 0\\ 0, x-y=0\end{array}$ 0(0,0) **va** A $\frac{\pi}{3};-\frac{\pi}{3}$ **.**

M6.13. $f(x,y)=\begin{array}{l}\dfrac{1}{x^2+y^2}, (x,y) \quad (0,0)\\ 0, (x,y)=(0,0)\end{array}$ 0(0,0) **va** A(2;0).

M6.14. $f(x,y)=\begin{array}{l}\dfrac{x^2y}{x^4+y^2}, x^2+y^2 \quad 0\\ 0, x^2+y^2=0\end{array}$ 0(0,0) **va** A(2;0).

M6.15. $f(x,y)=\begin{array}{l}\dfrac{2xy}{x^2+y^2}, x^2+y^2 \quad 0\\ 0, x^2+y^2=0\end{array}$ 0(0,0) **va** A(2;2).

II. *f(x,y)* funksiyani M to`plamda tekis uzluksiz bo`lishi ta`rif yordamida isbotlansin $(\delta = \delta(\varepsilon))$.

2.1. $f(x, y) = 2x + 3y + 5$, $M = R^2$.

2.2. $f(x, y) = x^2 + y^2$, $M = \{(x, y) \quad R^2, x^2 + y^2 < 4\}$.

2.3. $f(x, y) = \sqrt{x^2 + y^2}$, $M = R^2$.

M6. 16. $f(x, y) = x - 2y + 3$, $M = R^2$.

M.6. 17. $f(x, y) = -2x - 5y + 7$, $M = R^2$.

III. Quyidagi funksiyalarning ko`rsatilgan to`plamda tekis uzluksizlikka tekshiring.

3. 1. $f(x, y) = \dfrac{x^2 + y^2}{x^4 + y^4}$, $M = \{(x, y) \quad R^2, 0 \; x^2 + y^2 \; 1\}$.

3. 2. $f(x, y) = \dfrac{\sqrt{x^4 + y^4}}{x^2 + y^2}$, $M = \{(x, y) \quad R^2, 0 \; x^2 \quad 1\}$.

3. 3. $f(x, y) = x \; \sin\dfrac{1}{y}$, $M = \{(x, y) \quad R^2, 0 \; x \; 1, 0 \; y \; 1\}$.

3.4. $f(x, y) = xy \; \sin\dfrac{1}{y}$, $M = \{(x, y) \quad R^2, 0 \; x \; 1, 0 \; y \; 1\}$.

M6.18. $f(x, y) = xy \; \sin\dfrac{1}{x}$, $M = \{(x, y) \quad R^2, 0 \; x \; 1, 0 \; y \; 1\}$

M6.19. $f(x, y) = y \; \sin\dfrac{1}{x}$, $M = \{(x, y) \quad R^2, 0 \; x \; 1, 0 \; y \; 1\}$.

M6. 20. $f(x,y) = x^3 - y^3$ funksiyaning $M = \{(x,y) \quad R^2, 1 \quad x \quad 2, 0 \quad y \quad 1\}$ to`plamda tekis uzluksiz ekanligi ta`rif yordamida isbotlansin.

4-§. Ko`p o`zgaruvchili funksiyaning hosila va differensiali

$f(x)=f(x_1,\ldots,x_m)$ funksiya ochiq M to`plamda ($M \quad R^m$) berilgan bo`lib, $\left(x_1^0,\ x_2^0,\ldots,x_m^0\right) \quad M$ bo`lsin. Bu funksiyaning $x^0$ nuqtadagi $x_k$ o`zgaruvchi bo`yicha orttirmasi

$$\Delta_{x_k} f = f\left(x_1^0,\ x_2^0,\ldots,x_k^0 + \Delta x_k,\ldots,x_m\right) - f\left(x_1^0,\ x_2^0,\ldots,x_m^0\right)$$

bo`lsin.

Ko`p o`zgaruvchili funksiyaning xususiy hosilalarini hisoblashda bir o`zgaruvchili funksiyaning hosilasini hisoblashdagi ma’lum qoida va jadvallardan to`liq foydalanish mumkin.

Ikki o`zgaruvchili *f(x,y)* funksiyaning $f_x^{`}$ hosilasini hisoblashda y ni o`zgarmas, $f_y^{`}$ ni hisoblashda x ni o`zgarmas deb qaraymiz.

1-Ta`rif. *Ushbu*

$$\lim_{\Delta x_k \to 0} \frac{\Delta_{x_k} f(x^0)}{\Delta x_k}, (k = \overline{1,m})$$

limitga $f(x)=f(x_1,\ldots,x_m)$ *funksiyaning* $x^0 = \left(x_1^0,\ x_2^0,\ldots,x_m^0\right)$ *nuqtadagi* x_k *o`zgaruvchi bo`yicha xususiy hosila deyiladi va u* $\frac{\partial f(x^0)}{\partial x_k}$ *yoki* $f_{x_k}\left(x_1^0,\ x_2^0,\ldots,x_m^0\right)$ *kabi belgilanadi.*

16 – misol. Ushbu

$$f(x,y)=e^{x+y}$$

funksiyaning (2,2) nuqtada $f'_x,\ f'_y$ xususiy hosilalarini hisoblang.

◀Ta`rifdan

$$\frac{\partial f(2,2)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(2+\Delta x,2)-f(2,2)}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{2+\Delta x+2}-e^{2+2}}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{e^{4+\Delta x}-e^{4}}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{4}\left(e^{\Delta x}-1\right)}{\Delta x}=e^{4}\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=e^{4}.$$

Xuddi shunga o`xshash,

$$\frac{\partial f(2,2)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(2,2+\Delta y)-f(2,2)}{\Delta y}=\lim_{\Delta y\to 0}\frac{e^{2+\Delta y+2}-e^{2+2}}{\Delta y}=e^{4}.$$

Demak,

$$\frac{\partial f(2,2)}{\partial x}=e^{4},\qquad \frac{\partial f(2,2)}{\partial y}=e^{4}$$ ▶

17 – misol. Ushbu

$$f(x,y)=\ln\left(x^2+y^2+1\right)+\sin^2 xy$$

funksiyaning xususiy hosilalarini hisoblang.

◀ Hosila olishni qoidalaridan foydalanib topamiz

$$\frac{\partial f}{\partial x}=\frac{2x}{x^2+y^2+1}+2\sin xy\ \cos xy\ \ y=\frac{2x}{x^2+y^2+1}+y\sin 2xy,$$

$$\frac{\partial f}{\partial y}=\frac{2y}{x^2+y^2+1}+2\sin xy\ \cos xy\ \ x=\frac{2y}{x^2+y^2+1}+x\sin 2xy$$ ▶

Xususiy hosilaning **geometrik ma`nosini** bilish uchun $M\quad R^2$ to`plamda aniqlangan *z=f(x,y)* funksiyani qaraymiz.

Aytaylik *(x₀,y₀)* M bo`lib, bu nuqtada $\frac{\partial f(x_0,y_0)}{\partial x}$ va $\frac{\partial f(x_0,y_0)}{\partial y}$ lar mavjud bo`lsin. *z=f(x,y₀)* funksiya grafigi R^3 da biror sirtni aniqlaydi.

Bundan , $z=f(x,y_0)$ ning grafigi sirt bilan $y=y_0$ tekislikning kesishishida hosil bo`lgan $G_1$ chiziq bo`ladi. $z=f(x_0,y)$ ning grafigi G_2 chiziq bo`ladi. Agar $G_1$ va $G_2$ chiziqlarning $(x_0,y_0,f(x_0,y_0))$ nuqtasiga o`tkazilgan urinmaning Oxy tekisligi bilan hosil qilgan burchaklarini mos ravishda α va β deb belgilasak, unda

$$\frac{\partial f(x_0,y_0)}{\partial x}=tg\alpha \quad va \quad \frac{\partial f(x_0,y_0)}{\partial y}=tg\beta$$

bo`ladi. Bundan $z=f(x,y)$ sirtning $(x_0,y_0,z_0)$ nuqtasiga o`tkazilgan urinma tekislik tenglamasi ushbu

$$z-z_0=\frac{\partial f(x_0,y_0)}{\partial x}(x-x_0)+\frac{\partial f(x_0,y_0)}{\partial y}(y-y_0) \qquad (3)$$

ko`rinishida bo`lishini hosil qilamiz.

1-Teorema. *Agar $f(x)$ funksiya x^0 nuqtada chekli* $\frac{\partial f(x^0)}{\partial x_k},(k=\overline{1,m})$ *xususiy hosilaga ega bo`lsa, unda $f(x)$ funksiya shu nuqtada mos $x_k$ o`zgaruvchi bo`yicha **xususiy uzluksiz** bo`ladi.*

2-Ta`rif. *Agar $f(x)$ funksiya x^0 nuqtadagi $\Delta f(x^0)$ orttirmasini*

$$\Delta f(x^0)=A_1\,\Delta x_1+...+A_m\,\Delta x_m+\alpha_1\,\Delta x_1+...+\alpha_m\,\Delta x_m \qquad \mathbf{(4)}$$

*ko`rinishida ifodalash mumkin bo`lsa, $f(x)$ funksiya x^0 **nuqtada differensiallanuvchi** deyiladi. Bunda $A_1,...,A_m$ lar $\Delta x_1,...,\Delta x_m$ ga bog`liq bo`lmagan o`zgarmaslar va $\alpha_1,\ \alpha_2,...,\alpha_m$ lar esa $\Delta x_1,\ \Delta x_2,...,\Delta x_m$ larga bog`liq va $\Delta x_1\to 0,\ \Delta x_2\to 0,...,\Delta x_m\to 0$ da $\alpha_1\to 0,\ \alpha_2\to 0,...,\alpha_m\to 0$ $(\Delta x_1=...=\Delta x_m=0$ da $\alpha_1=\alpha_2=...=\alpha_m=0$ deb olinadi.)*

(4) – tenglik ushbu

$$\Delta f(x^0) = A_1 \ \ \Delta x_1 + \ldots + A_m \ \ \Delta x_m + 0(\rho) \qquad \textbf{(5)}$$

tenglikka ekvivalent. Bu yerda

$$\rho = \sqrt{(\Delta x_1)^2 + \ldots + (\Delta x_m)^2}$$

Agar $f\left(x_1, \ x_2, \ldots, x_m\right)$ funksiya M to`plamning har bir nuqtasida differensiallanuvchi bo`lsa, funksiya M to`plamda differensiallanuvchi deyiladi.

2-Teorema. *Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo`lsa, u holda bu funksiya shu nuqtada uzluksiz bo`ladi.*

3-Teorema. *Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo`lsa, unda bu funksiyaning shu nuqtadagi xususiy hosilalari mavjud va*

$$\frac{\partial f(x^0)}{\partial x_1} = A_1, \ldots, \frac{\partial f(x^0)}{\partial x_m} = A_m$$

tengliklar o`rinli bo`ladi.

Izoh: Teoremaning aksi har doim ham o`rinli bo`lavermaydi, ya`ni barcha xususiy hosilalari mavjud bo`lgan funksiya differensiallanuvchi bo`lishi shart emas.

18– misol. Ushbu

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \quad (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

funksiyaning (0,0) nuqtadagi xususiy hosilalari mavjud, lekin u bu nuqtada differensiallanuvchi emas.

◀Berilgan funksiya (0,0) nuqtadagi orttirmasini topamiz:

$$\Delta f(0,0) = f(0+\Delta x, 0+\Delta y) - f(0,0) = f(\Delta x, \ \Delta y) = \frac{\Delta x \ \ \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

Berilgan funksiya (0,0) nuqtada xususiy hosilalarga ega:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{\Delta x \ \ 0}{\sqrt{\Delta x^2 + 0}} - 0}{\Delta x} = 0$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{\frac{0\ \ \Delta y}{\sqrt{0+\Delta y^2}}-0}{\Delta y}=0.$$

Funksiyaning (0,0) nuqtadagi orttirmasi

$$\Delta f(0,0)=f(0+\Delta x,0+\Delta y)-f(0,0)=\frac{\Delta x\ \ \Delta y}{\sqrt{\Delta x^2+\Delta y^2}}.$$

bo`lib, uni (4) yoki (5) ko`rinishida yozib bo`lmaydi. Buni ko`rsatish uchun, teskarisini ya`ni *f(x,y)* funksiya (0,0) nuqtada differensiallanuvchi bo`lsin deb faraz qilamiz. Unda

$$\Delta f(0,0)=f_x(0,0)\Delta x+f_y(0,0)\Delta y+\alpha_1\Delta x+\alpha_2\Delta y=\alpha_1\Delta x+\alpha_2\Delta y$$

bo`lib, bunda $\Delta x\to 0,\ \Delta y\to 0$ *da* $\alpha_1\to 0,\ \alpha_2\to 0$ bo`ladi.

Demak,

$$\frac{\Delta x\ \ \Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\alpha_1\Delta x+\alpha_2\Delta y$$

Ma`lumki, $\Delta x$ *va* $\Delta y$ lar ixtiyoriy orttirmalar. Keyingi tenglikdan $\Delta x=\Delta y$ bo`lganda

$$\frac{\Delta x^2}{\sqrt{\Delta x^2+\Delta x^2}}=\alpha_1\Delta x+\alpha_2\Delta x \quad \text{yoki} \quad \frac{\Delta x}{\sqrt{2}}=\Delta x(\alpha_1+\alpha_2)$$

ko`rinishiga kelib, undan

$$\alpha_1+\alpha_2=\frac{\sqrt{2}}{2}$$

bo`lishi kelib chiqadi.

Natijada, $\Delta x\to 0,\ \Delta y\to 0$ *da* $\alpha_1\to 0,\ \alpha_2\to 0$ miqdorlarning nolga intilmasligini topamiz. Bu esa *f(x,y)* funksiya (0,0) nuqtada differensiallanuvchi bo`lsin deb qilgan farazga zid. Demak, berilgan funksiya (0,0) nuqtada xususiy hosilalarga ega, ammo bu nuqtada differensiallanuvchilik sharti bajarilmaydi►

Demak xususiy hosilalar mavjud bo`lishi funksiyaning differensiallanuvchi bo`lishi uchun zaruriy shart ekan.

4-**Teorema** (yetarli shart). *Agar f(x) funksiya x^0 nuqtaning biror atrofida barcha o`zgaruvchilari bo`yicha xususiy hosilalarga ega bo`lib bu xususiy hosiladan $x^0$ nuqtada uzluksiz bo`lsa, unda f(x) funksiya shu x^0 nuqtada differensiallanuvchi bo`ladi.*

3 – Ta`rif. $f(x_1, x_2, ..., x_m)$ funksiya orttirmasi $\Delta f(x_1^0, x_2^0, ..., x_m^0)$ ning $\Delta x_1, \Delta x_2, ..., \Delta x_m$ larga nisbatan chiziqli bosh qismi

$$A_1\Delta x_1 + A_2\Delta x_2 + ... + A_m\Delta x_m = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + ... + \frac{\partial f}{\partial x_m}\Delta x_m$$

$f(x_1, x_2, ..., x_m)$ funksiyaning $(x_1^0, x_2^0, ..., x_m^0)$ nuqtadagi differensiali deyiladi va df yoki $df(x_1^0, x_2^0, ..., x_m^0)$ kabi belgilanadi.

Ushbu

$$df(x^0) = \frac{\partial f(x^0)}{\partial x_1}dx_1 + ... + \frac{\partial f(x^0)}{\partial x_m}dx_m \qquad \textbf{(6)}$$

va $$d_{x_k} f(x_0) = \frac{\partial f(x^0)}{\partial x_k}dx_k, (\Delta x_k = dx_k). \quad (k = \overline{1, m})$$

ifodalar mos ravishda *f(x)* funksiyaning x^0 nuqtada **differensiali (to`liq differensiali)** va $x_k$ **o`zgaruvchi bo`yicha xususiy differensiali** deyiladi.

19 – misol. Ushbu

$$f(x, y) = \ln(x^2 + y^2 + xy + 1)$$

funksiyaning differensialini toping.

◄(6) formulaga ko`ra ikki o`zgaruvchili funksiya uchun

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

bo`ladi.

Endi funksiyaning xususiy hosilalarini topamiz.

$$\frac{\partial f}{\partial x} = \frac{1}{x^2 + y^2 + xy + 1}\left(2x + y\right) = \frac{2x + y}{x^2 + y^2 + xy + 1},$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2 + y^2 + xy + 1}\left(2y + x\right) = \frac{2y + x}{x^2 + y^2 + xy + 1}.$$

Demak,

$$df = \frac{2x + y}{x^2 + y^2 + xy + 1}dx + \frac{2y + x}{x^2 + y^2 + xy + 1}dy \ ►$$

20 – misol. Ushbu

$$f(x, y, z) = \frac{z}{\sqrt{x^2 + y^2}}$$

funksiyaning (3;4;5) nuqtadagi differensialini, ya`ni *df(3;4;5)* ni toping.

◄ (6) formulaga ko`ra uch o`zgaruvchili funksiya uchun

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

bo`ladi.

Funksiyaning xususiy hosilalarini (3;4;5) nuqtada hisoblaymiz:

$$\frac{\partial f(3,4,5)}{\partial x} = -\frac{2xz}{2\left(\sqrt{x^2 + y^2}\right)^3}\Bigg|_{(3,4,5)} = -\frac{3\ \ 5}{\left(\sqrt{3^2 + 4^2}\right)^3} = -\frac{3}{25},$$

$$\frac{\partial f(3,4,5)}{\partial y} = -\frac{2yz}{2\left(\sqrt{x^2 + y^2}\right)^3}\Bigg|_{(3,4,5)} = -\frac{4\ \ 5}{\left(\sqrt{3^2 + 4^2}\right)^3} = -\frac{4}{25},$$

$$\frac{\partial f(3,4,5)}{\partial z} = \frac{1}{\left(\sqrt{x^2 + y^2}\right)}\Bigg|_{(3,4,5)} = \frac{1}{5},$$

Natija,

$$df(3;4;5) = -\frac{3}{25}dx - \frac{4}{25}dy + \frac{1}{5}dz = \frac{1}{25}\left(5dz - 3dx - 4dy\right) ▶$$

Amaliy mashg'ulotlar uchun misol va masalalar

I. Quyidagi funksiya O(0,0) nuqtada xususiy hosilalarga egami va bu nuqtada differensiallanuvchimi?

1.1. $u(x,y) = \sqrt{x^2 + y^2}$. **1. 2.** $u(x,y) = \begin{cases} \dfrac{x^3 + y^3}{|x| + |y|}, |x| + |y| \quad 0 \\ 0, |x| + |y| = 0 \end{cases}$.

1. 3. $u(x,y) = \sqrt[3]{xy}$. **1. 4.** $u(x,y) = \sqrt{xy} \quad \sin x$.

1. 5. $u(x,y) = \sqrt[3]{x^4 + y^4}$. **1. 6.** $u(x,y) = \sqrt[3]{x^2 y} \quad tgx$.

1.7. $u(x,y) = \sqrt[3]{x} \sin y$. **1. 8.** $u(x,y) = \sqrt[4]{x^3 + y^3}$.

1. 9. $u(x,y) = \sqrt[4]{x^4 + y^4}$. **1. 10.** $u(x,y) = \sqrt{2x^2 - 3y^2}$.

II. Quyidagi funksiyalarning birinchi va ikkinchi tartibli xususiy hosilalari va differensiali topilsin.

2.1. $u = \frac{1}{x} + x^4 + y^4 - 4x^2y^2 + \sqrt{x}$. **2.2.** $u = xy + \frac{x}{y} + x^4$.

2.3. $u = \sin xy + \frac{x}{y^2}$ **2.4.** $u = \cos xy + \frac{y}{x^2}$.

2.5. $u = \frac{x}{\sqrt{x^2 + y^2}}$. **2.6.** $u = \frac{y}{\sqrt{x^2 + y^2}}$.

2.7. $u = y\sin(x+y)$.

2.8. $u = x\sin(x+y)$.

2.9. $u = \dfrac{\sin x^2}{y}$

2.10. $u = \dfrac{\cos y^2}{x}$.

III. Funksiya differentsialini ko`rsatilgan nuqtalarda toping.

3.1. $u = \dfrac{yz}{x}$, $M(1,1,1)$ **va** $M_0(1,2,3)$.

3.2. $u = \cos(xy + xz)$, $M\left(1, \dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$ **va** $M_0(1, \dfrac{\pi}{6}, \dfrac{\pi}{6})$.

3.3. $u=x^y$, $M(2,2)$ va $M_0(2,3)$.

3.4. $u = x\ln(xy)$, $M(e,e)$ **va** $M_0(-1,-1)$.

3.5. $u = \sqrt[z]{\dfrac{x}{y}}$, $M(1,8,3)$ **va** $M_0(1,1,1)$.

Mustaqil mashg'ulot uchun misol va masalalar

I. Quyidagi funksiya O(0,0) nuqtadagi xususiy hosilalarga egami va bu nuqtada differensiallanuvchimi?

M7.1. $u(x,y) = \sqrt{x^4 + y^4}$.

M7.2. $u(x,y) = \begin{cases} e^{-\frac{1}{x^4+y^4}}, x^4 + y^4 \quad 0 \\ 0, x^4 + y^4 = 0 \end{cases}$.

M7.3. $u(x,y) = \sqrt[3]{x^2 y^2}$.

M7.4. $u(x,y) = \sqrt{x^3 + y^4}$.

M7.5. $u(x,y) = \sqrt[3]{x^3 + y^3}$.

M7.6. $u(x,y) = \begin{cases} \dfrac{x^4 + y^4}{|x| + |y|}, |x| + |y| \quad 0 \\ 0, |x| + |y| = 0 \end{cases}$

M7.7. $u(x,y)= \begin{matrix} e^{\frac{1}{x^2+y^2}}, x^2+y^2 \quad 0 \\ 0, x^2+y^2=0 \end{matrix}$

M7.8. $u(x,y)=\sqrt[3]{xy^2}\ \sin x.$

M7.9. $u(x,y)=\sqrt[3]{y}\ \ tgx.$

M7.10. $u(x,y)=\sqrt[3]{xy\sin x}.$

M7.11. $f(x,y)= \begin{matrix} \dfrac{x^4+y^4}{x^2+y^2}, x^2+y^2 \quad 0 \\ 0, x^2+y^2=0 \end{matrix}$

.**M7.12**. $u(x,y)=\sqrt[3]{x}\ \ tgx$

M7.13. $u(x,y)=\sqrt[3]{y^2x}\ \ tgy.$

M7.14. $u(x,y)=\sqrt{2y^2-3x^2}$

M7.15. $u(x,y)= \begin{matrix} e^{-\frac{1}{x^2+y^2}}, x^2+y^2 \ 0 \\ 0 \ \ , \ \ x^2+y^2=0 \end{matrix}$

M7.16. $u(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$

M7.17. $u(x,y)=\arcsin\dfrac{x}{\sqrt{x^2+y^2}}$

M7.18. $u(x,y)=\sqrt[3]{y}\ \ \sin x$

M7.19. $u(x,y)=\sqrt{x^4+y^3}$

M7.20. $u(x,y)=(3x)^{3y}$

II. Quyidagi funksiyalarning birinchi tartibli xususiy hosilalari va differensiali topilsin.

M8.1. $u=tg\dfrac{x}{y}.$

M8.2. $u=tg\dfrac{x^2}{y}.$

M8.3. $u=x^y$

M8.4. $u=2^{xy}.$

M 8.5. $u=\ln(x+y).$

M8.6. $u=\ln(x+y^2).$

M8.7. $u=\ln(x^2+y^2).$

M8.8. $u=\arcsin\dfrac{y}{x}.$

M8.9. $u=\arccos\dfrac{x}{y}.$

M8.10. $u=\log_a(x+y) \quad \begin{matrix} a>0 \\ a \quad 1 \end{matrix}.$

M8.11. $u = \lg(y^2 + x)$.

M8.12. $u = \sin x + \cos y$.

M8.13. $u = \frac{x}{y} + \frac{y}{x}$.

M8.14. $u = \frac{1}{x+y}$.

M8.15. $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

M8.16. $u = \sqrt{x^2 + y^2 + z^2}$.

M8.17. $u = \sin x + \sin y + \sin z$

M8.18. $u = \frac{x}{y}^{\,z}$.

M8.19. $u = x^{y/z}$.

M8.20. $u = x^{y^z}$.

5-§. Funksiya differensialini hisoblashning sodda qoidalari

$u= f(x_1, \ x_2, ..., x_m)$ va $v= g(x_1, \ x_2, ..., x_m)$ funksiyalar ochiq $M \quad R^2$ to\`plamda berilgan bo\`lib, $x^0 \quad M$ nuqtada differensiallanuvchi bo\`lsin. U holda $u \pm v,$ $\frac{u}{v},$ u v$(v \quad 0)$ funksiyalar ham shu $x^0$ nuqtada differensiallanuvchi bo\`ladi va ularning differensiallari uchun quyidagi formulalar o\`rinli bo\`ladi:

1) $dcu = cdu \quad (c = const)$;

2) $d(u \pm v) = du \pm dv$;

3) $d(u \quad v) = udv + vdu$;

4) $d(\frac{u}{v}) = \frac{vdu - udv}{v^2} \qquad (v \quad 0)$.

§6. Taqribiy hisoblashlarda to`liq differensialning tadbig`i

Agar $f(x)$ funksiya x^0 nuqtada differensiallanuvchi bo`lib, $df(x^0)$ 0 bo`lsa

$$\Delta f(x^0) = df(x^0) + 0(\rho) \ \textit{va} \ \lim_{\rho \to 0} \frac{\Delta f(x^0)}{df(x^0)} = 1 \ \textit{bo`ladi.}$$

$$\textit{Bulardan,} \quad \Delta f(x^0) \approx df(x^0) \quad \textit{yoki}$$

$$f(x_1^0 + \Delta x_1, ..., x_m^0 + \Delta x_m) \approx f(x_1^0, ..., x_m^0) +$$
$$+ \frac{\partial f(x^0)}{\partial x_1} \Delta x_1 + ... + \frac{\partial f(x^0)}{\partial x_m} \Delta x_m \qquad (7)$$

bo`ladi (7) – formulaga **taqribiy hisoblash formulasi** deyiladi.

21 – misol. Ushbu

$$1{,}03^{1{,}98}$$

miqdorning taqribiy qiymatini toping.

◄Taqribiy qiymatini topish uchun

$$f(x, y) = x^y$$

funksiyani qaraymiz. Bu funksiya (1,2) nuqtada differensiallanuvchi va berilgan funksiyani (1,03; 1,98) nuqtada qiymatini topish uchun (7) formuladan foydalanamiz:

$$\Delta f(1,2) \approx \frac{\partial f(1,2)}{\partial x} \Delta x + \frac{\partial f(1,2)}{\partial y} \Delta y$$

Berilgan miqdorni quyidagicha yozsak

$$1{,}03^{1{,}98}=(1+0{,}03)^{2-0{,}02}$$

unda $\Delta x = 0{,}03$, $\Delta y = -0{,}02$ deyish mumkin.

Funksiya qiymatini va xususiy hosilalarini (1;2) nuqtada hisoblaymiz:

$$f(1,2)=1,$$

$$\frac{\partial f(1,2)}{\partial x} = yx^{y-1}\Big|_{(1,2)} = 2\ 1^{2-1} = 2,$$

$$\frac{\partial f(1,2)}{\partial y} = x^{y}\ln x\Big|_{(1,2)} = 0.$$

Natijada berilgan miqdorni qiymati quyidagicha hisoblanadi .

$$f(1+0{,}03;2-0{,}02) \approx f(1,2) + \frac{\partial f(1,2)}{\partial x}\Delta x + \frac{\partial f(1,2)}{\partial y}\Delta y \approx$$

$$\approx 1 + 2\ 0{,}03 + 0\ (-0{,}02) = 1{,}06.$$

Demak,

$$1{,}03^{1{,}98} \approx 1{,}06. ►$$

22 – misol. Ushbu

$$\frac{2{,}02^2}{\sqrt[3]{7{,}98\ \sqrt[4]{1{,}03^3}}}$$

miqdorning taqribiy qiymatini toping.

◄Taqribiy qiymatini topish uchun

$$f(x,y,z) = \frac{x^2}{\sqrt[3]{y\ \sqrt[4]{z^3}}} = x^2\ \ y^{-\frac{1}{3}}\ \ z^{-\frac{1}{4}}$$

funksiyani qaraymiz.

Bu funksiya (2;8;1) nuqtada differensiallanuvchi va berilgan funksiyani (2,02; 7,98; 1,03) nuqtadagi qiymatini hisoblash uchun (7) formuladan foydalanamiz:

$$\Delta f(2,8,1) \approx \frac{\partial f(2,8,1)}{\partial x}\Delta x + \frac{\partial f(2,8,1)}{\partial y}\Delta y + \frac{\partial f(2,8,1)}{\partial z}\Delta z$$

Berilgan miqdorni quyidagi ko`rinishda yozsak

$$\frac{2{,}02^2}{\sqrt[3]{7{,}98\ \sqrt[4]{1{,}03^3}}} = \frac{(2+0{,}02)^2}{\sqrt[3]{(8-0{,}02)\ \sqrt[4]{(1+0{,}03)^3}}}$$

unda $\Delta x = 0{,}02$, $\Delta y = -0{,}02$, $\Delta z = 0{,}03$ deyish mumkin.

Funksiyaning qiymatini va xususiy hosilalarni (2;8;1) nuqtada hisoblaymiz:

$$f(2,8,1) = \frac{2^2}{\sqrt[3]{8\ \sqrt[4]{1}}} = \frac{4}{2} = 2,$$

$$\frac{\partial f(2,8,1)}{\partial x} = 2x\ \ y^{-\frac{1}{3}}\ \ z^{-\frac{1}{4}}\Big|_{(2,8,1)} = 2\ \ 2\ \ 8^{-\frac{1}{3}}\ \ 1^{-\frac{1}{4}} = 2,$$

$$\frac{\partial f(2,8,1)}{\partial y} = -\frac{1}{3}x^2\ \ y^{-\frac{4}{3}}\ \ z^{-\frac{1}{4}}\Big|_{(2,8,1)} = -\frac{1}{3}\ \ 2^2\ \ 8^{-\frac{4}{3}}\ \ 1^{-\frac{1}{4}} = -\frac{1}{12},$$

$$\frac{\partial f(2,8,1)}{\partial z} = -\frac{1}{4}x^2\ \ y^{-\frac{1}{3}}\ \ z^{-\frac{5}{4}}\Big|_{(2,8,1)} = -\frac{1}{4}\ \ 2^2\ \ 8^{-\frac{1}{3}}\ \ 1^{-\frac{5}{4}} = -\frac{1}{2},$$

Natijada berilgan qiymat quyidagicha hisoblanadi.

$$f(2+0{,}02,\ 8-0{,}02,\ 1+0{,}03) \approx f(2,8,1) +$$

$$+\frac{\partial f(2,8,1)}{\partial x}\Delta x + \frac{\partial f(2,8,1)}{\partial y}\Delta y + \frac{\partial f(2,8,1)}{\partial z}\Delta z =$$

$$= 2 + 2\ \ 0{,}02 - \frac{1}{12}\ \ (-0{,}02) - \frac{1}{4}\ \ 0{,}03 \approx 2{,}034. \blacktriangleright$$

Amaliy mashg'ulot uchun mashqlar

Quyidagi miqdorlarning taqribiy qiymatlarini hisoblang

1. $1,002 \cdot 2,003^2 \cdot 3,004^3$.

2. $\sin 29^\circ \cdot tg 46^\circ$.

3. $\dfrac{1,03^2}{\sqrt[3]{0,98} \cdot \sqrt[4]{1.05^3}}$.

4. $2,67^{\sin 0,07}$.

5. $\sqrt{1,02 + 1,97^3}$.

6. $\sin 1,59 \cdot tg 3,09$.

7. $\sin 31^\circ \cdot tg 44^\circ$.

8. $\cos 29^\circ \cdot ctg 46^\circ$.

9. $1,003 \cdot 2,002^2 \cdot 3,003^3$.

10. $\dfrac{1,03^2}{\sqrt[3]{0,99} \cdot \sqrt[4]{1,04^3}}$.

Mustaqil mashg'ulot uchun mashqlar

Quyidagi miqdorlarning taqribiy qiymatlarini hisoblang

M9.1. $\sqrt{1,03^3 + 1,98^3}$.

M9. 2. $\sin 27^\circ \cdot tg 46^\circ$.

M9. 3. $2,65^{\sin 0,06}$.

M9. 4. $2,66^{\sin 0,08}$.

M9. 5. $\dfrac{1,02^2}{\sqrt[3]{0,98} \cdot \sqrt[4]{1,03^3}}$

M9. 6. $tg 3,08 \cdot ctg 1,42$.

M9. 7. $\sin 3,08 \cdot \cos 0,05$.

M9. 8. $\dfrac{1,03^3}{\sqrt{0,99} \cdot \sqrt[4]{1,07^3}}$.

M9. 9. $3,17^{\cos 0,06}$.

M9. 10. $\sqrt[3]{1,02^3 + 1,98^3}$.

M9. 11. $\cos 32^\circ \cdot \sin 28^\circ$.

M9. 12. $tg 46^\circ \cdot ctg 44^\circ$.

M9. 13. $\dfrac{1,02^3}{\sqrt[3]{0,98} \cdot \sqrt[4]{1,02^3}}$.

M9. 14. $\sqrt[5]{1,03^3 + 1,99^3}$.

M9. 15. $\sqrt[3]{1,03} \cdot \sqrt[4]{1,03}$

M9. 16. $3,02^{\sin 0,07}$.

M9. 17. $\cos 59^\circ \cdot ctg 44^\circ$.

M9. 18. $\sin 59^\circ \cdot \cos 1,2$

M9. 19. $tg 1,3 \cdot \sin 0,6$

M9. 20. $\sin 3,08 \cdot \cos 0,06$.

7-§. Yo`nalish bo`yicha hosila .Funksiya gradienti .

Endi yo`nalishi bo`yicha hosila tushunchasini kiritamiz.

Ikki o`zgaruvchili *z=f(x,y)* funksiya ochiq $M \quad R^2$ to`plamda berilgan bo`lsin. $\forall A_0(x_0, y_0) \quad M$ nuqta olib, bu nuqtadan biror $l$ to`gri chiziq o`tkazaylik. Bu to`gri chiziqning OX va OY koordinata o`qlari bilan hosil qilingan burchaklari $\alpha$ va $\beta$ bo`lsin (OZ o'qi bilan hosil qilgan burchagi γ bo'lsin).

**1 - Ta`rif.** *Agar A nuqta l to`gri chiziq bo`ylab A_0 nuqtaga intilganda ushbu*

$$\lim_{A\to A_0}\frac{f(A)-f(A_0)}{\rho(A_0, A)}$$

*limit mavjud bo`lsa, uning qiymatiga f(x,y)=f(A) funksiyaning $A_0=(x_0,y_0)$ nuqtadagi $l$* ***yo`nalish bo`yicha hosilasi*** *deyiladi va*

$$\frac{\partial f(A_0)}{\partial l} \quad \textit{yoki} \quad \frac{\partial f(x_0, y_0)}{\partial l} \quad \textit{kabi belgilanadi.}$$

Demak,

$$\frac{\partial f(A_0)}{\partial l} = \lim_{A\to A_0}\frac{f(A)-f(A_0)}{\rho(A_0, A)} \qquad \textbf{(8)}$$

Teorema. *Agar f(x,y) funksiya $A_0=(x_0,y_0)$ nuqtada differensiallanuvchi bo`lsa, u holda shu funksiya $A_0$ nuqtada $\forall l$ yo`nalish bo`yicha hosilaga ega va*

$$\frac{\partial f(A_0)}{\partial l}=\frac{\partial f(x_0,y_0)}{\partial x}\cos\alpha+\frac{\partial f(x_0,y_0)}{\partial y}\cos\beta \qquad \textbf{(9)}$$

tenglik o`rinli

Uch o`lchovli *u=f(x,y,z)* funksiyaning $l$ yo`nalish bo`yicha hosilasi ham xuddi shunday aniqlanadi. Bu holda

$$\frac{\partial f(A_0)}{\partial l}=\frac{\partial f(x_0,y_0,z_0)}{\partial x}\cos\alpha+\frac{\partial f(x_0,y_0,z_0)}{\partial y}\cos\beta+\frac{\partial f(x_0,u_0,z_0)}{\partial z}\cos\gamma \qquad \textbf{(10)}$$

bo`lib, $\alpha$, $\beta$, $\gamma$ burchaklar $l$ yo`nalishning mos koordinata o`qlari bilan tashkil qilgan burchaklari $\cos\alpha$, $\cos\beta$, $\cos\beta$ uning yo`naltiruvchi kosinuslari deyiladi.

Funksiya gradienti . *u=f(x,y,z)* funksiyaning gradienti deb, mos koordinatalar o`qlarida proyeksiyasi shu funksiya xususiy hosilalaridan iborat bo`lgan vektorlarga aytiladi va quyidagicha belgilanadi

$$grad u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k} \; . \qquad \textbf{(11)}$$

Berilgan nuqtada funksiyaning eng katta o`zgarish tezligi miqdori va yo`nalishi funksiya gradienti bilan aniqlanadi .

$$|grad u|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2} \; . \; . \qquad \textbf{(12)}$$

$\forall(x_0,y_0,z_0)$ *nuqtada* yo`nltiruvchi kosinuslar quyidagicha aniqlanadi.

$$\cos\alpha=\frac{\frac{\partial u(x_0,y_0,z_0)}{\partial x}}{|grad u|}, \quad \cos\beta=\frac{\frac{\partial u(x_0,y_0,z_0)}{\partial y}}{|grad u|}, \quad \cos\gamma=\frac{\frac{\partial u(x_0,y_0,z_0)}{\partial z}}{|grad u|}. \qquad (13)$$

23 – misol.

$z = x^2 - xy - 2y^2$ funksiyaning A(1;2) nuqtada OX o`qining musbat yo`nalish bilan 60^0 burchak tashkil qilgan yo`nalish bo`yicha hosilasini va o`zgarish tezligini toping:

◄Yo`nalish bo`yicha hosilani (9) formuladan topamiz. Funksiya (1;2) nuqtada differensiallanuvchi.

Agar l chiziq OX o`qning musbat yo`nalishi bo`yicha 60^0 burchak tashkil qilsa, OY bilan 30^0 burchak tashkil qiladi $\left(\beta = 30^0\right)$
Quyidagini hisoblashimiz kerak

$$\frac{\partial z(1;2)}{\partial l} = \frac{\partial z(1;2)}{\partial x}\cos 60^0 + \frac{\partial z(1,2)}{\partial x}\cos 30^0 .$$

Ravshanki,

$$\frac{\partial z(1;2)}{\partial l} = (2x - y)\Big|_{(1,2)} = 0,$$

$$\frac{\partial z(1;2)}{\partial l} = (-x - 4y)\Big|_{(1,2)} = -9$$

Demak,

$$\frac{\partial z}{\partial l} = 0\ \frac{1}{2} - 9\ \frac{\sqrt{3}}{2} = -\frac{9\sqrt{3}}{2} .$$

24-misol. $z = 3x^2 - 3y^2 + x + y$ funksiyaning $P(2,0)$ nuqtada shu nuqtadan OX o`qi bilan $120^0$ burchak tashkil qilgan yo`nalish bo`yicha hosilasi, gradienti va eng katta o`zgarish tezligi topilsin.

◄Xususiy hosilalari va ularni $P(2,0)$ nuqtadagi qiymatini topamiz:

$$\frac{\partial z}{\partial x} = 6x + 1, \qquad \frac{\partial z}{\partial x}\Big|_p = 6\ 2 + 1 = 7;$$

$$\frac{\partial z}{\partial y} = -6y + 1, \qquad \frac{\partial z}{\partial y}\Big|_p = -6\ 0 + 1 = 1.$$

Agar yo`nalish $Ox$ o`qining musbat yo`nalishi bilan $120^0$ burchak tashkil qilgan bo`lsa $(\alpha = 120^0)$, Oy o`qining musbat yo`nalishi bilan 30^0 burchak tashkil qiladi. Demak, $\beta = 30^0$. Bulardan, $\cos 120^0 = -\frac{1}{2}$, $\cos 30^0 = \frac{\sqrt{3}}{2}$ bo`lib, (9)-formuladan quyidagiga ega bo`lamiz

$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\cos\beta = 7\ (-\frac{1}{2}) + 1\ \frac{\sqrt{3}}{2} = \frac{-7+\sqrt{3}}{2}.$$

(11)-formuladan funksiya gradientini topamiz

$$gradz = \frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} = 7\vec{i} + \vec{j}.$$

Funksiyaning $P(2,0)$ nuqtadagi o`zgarish tezligi

$$|gradz| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\Bigg|_P = \sqrt{7^2 + 1^2} = 5\sqrt{2}. ►$$

25-misol. $u = x^2 + y^2 + z^2 + x + y + z$ funksiyaning $A(1,1,1)$ nuqtada shu nuqtadan $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ vektor yo`nalishi bo`yicha hosilasi hisoblansin.

◄Yo`naltiruvchi kosinuslarini topamiz. Buning uchun oldin $\vec{a}$ vektor modulini va yo'naltiruvchi kosinuslarini topamiz.

$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14},\ \cos\alpha = \frac{1}{\sqrt{14}},\ \cos\beta = \frac{2}{\sqrt{14}},\ \cos\gamma = \frac{3}{\sqrt{14}}.$$

Natijada, (10)-formuladan ixtiyoriy nuqtadagi hosilasini topamiz

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\ \frac{1}{\sqrt{14}} + \frac{\partial u}{\partial y}\ \frac{2}{\sqrt{14}} + \frac{\partial u}{\partial z}\ \frac{3}{\sqrt{14}}.$$

Xususiy hosilalarni A(1,1,1) nuqtada hisoblaymiz

$$\left.\frac{\partial u}{\partial x}\right|_A = (2x+1)|_{A(1,1,1)} = 3,$$

$$\left.\frac{\partial u}{\partial y}\right|_A = (2y+1)|_{A(1,1,1)} = 3,$$

$$\left.\frac{\partial u}{\partial z}\right|_A = (2z+1)|_{A(1,1,1)} = 3.$$

Natijada, berilgan nuqtadagi yo`nalish bo`yicha hosila quyidagiga teng bo`ladi

$$\frac{\partial u}{\partial l}=3\ \frac{1}{\sqrt{14}}+3\frac{2}{\sqrt{14}}+3\frac{3}{\sqrt{14}}=\frac{3}{\sqrt{14}}(1+2+3)=\frac{18}{\sqrt{14}}. \ ▶$$

26-misol. $z=x^2+y^2$ funksiyaning sath chizig`iga perpendikulyar va $A(3,4)$ nuqtadan o`tgan $\vec{l}$ yo`nalish bo`yicha hosilasi topilsin.

◁ $gradz$ vektor A nuqtada $c=x^2+y^2$ sath chizig`iga ortogonal bo`lgani uchun, shu A nuqtadan o`tgan $\vec{l}$ vector yo`naltiruchi kosinuslari $gradu$ vektorning A nuqtadagi yo`naltiruvchi kosinuslariga teng, ya`ni

Agar,

$$\cos\alpha=\frac{\frac{\partial z(A)}{\partial x}}{|gradu(A)|},\quad \cos\beta=\frac{\frac{\partial z(A)}{\partial y}}{|gradu(A)|}.$$

Agar, $\frac{\partial z(A)}{\partial x}=2x\big|_{(3,4)}=6,\ \frac{\partial z(A)}{\partial y}=2y\big|_{(3,4)}=8,$

$$|gradu|=\sqrt{\ \frac{\partial z(A)}{\partial x}\ ^2+\ \frac{\partial z(A)}{\partial y}\ ^2}=\sqrt{6^2+8^2}=10$$

larni e`tiborga olsak

$$\cos\alpha=\frac{6}{10}=\frac{3}{5},\quad \cos\beta=\frac{8}{10}=\frac{4}{5}.$$

Natijada, $\frac{\partial z(A)}{\partial l}=\frac{\partial z(A)}{\partial x}\cos\alpha+\frac{\partial z(A)}{\partial y}\cos\beta=6\ \frac{3}{5}+8\frac{4}{5}=10.$ ▶

Izoh: Funksiya biror nuqtada differensiallanuvchi bo`lmasa ham u shu nuqtada biror yo`nalish bo`yicha hosilaga ega bo`lishi mumkin.

Amaliy mashg'ulot uchun misol va masalalar

u=(x,y,z) **funksiyaning** ***(x_0,y_0,z_0)*** **nuqtadagi koordinata o`qlari musbat yo`nalish bo`yicha hosilasi va o`zgarish tezligi topilsin.**

T/r	$u(x,y,z)$	α	β	γ	(x_0,y_0,z_0)
1	$\sqrt{xyz}$	α_0	β_0	γ_0	(x_0,y_0,z_0)
2	$\sqrt{x^2+y^2+z^2}$	α_0	β_0	γ_0	$(1,1,1)$
3	xyz	α_0	β_0	γ_0	$(1,1,1)$
4	$x^3-3x+2y^2+z^3$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$(1,1,1)$
5	$\ln yxz$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	(e,e,e)
6	$\ln(x^2+y^2+z^2)$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$(2,2,2)$
7	$\ln\sqrt{x^2+y^2+z^2}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$(1,1,1)$
8	a^{xyz} $(a>0)$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$(0,0,0)$
9	$e^{xyz}\ln(x+y+z)$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$(1,1,1)$
10	$x\sin y+z\cos 2x+y\sin 3z$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$(0,0,0)$

11	$\frac{1}{\sqrt{x^2+y^2+z^2}}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	(1,1,1)
12	$ln\sqrt{x+y+z}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	(1,1,1)
13	x^2-xy+$3y^2$	$\frac{\pi}{3}$		-	(1,2)
14	arctgxy	$\frac{\pi}{4}$		-	(1,1)
15	$ln\sqrt{x^2+y^2}$	$\frac{\pi}{4}$		-	(1,1)
16	xsin2y+zsin2x+ysin3x	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	(1,2,2)
17	x+y+z+sinx+siny+sinz	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	(1,2,1)
18	e^{xyz}	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	
19	3^{xyz}	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	(1,2,1)
20	x^3-$3x^2y$+$3xy$+1+z^2	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	(3,1,2)
21	arctgxyz	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	
22	arcsinxy	$\frac{\pi}{4}$		-	(1,1)

23	$arccosxy$	$\frac{\pi}{3}$		-	$(1,1)$
24	$arctgxy$	$\frac{\pi}{6}$	-	-	$(1,2)$
25	$z=x^3-3x^2y+3xy^2+1$	M(2;1) nuqtada shu nuqtadan (5;2) nuqtaga qarab yo\`nalish bo\`yicha hosilasi topilsin.			
26	$z=x^2y^2-xy^3-3y-1$	M(2;1) nuqtada shu nuqtadan koordinata boshiga qarab yo’nalgan yo\`nalishi bo\`yicha hosilasi topilsin.			
27	$z=arctg(xy)$	M(1;1) nuqtada birinchi chorakning bissektrissasi yo\`nalishi bo\`yicha hosilasi topilsin			
28	$z=x^2-xy+y^2$	M(1,1)nuqtada OX o\`qning 1 ,bilan $\alpha$ burchak tashkil qiladigan hosilasi topilsin va qaysi yo\`nalishda bu hosila eng katta qiymatga ega bo\`ladi.			
29	$z=x^2-xy+y^2$	M(1,1)nuqtada OX o\`qning l ,bilan $\alpha$ burchak tashkil qiladigan hosilasi topilsin va qaysi yo\`nalishda bu hosila eng kichik qiymatga ega bo\`ladi.			
30	$z=x^2-xy+y^2$	M(1,1)nuqtada OX o\`qning l ,bilan $\alpha$ burchak tashkil qiladigan hosilasi topilsin va qaysi yo\`nalishda bu hosila nolga teng bo\`ladi.			

8-§. Ko\`p o\`zgaruvchili murakkab funksiyaning hosilasi va differensiali

1. ***Murakkab funksiyaning hosilasi.***

$y = f(x_1, x_2, ..., x_m)$ funksiya M R^m to`plamda berilgan bo`lib, $x_1, x_2, ..., x_m$ o`zgaruvchilarning har biri o`z navbatida $t_1, t_2, ..., t_k$ o`zgaruvchilarning $T$ $R^k$ to'plamda berilgan funksiyasi bo`lsin:

$$\begin{aligned} x_1 &= \varphi_1(t_1, t_2, ..., t_k), \\ x_2 &= \varphi_2(t_1, t_2, ..., t_k), \\ &.......... \quad .... \quad .... \\ x_m &= \varphi_m(t_1, t_2, ..., t_k) \end{aligned} \qquad (8.1)$$

Bunda $(t_1, t_2, ..., t_k)$ T bo`lganda unga mos $(x_1, x_2, ..., x_m)$ $M$ bo`lsin. Natijada $y = f(\varphi_1(t_1, t_2, ..., t_k), \varphi_2(t_1, t_2, ..., t_k), ..., \varphi_m(t_1, t_2, ..., t_k))$ murakkab funksiya hosil bo`ladi. Murakkab funksiyaning $t_1, t_2, ..., t_k$ o`zgaruvchilar bo`yicha xususiy hosilalarini topamiz.

8.1-Teorema. Agar (8.1)-funksiyalarning har biri $(t_1^0, t_2^0, ..., t_k^0)$ nuqtada differensiallanuvchi bo`lib, $f(x_1, x_2, ..., x_m)$ funksiya esa mos $(x_1^0, x_2^0, ..., x_m^0)$ nuqtada differensiallanuvchi bo`lsa, u holda murakkab funksiyua ham $(t_1^0, t_2^0, ..., t_k^0)$ nuqtada differensiallanuvchi bo`lib,

$$\begin{aligned} \frac{\partial f}{\partial t_1} &= \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_1} + ... + \frac{\partial f}{\partial x_m} \frac{\partial x_m}{\partial t_1}, \\ \frac{\partial f}{\partial t_2} &= \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_2} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_2} + ... + \frac{\partial f}{\partial x_m} \frac{\partial x_m}{\partial t_2}, \\ &.......... \; .... \; .... \; .... \; .... \; .... \; .... \\ \frac{\partial f}{\partial t_k} &= \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2} \frac{\partial x_2}{\partial t_k} + ... + \frac{\partial f}{\partial x_m} \frac{\partial x_m}{\partial t_k} \end{aligned} \qquad \mathbf{(8.2)}$$

bo`ladi.

Agar differensiallanuvchi *w=f(x,y,z)* va $x = \varphi(u, v)$, $y = \psi(u, v)$, $z = \chi(u, v)$ funksiyalar berilgan bo`lib, ular yordamida $w = f[\varphi(u, v), \psi(u, v), \chi(u, v)] = F(u, v)$ murakkab funksiya aniqlangan bo'lsa, unda murakkab funksiya ham differensiallanuvchi bo`ladi va

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\;\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\;\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\;\frac{\partial z}{\partial u},$$
$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\;\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\;\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\;\frac{\partial z}{\partial v} \qquad \textbf{(14)}$$

tenglik o`rinli bo`ladi.

Izoh. $w=f(x,y)$ va $x=\varphi(u,v)$, y=ψ(u,v) bo`lsa, (14)-formuladagi xususiy hosilalarga oxirgi qo'shiluvchi bo`lmaydi.

Agar x,y,z lar faqat bitta t o`zgaruvchining funksiyasi bo`lsa, (14)-formula quyidagi ko`rinishga bo`ladi.

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\;\frac{dx}{dy}+\frac{\partial w}{\partial y}\;\frac{dy}{dt}+\frac{\partial w}{\partial z}\;\frac{dz}{dt} \qquad \textbf{(15)}$$

27-misol. $f(x,y)=\sin^2 x+\cos^2 y$ funksiyaning x=2t, y=t^3 bo`lgandagi hosilasini toping.

◄**(15)**-formuladan foydalanamiz

$$\frac{\partial f}{\partial x}=2\sin x\cos x=\sin 2x,\;\;\frac{dx}{dt}=2$$
$$\frac{\partial f}{\partial y}=-2\cos x\sin x=-\sin 2x,\;\;\frac{dy}{dt}=3t^2$$

Natija $\frac{\partial f}{\partial t}=2\sin 2x-3t^2\sin 2x=(2-3t^2)\sin 2x.$ ►

28-misol. $\omega=\sin xy$, bunda $x=u^2+v^2$, y=u v bo`lsa, ω funksiyaning xususiy hosilalarini toping.

◄**(14)**-formuladan foydalanamiz, ya`ni quyidagilarni topishimiz kerak

$$\frac{\partial \varpi}{\partial u}=\frac{\partial \varpi}{\partial x}\;\frac{\partial x}{\partial u}+\frac{\partial \varpi}{\partial y}\;\frac{\partial y}{\partial u},$$
$$\frac{\partial \varpi}{\partial v}=\frac{\partial \varpi}{\partial x}\;\frac{\partial x}{\partial v}+\frac{\partial \varpi}{\partial y}\;\frac{\partial y}{\partial v}$$

ya`ni

$$\frac{\partial \varpi}{\partial u}=y\cos xy\;\;2u+x\cos xy\;\;v=(2yu+xv)\cos xy$$
$$\frac{\partial \varpi}{\partial v}=y\cos xy\;\;2v+x\cos xy\;\;u=(2yv+xu)\cos xy.$$ ►

29-misol. Ushbu $F = f(x+y, x^2+y^2)$ funksiyaning hosilalarini toping.

◃ Berilgan funksiyani $F = f(u,v)$, bu yerda $u = x+y,\quad v = x^2+y^2$ deb qarash mumkin.

Unda (14)-formuladan foydalanib topamiz.

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial F}{\partial u}\, 1 + \frac{\partial F}{\partial v}\, 2x,$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial F}{\partial u}\, 1 + \frac{\partial F}{\partial y}\, 2y.$$

Agar $\varpi = f(x,y,z)$ funksiyada *x* ni erkli o`zgaruvchi sifatida qoldirib, $y = y(x)$ va $z = z(x)$ lar *x* bo`yicha differensiallanuvchi bo`lsa, $\varpi$ funksiya *x* argument bo`yicha murakkab funksiya bo`ladi va quyidagi hosilaga ega bo`ladi.

$$\frac{d\bar{\omega}}{dx} = \frac{\partial\bar{\omega}}{\partial x}\, 1 + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx} \qquad \textbf{(8.3)}$$

Agar $\varpi = f(x,y,z)$ funksiyada *x,y* lar erkli o`zgaruvchilar sifatida qoldirilgan bo`lib, $z = z(x,y)$ funksiya *x* va *y* lar bo`yicha hosilaga ega bo`lsa, u holda $\varpi = f(x,y,z(x,y))$ murakkab funksiya quyidagi xususiy hosilalarga ega

$$\frac{\partial u}{\partial x} = f_x(x,y,z(x,y)) + f_z(x,y,z(x,y))z_x(x,y),$$

$$\frac{\partial u}{\partial y} = f_y(x,y,z(x,y)) + f_z(x,y,z(x,y))z_y(x,y)$$ ►

30-misol. Ushbu $u = f(x, xy, xyz)$ funksiyaning x,y va z argumentlar bo`yicha hosilasini toping

◄Bu funksiya *x,y* va *z* o`zgaruvchilarning murakkab funksiyasi: $u = f(x_1, x_2, x_3)$ bu yerda $x_1 = x_1$, $x_2 = xy$, $x_3 = xyz$. *u=f(* $x_1, x_2, x_3$ ) funksiyaning $x_1, x_2, x_3$ argumentlar bo`yicha hosilasini $f_1^`, f_2^`, f_3^`$ bilan belgilaymiz.

Bu funksiyalar argumentlari ham xuddi f funksiyaning argumentlaridek (8.2)-formulani qo`llab quyidagilarga ega bo`lamiz.

$$\frac{\partial u}{\partial x}=f_1^{`}\;1+f_2^{`}y+f_3^{`}yz,$$
$$\frac{\partial u}{\partial y}=f_2^{`}x+f_3^{`}xz \qquad ►$$
$$\frac{\partial u}{\partial z}=f_3^{`}xy.$$

31-misol. $z=f(x^2+y^2)$ differensiallanuvchi funksiya quyidagi tenglamani qanoatlantirishini ko`rsating

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$$

◄Agar $x^2+y^2=t$ desak, f funksiya x va y larga yordamchi t argument orqali bog`liq bo`ladi. Shuning uchun

$$\frac{\partial z}{\partial x}=\frac{dz}{dt}\,\frac{\partial t}{\partial x}=f\left(x^2+y^2\right)2x,$$
$$\frac{\partial z}{\partial y}=\frac{dz}{dt}\,\frac{\partial t}{\partial y}=f\left(x^2+y^2\right)2y.$$

Bu xususiy hosilalarni tenglamaning o`ng tomoniga qo`yib, quyidagiga ega bo`lamiz:

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=y\;f\left(x^2+y^2\right)2x-xf\;\;x^2+y^2\;\;2y=2xyf\left(x^2+y^2\right)-2xyf\;\;x^2+y^2\;\;=0 \;►$$

2. Murakkab funksiyaning differensiali

Faraz qilaylik (8.1)-funksiyalarning har biri $(t_1^0,t_2^0,...,t_k^0)\;T$ nuqtada differensiallanuvchi bo`lib, $y=f(x_1,x_2,...,x_m)$ funksiya esa $(x_1^0,x_2^0,...,x_m^0)\;M$ nuqtada differensiallanuvchi bo`lsin. U holda 8.1-teoremaga ko`ra murakkab funksiya $(t_1^0,t_2^0,...,t_k^0)$ nuqtada differensiallanuvchi bo`ladi. Unda murakkab funksiyaning shu nuqtadagi differensiali

$$df=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+...+\frac{\partial f}{\partial x_m}dx_m, \qquad \textbf{(8.3)}$$

bo`lib, bunda

$$
\begin{aligned}
dx_1 &= \frac{\partial x_1}{\partial t_1}dt_1 + \frac{\partial x_1}{\partial t_2}dt_2 + \dots + \frac{\partial x_1}{\partial t_k}dt_k, \\
dx_2 &= \frac{\partial x_2}{\partial t_1}dt_1 + \frac{\partial x_2}{\partial t_2}dt_2 + \dots + \frac{\partial x_2}{\partial t_k}dt_k, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
dx_m &= \frac{\partial x_m}{\partial t_1}dt_1 + \frac{\partial x_m}{\partial t_2}dt_2 + \dots + \frac{\partial x_m}{\partial t_k}dt_k,
\end{aligned}
$$

Murakkab funksiya differensialini ifodolovchi (8.4) formula avval qarab o`tilgan (6)-formula bilan solishtirganda, funksiya murakkab bo`lgan holda ham funksiya differensiali funksiya xususiy hosilalari

$\frac{\partial f(x_1, x_2, ..., x_m)}{\partial x_i}$ (i=$\overline{1,m}$) bilan mos argument differensiallari dx_i $\left(i = \overline{l,m}\right)$

ko`paytmasi yig'indisidan iborat ekanini ko`ramiz.

Demak, qaralayotgan funksiyalar murakkab ko`rinishda bo`lsa ham, bu funksiyalarning differensiallari bir xil (8.4)-formaga ega bo`ladi, ya`ni differensial formasi saqlanadi.

Bu xossa differensial formasining *invariantligi* deyiladi.

Agar $u = f(x, y, z)$ funksiya $u_x^`, u_y^`, u_z^`$ uzluksiz xususiy hosilalarga ega, shu bilan birga *x,y,z* lar o`z navbatida t va *v* o`zgaruvchilarning

$$x = \varphi(t, v), \quad y = \psi(t, v), \quad z = \chi(t, v),$$

funksiyalari bo`lib, uzluksiz $x_t^`, x_v^`, y_t^`, y_v^`, z_t^`, z_v^`$ xususiy hosilalarga ega bo`lsa

$$du = u_x dx + u_y dy + u_z dz \qquad \textbf{(8.5)}$$

O`rinli bo`lib, bu yerda

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial v}dv, \\
dy &= \frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial v}dv, \\
dz &= \frac{\partial z}{\partial t}dt + \frac{\partial z}{\partial v}dv
\end{aligned}
\qquad \textbf{(8.6)}
$$

x,y,z funksiyalar har xil o`zgaruvchilarga bog`liq bo`lgan holda, masalan

$x = \varphi(t), \quad y = \psi(t,v), \quad z = \chi(v,\varpi),$ bo`lganda, biz doimo $x = \varphi_1(t,v,\varpi), \quad y = \psi_t(t,v,\varpi), \quad z = \chi_t(t,v,\varpi),$ deyishimiz mumkin va oldingi hamma mulohazalar bu holda ham qo`llaniladi.

32-misol. Ushbu

$$F = f(u,v), \quad u=2t, \quad v=3t+t^2$$

murakkab funksiyaning differensialini toping.

◀ Murakkab funksiyaning differensiali invariantlik xossasidan

$$dF = \frac{\partial f}{\partial u}\, du + \frac{\partial f}{\partial v}\, dv$$

ko`rinishda bo`ladi. Bunda du va dv lar erkli orttirmalar emas, ular t ga bog`liq bo`ladi.

Shuni e`tiborga olsak

$$du = d\,(2t) = (2t)'\,dt = 2dt,$$

$$dv = d\,(3t + t^2) = (3t + t^2)'\,dt = (3 + 2t)dt,$$

Demak,

$$dF = \frac{\partial f}{\partial u}\, 2dt + \frac{\partial f}{\partial v}\,(3 + 2t)dt$$ ▶

33-misol. Ushbu $f(x,y) = y\sin x + x\cos y,\ x = t + \cos t,\ y = t^2 + \sin t$ murakkab funksiyaning differensialini toping.

◀**Q**uyidagi differensialni topishimiz kerak

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

Bunda

$$\frac{\partial f}{\partial x} = y\cos x + \cos y, \quad \frac{\partial f}{\partial y} = \sin x - x\sin y$$

$$dx = d(t + \cos t) = (t + \cos t)'\, dt = (1 - \sin t)dt$$

$$dy = d(t^2 + \sin t) = (t^2 + \sin t)'\, dt = (2t + \cos t)dt.$$

Natijada,

$$df = (y\cos x + \cos y)\,(1 - \sin t)dt + (\sin x - x\sin y)\,(2t + \cos t)dt$$ ▶

34-misol. Ushbu $f(x,y)=\ln\sqrt{x+y}$, x=$\frac{u}{v}$, y=$\frac{v}{u}$ murakkab funksiyaning differensialini toping.

◀ Quyidagi differensialni topishimiz kerak

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy$$

Bu yerda dx va dy lar erkli orttirmalar emas, ular u va v ga bog`liq bo`ladi. Avvalo berilgan funksiyaning xususiy hosilalarini topamiz

$$\frac{\partial f}{\partial x}=\frac{1}{\sqrt{x+y}}\left(\sqrt{x+y}\right)=\frac{1}{2\sqrt{x+y}\sqrt{x+y}}=\frac{1}{2(x+y)},$$

$$\frac{\partial f}{\partial y}=\frac{1}{2(x+y)},\quad \frac{\partial x}{\partial u}=\frac{1}{v},\quad \frac{\partial x}{\partial v}=\frac{u}{v^2},\quad \frac{\partial y}{\partial u}=-\frac{v^2}{u^2},\quad \frac{\partial u}{\partial v}=\frac{1}{u}.$$

Endi dx va dy differensiallarni topamiz

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv=\frac{1}{v}du-\frac{u}{v^2}dv$$

$$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv=-\frac{v}{u^2}du+\frac{1}{u}dv.$$

Demak,

$$df=\frac{1}{2(x+y)}\ \frac{1}{v}du-\frac{u}{v^2}dv\ +\frac{1}{2(x+y)}\ -\frac{v}{u^2}du+\frac{1}{u}dv\ =$$

$$=\frac{1}{2(x+y)}\ \frac{1}{v}-\frac{v}{u^2}\ du+\ \frac{1}{u}-\frac{u}{v^2}\ dv\ .$$ ▶

Amaliy mashg'ulot uchun misl va masalalar

I. Quyidagi murakkab funksiyalarning birinchi tartibli xususiy hosilalari va differensiali topilsin.

1.1. $u=\sin xyz,\quad x=\xi+\eta,\quad y=\eta,\quad z=\xi+5$.

1.2. $u=\ln(x+y+z),\quad x=2\xi,\quad y=3\xi+\eta,\quad z=\xi+\eta.$

1.3. $u = e^{xyz}, \quad x = \sin\xi, \quad y = \cos\eta, \quad z = \sin\eta.$

1.4. $u = x^y + z^x, \quad x = e^t, \quad y = 2^t, \quad z = t^2.$

1.5. $u = e^{x+y+z}, \quad x = \xi + \eta, \quad y = \xi\eta, \quad z = \xi^2.$

1.6. $u = x^2 + y^2 + z^2, \quad x = \xi^2 + \eta, \quad y = \eta^2 + \xi, \quad z = \xi.$

1.7. $u = \sin x + \sin y + \sin z, \quad x = \sin\xi, \quad y = \xi, \quad z = \cos\eta.$

1.8. $u = x^y, \quad x = \sin\xi + \eta, \quad y = \cos\eta + \xi, \quad .$

1.9. $u = f(2x + 2y + 2z).$

1.10. $u = f(x^2 + y^2 + z^2).$

II. Agar *f* - ixtiyoriy differentsiallanuvchi funksiya bo`lsa, *u(x;y)* funksiya mos tenglamani qanoatlantirishini tekshiring.

2.1. $u = f\left(x^2 + y^2\right); y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$.

2.2. $u = x^n f\ \frac{y}{x}\ ; x\frac{\partial u}{\partial y} - 2y\frac{\partial u}{\partial y} = nu.$

2.3. $u = yf\left(x^2 - y^2\right); y^2\frac{\partial u}{\partial x} + xy\frac{\partial u}{\partial y} = xu$.

2.4. $u = \frac{y^2}{3x} + f(xy); x^2\frac{\partial u}{\partial x} - xy\frac{\partial u}{\partial y} + y^2 = 0$.

2.5. $u = x^n f\ \frac{y}{x^\alpha}, \frac{z}{x^\beta}\ ; x\frac{\partial u}{\partial x} + \alpha y\frac{\partial u}{\partial y} + \beta z\frac{\partial u}{\partial z} = nu.$

2.6. $u = \frac{xy}{z}\ln x + xf\ \frac{y}{z}, \frac{z}{x}\ ; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = u + \frac{xy}{z}.$

Mustaqil mashg’ulot uchun misol va masalalar.

Quyidagi murakkab funksiyalarning birinchi

tartibli xususiy hosilalari va differensial topilsin.

M10.1. $u = f(x, xy, xyz).$

M10.2. $u = f(\sqrt{x^2 + y^2}).$

M10.3. $u = f(t),\ t = x + y$

M10.4. $u = f(t),\ t = \frac{y}{x}$**.**

M10.5. $u = f(t),\ t = xyz$**.**

M10.6. $u = f(\xi, \eta),\ \xi = 3x,\ \eta = 5y$**.**

M10.7. $u = f(\xi, \eta),\ \xi = x + y,\ \eta = x - y$**.**

M10.8. $u = f(\xi, \eta),\ \xi = xy,\ \eta = \frac{x}{y}$**.**

M109.9. $u = f(x + y, z).$

M10.20. $u = f(x + y + z,\ x^2 + y^2 + z^2).$

M10.11. $u = f\ \ \frac{x}{y}, \frac{y}{x}\ \ .$

M10.12. $u = f(x, y),\ x = t^2,\ y = t^3.$

M10.13. $u = f(x, y, z),\ x = t,\ y = t^2,\ z = t^3.$

M10.14. $u = f(x, y),\ x = t + 8,\ y = \sqrt{t}.$

M10.15. $u = f(x, y),\ x = \xi + \eta,\ y = \xi - \eta.$

M10.16. $u = f(\xi, \eta, \zeta),\ \xi = Sx,\ \eta = 2y,\ \zeta = 3z.$

M10.17. $u = f(\xi, \eta, \zeta),\ \xi = ax,\ \eta = x + y,\ \zeta = x + z.$

M10.18. $u = f(\xi, \eta, \zeta),\ \xi = \sin x,\ \eta = \sin y,\ \zeta = \sin z.$

M10.19. $u = f(\xi,\eta,\zeta),\ \xi = \cos x,\ \eta = \cos y,\ \zeta = \cos z.$

M10.20. $u = f(\xi,\eta,\zeta),\ \xi = x + y,\ \eta = x^2,\ \zeta = xy.$

9-§. Ko`p o`zgaruvchili funksiyalarning yuqori tartibli hosilasi

$f(x) = f(x_1, x_2, ..., x_m)$ funksiya ochiq $M \quad R^m$ to`plamda berilgan bo`lib, uning har bir $(x_1, x_2, ..., x_m)$ nudtasida $f'_{x_1}, f'_{x_2}, ..., f'_{x_m}$ hosilalarga ega bo`lsin. Bu xususiy hosilalarning o`zlari ham, o`z navbatida $x_1, x_2, ..., x_m$ o`zgaruvchilarning funksiyasi bo`lishi mumkin.

**Ta`rif:** $f(x) = f(x_1, x_2, ..., x_m)$ funksiya xususiy hosilalari $f'_{x_1}, f'_{x_2}, ..., f'_{x_m}$ larning $x_k$ $(k=\overrightarrow{1,m})$ o`zgaruvchi bo`yicha xususiy hosilalari berilgan funksiyaning ikkinchi tartibli xususiy hosilalari deb ataladi va

$$f_{x_i x_k} \text{ yoki } \frac{\partial^2 f(x)}{\partial x_k \partial x_i} \quad (i,k=1,2,\dots,m)$$

kabi belgilanadi.

Agar *i=k* bo`lsa, ikkinchi tartibli xususiy hosilalar

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_i} = f''_{x_i x_k}$$

qisqacha quyidagicha yoziladi

$$\frac{\partial^2 f(x)}{\partial x_i^k}, \text{ yoki } f_{x_i^2}. \qquad \textbf{(16)}$$

Agar $i \quad k$ bo`lsa, $\frac{\partial^2 f(x)}{\partial x_k \partial x_i}$ ikkinchi tartibli xususiy hosila *aralash hosilalar* deyiladi.

Umuman $f(x_1, x_2, ..., x_m)$ funksiya (n-1) tartibli xususiy hosilasining xususiy hosilasiberilgan funksiyaning n – tartibli xususiy hosilasi deyiladi.

Agar faqat x_k argument bo`yicha $n$ – tartibli hosila olingan bo`lsa, u quyidagicha

$$\frac{\partial^n f(x)}{\partial x_k^n}$$

yoziladi.

$f(x_1, x_2, ..., x_m)$ funksiyaning turli o`zgaruvchilar bo`yicha turli tartibda olingan xususiy hosilalari berilgan funksiyaning turli aralash hosilalarini yuzaga keltiradi.

Ikki argumentli $f(x, y)$ funksiya uchun ikkinchi tartibli hosilalar

$$\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}$$

bo`ladi.

35-misol. $f(x, y) = 3xy^2 - 5x^3y + x - y$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping.

◃ Oldin birinchi tartibli xususiy hosilalarini topamiz:

$$\frac{\partial f(x,y)}{\partial x} = 3y^2 - 15x^2y + 1; \quad \frac{\partial f(x,y)}{\partial y} = 6xy - 5x^3 - 1.$$

Endi ikkinchi tartibli xususiy hosilalarini topamiz:

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}(3y^2 - 15x^2y + 1) = -30xy;$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial}{\partial y} \frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}(6xy - 5x^3 - 1) = 6x;$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial x}(6xy - 5x^3 - 1) = 6y - 15x^2;$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y} \frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial y}(3y^2 - 15x^2y + 1) = 6y - 15x^2 \blacktriangleright$$

E`tabor bering, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$ va $\frac{\partial^2 f(x,y)}{\partial x\partial y}$ aralash hosilalar teng.

Uch o`lchovli $f(x,y,z)$ funksiya ikkinchi tartibli xususiy hosilalari quyidagilar bo`ladi:

$$\frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial y^2},\ \frac{\partial^2 f}{\partial z^2},\ \frac{\partial^2 f}{\partial x\partial y},\ \frac{\partial^2 f}{\partial x\partial z},\ \frac{\partial^2 f}{\partial y\partial z},\ \frac{\partial^2 f}{\partial y\partial x},\ \frac{\partial^2 f}{\partial z\partial x},\ \frac{\partial^2 f}{\partial z\partial y}.$$

36-misol. $f(x,y,z) = x^3y^3z^3 + \sin x + \sin y + \sin z + xyz$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping.

◀Oldin birinchi tartibli xususiy hosilalarini topamiz.

$$\frac{\partial f}{\partial x} = 3x^2y^3z^3 + \cos x + yz$$

$$\frac{\partial f}{\partial y} = 3x^3y^2z^3 + \cos y + xz$$

$$\frac{\partial f}{\partial z} = 3x^3y^3z^2 + \cos z + xy$$

Endi ikkinchi tartibli xususiy hosilalarini topamiz:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^2y^3z^3 + \cos x + yz) = 6xy^3z^3 - \sin x;$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^3y^2z^3 + \cos y + xz) = 6x^3yz^3 - \sin y;$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(3x^3y^3z^2 + \cos z + xy) = 6x^3y^3z - \sin z$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}(3x^2y^3z^3 + \cos x + yz) = 9x^2y^2z^3 + z;$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(3x^3y^2z^3 + \cos y + xz) = 9x^2y^2z^3 + z;$$

$$\frac{\partial^2 f}{\partial x\partial z} = \frac{\partial}{\partial z}\frac{\partial f}{\partial x} = \frac{\partial}{\partial z}(3x^2y^3z^3 + \cos x + yz) = 9x^2y^3z^2 + y;$$

$$\frac{\partial^2 f}{\partial z\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \frac{\partial}{\partial x}(3x^3y^3z^2 + \cos z + xy) = 9x^2y^3z^2 + y;$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial z}\ \frac{\partial f}{\partial z}\ = \frac{\partial}{\partial z}(3x^3y^2z^3 + \cos y + xz) = 9x^3y^2z^2 + x;$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial y}\ \frac{\partial f}{\partial z}\ = \frac{\partial}{\partial y}(3x^3y^3z^2 + \cos z + xy) = 9x^3y^2z^2 + x\,.$$

Bu natijalardan ko`rinib turibdiki, oldin $x$ bo`yicha olingan $\frac{\partial^2 f}{\partial x \partial y}$ hosila oldin y bo`yicha $\frac{\partial^2 f}{\partial y \partial x}$ hosilaga teng, shuningdek

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}, \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$

tengliklar o`rinli.

Yuqoridagi 35- va 36- misollardan ko`rinadiki, bir xil o`zgaruvchilar bo`yicha, lekin turli tartibda olingan aralash hosilalar bir-biriga teng ekan.

Aralash hosilalar bir-biriga teng bo`lmaslig ham mumkin.

37-misol. Ushbu

$$f(x,y) = \begin{cases} xy\ \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{agar } x^2 + y^2 > 0 \text{ bo`lsa,} \\ 0 \quad , & \text{agar } x^2 + y^2 = 0 \text{ bo`lsa} \end{cases}$$

funksiyalarning aralash hosilalarini topamiz.

◀Agar x=0 bo`lsa, ixtiyoriy y da (shu bilan y=0 da ham)

$$\frac{\partial f(0,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, y) - f(0,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(0+\Delta x)y\ \dfrac{(0+\Delta x)^2 - y^2}{(0+\Delta x)^2 + y^2} - 0\ \ y\ \dfrac{0^2 - y^2}{0^2 + y^2}}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x y(\Delta x^2 - y^2)}{\Delta x(\Delta x^2 + y^2)} = \lim_{\Delta x \to 0} \frac{y(\Delta x^2 - y^2)}{(\Delta x^2 + y^2)} = -y$$

ga ega bo`lamiz, ya`ni $f_x'(0,y) = -y$.

Bu funksiyani y bo`yicha differensiallab,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\ \frac{\partial f}{\partial x} \quad = \frac{\partial}{\partial y}(-y) = -1$$

ni hosil qilamiz.

Bundan xususiy holda, ya`ni (0,0) nuqtada ham $f_{xy}(0,0) = -1$ bo`lishini topamiz.

Agar y=0 bo`lsa, ixtiyoriy x da

$$\frac{\partial f(x,0)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,0+\Delta y) - f(x,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{x\ (0+\Delta y)\ \frac{x^2 - (0+\Delta y)^2}{x^2 + (0+\Delta y)^2} - x\ \ 0\ \ \frac{x^2 - 0^2}{x^2 + 0^2}}{\Delta y} =$$

$$= \lim_{\Delta y \to 0} \frac{x \Delta y (x^2 - \Delta y^2)}{\Delta y (x^2 + \Delta y^2)} = x$$

ga ega bo`lamiz. Bundan $f''_{yx} = 1$.

Shuningdek f''_{yx} ni ham (0,0) nuqtada hisoblaymiz, $f''_{yx}(0,1) = 1$ bo`ladi.

Demak, berilgan funksiya uchun aralash hosilalar teng emas:

$$f''_{xy}(0,0) \quad f''_{yx}(0,0). \ \blacktriangleright$$

Aralash hosilalar qachon teng bo`ladi degan savolga quyidagi teorema javob beradi.

1-Teorema. $f(x,y)$ funksiya ochiq M (M R^2) to`plamda berilgan bo`lib, shu to`plamda $f'_x, f'_y$ hamda $f''_{xy},\ f''_{yx}$ aralash hosilalarga ega bo`lib, ular x va y ning funksiyasi sifatida $(x_0, y_0)\ \ M$ nuqtada uzluksiz bo`lsa, u holda shu nuqtada $f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0)$ bo`ladi.

2-Teorema. Faraz qilaylik m ta o`zgaruvchili $f(x_1, x_2, ..., x_m)$ funksiya $x^0 = (x_1^0, x_2^0, ..., x_m^0)\ \ M\ \ R^m$ nuqtada n marta differensiallanuvchi bo`lsin. U holda x^0 nuqtada $f(x_1, x_2, ..., x_m)$ funksiyaning ixtiyoriy n – tartibli aralash hosilalarning qiymati $x_1, x_2, ..., x_m$ o`zgaruvchilar bo`yicha qanday tartibda differensiallanishiga bog`liq bo`lmaydi.

38-misol. Agar $f(x,y) = e^{xy}$ bo`lsa, $\frac{\partial^{10} f}{\partial x^2 \partial y^8}$ topilsin.

◀Berilgan funksiya x va y ning funksiyasi sifatida 10 chi tartibli xususiy hosilasi uning differensiallash tartibiga bog`liq emas. Birin ketin $y$ bo`yicha xususiy hosilasini topamiz:

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(e^{xy})=xe^{xy};$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(xe^{xy})=x^2e^{xy};$$

$$\frac{\partial^3 f}{\partial y^3}=\frac{\partial}{\partial y}(x^2e^{xy})=x^3e^{xy}$$

va hokazo. Ko`rinib turibdiki, $\frac{\partial^8 f}{\partial y^8}=x^8e^{xy}.$

Endi $\frac{\partial^8 f}{\partial y^8}=x^8e^{xy}$ funksiyadan Leybnits formulasini qo`llab $x$ bo`yicha ikkinchi tartibli xususiy hosilasini topamiz:

$$\frac{\partial^2}{\partial x^2}\ \frac{\partial^8 f}{\partial y^8}\ =\frac{\partial^2}{\partial x^2}(x^8e^{xy})=(x^8)_{xx}e^{xy}+2(x^8)_x(e^{xy})_x+x^8(e^{xy})_{xx}=56x^6e^{xy}+16x^7ye^{xy}+x^8y^2e^{xy}$$

Javob: $\frac{\partial^{10} f}{\partial x^2\partial y^8}=e^{xy}(56x^6+16x^7y+x^8y^2).$ ▶

39-misol. Ushbu $f(x,y)=\frac{x+y}{x-y}$ funksiyaning $\frac{\partial^{m+n} f}{\partial x^m\partial y^n}$ hosilasi topilsin.

◀Funksiya $x=y$ shartni bajaruvchi qiymatlaridan tashqari barcha (x,y) qiymatlarida uzluksiz. Oldin y bo`yicha xususiy hosilalarini topamiz:

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\ \frac{x+y}{x-y}\ =\frac{x-y-(-1)(x+y)}{(x-y)^2}=\frac{2x}{(x-y)^2};$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\ \frac{2x}{(x-y)^2}\ =-\frac{2x\ \ 2(x-y)(-1)}{(x-y)^4}=\frac{4x}{(x-y)^3}=\frac{2\ \ 2x}{(x-y)^3};$$

$$\frac{\partial^3 f}{\partial y^3}=\frac{\partial}{\partial y}\ \frac{2\ \ 2x}{(x-y)^3}\ =-\frac{2\ \ 2\ \ 3x\ \ (x-y)^2(-1)}{(x-y)^6}=\frac{2\ \ 2\ \ 3x}{(x-y)^4}$$

va h.k

$$\frac{\partial^n f}{\partial y^n}=\frac{2\ \ 2\ \ 3\ \ ...\ \ n\ \ x}{(x-y)^{n+1}}=\frac{2\ \ 1\ \ 2\ \ 3\ \ ...\ \ n\ \ x}{(x-y)^{n+1}}=\frac{2\ \ n!\,x}{(x-y)^{n+1}}$$

Endi shu xususiy hosilalardan birin – ketin x o`zgaruvchi bo`yicha hosila olamiz:

$$\frac{\partial}{\partial x}\frac{\partial^n f}{\partial y^n}=\frac{\partial}{\partial x}\frac{2\ n!\ x}{(x-y)^{n+1}}=2n!\frac{\partial}{\partial x}\frac{x}{(x-y)^{n+1}}=\frac{-2\ n!\ (nx+y)}{(x-y)^{n+2}};$$

Bundan

$$\frac{\partial^{n+1} f}{\partial x\partial y^n}=\frac{-2\ n!\ (nx+y)}{(x-y)^{n+2}};$$

x o`zgaruvchi bo`yicha 2-tartibli xususiy hosilani topamiz:

$$\frac{\partial}{\partial x}\frac{\partial^{n+1} f}{\partial x\partial y^n}=\frac{\partial}{\partial x}\frac{-2\ n!\ (nx+y)}{(x-y)^{n+2}}=\frac{n!\ (n+1)\ [nx+2y]}{(x-y)^{n+3}}$$

Bundan

$$\frac{\partial^{n+2} f}{\partial x^2\partial y^n}=\frac{2n!\ (n+1)\ [nx+2y]}{(x-y)^{n+3}}$$

x o`zgaruvchi bo`yicha 3-tartibli xususiy hosilani olamiz:

$$\frac{\partial}{\partial x}\frac{\partial^{n+2} f}{\partial x^2\partial y^n}=\frac{\partial}{\partial x}\frac{2n!\ (n+1)\ [nx+2y]}{(x-y)^{n+3}}=-\frac{2n!\ (n+1)(n+2)\ [nx+3y]}{(x-y)^{n+4}}.$$

Bundan

$$\frac{\partial^{n+3} f}{\partial x^3\partial y^n}=-\frac{2n!\ (n+1)(n+2)\ [nx+3y]}{(x-y)^{n+4}}.$$

Natijalarga e`tabor bersak, $x$ – bo`yicha m – tartibli xususiy hosila shunday ko`rinishda bo`ladi:

$$\frac{\partial^{n+m} f}{\partial x^m\partial y^n}=\frac{(-1)^m 2n!(n+1)(n+2)...(n+m-1)(nx+my)}{(x-y)^{n+m+1}}$$

Javob: $\frac{\partial^{n+m} f}{\partial x^m\partial y^n}=\frac{(-1)^m 2(n+m-1)!(nx+my)}{(x-y)^{n+m+1}}$

Agar $F=f(u,v)$ da u=φ(x,y), v=ψ(x,y) bo`lib, $\varphi$, $\psi$ differensiallanuvchi funksiyalar bo`1sa, u holda murakkab funksiyaning ikkinchi tartibli xususiy hosilalarini topishda quyidagi simvolik formuladan foydalanshimiz ma`qul:

$$\frac{\partial^2 F}{\partial x^2} = \left(\frac{\partial}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\frac{\partial v}{\partial x}\right)^2 F + \frac{\partial^2 u}{\partial x^2}\frac{\partial F}{\partial u} + \frac{\partial^2 v}{\partial x^2}\frac{\partial F}{\partial v},$$

$$\frac{\partial^2 F}{\partial y^2} = \left(\frac{\partial}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\frac{\partial v}{\partial y}\right)^2 F + \frac{\partial^2 u}{\partial y^2}\frac{\partial F}{\partial u} + \frac{\partial^2 v}{\partial y^2}\frac{\partial F}{\partial v} \qquad \textbf{(9.1)}$$

$$\frac{\partial^2 F}{\partial x \partial y} = \left(\frac{\partial}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\frac{\partial v}{\partial x}\right)\left(\frac{\partial}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\frac{\partial v}{\partial y}\right) F + \frac{\partial^2 u}{\partial y \partial x}\frac{\partial F}{\partial u} + \frac{\partial^2 v}{\partial y \partial x}\frac{\partial F}{\partial v}.$$

40-misol. Ushbu $F = f(x+y, x^2+y^2)$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping.

◄ f funksiya ikki marta differensiallanuvchi deb faraz qilamiz. Quyidagi $u = x + y$ va $v = x^2 + y^2$ almashtirishlardan so`ng funksiya $F = f(u, v)$ ko`rinishda bo`ladi. Bu funksiyaning birinchi tartibli xususiy hosilalari 29 – misolda (14) formuladan foydalanib topilgan edi. (9.1) formuladan foydalanamiz. Buning uchun oldinquyidagi xususiy hosilalarni topamiz:

$$\frac{\partial u}{\partial x}=1;\quad \frac{\partial^2 u}{\partial x^2}=0;\quad \frac{\partial u}{\partial y}=1;\quad \frac{\partial^2 u}{\partial y^2}=0;$$

$$\frac{\partial v}{\partial x}=2x;\quad \frac{\partial^2 v}{\partial x^2}=2;\quad \frac{\partial v}{\partial y}=2y;\quad \frac{\partial^2 v}{\partial y^2}=2;$$

$$\frac{\partial^2 u}{\partial y\partial x}=0;\quad \frac{\partial^2 v}{\partial y\partial x}=0.$$

$$\frac{\partial^2 F}{\partial x^2}= \; 1 \; \frac{\partial}{\partial u}+2x\;\frac{\partial}{\partial v}\;^2 \; F+0\;\frac{\partial F}{\partial u}+2\;\frac{\partial F}{\partial v}=$$

$$= \; \frac{\partial}{\partial u}+2x\;\frac{\partial}{\partial v}\;^2 \; F+2\;\frac{\partial F}{\partial v}=\frac{\partial^2 F}{\partial u^2}+4x\;\frac{\partial F}{\partial u\partial v}+4x^2\frac{\partial^2 F}{\partial v^2}+2\;\frac{\partial F}{\partial v};$$

$$\frac{\partial^2 F}{\partial y^2}= \; 1 \; \frac{\partial}{\partial u}+2y\;\frac{\partial}{\partial v}\;^2 \; F+0\;\frac{\partial F}{\partial u}+2\;\frac{\partial F}{\partial v}=$$

$$= \; \frac{\partial}{\partial u}+2y\;\frac{\partial}{\partial v}\;^2 \; F+2\;\frac{\partial F}{\partial v}=\frac{\partial^2 F}{\partial u^2}+4y\;\frac{\partial F}{\partial u\partial v}+4y^2\frac{\partial^2 F}{\partial v^2}+2\;\frac{\partial F}{\partial v};$$

$$\frac{\partial^2 F}{\partial u\partial v}= \; \frac{\partial}{\partial u}+2x\;\frac{\partial}{\partial v}\;\;\frac{\partial}{\partial u}+2y\;\frac{\partial}{\partial v}\;F+0\;\frac{\partial F}{\partial u}+0\;\frac{\partial F}{\partial v}=$$

$$=\frac{\partial^2 F}{\partial u^2}+2y\;\frac{\partial^2 F}{\partial u\partial v}+2x\;\frac{\partial^2 F}{\partial v\partial u}+4xy\;\frac{\partial^2 F}{\partial v^2}=$$ ►

$$=\frac{\partial^2 F}{\partial u^2}+2(x+y)\frac{\partial^2 F}{\partial u\partial v}+4xy\;\frac{\partial^2 F}{\partial v^2}.$$

Amaliy mashg'ulotlar uchun misol va masalalar

1. Quyidagi funksiyalarning 2-tartibli xususiy hosilalari topilsin:

1) f(x,y)=$x^4+y^4+\frac{1}{2}x^2y^6$;

2) f(x,y)=cos(x^2y^2);

3) f(x,y)=$\sqrt{x^2+y^2}\;\; e^{x+y}$;

4) f(x,y,z)=$xy^2z^3+\frac{x}{y^2z^3}$;

5) f(x,y,z)=x^{yz}.

2. Ushbu funksiya

$$f(x,y)=\begin{cases} xy, & |y| \quad |x|, \\ -xy, & |y|>|x| \end{cases}$$

O(0,0) nuqtada aralash hosilalarga ega, lekin $f_{xy}(0,0) \quad f_{yx}(0,0)$.

3. Quyidagi funksiya;arning ko`rsatilgan tartibdagi xususiy hosilalarini toping:

1) f(x,y)=cosxy, $\frac{\partial^3 f}{\partial x^2 \partial y}$, $\frac{\partial^3 f}{\partial x \partial y^2}$;

2) f(x,y)=$x^4 \cos y + y^4 \cos x$, $\frac{\partial^8 f}{\partial x^4 \partial y^4}$;

3) f(x,y,z)=e^{xyz}, $\frac{\partial^3 f}{\partial y \partial z \partial x}$;

4) f(x,y)=$(x^2+y)^{10} tgx$, $\frac{\partial^{10} f}{\partial x \partial y^9}$;

5) f(x,y)=$x^m y^n$, $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$;

5) f(x,y)=$e^{2x} \sin y + e^x \cos \frac{y}{2}$, $\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$.

4. Quyidagi funksiyalarning 2-tartibli xususiy hosilalarini toping: (F ikki marta differensiallanuvchi funksiya).

1) F=f(x+2y,x^3+y^3);

2) F=f(xy,$\frac{x}{y}$);

3) F=f(x,x+y);

4) F=f(sinx,cosy).

5. *f* va *g* funksiyalar differensiallanuvchi bo'lganda tanglamalarni o'rinliligi tekshirilsin.

1) $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$ u=xf(x+y)+yg(x+y);

2) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0,$ u=f($\frac{y}{x}$)+xg($\frac{y}{x}$).

10-§. Ko`p o`zgaruvchili funksiyalarning yuqori tartibli differensiallari

Faraz qilaylik, $f(x)=f(x_1,x_2,....,x_m)$ funksiya ochiq $M \quad R^m$ to`plamda berilgan bo`lib, uning n – tartibli xususiy hosilalari mavjud bo`lsin.

Agar $f(x_1,x_2,....,x_m)$ funksiya $x \quad R^m$ nuqtada diffeensiallanuvchi bo`lsa, uning shu nuqtadagi differensiali

$$df=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+...+\frac{\partial f}{\partial x_m}dx_m$$

edi.

1-ta`rif. $f(x)$ funksiya differensiali $df(x)$ ning differensiali berilgan funksiyaning $x \quad R^m$ nuqtadagi ikkinchi tartibli differensiali deyiladi va $d^2 f(x)$ kabi belgilanadi: yoki

$$d^2 f= \quad \frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+...+\frac{\partial}{\partial x_n}dx_n \quad ^2 f \qquad \textbf{(10.1)}$$

oxirgi yozuv, qavs ichidagi yig`indi kvadratga ko`paytirilib, so'ng go'yoki f ga ko`paytiriladi va daraja ko`rsatkichlari xususiy hosilalar tartibi deb hisoblanadi.

Ikki o'zgaruvchili $f(x,y)$ funksiya uchun (10.1) formula quyidagi ko'rinishda bo'ladi:

$$d^2 f= \quad \frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy \quad ^2 f=\frac{\partial^2 f}{\partial x^2}dx^2+2\frac{\partial^2 f}{\partial x\partial y}dxdy+\frac{\partial^2 f}{\partial y^2}dy^2 \qquad \textbf{(10.2)}$$

Xuddi shunga o`xshash

$$d^n f=d(d^{n-1}f)= \quad \frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+...+\frac{\partial}{\partial x_m}dx_m \quad ^n f \qquad \textbf{(17)}$$

Masalan, $2x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y} \quad ^2 f=4x^2\frac{\partial^2 f}{\partial x^2}dx^2+4xy\frac{\partial^2 f}{\partial x\partial y}dxdy+y^2\frac{\partial^2 f}{\partial y^2}dy^2.$

41-misol. $f(x,y)=(x+2)^{y+1}$ funksiyaning $M(0,1)$ nuqtadagi ikkinchi tartibli differensialini toping.

◄Oldin funksiyaning ikkinchi tartibli xususiy hosilalarini topamiz:

$$\frac{\partial f}{\partial x}=(y+1)(x+2)^{y};\quad \frac{\partial f}{\partial y}=(x+2)^{y+1}\ln(x+2);$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left((y+1)(x+2)^{y}\right)=y(y+1)(x+2)^{y-1};$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left((x+2)^{y+1}\ln(x+2)\right)=(x+2)^{y+1}\ln^2(x+2);$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left((y+1)(x+2)^{y}\right)=(x+2)^{y}+(y+1)(x+2)^{y}\ln(x+2)=(x+2)^{y}\left(1+(y+1)\ln(x+2)\right).$$

Endi ikkinchi tartibli xususiy hosilalarini x=0, y=1 dagi qiymatlarini hisoblaymiz.

$$\frac{\partial^2 f}{\partial x^2}(M)=1\cdot(1+1)\cdot(0+2)^{1-1}=2,$$

$$\frac{\partial^2 f}{\partial y^2}(M)=(0+2)^{1+1}\cdot\ln^2(0+2)=4\cdot\ln^2 2,$$

$$\frac{\partial^2 f}{\partial x\partial y}(M)=(0+2)\left[1+(1+1)\cdot\ln(0+2)\right]=2(1+2\ln 2).$$

Bularni ikki (10.2) formulaga qo`yamiz ya`ni

$$d^2 f=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^2 f=\frac{\partial^2 f}{\partial x^2}dx^2+2\frac{\partial^2 f}{\partial x\partial y}dxdy+\frac{\partial^2 f}{\partial y^2}dy^2 \text{ ga}$$

$$d^2 f(M)=2dx^2+4(1+2\ln 2)dxdy+4\ln^2 2dy^2$$ ►

42-misol. Agar $f(x,y,z)=e^{ax+by+cz}$ bo`lsa, $d^n f$ ni toping.

◄Birin – ketin x bo`yicha xususiy hosilalarini topamiz:

$$\frac{\partial f}{\partial x}=ae^{ax+by+cz},\quad \frac{\partial^2 f}{\partial x^2}=a^2e^{ax+by+cz},$$

$$\frac{\partial^3 f}{\partial x^3}=a^3e^{ax+by+cz},$$ va h.k, bu natijalardan ko`rinib turibdiki

$$\frac{\partial^n f}{\partial x^n}=a^n e^{ax+by+cz}.$$

Xuddi shunga o`xshash y va z bo’yicha hosilalarni topsak

$$\frac{\partial^n f}{\partial y^n} = b^n e^{ax+by+cz}, \quad \frac{\partial^n f}{\partial z^n} = c^n e^{ax+by+cz}$$

Endi bir nechta aralash hosilalarinii topaylik:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \quad \frac{\partial f}{\partial x} \quad = abe^{ax+by+cz},$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial y} \quad \frac{\partial^2 f}{\partial x \partial y} \quad = ab^2 e^{ax+by+cz},$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial z} \quad \frac{\partial f}{\partial x} \quad = ace^{ax+by+cz},$$

$$\frac{\partial^3 f}{\partial x \partial z^2} = \frac{\partial}{\partial z} \quad \frac{\partial^2 f}{\partial x \partial z} \quad = ac^2 e^{ax+by+cz},$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = abce^{ax+by+cz}.$$

Bu natijalardan ko`rinib turibdiki hosilalarni olish tartibiga qarab faqat koefitsentlariga qonuniyat sezilmoqda.

Berilgan funksiya (*x,y,z*) o`zgaruvchilarning funksiyasi sifatida uzluksizligini e`tiborga olsak, bir xil o`zgaruvchilar bo`yicha aralash hosilalar teng bo`lishidan ikkinchi tartibli differensial quyiagicha bo`ladi.

$$d^2 f = \quad \frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz \quad {}^2 f =$$

$$= \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial^2 f}{\partial z^2}dz^2 + 2 \quad \frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial x \partial z}dxdz + \frac{\partial^2 f}{\partial y \partial z}dydz \quad =$$

$$= a^2 e^{ax+by+cz}dx^2 + b^2 e^{ax+by+cz}dy^2 + c^2 e^{ax+by+cz}dz^2 +$$

$$+2\left(abe^{ax+by+cz}dxdy + ace^{ax+by+cz}dxdz + bce^{ax+by+cz}dydz\right) =$$

$$= e^{ax+by+cz}\left(a^2dx^2 + bdy^2 + cdz^2 + 2abdxdy + 2acdxdz + 2bcdydz\right).$$

Xuddi shunday yuqori tartibli hosilalarni yozish mumkin.

Javob: $d^n f = e^{ax+by+cz}(adx + bdy + cdz)^n$.

Amaliy mashg'ulot uchun misol va masalalar

1. Quyidagi funksiyalarning ko`rsatilgan tartibdagi differensiallarini toping:

1) $f(x,y)=x^5 y^5 + x^3 y^3,\ \ d^3 f.$

2) $f(x,y)=\sin(x^2+y^2),\ \ d^2 f.$

3) $f(x,y)=e^{2x+3y},\ \ d^{10} f.$

4) $f(x,y)=e^{5x} y^4,\ \ d^{10} f.$

5) $f(x,y)=e^{ax+by},\ \ d^n f.$

6) $f(x,y,z)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ \ d^n f.$

11-§. Ko`p o`zgaruvchili murakkab funksiyaning yuqori tartibli differensiallari

$f(x)=f(x_1,x_2,...,x_m)$ funksiyada $x_1,x_2,...,x_m$ o`zgaruvchilarning har biri $t_1,t_2,...,t_k$ o`zgaruvchilarning funksiyasi bo`lsin, ya`ni

$$x_i=\varphi_i(t_1,t_2,...,t_k),\quad (i=\overline{1,m}) \qquad (11.1)$$

$f(x)$ va $x_i=\varphi_i(t)\ \ (i=\overline{1,m})$ funksiyalar n–marta differensiallanuvchi deb faraz qilamiz. Differensial shaklining invariantligi xossasidan, murakkab funksiyaning differensiali

$$df=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+...+\frac{\partial f}{\partial x_m}dx_m$$

bo`ladi. Differensiallash qoidalaridan foydalanib funksiyaning ikkinchi tartibli differensiali topiladi:

$$d^2 f=d(df)=d\left(\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+...+\frac{\partial f}{\partial x_m}dx_m\right)=$$

$$=d\left(\frac{\partial f}{\partial x_1}\right)dx_1+\frac{\partial f}{\partial x_1}d(dx_1)+d\left(\frac{\partial f}{\partial x_2}\right)dx_2+\frac{\partial f}{\partial x_2}d(dx_2)+...+$$

$$+d\left(\frac{\partial f}{\partial x_m}\right)dx_m+\frac{\partial f}{\partial x_m}d(dx_m)=d\left(\frac{\partial f}{\partial x_1}\right)dx_1+d\left(\frac{\partial f}{\partial x_2}\right)dx_2+\ldots+$$

$$+d\left(\frac{\partial f}{\partial x_m}\right)dx_m+\frac{\partial f}{\partial x_1}d^2x_1+\frac{\partial f}{\partial x_2}d^2x_2+\ldots+\frac{\partial f}{\partial x_m}d^2x_m.$$

$$d^2f=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+\ldots+\frac{\partial}{\partial x_m}dx_m\right)^2 f+\frac{\partial f}{\partial x_1}d^2x_1+\frac{\partial f}{\partial x_2}d^2x_2+\ldots+\frac{\partial f}{\partial x_m}d^2x_m \qquad (18)$$

bunda,

$$dx_1=\frac{\partial x_1}{\partial t_1}dt_1+\frac{\partial x_1}{\partial t_2}dt_2+\ldots+\frac{\partial x_1}{\partial t_k}dt_k$$

$$dx_2=\frac{\partial x_2}{\partial t_1}dt_1+\frac{\partial x_2}{\partial t_2}dt_2+\ldots+\frac{\partial x_2}{\partial t_k}dt_k$$

....

$$dx_m=\frac{\partial x_m}{\partial t_1}dt_1+\frac{\partial x_m}{\partial t_2}dt_2+\ldots+\frac{\partial x_m}{\partial t_k}dt_k$$

Shu yo`l bilan berilgan murakkab funksiyaning keyingi yuqori tartibli differensiallari topiladi. (17) va (18) formulalarni solishtirib , ikkinchi tartibli differensiallarda differensial shaklining invariantligi xossasi o`rinli emasligi ko`rinadi.

Eslatma. (11.1) funksiyalarning har biri $t_1,t_2,\ldots,t_k$ o`zgaruvchilarning chiziqli funksiyasi

$$\begin{aligned}
x_1&=\alpha_{11}t_1+\alpha_{12}t_2+\ldots+\alpha_{1k}t_k+\beta_1,\\
x_2&=\alpha_{21}t_1+\alpha_{22}t_2+\ldots+\alpha_{2k}t_k+\beta_2,\\
&.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\\
x_m&=\alpha_{m1}t_1+\alpha_{m2}t_2+\ldots+\alpha_{mk}t_k+\beta_m
\end{aligned}$$

bo`lsa, $(\alpha_{ji},\beta_{ji}$ $(i=\overline{1,k}, j=\overline{1,m})$-o`zgarmas sonlar), $dx_1,dx_2,..,dx_m$ larning har biri $t_1,t_2,\ldots,t_k$ o`zgaruvchilarga bog`liq bo`lmaydi. Unda $d^2x_1=d^2x_2=\ldots=d^2x_m=0$ bo`lib,

$$d^2f=\left(\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+\ldots+\frac{\partial f}{\partial x_m}dx_m\right)^2 \qquad (17)$$

bo`ladi.

Demak, bu holda ikkinchi (va undan yuqori) tartibli differensiallar differensial shaklining invariantligi xossasiga ega bo`ladi.

43-misol. Ushbu $F = f(x,y)$, x=t+sint, y=$t^2+\cos t$ funksiyaning ikkinchi tartibli differensialini toping.

◀Bu holda (18) formula quyidagi ko`rinishda bo`ladi

$$d^2F = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 f + \frac{\partial f}{\partial x}d^2x + \frac{\partial f}{\partial y}d^2y \qquad (11.2)$$

yoki

$$d^2F = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x\partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial x}d^2x + \frac{\partial f}{\partial y}d^2y\ . \qquad (11.3)$$

Oldin dx, dy, d^2x, d^2y larni topamiz:

$$dx = d(t+\sin t) = (1+\cos t)dt;$$
$$dy = d(t^2+\cos t) = (2t-\sin t)dt;$$
$$d^2x = d((1+\cos t)dt) = -\sin t dt^2;$$
$$d^2y = d((2t-\sin t)dt) = (2-\text{cost})dt^2$$

Bularni (11.3) ga qo`yamiz

$$d^2F = \frac{\partial^2 f}{\partial x^2}((1+\cos t)dt)^2 + 2\frac{\partial^2 f}{\partial x\partial y}(1+\cos t)(2t-\sin t)dt^2 +$$

$$+\frac{\partial^2 f}{\partial y^2}((2t-\sin t)dt)^2 - \frac{\partial f}{\partial x}\sin t dt^2 + \frac{\partial f}{\partial y}(2-\cos t)dt^2 =$$

$$= (1+\cos t)^2\frac{\partial^2 f}{\partial x^2}dt^2 + 2(1+\cos t)(2t-\sin t)\frac{\partial^2 f}{\partial x\partial y}dt^2 + (2t-\sin t)\frac{\partial^2 f}{\partial x^2}dt^2 -$$

$$-\sin t\frac{\partial f}{\partial x}dt^2 + (2-\cos t)\frac{\partial f}{\partial y}dt^2.$$ ▶

44-misol. Ushbu $F = f(x,y,z)$, x=sint, y=cost, z=π+t^3 funksiyaning ikkinchi tartibli differensialini toping.

◀Bu holda (18) formula quyidagi ko`rinishda bo`ladi

$$d^2F = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)^2 f + \frac{\partial f}{\partial x}d^2x + \frac{\partial f}{\partial y}d^2y + \frac{\partial f}{\partial z}d^2z \qquad \mathbf{(11.4)}$$

Oldin dx, dy, dz, d^2x, d^2y, d^2z larni topamiz:

$$dx = d(\sin t) = \cos tdt,$$
$$dy = d(\cos t) = -\sin tdt,$$
$$dz = d(\pi + t^2) = 3t^2 dt^2,$$
$$d^2x = d(\cos tdt) = -\sin tdt^2,$$
$$d^2y = d(-\sin tdt) = -\cos tdt^2,$$
$$d^2z = d(3t^2 dt) = 6tdt^2$$

Bularni (11.4) ga qo`yamiz

$$d^2F = \left(\frac{\partial}{\partial x}\cos tdt + \frac{\partial}{\partial y}\sin tdt + \frac{\partial}{\partial z}3t^2 dt\right)^2 f +$$
$$+\frac{\partial f}{\partial x}(-\sin tdt^2) + \frac{\partial f}{\partial y}(-\cos tdt^2) + \frac{\partial f}{\partial z}(6tdt^2) =$$
$$= \cos^2 t\frac{\partial^2 f}{\partial x^2}dt^2 + \sin^2 t\frac{\partial^2 f}{\partial y^2}dt^2 + 9t^4\frac{\partial^2 f}{\partial z^2}dt^2 -$$
$$-\sin 2t\frac{\partial^2 f}{\partial x\partial y}dt^2 + 6t^2\cos t\frac{\partial^2 f}{\partial x\partial z}dt^2 - 6t^2\sin t\frac{\partial^2 f}{\partial y\partial z}dt^2 -$$
$$-\sin t\frac{\partial f}{\partial x}dt^2 - \cos t\frac{\partial f}{\partial y}dt^2 + 6t\frac{\partial f}{\partial z}dt^2.$$ ▶

45-misol. Ushbu

$$F = f(\sqrt{x^2+y^2}) \quad (x^2+y^2 \quad 0)$$

funksiyaning ikkinchi tartibli differensialini toping.

◀ F ni murakkab funksiya deb differensiallaymiz:

$$dF = df\left(\sqrt{x^2+y^2}\right) = f\,d\sqrt{x^2+y^2} = f\left(\left(\sqrt{x^2+y^2}\right)_x dx + \left(\sqrt{x^2+y^2}\right)_y dy\right) =$$

$$= f\left(\frac{2x}{2\left(\sqrt{x^2+y^2}\right)}dx + \frac{2y}{2\left(\sqrt{x^2+y^2}\right)}dy\right) = f\,\frac{xdx+ydy}{\sqrt{x^2+y^2}}.$$

$$d^2F = d\left(f\,\frac{xdx+ydy}{\sqrt{x^2+y^2}}\right) = df\,\frac{xdx+ydy}{\sqrt{x^2+y^2}} + f\,d\frac{xdx+ydy}{\sqrt{x^2+y^2}}$$

bunda

$$df = f\,d\sqrt{x^2+y^2} = f\,\frac{xdx+ydy}{\sqrt{x^2+y^2}}.$$

$$d\ \frac{xdx+ydy}{\sqrt{x^2+y^2}}\ =\frac{d(xdx+ydy)\ \sqrt{x^2+y^2}-d\sqrt{x^2+y^2}\ \ (xdx+ydy)}{\left(\sqrt{x^2+y^2}\right)^2}=$$

$$=\frac{(dx^2+dy^2)\ \sqrt{x^2+y^2}-\dfrac{xdx+ydy}{\sqrt{x^2+y^2}}\ \ (xdx+ydy)}{x^2+y^2}=\frac{(ydx-xdy)^2}{\sqrt{\left(x^2+y^2\right)^3}}.$$

Natijada quyidagiga ega bo`lamiz

$$d^2F=f\ \frac{(xdx+ydy)^2}{x^2+y^2}+f\ \frac{(ydx-xdy)^2}{\sqrt{(x^2+y^2)^3}}.\ ▶$$

46-misol. Ushbu

$$F=\sin(x^2+y^2),\ \ \text{x=u}^2+v^2,\ \ \text{y=sinuv}$$

funksiyaning ikkinchi tartibli differensialini toping.

◀(11.3) dan foydalanamiz. Buning uchun avvalo $\frac{\partial x}{\partial u},\frac{\partial x}{\partial v},\frac{\partial^2 x}{\partial u^2},\frac{\partial^2 x}{\partial v^2},\frac{\partial^2 x}{\partial u\partial v},\frac{\partial y}{\partial u},\frac{\partial y}{\partial v},\frac{\partial^2 y}{\partial u^2},\frac{\partial^2 y}{\partial v^2},\frac{\partial^2 y}{\partial u\partial v}$ larni so`ng dx, dy, d^2x, d^2y larni topamiz:

$$\frac{\partial x}{\partial u}=2u;\ \ \frac{\partial x}{\partial v}=2v;\ \ \frac{\partial^2 x}{\partial u\partial v}=\frac{\partial}{\partial v}\ \frac{\partial x}{\partial u}\ \ =\frac{\partial}{\partial v}(2u)=0;$$

$$\frac{\partial^2 x}{\partial u^2}=2;\ \ \frac{\partial^2 x}{\partial v^2}=2,$$

$$\frac{\partial y}{\partial u}=v\cos uv;\ \ \frac{\partial y}{\partial v}=u\cos uv;$$

$$\frac{\partial^2 y}{\partial u\partial v}=\frac{\partial}{\partial v}\ \frac{\partial y}{\partial u}\ \ =\frac{\partial}{\partial v}(v\cos uv)=\cos uv-uv\sin uv;$$

$$\frac{\partial^2 y}{\partial v^2}=-u^2\sin uv,\ \ \frac{\partial^2 y}{\partial u^2}=-v^2\sin uv;$$

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial^2 y}{\partial v}dv=2udu+2vdv=2(udu+vdv),$$

$$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv=v\cos uvdu+u\cos uvdv,$$

$$d^2x=\frac{\partial^2 x}{\partial u^2}du^2+2\frac{\partial^2 x}{\partial u\partial v}dudv+\frac{\partial^2 x}{\partial v^2}dv^2\ =$$

$$= 2du^2 + 0 + 2dv^2 = 2(du^2 + dv^2),$$

$$d^2 y = \frac{\partial^2 y}{\partial u^2} du^2 + 2\frac{\partial^2 y}{\partial u \partial v} dudv + \frac{\partial^2 y}{\partial v^2} dv^2 = -v^2 \sin uvdu^2 +$$

$$+ 2(\cos uv - vu \sin uv)dudv - u^2 \sin uvdv^2.$$

Natija uchun (11.3) dan foydalanamiz. Buning uchun

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial^2 f}{\partial x^2}, \ \frac{\partial^2 f}{\partial x \partial y}, \ \frac{\partial^2 f}{\partial y^2}$$

larni topamiz:

$$\frac{\partial f}{\partial x} = 2x\cos(x^2 + y^2), \ \frac{\partial f}{\partial y} = 2y\cos(x^2 + y^2),$$

$$\frac{\partial^2 f}{\partial x^2} = 2\left(\cos(x^2 + y^2) - 2x^2 \sin(x^2 + y^2)\right),$$

$$\frac{\partial^2 f}{\partial y^2} = 2\left(\cos(x^2 + y^2) - 2y^2 \sin(x^2 + y^2)\right),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \ \frac{\partial f}{\partial x} \ = \frac{\partial}{\partial y}\left(2x\cos(x^2 + y^2)\right) =$$

$$= -4xy\sin(x^2 + y^2).$$

Endi yuqorida topilgan hosila va differensiallarni (11.3) ga qo`yamiz:

$$d^2F = 2\left(\cos(x^2 + y^2) - 2x^2 \sin(x^2 + y^2)\right)dx^2 - 8xy\sin(x^2 + y^2)dxdy +$$

$$+2\left(\cos(x^2 + y^2) - 2y^2 \sin(x^2 + y^2)\right)dy^2 + 4x\cos(x^2 + y^2)(du^2 + dv^2) +$$

$$+2y\cos(x^2 + y^2)\left(-v^2 \sin uvdu^2 + 2(\cos uv - vu\sin uv)dudv - u^2 \sin uvdv^2\right).$$ ►

47-misol. Ushbu

f(x,y)=e^{xy}, $x = 2u + 3v$, y=3u-5v

funksiyaning ikkinchi tartibli differensialini toping.

◅ x va y funksiyalarning har biri u va v o`zgaruvchilarning chiziqli funksiyasi bo`lganligi sababli dx, dy lar u va v o`zgaruvchilarga bog`liq bo`lmaydi. Shu sababli $d^2x = 0$, $\mathrm{d}^2 y = 0$ bo`ladi va ikkinchi tartibli differensial shaklining invariantligi saqlanadi, ya`ni bu holda

$$d^2F=\ \frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\ {}^{2}\ f=\frac{\partial^2 f}{\partial x^2}dx^2+2\frac{\partial^2 f}{\partial x\partial y}dxdy+\frac{\partial^2 f}{\partial y^2}dy^2$$

bo`ladi. Oxirgi formuladagi xususiy hosila va differensiallarni topamiz:

$$\frac{\partial f}{\partial x}=ye^{xy},\ \ \frac{\partial^2 f}{\partial x^2}=y^2e^{xy},\ \ \frac{\partial f}{\partial y}=xe^{xy},\ \ \frac{\partial^2 f}{\partial y^2}=x^2e^{xy}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(ye^{xy}\right)=e^{xy}+yxe^{xy},$$

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv=2du+3dv,$$

$$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv=3du-5dv.$$

Bularni barchasini yuqoridagi formulaga qo`yamiz

$$d^2f=y^2e^{xy}(2du+3dv)^2+2(e^{xy}+yxe^{xy})(2du+3dv)(3du-5dv)+$$
$$+x^2e^{xy}(3du-5dv)^2=e^{xy}\big[(9x^2+4y^2+12xy+12)du^2+(12y^2-$$
$$-30x^2-2-2xy)dudv+(25x^2+9y^2-30-30xy)dv^2\big].\ \blacktriangleright$$

Amaliy mashg'ulotlar uchun misol va masalalar

Quyidagi murakkab funksiyalarning ikkinchi tartibli differensialini toping.

1) $f(x,y)=x^2y^2,\ \ x=t^2,\ \ y=t^3$

2) $f(x,y,z)=x^3y^3z^3,\ \ x=t,\ \ y=t^2,\ \ z=t^3$

3) $f(x,y)=x^y,\ \ x=\frac{u}{v},\ \ y=u^2+v^2$

4) $f(x,y)=\ln xy,\ \ x=u^4+v^4,\ \ y=u^2v^2.$

5) $F=f(x,y),\ \ x=ue^v,\ \ y=ve^u$

6) $F=f(x,y,z),\ \ x=u^2+v^2,\ \ y=x^2-y^2,\ \ z=2xy.$

7) $F=f(x,y,z),$

$$x=2x+2y+2z,$$
$$y=3x+3y+3z,$$
$$z=4x+4y+4z.$$

Mustaqil mashg'ulot uchun misol va masalalar

Ko`rsatilgan tartibdagi xususiy hosilalar va differensiallar hisoblansin

M11.1. $u = x^4 \cos y + y^4 \sin x; \dfrac{\partial^8 u}{\partial x^4 \partial y^4}$.

M11.2. $u = \sin x \ \cos 2y; \dfrac{\partial^{10} u}{\partial x^4 \partial y^6}$.

M11.3. $u = \left(x^2 + y\right)^{10} tgx; \dfrac{\partial^{10} u}{\partial x \ \partial y^9}$

M11.4. $u = \sin xy; \dfrac{\partial^3 u}{\partial x^2 \partial y}$ **va** $\dfrac{\partial^3 u}{\partial x \partial y^2}$

M11.5. $u = x \ln(xy) + x^4 - y^4; \dfrac{\partial^4 u}{\partial x^2 \partial y^2}$

M11.6. $u = arctg \dfrac{x + y + z - xyz}{1 - xy - xz - yz}; \dfrac{\partial^3 u}{\partial x \partial y \partial z}$

M11.7. $u = e^{xyz}; \dfrac{\partial^3 u}{\partial x \partial y \partial z}$.

M11.8. $u = \ln \dfrac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}; \dfrac{\partial^4 u}{\partial x \partial y \partial \xi \partial \eta}$

M11.9. $u = (x-a)^3 (y-b)^4; \dfrac{\partial^7 u}{\partial x^3 \partial y^4}$.

M11.10. $u = \frac{x+y}{x-y}; \frac{\partial^6 u}{\partial x^3 \partial y^3}$.

M11.11. $u = \left(x^2 + y^2\right)e^{x+y}; \frac{\partial^{10} u}{\partial x^5 \partial y^5}$.

M11.12. $u = x^m y^n; \frac{\partial^{m+n} u}{\partial x^m \partial y^n}$.

M11.13. $u = \frac{x+y}{x-y}; \frac{\partial^{m+n} u}{\partial x^m \partial y^n}$.

M11.14. $u = e^{2x} \sin y + e^x \cos\frac{y}{2}; \frac{\partial^{m+n} u}{\partial x^m \partial y^n}$.

M11.15. $u = e^y \sin x; \frac{\partial^{m+n} u(0,0)}{\partial x^m \partial y^n}$

M11.16. $u = e^x \sin y; \frac{\partial^{12} u(0,0)}{\partial x^6 \partial y^6}$

M11.17. $u = \sqrt{x^2 + y^2}\; e^{xy}; d^2u$

M11.18. $u = \frac{x}{y}^{\,z}; d^2u$.

M11.19. $u = x^{yz}; d^2u$.

M11.20. $u = \sin\left(x^2 + y^2\right); d^3u$.

M11.21. $u = \ln(x+y); d^{10}u$.

M11.22. $u = \ln\left(x^x y^y z^z\right); d^4u$.

M11.23. $u = \sin(xy); d^{10}u$

M11.24. $u = \sin x \; \sin y; d^4u$

M11.25. $u = f\left(x+y, x^2+y^2\right); d^2u$.

M11.26. $u = f(xy)\ \ g(xz); d^2u$.

M11.27 $u = f(\sin x + \cos y); d^2u$.

M11.28 $u = f(2x, 3y, 2z); d^n u$.

M11.29. $u = f(2x - 3y + 4z); d^n u$.

M11.30. $u = e^{ax+by+cz}; d^n u$.

12-§. Ko`p o`zgaruvchili funksiyaning Teylor formulasi

$f(x) = f(x_1, x_2, ..., x_m)$ funksiya ochiq M R^m to`plamda berilgan va $(x_1^0, x_2^0, ...., x_m^0)$ $M$ nuqtaning atrofida ($n$+1) marta differensiallanuvchi bo`lsin. Quyidagi formula ko`p o`zgaruvchili funksiyaning Teylor formulasi deyiladi:

$$f(x_1, x_2, ..., x_m) = f(x_1^0, x_2^0, ..., x_m^0) + \frac{\partial f}{\partial x_1}(x - x_1^0) + \frac{\partial f}{\partial x_2}(x - x_2^0) + ... + \frac{\partial f}{\partial x_m}(x - x_m^0) +$$

$$+\frac{1}{2!}\left(\frac{\partial}{\partial x_1}(x - x_1^0) + \frac{\partial}{\partial x_1}(x - x_2^0) + ... + \frac{\partial}{\partial x_m}(x - x_m^0)\right)^2 f + ... + \qquad \textbf{(19)}$$

$$+\frac{1}{n!}\left(\frac{\partial}{\partial x_1}(x - x_1^0) + \frac{\partial}{\partial x_2}(x - x_2^0) + ... + \frac{\partial}{\partial x_m}(x - x_m^0)\right)^n f + R_n(f)$$

Bu yerda $R_n(f)$ Lagranj ko`rinishidagi qoldiq had.

$$R_n(f) = \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x_1}(x - x_1^0) + \frac{\partial}{\partial x_2}(x - x_2^0) + ... + \frac{\partial}{\partial x_m}(x - x_m^0)\right)^{n+1} f(x_1^0 + \theta(x - x_1^0), x_2^0 + \theta(x - x_2^0), ..., x_m^0 + \theta(x - x_m^0))$$

bunda 0<θ<1.

Ikki o`zgaruvchili $f(x,y)$ funksiyaning Teylor formulasi quyidagicha bo`ladi:

$$f(x,y) = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x_0 - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y_0 - y_0) +$$

$$+\frac{1}{2!}\left(\frac{\partial f^2(x_0, y_0)}{\partial x^2}(x_0 - x_0)^2 + 2\frac{\partial f^2(x_0, y_0)}{\partial x \partial y}(x_0 - x_0)(y_0 - y_0) + \right. \qquad \textbf{(12.2)}$$

$$+\frac{\partial f^2(x_0,y_0)}{\partial y^2}(y-y_0)^2 \quad +\frac{1}{3!}\ \frac{\partial f^3(x_0,y_0)}{\partial x^3}(x-x_0)^3+$$

$$+3\frac{\partial f^3(x_0,y_0)}{\partial x^2\partial y}(x-x_0)^2+(y-y_0)+3\frac{\partial f^3(x_0,y_0)}{\partial x\partial y^2}(x-x_0)(y-y_0)^2+$$

$$+\frac{\partial f^2(x_0,y_0)}{\partial y^3}(y-y_0)^3 \quad +R_3(f)$$

47-misol. Ushbu $f(x,y)=e^{x-y}$ funksiyaning n=3 bo`lganda $(x_0,y_0)=(1,1)$ nuqta atrofida Teylor formulasini yozing.

◀Bu holda (12.1) quyidagi ko`rinishda bo`ladi:

$$f(x,y)=f(x_0,y_0)+\frac{\partial f(x_0,y_0)}{\partial x}(x-x_0)+\frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)+$$

$$+\frac{1}{2!}\ \frac{\partial f^2(x_0,y_0)}{\partial x^2}(x-x_0)^2+2\frac{\partial f^2(x_0,y_0)}{\partial x\partial y}(x-x_0)(y-y_0)+$$

$$+\frac{\partial f^2(x_0,y_0)}{\partial y^2}(y-y_0)^2\Big]++\frac{1}{3!}\ \frac{\partial f^3(x_0,y_0)}{\partial x^3}(x-x_0)^3+ \qquad \textbf{(12.2)}$$

$$+\frac{1}{3!}\ \frac{\partial f^3(x_0,y_0)}{\partial x^3}(x-x_0)^3+3\frac{\partial f^3(x_0,y_0)}{\partial x^2\partial y^2}(x-x_0)^2(y-y_0)+$$

$$+3\frac{\partial f^3(x_0,y_0)}{\partial x\partial y^2}(x-x_0)(y-y_0)^2++\frac{\partial f^2(x_0,y_0)}{\partial y^3}(y-y_0)^3\Big]+R_3(f)$$

Funksiyaning (2,2) nuqtadagi qiymati $f(2,2)=1$.

Teylor formulasini yozish uchun berilgan funksiyaning xususiy hosilalarini va ularning (2,2) nuqtadagi qiymatlarini topamiz:

$$\frac{\partial f}{\partial x}=e^{x-y},\quad \frac{\partial f(2,2)}{\partial x}=1,$$

$$\frac{\partial f}{\partial y}=-e^{x-y},\quad \frac{\partial f(2,2)}{\partial y}=-1,$$

$$\frac{\partial^2 f}{\partial x^2}=e^{x-y},\quad \frac{\partial f^2(2,2)}{\partial x^2}=1,$$

$$\frac{\partial^2 f}{\partial x\partial y}=-e^{x-y},\quad \frac{\partial^2 f(2,2)}{\partial x\partial y}=-1$$

$$\frac{\partial^2 f}{\partial y^2}=e^{x-y},\quad \frac{\partial f^2(2,2)}{\partial y^2}=1,$$

$$\frac{\partial^3 f}{\partial x^3}=e^{x-y},\quad \frac{\partial f^3(2,2)}{\partial x^3}=1,$$

$$\frac{\partial^3 f}{\partial x^2\partial y}=-e^{x-y},\quad \frac{\partial f^3(2,2)}{\partial x^2\partial y}=-1,$$

$$\frac{\partial^3 f}{\partial x\partial y^2}=e^{x-y},\quad \frac{\partial f^3(2,2)}{\partial x\partial y^2}=1,$$

$$\frac{\partial^3 f}{\partial y^3}=-e^{x-y},\quad \frac{\partial f^3(2,2)}{\partial y^3}=-1.$$

Yuqorida topilgan funksiya qiymati va xususiy hosilalarni (12.2) ga qo`yib quyidagi natijaga ega bo`lamiz:

$$f(x,y)=1+1\ (x-2)-1\ (y-2)+\frac{1}{2}\Big(1\ (x-2)^2+2\ (-1)\ (x-2)\ (y-2)+1\ (y-2)^2\Big)+$$
$$+\frac{1}{6}\Big(1\ (x-2)^3+3\ (-1)\ (x-2)^2\ (y-2)+3\ 1\ (x-2)\ (y-2)^2-1\ (y-2)^3\Big)+R_3(f).$$

$$f(x,y)=1+x-y+\frac{1}{2}\Big((x-2)^2-2(x-2)(y-2)+(y-2)^2\Big)+$$
$$+\frac{1}{6}\Big((x-2)^3-3(x-2)^2(y-2)+3(x-2)(y-2)^2-(y-2)^3\Big)+R_3(f),$$

yoki

$$f(x,y)=1+x-y+\frac{1}{2}(x-y)^2+\frac{1}{6}(x-y)^3+R_3(f).\blacktriangleright$$

Amaliy mashg'ulot uchun misol va masalalar

Quyidagi funksiyalarning *n*=3 bo`lganda keltirilgan nuqta atrofida Teylor formulasini yozing

1) $f(x,y)=2x^2-xy-y^2-6x-3y+5$, $(x_0,y_0)=(1,-2)$.

2) $f(x,y)=\ln(1+x+y)$, $(x_0,y_0)=(0,0)$.

3) $f(x,y)=x^y$, $(x_0,y_0)=(1,1)$.

4) $f(x,y)=e^x\cos y$, $(x_0,y_0)=(0,0)$.

5) $f(x,y)=\sqrt{1-x-y}$, $(x_0,y_0)=(0,0)$.

6) $f(x,y)=x^3+y^3+z^3-3xyz$, $(x_0,y_0,z_0)=(1,1,1)$.

Mustaqil mashg'ulot uchun misol va masalalar

1. Ushbu

$$f(x,y)=2x^2-xy-y^2-6x-3y+5$$

funksiyani *A(1,-2)* nuqta atrofida Teylor formulasini yozish.

2. Ushbu

$$f(x,y)=x^3+y^3-z^3-3xyz$$

funksiyani *A(1,1)* nuqta atrofida Teylor formulasini yozish.

3. Ushbu

$$f(x,y)=x^4$$

funksiyani *A(1,1)* nuqta atrofida uchinchi darajali hadlarigacha yozing.

4. Ushbu

$$f(x,y)=\sqrt{1-x^2-y^2}$$

funksiyani Makloren formulasi bo`yicha to`rtinchi tartibli hadgacha yozing.

13-§. Ko'p o'zgaruvchili funksiyaning ekstremum qiymatlari

$f(x) = f(x_1, x_2, ..., x_m)$ funksiya $M \quad R^m$ to'plamda berilgan bo'lib, $x^0 = (x_1^0, x_2^0, ..., x_m^0) \quad M$ bo'lsin.

1-ta'rif. Agar shunday $\delta > 0$ son topilsaki, $U_\delta(x^0) \quad M$ bo'lib, $\forall x \quad U_\delta(x^0)$ da $f(x) \quad f(x^0)$ bo'lsa, $f(x)$ funksiya x^0 nuqtada lokal maksimumga, $f(x) \quad f(x^0)$ bo'lsa, $f(x)$ funksiya x^0 nuqtada local minimumga erishadi deyiladi.

2-tarif. Agar x^0 nuqtaning shunday $U_\delta(x^0)$ atrofi mavjud bo'lsaki, $\forall x \quad U_\delta(x^0) \setminus \{x_0\}$ uchun $f(x) < f(x^0) \ (f(x) > f(x^0))$ bo'lsa, $f(x)$ funksiya x^0 nuqtada lokal qat'iy maksimumga (minimumga) erishadi deyiladi.

x^0 nuqta $f(x)$ funksiyaga maksimum (minimum) qiymat beradigan nuqta deyiladi.

Funksiyaning maksimum va minimumi umumiy nom bilan uning ekstremumi deb ataladi.

Funksiya ekstremumga erishish zaruriy sharti.

1-teorema. Agar $f(x)$ funksiya x^0 nuqtada ekstremumga erishsa va shu nuqtada barcha $f_{x_1}, f_{x_2}, ..., f_{x_m}$ xususiy hosilalarga ega bo'lsa, u holda

$$f_{x_1}(x^0) = 0, f_{x_2}(x^0) = 0, ..., f_{x_m}(x^0) = 0 \qquad (20)$$

bo'ladi.

Demak, birinchi tartibli xusuiy hosilalarning nolga teng bo'lishi ekstremum mavjad bo'lishining zaruriy sharti bo'ladi.

$f(x)$ funksiyaning xususiy hosilalari nolga aylanadigan nuqtalar uning ***statsionar nuqtalari*** deyiladi.

Eslatma. $f(x)$ funksiyaning biror x^0 nuqtada barcha xususiy hosilalarga ega bo'lishi va (20) shartning bajarilishidan berilgan funksiyaning shu nuqtada ekstremumga erishishi har doim kelib chiqavermaydi.

Funksiya ekstremumining yetarli sharti. Funksiya ekstremum yetarli shartlarini ikki va uch o'lchovli funksiyalar uchun keltirish bilan cheklanamiz.

1-teorema. $f(x,y)$ funksiya (x_0,y_0) R^2 nuqtaning biror U_δ atrofida ($\delta > 0$) berilgan va bu atrofda barcha birinchi, ikkinchi tartibli uzluksiz xususiy hosilalarga ega bo'lsin. (x_0,y_0) nuqta $f(x,y)$ funksiyaning statsionar nuqtasi

$$f_x(x_0,y_0)=0, \quad f_y(x_0,y_0)=0$$

va

$$a_{11}=\frac{\partial f^2(x_0,y_0)}{\partial x^2}; \; a_{12}=\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}; \qquad a_{22}=\frac{\partial^2 f(x_0,y_0)}{\partial y^2}$$

bo'lsin.

1) Agar $a_{11}>0$ va $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22}-a_{12}{}^2>0$ bo'lsa, $f(x,y)$ funksiya (x_0,y_0) nuqtada minimumga ega bo'ladi.

2) Agar $a_{11}<0$ va $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22}-a_{12}{}^2>0$ bo'lsa, $f(x,y)$ funksiya (x_0,y_0) nuqtada maksimumga ega bo'ladi.

3) Agar $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}{}^2 < 0$ bo'lsa, $f(x,y)$ funksiya (x_0, y_0) nuqtada ekstremumga ega bo'lmaydi.

4) Agar $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}{}^2 = 0$ bo'lsa, $f(x,y)$ funksiya (x_0, y_0) nuqtada ekstremumga erishishi ham, erishmasligi ham mumkin. Bu hol *"shubhali"* hol deyiladi va qo'shimcha tekshirish talab qiladi.

48-masala. Ushbu

$$z = x^3 + y^3 - 3xy + 10$$

funksiyani ekstremumga tekshiring.

◄Xususiy hosilalarini topamiz:

$$z_x = 3x^2 - 3y; \qquad z_y = 3y^2 - 3x.$$

Statsionar nuqtalarini quyidagi sistemadan tapamiz:

$$\begin{matrix} 3x^2 - 3y = 0, & x^2 - y = 0, & y = x^2, \\ 3y^3 - 3x = 0. & y^3 - x = 0. & x^6 - x = 0. \end{matrix}$$

Bu sistema $x_1 = 0, y_1 = 0$ va $x_2 = 1, y_1 = 1$ yechimlarga ega bo'ladi. Demak, (0,0) va (1,1) statsionar nuqtalarga ega bo'ladi.

Endi ekstremumga erishishning yetarli shartini tekshirish uchun ikkinchi tartibli xususiy hosilalarni topamiz:

$$\frac{\partial^2 z}{\partial x^2} = 6x, \qquad \frac{\partial^2 z}{\partial x \partial y} = -3, \qquad \frac{\partial^2 z}{\partial y^2} = 6y.$$

Yetarli shartni (0,0) nuqtada tekshiramiz. Buning uchun ikkinchi tartibli xususiy hosilalarni bu nuqtada hisoblaymiz:

$$a_{11} = \frac{\partial^2 z(0,0)}{\partial x^2} = 0, \qquad a_{12} = \frac{\partial^2 z(0,0)}{\partial x \partial y} = -3, \qquad \frac{\partial^2 z(0,0)}{\partial y^2} = 0.$$

Unda

$$a_{11}a_{22} - a_{12}^2 = 0 - (-3)^2 = -9 < 0$$

bo'lib, 3) shartdan funksiya (0,0) nuqtada ekstremumga ega bo'lmaydi.

Endi yetarli shartni (1,1) nuqtada tekshiramiz:

$$a_{11} = \frac{\partial^2 z(1,1)}{\partial x^2} = 6, \qquad a_{12} = \frac{\partial^2 z(1,1)}{\partial x \partial y} = -3, \qquad a_{22} = \frac{\partial^2 z(1,1)}{\partial y^2} = 6,$$

$$a_{11}a_{22} - a_{12}^2 = 6 \cdot 6 - (-3)^2 = 36 - 9 = 27 > 0.$$

Bulardan, $a_{11} > 0$, $a_{11} \cdot a_{22} - a_{12}^2 > 0$ bo'lib, 1) shart o'rinli bo'lmoqda. Demak, funksiya (1,1) nuqtada minimumga ega bo'ladi va u quyidagiga teng

$$\min_{x=1,\, y=1} z = 1^3 + 1^3 - 3 \cdot 1 \cdot 1 + 10 = 9. \blacktriangleright$$

49-misol. Ushbu

$$z = x^4 + y^4 - x^2 - 2xy - y^2$$

funksiyani ekstremumga tekshiring.

◀Xususiy hosilalarini topamiz:

$$z_x = 4x^3 - 2x - 2y, \qquad z_y = 4y^3 - 2y - 2x.$$

Statsionar nuqtalarini quyidagi sitemadan topamiz:

$$\begin{aligned} 4x^3 - 2x - 2y &= 0, \\ 4y^3 - 2y - 2x &= 0. \end{aligned} \qquad \begin{aligned} 2x^3 - x - y &= 0, \\ 2y^3 - y - x &= 0. \end{aligned}$$

Bu sistema uchta ildizga ega: $x_1 = 0, y_1 = 0;\ x_2 = -1, y_2 = -1;\ x_3 = 1, y_3 = 1.$

Yetarli shartni tekshirish uchun ikkinchi tartibli xususiy hosilalarni topamiz:

$$z_{x^2} = 12x^2 - 2, \quad z_{xy} = -2, \quad z_{y^2} = 12y^2 - 2$$

va statsionar nuqtalarda $a_{11}a_{22} - a_{12}^2$ ifodaning qiymatini hisoblaymiz.

(0,0) statsionar nuqtada $a_{12} = z_{x^2}(0,0) = -2,\ a_{12} = z_{xy}(0,0) = -2,\ a_{22} = z_{y^2}(0,0) = -2$

va

$$a_{11}a_{22} - a_{12}^2 = -2\ \ (-2) - (-2)^2 = 0.$$

4) shartdan bu (0,0) statsionar nuqta ”shubhali” hol bo’lyapti. Qo’shimcha tekshirish talab qilinadi.

Buning uchun z funksiyaning orttirmasi

$$\Delta z(x, y) = z(x + h, y + h) - z(x, y)$$

ni (0,0) nuqtada qaraymiz:

$$\Delta z(0,0) = z(h, k) - z(0,0) = h^4 + k^4 - h^2 - 2hk - k^2.$$

Agar $h = k$ desak, $\Delta z(0,0) = h^4 + h^4 - h^2 - 2h^2 - h^2 = 2h^2(h^2 - 2)$ bunda $0 < h < \sqrt{2}$ bo’lib, $\Delta z(0,0) < 0$ bo’ladi.

Agar $k = -h\ \ (h > 0)$ desak,

$$\Delta z(0,0) = h^4 + h^4 - h^2 + 2h^2 - h^2 = 2h^4 > 0$$

bo’ladi.

Natijada $\Delta z(0,0)$ (0,0) nuqtada atrofida har xil ishora qabul qiladi, shu sababli (0,0) nuqtada ekstremum mavjud emas.

(-1,-1) statsionar nuqtada

$a_{12} = z_{x^2}(-1,-1) = 12\ \ (-1)^2 - 2 = 10,\ \ a_{12} = z_{xy}(-1,-1) = -2,\ \ a_{22} = z_{y^2}(-1,-1) = 12\ \ (-1)^2 - 2 = 10$ va

$a_{11}a_{22} - a_{12}^2 = 10\ \ 10 - (-2)^2 = 96 > 0.$ $a_{11} > 0$ va $a_{11}\ \ a_{22} - a_{12}^2 > 0$ ekanligini e’tiborga olsak, z funksiya (-1,-1) nuqtada minimumga ega bo’ladi va

$$\min_{x=-1, y=-1} z = (-1)^4 + (-1)^4 - (-1)^2 - 2(-1)(-1) - 1^2 = -2.$$

Xuddi shuningdek funksiya (1,1) da ham minimumga ega bo’lshi va -2 ga tengligi ko’rsatiladi.▶

$f(x, y, z)$ funksiya (x_0, y_0, z_0) R^3 nuqtaning biror U_δ atrofida ($\delta > 0$) berilgan va bu atrofda barcha birinchi, ikkinchi tartibli uzluksiz hosilalarga ega bo’lsin. (x_0, y_0, z_0) nuqta $f(x, y, z)$ funksiyaning statsionar nuqtasi

$$f_x(x_0, y_0, z_0) = 0,$$
$$f_y(x_0, y_0, z_0) = 0,$$
$$f_z(x_0, y_0, z_0) = 0.$$

va

$$A_1 = a_{11}, \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (21)$$

bu determinantlarda,

$$a_{11} = f_{x^2}(x_0, y_0, z_0), \; a_{22} = f_{y^2}(x_0, y_0, z_0), \; a_{33} = f_{z^2}(x_0, y_0, z_0),$$

$$a_{12} = a_{21} = f_{xy}(x_0, y_0, z_0) = f_{yx}(x_0, y_0, z_0), \; a_{13} = a_{31} = f_{xz}(x_0, y_0, z_0) = f_{zx}(x_0, y_0, z_0),$$

$$a_{23} = a_{32} = f_{yz}(x_0, y_0, z_0) = f_{zy}(x_0, y_0, z_0).$$

1) Agar $A_1 > 0, \; A_2 > 0, \; A_3 > 0$ bo'lsa $f(x, y, z)$ funksiya (x_0, y_0, z_0) nuqtada minimumga ega bo'ladi.

2) Agar $A_1 < 0, \; A_2 > 0, \; A_3 < 0$ bo'lsa $f(x, y, z)$ funksiya (x_0, y_0, z_0) nuqtada maksimumga ega bo'ladi.

3) Agar 1) va 2) gruhdagi shartlarning birortasi bajarilmasa qo'shimcha tekshirish talab qilinadi.

50-misol. Ushbu

$$f(x, y, z) = x^2 + y^2 + z^2 + 2x + 4y - 6z.$$

funksiyani ekstremumga tekshiring.

◄Xususiy hosilalarini topamiz:

$$f_x = 2x + 2, \; f_y = 2y + 4, \quad f_z = 2z - 6.$$

Statsionar nuqtalarini topamiz:

$$\begin{matrix} 2x + 2 = 0, & \qquad x = -1, \\ 2y + 4 = 0, & \qquad y = -2, \\ 2z - 6 = 0. & \qquad z = 3. \end{matrix}$$

Funksiya bitta (-1;-2;3) statsionar nuqtaga ega. Ikkinchi tartibli xususiy hosilalarini topamiz: $f_{x^2} = 2,\ f_{y^2} = 2,\ f_{z^2} = 2,\ f_{xy} = f_{xz} = f_{yz} = 0$ (aralash hosilalar teng).

(21) dagi determinantlarni hisoblaymiz:

$$A_1 = f_{x^2} = 2 > 0, \quad A_2 = \begin{vmatrix} f_{x^2} & f_{xy} \\ f_{yx} & f_{y^2} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0,$$

$$A_3 = \begin{vmatrix} f_{x^2} & f_{xy} & f_{xz} \\ f_{yx} & f_{y^2} & f_{yz} \\ f_{zx} & f_{zy} & f_{z^2} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0,$$

1) gruhdagi shartlar statsionar (-1,-2,3) nuqtada o'rinli bo'ladi, demak, bu nuqtada funksiya minimumga ega va quyidagiga teng:

$$\min_{(x=-1,y=-2,z=3)} f(x,y,z) = (-1)^2 + (-2)^2 + 3^2 + 2(-1) + 4(-2) - 6\ \ 3 = -14. \blacktriangleright$$

51-misol. Ushbu

$$f(x,y,z) = \sin x + \sin y + \sin z - \sin(x+y+z),$$
$$0 \quad x \quad \pi \quad 0 \quad y \quad \pi \quad 0 \quad z \quad \pi$$

funksiyani ekstremumga tekshiring.

◀Xususiy hosilalarini topamiz:

$$f_x = \cos x - \cos(x+y+z), \quad f_y = \cos y - \cos(x+y+z), \quad f_z = \cos z - \cos(x+y+z).$$

Statsionar nuqtalarini topamiz:

$$\cos x - \cos(x+y+z) = 0,$$
$$\cos y - \cos(x+y+z) = 0,$$
$$\cos z - \cos(x+y+z) = 0.$$

Ayirmalarni ko'paytuvchiga keltirib yechamiz:

$$2\sin\frac{x+x+y+z}{2}\ \ \sin\frac{x-x-y-z}{2} = 0,$$
$$2\sin\frac{y+x+y+z}{2}\ \ \sin\frac{y-x-y-z}{2} = 0,$$
$$2\sin\frac{z+x+y+z}{2}\ \ \sin\frac{z-x-y-z}{2} = 0.$$

$$\sin\frac{2x+y+z}{2}\sin\frac{y+z}{2}=0,$$
$$\sin\frac{2y+x+z}{2}\sin\frac{x+z}{2}=0,$$
$$\sin\frac{2z+y+x}{2}\sin\frac{x+y}{2}=0.$$

Oxirgi sistema yechimi quyidagicha bo'lib

$$\frac{2x+y+z}{2}=\pi m,\quad \frac{y+z}{2}=\pi n,\quad m,n\quad Z,$$
$$\frac{2y+x+z}{2}=\pi k,\quad \frac{x+z}{2}=\pi l,\quad k,l\quad Z,$$
$$\frac{2z+y+x}{2}=\pi i,\quad \frac{x+y}{2}=\pi j,\quad i,j\quad Z.$$

Masala shartini qanoatlantiradiganlari $x=0,\ y=0,\ z=0;\ x=\frac{\pi}{2},\ y=\frac{\pi}{2},$ $z=\frac{\pi}{2}$ va $x=\pi,\ y=\pi,\ z=\pi$ bo'ladi (bular i,j,k,l,m,n -larga mos ravishda 0 va 1 qiymatlar berib topiladi).

Funksiya uchta statsionar nuqtaga ega ekan: (0,0,0), $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$, (π,π,π).

Endi shu nuqtalarda funksiya ekstremumga ega bo'lish yoki bo'lmasligini tekshiramiz.

Funksiyaning ikkinchi tartibli xususiy hosilalari:

$$f_{x^2}=-\sin x+\sin(x+y+z),\qquad f_{xy}=\sin(x+y+z),$$
$$f_{y^2}=-\sin y+\sin(x+y+z),\qquad f_{yz}=\sin(x+y+z),$$
$$f_{z^2}=-\sin z+\sin(x+y+z),\qquad f_{zx}=\sin(x+y+z).$$

$(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ statsionar nuqtada hisoblaymiz

$$f_{x^2}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})=-\sin\frac{\pi}{2}+\sin(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2})=-2,$$
$$f_{xy}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})=\sin(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2})=-1$$
,

$$f_{y^2}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) = -\sin\frac{\pi}{2} + \sin(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}) = -2,$$

$$f_{zx}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) = \sin(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}) = -1,$$

$$f_{z^2}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) = -\sin\frac{\pi}{2} + \sin(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}) = -2,$$

$$f_{yz}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) = \sin(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}) = -1.$$

Bulardan

$$A_{11} = f_{x^2}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) = -2 < 0,\ \ A_2 = \begin{vmatrix} f_{x^2} & f_{xy} \\ f_{yx} & f_{y^2} \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = (-2)(-2)-(-1)(-1) = 3 > 0,$$

$$A_3 = \begin{vmatrix} f_{x^2} & f_{xy} & f_{xz} \\ f_{yx} & f_{y^2} & f_{yz} \\ f_{zx} & f_{zy} & f_{z^2} \end{vmatrix} = \begin{vmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{vmatrix} = -4 < 0.$$

Shunday qilib, 2) shartdan $f(x,y,z)$ funksiya $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ nuqtada maksimumga ega bo'ladi va u

$$\max_{(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})} f(x,y,z) = \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} - \sin(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}) = 4.$$

(0,0,0) va $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ statsionar nuqtalarda barcha ikkinchi tartibli hosilalar mavjud va ular nolga teng, shuningdek $A_1 = 0,\ A_2 = 0,\ A_3 = 0$ o'rinli bo'ladi. Bu nuqtalar chegara nuqtalar bo'lib, ularda funksiyaning qiymati nolga teng. Bundan funksiya bu nuqtalarda chegaraviy minimumga ega bo'ladi.►

Amaliy mashg'ulot uchun mashqlar.

1. $u = x^3 + 8y^3 - 6xy + 5$.
2. $u = (x-1)^2 + 2y^2$.

3. $u = 2x^3 - x^2 + xy^2 - 4x + 3.$

4. $u = 2x^3 + 2y^3 - 36xy + 430.$

5. $u = x^2 - xy + y^2 + 9x - 6y + 20.$

6. $u = 3x + 6y - x^2 - xy - y^2.$

7. $u = e^{\frac{x}{2}}(x + y^2).$

8. $u = x^2 + y^2 + z^2 + 2x + 4y - 6z.$

9. $u = x^3 + y^2 + z^2 + 12xy + 2z - \pi - 2.$

10. $u = \frac{a^2}{x} + \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{b} \quad (x > 0, y > 0, z > 0, a > 0, b > 0).$

Javoblar.

1. $(1;\frac{1}{2})$ nuqtada minimumga ega ($u_{\min} = 4$); (0;0) da ekstremum yo'q.

2. (1;0) nuqtada minimumga ega ($u_{\min} = 0$).

3. ($-\frac{2}{3};0$) nuqtada maksimumga ega ($u_{\max} = 4\frac{17}{27}$); (1;0) nuqtada minimumga ega ($u_{\min} = 0$); (0;-2) va (0;2) nuqtalarda ekstremum yo'q.

4. (6;6) nuqtada minimumga ega ($u_{\min} = -2$); (0;0) nuqtada ekstremum yo'q.

5. (-4;1) da minimmga ega ($u_{\min} = -1$).

6. (0;3) nuqtada maksimumga ega ($u_{\max} = 9$).

7. (-2;0) nuqtada minimumga ega ($u_{\min} = -\frac{2}{e}$).

8. (-1;-2;3) nuqtada minimumga ega ($u_{\min} = -14$).

9. (24;-144;-1) nuqtada minimumga ega ($u_{\min} = -6915 - \pi$).

10. $(\frac{1}{2}\sqrt[15]{16a^{14}b}; \frac{1}{4}\sqrt[5]{16a^4b}; \frac{1}{2}\sqrt{\frac{a^8b^7}{4}})$ nuqtada minimumga ega ($u_{\min} = \frac{15a}{4}\sqrt[15]{\frac{a}{16b}}$).

Mustaqil mashg'ulot uchun misol va masalalar

Quyidagi funksiyalar ekstremumga tekshiring.

M12.1. $u = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$.

M12.2. $u = -x^2 - xy - y^2 + x + y$.

M12.3. $u = x^3 + y^3 - 3axy$.

M12.4. $u = x^4 + y^4 - 36xy$.

M12.5. $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

M12.6. $u = x^2 + y^4 - 2xy^2 - y^5$.

M12.7. $u = e^{2x} \left(x + y^2 + 2y\right)$.

M12.8. $u = 3x^2 y + y^3 - 18x - 30y$.

M12.9. $u = xy + yz + zx$.

M12.10. $u = \left(x^2 + y^2\right)e^{-\left(x^2 + y^2\right)}$

M12.11. $u = 4 - \left(x^2 + y^2\right)^{2/3}$.

M12.12. $u = x^2 + 2y^2 + z^2 - 2x + 4y - 6z + 1$.

M12.13. $u = 2x^2 + y^2 + z^2 - 2xy + 4z - x$.

M12.14. $u = x^3 + xy + y^2 - 2zx + 2z^2 + 3y - 1$.

M12.15. $u = \frac{ax + by + c}{\sqrt{x^2 + y^2 + 1}}; \quad a^2 + b^2 + c^2 \quad 0$.

M12.16. $u = (x - y + 1)^2$.

M12.17. $u = x^4 + y^4 - x^2 - 2y^2$.

M12.18. $u = x^2 y^3 \ (6 - x - y)$.

M12.19. $u = x^4 + y^4 - x^2 - 2xy - y^2$.

M12.20. $u = x^2 - (y-1)^2$.

M12.21. $u = x^2 - 2xy + 4y^2 + 6z^2 + 6yz - 6z$.

M12.22. $u = x^2 - xy + y^2 - 2x + y$.

M12.23. $u = xy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$; *(a>0, b>0)*

M12.24. $u = e^{2x+3y}\left(8x^2 - 6xy + 3y^2\right)$

M12.25. $u = (5x + 7y - 25)e^{-\left(x^2+xy+y^2\right)}$

M12.26. $u = x^2 + xy + y^2 - 4\ln x - 10\ln y$

M12.27. $u = x^3 + y^2 + z^2 + 12xy + 2z$.

M12.28. $u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$; *(x>0, y>0, z>0)*

M12.29. $u = xy^2z^3(a - x - 2y - 3z)$; *(a>0)*

14-§. Ko`p o`zgaruvchili funksiyaning eng katta va eng kichik qiymati

$f(x) = f(x_1, x_2, ..., x_m)$ funksiya M R^m yopiq chegaralangan sohada aniqlangan va uzluksiz bo`lsin. $M$ sohaning chegarasi $L$ bo`lsin. U holda $f(x)$ funksiyaning M sohada (Veyershtrass teoremasiga ko`ra) eng katta hamda eng kichik qiymati mavjud bo`ladi. Funksiyaning M sohadagi eng katta (eng kichik) qiymati quyidagicha topiladi:

1) $f(x)$ funksiyaning M sohadagi maksimum (minimum) qiymatlari to`plami $\{\max f(x)\}$ $(\{\min f(x)\})$.

2) Funksiyaning M soha L chegaradagi qiymatlari hisoblanadi: f(L).

3){max $f(x)$} ({min $f(x)$}) to`plamning barcha elementlari bilan $f$(L) taqqoslanadi. Bu qiymatlar ichida eng kattasi (eng kichigi) f(x) funksiyaning eng katta (eng kichik) qiymati bo`ladi.

52-misol. Ushbu

$$f(x,y)=\sin x+\sin y-\sin(x+y)$$

funksiyaning *Ox*, *Oy* va $x+y=2\pi$ chiziqlar bilan chegaralangan sohadagi eng katta qiymati topilsin.

◀Shartada berilgan soha uchburchak bo`ladi (14.1-rasm). Berilgan funksiyaning xususiy hosilalarini olib

RASM BOR

$$f_x^{'}=\cos x-\cos(x+y),$$
$$f_y^{'}=\cos y-\cos(x+y).$$

quyidagi sistemadan statsionar nuqtalarni topamiz.

$$\cos x-\cos(x+y)=0,$$
$$\cos y-\cos(x+y)=0.$$

Bu sistema soha ichidagi yagona $\frac{2\pi}{3},\frac{2\pi}{3}$ nuqtada o`rinli bo`ladi. Bu nuqtada berilgan funksiya qiymati $f\ \frac{2\pi}{3},\frac{2\pi}{3}\ =\frac{3\sqrt{3}}{2}$ ga teng bo`ladi.

Sohaning chegaralarida, ya`ni x=0, y=0 va $x+y=2\pi$ chiziqlarda funksiya qiymati nolga teng $\left(f(0,y)=0,\ \text{f(x,0)=0, f(x,2}\pi\text{-x)=0}\right)$. Demak, berilgan sohada funksiyaning eng katta qiymati $\frac{2\pi}{3},\frac{2\pi}{3}$ nuqtada bo`lib, u $\frac{3\sqrt{3}}{2}$ ga teng.▶

53-misol. Radiusi *R* ga teng bo`lgan doiraga ichki chizilgan barcha uchburchaklardan yuzi eng katta bo`lganini toping.

◀Agar uchburchakning tomonlarini tortib turgan markaziy burchaklarni *x,y,z* desak (14.2-rasm), quyidagi tenglik o`rinli bo`ladi

$$x + y + z = 2\pi \ .$$

Bundan $z = 2\pi - x - y$.

Bular yordamida uchburchak yuzi quyidagicha ifodalanadi:

RASM BOR

$$S = \frac{1}{2}R^2 \sin x + \frac{1}{2}R^2 \sin y + \frac{1}{2}R^2 \sin z$$

yoki

$$S = \frac{1}{2}R^2 \sin x + \frac{1}{2}R^2 \sin y + \frac{1}{2}R^2 \sin(2\pi - x - y) =$$

$$= \frac{1}{2}R^2 \left[\sin x + \sin y - \sin(x + y)\right]$$

Bu yerda x va y o`zgaruvchilarning aniqlanish sohasi $x \quad 0$, y  0, x+y=2$\pi$ bo`ladi. Shu shartlarda masala yechimini topish

$$\sin x + \sin y - \sin(x + y)$$

sohadagi eng katta qiymar beradigan (*x,y*) nuqtani topishga keladi. Bu masala yechimi 52-masalada topilgan edi, ya`ni u $\frac{2\pi}{3}, \frac{2\pi}{3}$ nuqtada eng katta qiymatga ega bo`lgan edi, ya’ni

$$x = \frac{2\pi}{3}, \ \text{y=}\frac{2\pi}{3}, \ \text{da z=}\frac{2\pi}{3}$$

bo`ladi.

Demak, R radiusli doiraga ichki chizilgan uchburchakning yuzi eng katta bo`lishi uchun u teng tomonli bo`lishi kerak ekan.►

Amaliy mashg’ulot uchun misol va masalalar

1. Ushbu funksiyalarni berilgan shart bilan bog`liq eng katta va eng kichik qiymatlarini toping.

1) $f(x,y) = x + y,\ \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{16}.$

2) $f(x,y,z) = x^2 y^3 z^4,\ 2x + 3y + 4z = 10$

3) $z = x - 2y - 3,\ 0 \quad x \quad 1,\ 0 \quad y \quad 1,\quad 0 \quad x + y \quad 1$

2. Agar parallelepipedning qirralari yig`indisi $12a$ ga teng bo`lsa, parallelepipedning eng katta hajmi nimaga teng bo`ladi.

3. Peremetri $2p$ bo`lgan barcha uchburchaklardan yuzi eng katta bo`lgani topilsin.

4. Musbat a sonni 4 ta musbat sonni ko`paytmasi ko`rinishida shunday ifodalangki, uning yig`indisi eng kichuk bo`lsin.

Mustaqil mashg'ulot uchun misol va masalalar

Berilgan funksiyaning ko'rsatilgan to'plamdagi eng katta va eng kichik qiymatlari topilsin.

M13. 1. $u = xy - x^2 y - \frac{x^2 y}{2};\ (0 \quad x \quad 1;\ 0 \quad y \quad 2).$

M13. 2. $u = x^2 + 3y^2 - x + 18y - 4;\ (0 \quad x \quad 1;\ 0 \quad y \quad 1).$

M13. 3. $u = x^2 + 3y^2 - 3xy;\ (0 \quad x \quad 1;\ 0 \quad y \quad 1)$

M13. 4. $u = \frac{xy}{2} - \frac{x^2 y}{6} - \frac{xy^2}{8};\ (x \quad 0; y \quad 0; \frac{x}{3} + \frac{y}{4} \quad 1).$

M13. 5. $u = x^6 + y^6 - 3x^2 + 6xy - 3y^2;\ (0 \quad y \quad x \quad 2).$

M13. 6. $u = \cos x \ \ \cos y \ \ \cos(x+y);\ (0 \quad x \quad \pi;\ 0 \quad y \quad \pi).$

M13. 7. $u = (x - y^2)\ \sqrt[3]{(1-x)^2};\ (y^2 \quad x \quad 2).$

M13. 8. $u = x^3 + y^3 - 9xy + 27;\ (0 \quad x \quad 6;\ 0 \quad y \quad 6)$.

M13. 9. $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2;\ (0 \quad x \quad 2;\ 0 \quad y \quad 2)$.

M13. 10. $u = xy + yz + zx,\ \ x^2 + y^2 + z^2 \quad 9$.

M13. 11. $u = x + y + z,\ \ x^2 + y^2 \quad z \quad 1$.

M13. 12. $u = 2\sin x + 2\sin y + \sin(x+y), \quad 0 \quad x \quad \frac{\pi}{2},\ 0 \quad y \quad \frac{\pi}{2}$.

M13. 13. $u = \sin x + \cos y + \cos(x-y), \quad 0 \quad x \quad \frac{\pi}{2},\ 0 \quad y \quad \frac{\pi}{2}$.

M13. 14. $u = xy\ln(x^2 + y^2);\ (0 \quad x \quad 1,\ 1 \quad y \quad e)$.

M13. 15. $u = x^2 + 3y^2 - 3xy + \sin 2;\ (0 \quad x \quad 3,\ 0 \quad y \quad 2)$.

M13. 16. $u = \sin x \ \sin y \ \sin(x+y);\ (0 \quad x \quad \pi,\ 0 \quad y \quad \pi)$.

M13. 17. $u = x^2 + xy + y^2 - 6x - 9y;\ (0 \quad x \quad 1,\ 0 \quad y \quad 2)$.

M13. 18. $u = x^3 + xy^2 + 3xy;\ (0 \quad x \quad 1,\ 1 \quad y \quad 2)$.

M13. 19. $u = (x + y^2)\ \sqrt{e^x};\ (0 \quad x \quad 2,\ 1 \quad y \quad 2)$

M13. 20. $u = xe^{y + x\sin y},\ (0 \quad x \quad 1,\ 0 \quad y \quad \frac{\pi}{2})$.

M13. 21. $u = x^2 + (y-1)^2;\ (1 \quad x \quad 2,\ 1 \quad y \quad 2)$.

M13. 22. $u = x + y + 4\sin x \ \sin y;\ \ 0 \quad x \quad \frac{\pi}{2},\ 0 \quad y \quad \frac{\pi}{2}$.

Berilgan funksiyalar shartli ekstrimumga tekshirilsin

M13. 23. $u = xy;\ \ x + y = 1$.

M13. 24. $u = x^2 + 12xy + 2y^2;\ \ 4x^2 + y^2 = 25$.

M13. 25. $u = \cos^2 x + \cos^2 y;\ \ x - y = \frac{\pi}{4}$.

M13. 26. $u = x - 2y + 2z;\;\; x^2 + y^2 + z^2 = 1$.

M13. 27. $u = x^2 + y^2 + z^2,\;\; u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\; (a > b > c > 0)$.

M13. 28. $u = \sin x \;\; \sin y \;\; \sin z;\;\; x + y + z = \frac{\pi}{2}\;\; (x > 0,\; y > 0,\; z > 0)$

M13. 29. $u = xyz;\;\; x^2 + y^2 + z^2 = 1;\;\; x + y + z = 0$.

15-§. Oshkormas funksiyalar

<u>1. Oshkormas funksiya tushunchasi.</u> Faraz qilaylik, x va y o’zgaruvchilarning $F(x,y)$ funksiyasi $M = \{(x,y)\quad R^2 : a < x < b, c < y < d\}$ to’plamda berilgan bo’lsin. X to’plamdan o’ligan har bir x ga ($x \quad X \quad R$)

$$F(x,y) = 0 \qquad\qquad \textbf{(15.1)}$$

tenglamaning yagona yechimi y ($y \quad Y \quad R$) mos qo’yilgan bo’lsa, bunday aniqlangan funksiya oshkormas ko’rinishida berilgan funksiya (oshkormas) deyiladi va

$$x \to y : F(x,y) = 0$$

kabi belgilanadi.

(15.1) tenglama yechimi $y \quad x \quad X$ da aniqlangan $f(x)$ bo’ladi va

$$F(x, f(x)) = 0 \qquad\qquad \textbf{(15.2)}$$

bo’lsa $f(x)$ funksiya oshkormas ko’rinishda berilgan deyiladi.

Qisqasi, $y = f(x)$ funksiya (y ga nisbatan) yechilmagan (15.1) tenglama yordamida berilgan bo’lsa, unga <u>oshkormas</u> funksiya deyiladi; agar y ning x ga bevasita bog’lanishi berilgan bo’lsa <u>oshkor</u> funksiya deyiladi.

54-misol. Ushbu

$$xy - x^2 + 3 = 0 \quad \left(F(x,y) = xy - x^2 + 3 \right)$$

tenglama yordamida oshkormas funksiya aniqlanishi ko'rsatilsin.

◀Berilgan tenglama $x \quad (- \quad ,0) \quad (0,+ \quad)$ da yagona

$$y = f(x) = \frac{x^2 - 3}{x}$$

yechimga ega va

$$F(x, f(x)) = 0 .$$

Demak, berilgan tenglama $f(x)$ oshkormas funksiyani aniqlaydi.▶

Oshkormas funksiyalarni o'rganishda quyidagi masalalar muhimdir:

1) $F(x,y)$ funksiya biror $M \quad R^2$ to'plamda berilgan holda $y = f(x)$ oshkormas funksiya mavjud bo'ladimi va bu funksiyaning aniqlanish to'plami qanday bo'ladi?

2) (15.1) tenglama bilan aniqlangan oshkormas funksiya $y = f(x)$ qanday xossalarga ega va bu xossalar $F(x,y)$ funksiya bilan qanday bog'langan.

Bu masalalarga quyidagi teorema javob beradi.

2. Oshkormas funksiyaning mavjudligi.

1-teorema. Faraz qilaylik, $F(x,y)$ funksiya (x_0, y_0) nuqtaning biror atrofi

$$U_{hk}((x_0, y_0)) = \{(x,y) \quad R^2 : x_0 - h < x < x_0 + h, y_0 - k < y < y_0 + k\}$$

da ($h > 0, k > 0$) berilgan bo'lib, quyidagi shartlarni bajarsin:

1) $F(x,y)$ funksiya $U_{hk}((x_0, y_0))$ da uzluksiz;

2) har bir tayin $x \quad (x_0 - h, x_0 + h)$ da y o'zgaruvchining funksiyasi sifatida o'suvchi;

3) $F(x_0, y_0) = 0$.

U holda (x_0, y_0) nuqtaning shunday atrofi

$$U_{\delta\varepsilon}((x_0, y_0)) = \{(x, y) \quad R^2 : x_0 - \delta < x < x_0 + \delta, y_0 - \varepsilon < y < y_0 + \varepsilon\}$$

topiladiki ($0 < \delta < h, 0 < \varepsilon < k$),

a) $\forall x \quad (x_0 - \delta, x_0 + \delta)$ da

$$F(x, y) = 0$$

tenglama yagona y ($y \quad (y_0 - \varepsilon, y_0 + \varepsilon)$) yechimga ega, ya'ni $F(x, y) = 0$ tenglama yordamida oshkormas $y = f(x)$ funksiya aniqlanadi.

b) oshkormas korinishda aniqlangan

$$x \to y : F(x, y) = 0$$

funksiya $(x_0 - \delta, x_0 + \delta)$ oraliqda uzluksiz bo'ladi.

Bir necha o'zgaruvchining oshkormas funksiyasini (15.1) tenglamaga o'xshash ko'rish mumkin.

Masalan, uch o'zgaruvchili,

$$F(x, y, z) = 0 \qquad \textbf{(15.3)}$$

tenglama berilgan bo'lsin. Ma'lum shartlar bajarilganda, z bu tenglama orqali ikkita o'zgaruvchi x va y larning oshkormas funksiyasi sifatida aniqlanadi:

$$z = h(x, y)$$

va u ko'p qiymatli bo'ladi.

Agar uni z ni o'rniga qo'ysak, x va y ga nisbatan aynan bajariladigan ushbu

$$F(x, y, h(x, y)) = 0$$

munosabat hosil bo'ladi.

Bu holda ham 1-teoremega o'xshash teorema keltirish mumkin.

3. Oshkormas funksiyaning hosilalari.

2-teorema. $F(x,y)$ funksiya (x_0, y_0) nuqtaning biror atrofi $U_{hk}((x_0,y_0))$ da ($h>0, k>0$) berilgan bo'lib,, quyidagi shartlarni bajarsin:

1) $U_{hk}((x_0,y_0))$ da uzluksiz;

2) $U_{hk}((x_0,y_0))$ da uzluksiz $F_x(x,y), F_y(x,y)$ xususiy hosilalarga ega va $F_y(x_0,y_0) \quad 0$;

3) $F(x_0,y_0)=0$

u holda (x_0, y_0) nuqtaning shunday $U_{\delta\varepsilon}((x_0,y_0))$ atrofi ($0<\delta<h, 0<\varepsilon<k$) topiladiki, $F(x,y)=0$ tenglama y ni x ning oshkormas $y=f(x)$ funksiyasi sifatida aniqlaydi va $y=f(x)$ funkisya $(x_0-\delta, x_0+\delta)$ da uzluksiz differensiallanuvchi bo'lib,

$$y_x = -\frac{F_x(x,y)}{F_y(x,y)} \qquad \textbf{(15.4)}$$

bo'ladi.

55-misol. Ushbu

$$F(x,y) = 2^y + y^2x - x^2 \sin y - 1 = 0$$

tenglama (1,0) nuqtaning atrofida y ni x ning oshkormas funksiyasi sifatida aniqlashi va bu oshkormas funksiyaning hosilasi topilsin.

◄ $F(x,y) \quad R^2$ da aniqlangan va uzluksiz shu bilan birga (1,0) nuqtaning atrofida ham uzluksiz.

$F(x,y)$ funksiyaning xususiy hosilalari quyidagicha bo'ladi.

$$\frac{\partial F(x,y)}{\partial x} = \frac{\partial}{\partial x}(2^y + y^2x - x^2 \sin y - 1) = y^2 - 2x \sin y;$$

$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y}(2^y + y^2x - x^2 \sin y - 1) = 2^y \ln 2 + 2yx - x^2 \cos y.$$

$F(x,y)$ funksiyaning xususiy hosilalari ham R^2 da, shu bilan birga (1,0) nuqta atrofida uzluksiz.

$\frac{\partial F(x,y)}{\partial y}$ xususiy hosila qiymatini (1,0) nuqtada hisoblaymiz:

$$\frac{\partial F(1,0)}{\partial y} = (2^y \ln 2 + 2yx - x^2 \cos y)_{x=1;y=0} = 2^0 \;\; \ln 2 + 2 \;\; 0 \;\; 1 - 1^2 \;\; \cos 0 = \ln 2 - 1 \quad 0.$$

Va nihoyat,

$$F(1,0) = (2^y + y^2 x - x^2 \sin y - 1)_{x=1;y=0} = 0$$

o'rinli bo'ladi. Unda 2-teoremaga ko'ra

$$F(x,y) = 2^y + y^2 x - x^2 \sin y - 1 = 0$$

tenglama (1,0) nuqtaning atrofida y ni x ning oshkormas funksiyasi sifatida aniqlaydi va bu oshkormas $f(x)$ funksiyaning hosilasi

$$f \;\; (x) = -\frac{F_x(x,y)}{F_y(x,y)} = \frac{y^2 - 2x \sin y}{2^y \ln 2 + 2yx - x^2 \cos y}. \ ▶$$

<u>1-eslatma</u>. Oshkormas funksiyaning hosilasini quyidagicha ham hisoblash mumkin: $F(x,y) = 0$ da y o'zgaruvchi x ning funksiyasi ekanligini hisobga olsak uni differensiallab quyidagiga ega bo'lamiz

$$F_x(x,y) + F_y(x,y) \;\; y \;\; = 0$$

oxirgi tenglikdan

$$y \;\; = -\frac{F_x(x,y)}{F_y(x,y)}$$

bo'lishi kelib chiqadi.

56-misol. Ushbu

$$F(x,y) = x^2 + y^2 + 1 + 2x \sin xy + xy = 0$$

tenglama bilan berilgan y funksiyaning hosilasi topilsin.

◀2-teoremaning shartlari bajariladi deb faraz qilamiz.

Differensiallab topamiz:

$$(F(x,y))_x = (x^2 + y^2 + 1 + 2x\sin xy + xy)_x = 0,$$

$$2x + 2yy + 2(\sin xy + x\cos xy \ (xy)_x) + y + xy = 0,$$

$$2x + 2yy + 2\sin xy + 2x\cos xy \ (y + xy) + y + xy = 0,$$

$$2x + 2yy + 2\sin xy + 2xy\cos xy + 2x^2\cos xy \ y + y + xy = 0,$$

$$(2y + 2x^2 \ \cos xy + x)y + 2x + 2\sin xy + 2xy\cos xy + y = 0.$$

Keyingi tenglikdan

$$y = -\frac{2\sin xy + 2xy\cos xy + y}{2y + 2x^2 \ \cos xy + x}$$ ▶

<u>4. Oshkormas funksiyaning yuqori tartibli hosilasi.</u> Agar $F(x,y)$ funksiya $U_{\delta,\varepsilon}((x_0,y_0))$ da uzluksiz ikkinchi tartibli $F_{x^2}(x,y), F_{xy}(x,y), F_{y^2}(x,y)$ xususiy hosilalarga ega bo'lib, y o'zgaruvchi x ning funksiyasi bo'lsa uning ikkinchi tartibli hosilasi quyidagicha topiladi.

$$y = \frac{2F_xF_yF_{xy} - F_y^2 \ F_{x^2} - F_x^2 \ F_{y^2}}{F_y^3} \qquad \textbf{(15.5)}$$

57-misol. Ushbu

$$F(x,y) = x^2\ln y - y^2\ln x = 0$$

tenglama bilan berilgan y funksiyaning ikkinchi tartibli hosilasini toping.

◀(15.5) formuladan foydalanamiz. Buning uchun undagi barcha hosilalarni topamiz.

$$F_x = 2x\ln y - \frac{y^2}{x}, \qquad F_y = \frac{x^2}{y} - 2y\ln x, \qquad F_{xy} = (2x\ln y - \frac{y^2}{x})_y = \frac{2x}{y} - \frac{2y}{x},$$

$$F_{x^2} = (2x\ln y - \frac{y^2}{x})_x = 2\ln y + \frac{y^2}{x^2}, \qquad F_{y^2} = (\frac{x^2}{y} - 2y\ln x)_y = -\frac{x^2}{y^2} - 2\ln x.$$

Bularni barchasini (15.5) ga qo'yamiz.

$$y \;=\frac{2\ \ (2x\ln y-\dfrac{y^2}{x})(\dfrac{x^2}{y}-2y\ln x)(\dfrac{2x}{y}-\dfrac{2y}{x})-(\dfrac{x^2}{y}-2y\ln x)^2(2\ln y+\dfrac{y^2}{x^2})-(2x\ln y-\dfrac{y^2}{x})^2(-\dfrac{x^2}{y^2}-2\ln x)}{(\dfrac{x^2}{y}-2y\ln x)^3}$$

2-eslatma. Oshkormas funksiyaning yuqori tartibli hosilalarini quyidagicha ham olish mumkin. Buning uchun oldin $F(x,y)=0$ tenglamadan 1-eslatmadek hosila olib, uni yana bir marta differensiallab (va hokazo) ikkinchi tartibli hosilasini topamiz.

Yuqorida $F(x,y)=0$ ni differensiallab

$$F_x(x,y)+F_y(x,y)\ \ y\ =0$$

bo'lishini topgan edik.

Buni yana bir marta differensiallaymiz

$$\lfloor F_x(x,y)+F_y(x,y)\ \ y\ \rfloor_x\ =0\,,$$

$$(F_x(x,y))_x+(F_y(x,y))_x\ \ y\ +F_y(x,y)\ \ y\ =0\,.$$

Agar

$$(F_x(x,y))_x\ =F_{x^2}(x,y)+F_{xy}(x,y)\ \ y\ \ ,$$

$$(F_y(x,y))_x\ =F_{yx}(x,y)+F_{y^2}(x,y)y \qquad \textbf{(15.6)}$$

Tengliklarni e'tiborga olsak, yuqoridagi tenglik quyidagi ko'rinishni oladi

$$F_{x^2}(x,y)+2F_{xy}(x,y)y\ +F_{x^2}(x,y)y\ ^2+F_y(x,y)\ \ y\ =0$$

va bundan

$$y\ \ =-\frac{F_{x^2}(x,y)+2F_{xy}(x,y)y\ +F_{y^2}(x,y)y\ ^2}{F_y(x,y)} \qquad \textbf{(15.7)}$$

Oxirgi tenglikda y ni (15.4) bilan almashtirsak (15.5) tenglikka kelamiz.

58-misol. Ushbu

$$F(x, y) = x - y + \frac{1}{2}\sin y$$

tenglama bilan berilgan oshkormas funksiyaning ikkinchi tartibli hosilasini toping.

◀Differensiallab topamiz:

$$(F(x, y))_x = (x - y + \frac{1}{2}\sin y)_x = 1 - y + \frac{1}{2}\cos y \; y = 0 \qquad \textbf{(15.8)}$$

Bundan birinchi tartibli hosilani topamiz:

$$\frac{1}{2}\cos y - 1 \quad y = -1,$$

$$y = \frac{1}{1 - \frac{1}{2}\cos y} = \frac{2}{2 - \cos y} \qquad \textbf{(15.9)}$$

Endi (15.8) ni yana bir marta differensiallaymiz:

$$\left(1 - y + \frac{1}{2}\cos y \; y\right)_x = 0,$$

$$-y + \frac{1}{2}\left(-y \; \sin y \; y + y \; \cos y\right) = 0, \; -y - \frac{1}{2}y^2 \sin y + \frac{1}{2}y \; \cos y = 0,$$

$$\frac{1}{2}\cos y - 1 \; y = \frac{1}{2}y^2 \sin y, \qquad y = \frac{\frac{1}{2}y^2 \sin y}{\frac{1}{2}\cos y - 1} = \frac{y^2 \sin y}{\cos y - 2}.$$

Oxirgi natijada y ni (15.9) bilan almashtirsak

$$y = -\frac{4\sin y}{(\cos y - 2)^3}. \; ▶$$

5. Ko'p o'zgaruvchili oshkormas funksiya hosilasi

Faraz qilaylik,

$$F(x_1, x_2, ..., x_m, y) = 0 \qquad (15.10)$$

tenglama berilgan bo'lib, $F(x_1, x_2, ..., x_m, y)$ ko'p o'zgaruvchili funksiya uchun 2-teoremaning barcha shartlarini qanoatlantirsin. Bu yerda y

o’zgaruvchi $x_1, x_2,..., x_m$ larga bog’liq. Berilgan tenglama bilan aniqlagan oshkormas funksiyaning xususiy hosilalari quyidagicha topiladi:

$$
\begin{aligned}
y_{x_1} &= -\frac{F_{x_1}(x_1, x_2,..., x_m, y)}{F_y(x_1, x_2,..., x_m, y)},\\
y_{x_2} &= -\frac{F_{x_2}(x_1, x_2,..., x_m, y)}{F_y(x_1, x_2,..., x_m, y)}, \qquad (15.11)\\
&\dots\dots \quad \dots\dots \quad \dots\dots \quad \dots\dots\\
y_{x_m} &= -\frac{F_{x_m}(x_1, x_2,..., x_m, y)}{F_y(x_1, x_2,..., x_m, y)}.
\end{aligned}
$$

Agar $F(x,y)$ funksiya $\{(x_1,x_2,...,x_m) \quad R^m : x_1-\delta_1 < x_1 < x_1+\delta_1,\ x_2-\delta_2 < x_2 < x_2+\delta_2,...,$ $x_m-\delta_m < x_m < x_m+\delta_m\}$ da uzluksiz yuqori tartibli xususiy hosilalarga ega bo’lganda $F(x,y)=0$ tenglama bilan aniqlangan oshkormas ko’rinishdagi funksiyaning ham yuqori tartibli hosilalari mavjud bo’ladi.

59- misol. Ushbu

$$F(x,y,z) = x^2 - 2y^2 + 3z^2 - yz + y = 0$$

tenglama bilan berilgan oshkormas funksiyaning birinchi va ikkinchi tartibli xususiy hosilalarini toping.

◀*I-usul.* Berilgan $F(x,y,z)$ ko’p o’zgaruvchili oshkormas funksiya uzluksiz va uzluksiz xususiy hosilalari mavjud. Bu yerda $z=z(x,y)$ bo’lib, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$ xususiy hosilalarini topishimiz kerak.

Oldin F_x, F_y, F_z xususiy hosilalarini topamiz.

$$F_x(x,y,z) = 2x,\ \ F_y = -4y - z + 1,\ \ \ F_z(x,y,z) = 6z - y.$$

(15.11) formulani qo’llab, quyidagiga ega bo’lamiz;

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{2x}{6z-y}; \qquad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{1-4y-z}{6z-y}.$$

Berilgan oshkormas funksiyaning ikkinchi tartibli xususiy hosilalarini quyidagicha topish mumkin:

$$z_{x^2} = \frac{\partial}{\partial x} \quad -\frac{2x}{6z-y} \quad {}_x = -2 \quad \frac{6z-y-6xz_x}{(6z-y)^2},$$

$$z_{y^2} = \frac{\partial}{\partial y} \quad -\frac{1-4y-z}{6z-y} \quad = -\frac{(-4-z_y)(6z-y)-(6z_y-1)(1-4y-z)}{(6z-y)^2}$$

$$z_{xy} = \frac{\partial}{\partial y} \quad \frac{\partial z}{\partial x} \quad = \frac{\partial}{\partial y} \quad -\frac{2x}{6z-y} \quad = \frac{2x(6z_y-1)}{(6z-y)^2}.$$

Berilgan tengliklardagi z_x, z_y larning o'rniga ularning qiymatlarini qo'yib, soddalashtirish mumkin.

II-usul. Berilgan tenglamani differensiallab, quyidagiga ega bo'lamiz:

$$2xdx - 4ydy + 6zdz - ydz - zdy + dy = 0\,.$$

Bundan dz ni ya'ni oshkomas funksiyaning to'la differensialini topamiz:

$$dz = \frac{2xdx + (1-4y-z)dy}{y-6z} = -\frac{2x}{6z-y}dx - \frac{1-4y-z}{6z-y}dy.$$

$z(x,y)$ funksiyaning to'la differensiali:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

bilan taqqoslab,

$$\frac{\partial z}{\partial x} = -\frac{2x}{6z-y}, \qquad \frac{\partial z}{\partial y} = -\frac{1-4y-z}{6z-y}$$

natijalarga ega bo'lamiz.

Ikkinchi tartibli xususiy hosilalar

$$d^2z = d(dz) = d \quad -\frac{2x}{6z-y}dx - \frac{1-4y-z}{6z-y}dy \quad =$$

$$-\frac{2x}{6z-y}dx - \frac{1-4y-z}{6z-y}dy \quad {}_x dx + \quad -\frac{2x}{6z-y}dx - \frac{1-4y-z}{6z-y}dy \quad {}_y dx$$

tenglikdan topiladi.Bunda dx, dy differensiallar x va y o'zgaruvchilarga bog'liq emas debdifferensiallanadi.

6. Oshkormas funksiyaning ekstrimumi

Soddalik uchun oshkormas ko'rinishda berilgan $z = z(x, y)$ funksiyaning ekstrimumlarini topish bilan cheklanamiz. z funksiyani oshkormas ko'rinishda ifodalovchi $F(x, y, z) = 0$ funksiya ko'p o'zgaruvchili funksiya uchun 2-teorema barcha shartlarini qanoatlantirsin.

$z = z(x, y)$ funksiyaning ekstrimumini topish uchun quyidagi qadamlar bajariladi:

1) Oshkormas funksiyaning xususiy hosilalari z_x va z_y topiladi, ya'ni

$$z_x = -\frac{F_x(x, y, z(x, y))}{F_z(x, y, z(x, y))} \text{ va } z_y = -\frac{F_y(x, y, (x, y))}{F_z(x, y, z(x, y))};$$

2) $\begin{matrix} z_x = 0 \\ z_y = 0 \\ F \quad 0 \end{matrix}$ sistema yechilib statsionar nuqtalar topiladi;

3) Statsionar nuqtalarda $F_z \quad 0$ shart tekshiriladi;

4) Ekstrimumga erishishning yetarli shartini tekshirish uchun $a_{11} = z_{x^2}$, $a_{22} = z_{y^2}$ va $a_{12} = z_{xy}$ hosilalar topilib, statsionar nuqtalarda ularning qiymati hisoblanadi;

5) Har bir statasionar nuqta uchun $a_{11}a_{22} - a_{12}^2$ ifoda qiymati hisoblanadi. Funksiya ekstrimumga erishish haqidagi teorema (13-§. 1-teorema) shartlaridan foydalanib natija chiqariladi;

6) Ekstrimum mavjud bo'lgan nuqtalarda z funksiyaning qiymati hisoblanadi, u statsionar nuqtadagi z_0 bo'ladi.

60-misol. Ushbu oshkormas ko'rinishda berilgan $z = z(x, y)$ funksiyaning ekstrimumlari topilsin:

$$x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0.$$

◀Berilgan

$$F(x, y, z) \quad x^2 + y^2 + z^2 - 2x + 2y - 4z - 10$$

funksiya butun darajali ko'phad bo'lgani uchun u $\forall (x, y, z) \quad R^3$ da uzluksiz va ixtiyoriy tartibda differensiallanuvchi. Shu sababli $F(x_0, y_0, z_0) = 0$, $F_z(x_0, y_0, z_0) \quad 0$ o'rinli bo'lgan $\forall (x_0, y_0, z_0)$ nuqta atrofida $z = z(x, y)$ bo'lganda $F(x, y, z) = 0$ tenglama (x_0, y_0) nuqtada z_0 qiymatni qabul qiluvchi oshkormas funksiyani ifodalaydi.

Statsionar nuqtalarni topish uchun oshkormas funksiyaning birinchi tartibli xususiy hosilalarini topib

$$z_x = -\frac{F_x}{F_z} = -\frac{2x-2}{2z-4} = -\frac{x-1}{z-2}, \quad z_y = -\frac{F_y}{F_z} = -\frac{2y+2}{2z-4} = -\frac{y+1}{z-2},$$

quyidagi sistemani tuzamiz:

$$\begin{array}{ll} & -\dfrac{x-1}{z-2} = 0, \\ z_x = 0, & \\ z_y = 0, & -\dfrac{y+1}{z-2} = 0, \\ F = 0. & x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0. \end{array}$$

Oxirgi sistemani x, y, z o'zgaruvchilarga nisbatan yechib $M_1(1, -1, -2)$ va $M_2(1, -1, 6)$ statsionar nuqtalarni topamiz. Bu nuqtalarda $F_z = 2z - 4$ hosila nolga teng emas ($F_z(1, -1, -2) = -8, \ F_z(1, -1, 6) = 8$). Ekstrimumning yetarli shartini tekshirish uchun $z_{x^2}, \ z_{xy}, \ z_{y^2}$ xususiy hosilalarni topamiz

$$z_{x^2} = (z_x)_x = \ -\frac{x-1}{z-2} \quad {}_x = -\frac{z - 2 - (x-1) \ z_x}{(z-2)^2},$$

$$z_{y^2}=(z_y)_y=\left(-\frac{y+1}{z-2}\right)_y=-\frac{z-2-(y+1)\ z_y}{(z-2)^2},$$

$$z_{xy}=(z_x)_y=\left(-\frac{x-1}{z-2}\right)_y=\frac{(x-1)z_y}{(z-2)^2}.$$

Bu topilgan xususiy hosilalarni qiymatini statsionar nuqtalarda hisoblab ekstrimumga tekshiramiz:

1) $M_1(1,-1,-2)$ nuqtada ekstrimumga tekshiramiz ($z_x(1,-1)=0,\ z_y(1,-1)=0$).

$$a_{11}=z_{x^2}\Big|_{(1,-1,-2)}=-\frac{-2-2}{(-2-2)^2}=\frac{1}{4},\qquad a_{22}=z_{y^2}\Big|_{(1,-1,-2)}=-\frac{-2-2}{(-2-2)^2}=\frac{1}{4},$$

$$a_{12}=z_{xy}\Big|_{(1,-1,-2)}=0$$

Bulardan

$$\Delta=a_{11}a_{22}-a_{12}^2=\frac{1}{16}.$$

Demak, $a_{11}>0$, $\Delta>0$ bo'lib, z funksiya (1,-1) nuqtada minimumga ega bo'ladi va $z_{\min}=-2$.

2) $M_2(1,-1,6)$ nuqtada ekstrimumga tekshiramiz ($z_x(1,-1)=0,\ z_y(1,-1)=0$).

$$a_{11}=z_{x^2}\Big|_{(1,-1,6)}=-\frac{6-2}{(6-2)^2}=-\frac{1}{4},\qquad a_{22}=z_{y^2}\Big|_{(1,-1,6)}=-\frac{6-2}{(6-2)^2}=-\frac{1}{4},$$

$$a_{12}=z_{xy}\Big|_{(1,-1,6)}=0,$$

Bulardan

$$\Delta=a_{11}a_{22}-a_{12}^2=\frac{1}{16}.$$

Demak, $a_{11}<0$, $\Delta>0$ bo'lib, z funksiya (1,-1) nuqtada maksimumga ega bo'ladi va $z_{\max}=6$.

Amaliy mashg’ulotlar uchun misol va masalalar

1. Quyidagi oshkormas ko’rinishda beilgan funksiyalarning birinchi va ikkinchi tartibli hosilalarini toping.

1. $F(x,y)=x^2+xy+y^2-3=0$.

2. $F(x,y)=x^y-y^x=0.$

3. $F(x,y)=2xarctg\dfrac{x}{y}-y=0.$

4. $F(x,y)=1+y^x-y=0.$

5. $F(x,y)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0.$

6. $F(x,y)=x^2-xy+2y^2+x-y-1=0.$

7. $F(x,y,z)=x^2+y^2+z^2-1=0.$

8. $F(x,y,z)=z^3-3xyz-a^3=0.$

9. $F(x,y,z)=x+y+z-e^{-(x+y+z)}=0.$

10. $F(x,y,z)=x\cos y+y\cos z+z\cos x-1=0.$

2. Quyidagi oshkormas ko’rinishda berilgan funksiyalarning birinchi va ikkinchi tartibli hosilalarini qiymatini toping.

1) $F(x,y)=x^2-2xy+y^2+x+y-2=0,\quad x=1,\quad y=1.$

2) $F(x,y,z)=x^2+2y^2+3z^2+xy-z-9=0,\quad x=-1, y=-2, z=1.$

3) $F(x,y,z)=2x^2+2y^2+z^2-8xz-z+8=0,\quad x=2, y=0, z=1.$

3. Oshkormas ko’rinishda berilgan $z=z(x,y)$ funksiyaning ekstrimumlari topilsin.

1. $2x^2+2y^2+z^2+8yz-z+8=0$.

2. $x^4+y^4+z^4=2(x^2+y^2+z^2)$.

3. $5x^2+5y^2+5z^2-2xy-2xz-2yz-72=0$.

4. $z^2 + xyz - xy^2 - x^3 = 0$.

5. $5z^2 + 4yz + y^2 - 2y + 3x^2 - 6x + 4 = 0$.

6. $x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0$.

Mustaqil mashg'ulot uchun misol va masalalar.

Oshkormas ko`rinishda berilgan *u=u(x) (z=z(x,y))* funksiyalarning birinchi va ikkinchi tartibli hosilalari topilsin.

M14.1. $x^2 + 2xy - y^2 = 5, \quad \left(x^2 + 2xyz - y^2 + z^2 = 6\right)$

M14.2. $\ln\sqrt{x^2 + y^2} = arctg\frac{x}{y}, \quad \left(x^2 + y^2 + z^2 = a^2\right)$

M14.3. $x\ln y = y, \quad \left(x + xyz + \ln z = 3\right)$

M14.4. $x^2 + xy + y^2 = 3, \quad \left(x + y + z = e^z\right)$

M14.5. $x^2 - xy + 2y^2 + x - y - 5 = 0, \quad \left(z^3 - 3xyz = a^2\right)$

M14.6. $y - a\sin y = x \quad (0\ \ a\ \ 1), \qquad z = \sqrt{x^2 - y^2}\,tg\frac{z}{\sqrt{x^2 - y^2}}$

M14.7. $y - \cos y = 2x \qquad z = x^2 z + \frac{1}{\sqrt{x + z}}$

M14.8. $x^2 + x^2 y + xy - 3 + y^2 = 6, \quad \left(2x + 2z^2 y + 5 - z^3 = 0\right)$

M14.9. $x^y = y^x \quad (x \quad y), \qquad \frac{x}{z} = \ln\frac{z}{y} + 1$

M14.10. $y = 2x\,arctg\,\dfrac{y}{x}, \quad \left(z^3 + 3xyz = 10\right)$

M14.11. $\dfrac{y}{x} = \arcsin\dfrac{x}{y}, \quad \left(x^2 + y^2 + z^2 = 2z - 2\right)$

M14.12. $\sin xy + y\cos x = 5, \quad \left(x^3 + y^3 + z^3 - 3z = 0\right)$

M14.13. $x^2 + y^2 - 2xy + 7 = 0, \quad \left(x\cos y + y\cos z + z = 9\right)$

M14.14. $x + y = e^{x-y}, \quad \left(x^2 + 2y^2 + 3z^2 = z\right)$

M14.15. $2x + 2y = 2^{x-y}, \quad \left(x^3 + y^3 + z^3 = 3xyz\right)$

M14.16. $2xy = e^{x+y}, \quad \left(xyz - e^{xyz} = 0\right)$

M14.17. $x\cos y + y\cos x = a, \quad \left(x\sin y + z\sin x + \cos z = a\right)$

M14.18. $x^4 + y^4 = 4xy, \quad \left(x^4 + y^4 + z^4 = 4z\right)$

M14.19. $x^3 + y^3 = 3xy, \quad \left(x^3 + y^3 + z^3 = 3z\right)$

M14.20. $x^3 + y^3 = x + y, \quad \left(x^3 + y^3 = z^3 + x + z\right)$

M14.21. $x^2 + y^2 = x - y, \quad \left(x^3 - y^3 = z - z^2\right)$

M14.22. $x\sin y + y\sin x + \cos y = 1, \quad \left(\sin x + \cos z + zy = 5\right)$

M14.23. $\sqrt{xy} + \ln xy = 3, \quad \left(xy + xz + yz = z^2\right)$

M14.24. $x^3 + y^3 = xy + y^2, \quad \left(x^2 + z^3 + 3axz = y^2\right)$

M14.25. $x + \sqrt{xy} = y^2, \quad \left(\sin xyz + x + z = 5\right)$

M14.26. $F(x - y, xy) = 0, \quad \left(F(xy, yz, zx) = 0\right)$

M14.27. $F\left(\frac{x}{y},\frac{y}{x}\right)=0, \quad \left(F(xyz, x+y, zx)=0\right)$

M14.28. $F(x\sin y, y\sin x)=0, \quad \left(F(y-zx, x-yz, z-xy)=0\right)$

M14.29. $F(x+y, x^2+y^2)=0, \quad \left(F(x+y+z, xz, yz)=0\right)$

M14.30. $F(\sin, \sin y)=0, \quad \left(F(\cos x+y, \sin y+z, \sin z)=0\right)$

Mustaqil yechish uchun qo'shimcha misol va masalalar

1. Ko`p o`zgaruvchi funksiyalar topilsin.

1.1. a tomoni va uning qarshisidagi A burchagiga ko`ra belgilangan uchburchakning eng katta yuzani toping.

1.2. Uchburchakning a, b tomonlari va ular orasidagi C burchak ma`lum. Bu uchburchakning $a$ va $b$ tomonlaridan shunday kesma bilan teng ikkiga (yuzaga nisbatan) bo`lingki, natijada kesma uzunligi eng kichik bo`lsin.

1.3. $u=x^2$ parabola va $x-y-2=0$ to`gri chiziq orasidagi eng kichik masofani toping.

1.4. (x_0, y_0, z_0) nuqta bilan $Ax+Bu+Cz+D$ tekislik orasidagi eng kichik masofani toping.

1.5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoidga ichki chizilgan eng katta hajmli to`gri burchak parallelepipedning o`lchamlarini toping.

1.6. O`lchamlari qanday bo`lganda ko`ndalang kesimi yarim doira, sirtining yuzasi $3\pi м^2$ bo`lgan ochiq silindrik vanna eng katta hajmga ega bo`ladi?

1.7. O`lchamlari qanday bo`lganda usti ochiq, hajmi $32\ sm^3$ bo`lgan to`gri burchakli banka eng kichik sirtgi ega bo`ladi?

1.8. Hajmi 54π bo`lgan silindrik banka, asos diametri d va balanligi $h$ ning qanday qiymatlarida eng kichik sirtga ega bo`ladi.

1.9. Musbat a sonini 5 ta shunday musbat sonlarning yig`indisi ko`rinishida ifodalangki, ularning ko`paytmasi eng katta bo`lsin.

1.10. Qirralari uzunliklarining yig`indisi $a$ ga teng bo`lgan to`gri burchakli parallelepipedning o`lchamlari qanday bo`lganda uning hajmi eng katta bo`ladi?

1.11. Hajmi V ga teng bo`lgan to`gri burchakli parallelepipedning o`lchamlari qanday bo`lganda uning to`la sirti eng kichik bo`ladi.

2. Agar $u=f(x, y, z)$ n o`lchovli bir jinsli funksiya bo`lsa, Eyler teoremasini quyidagi funksiyalar uchun tekshiring.

a) $u=(x-2y+3z)^2$ b) $u=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$

c) $u=\left(\dfrac{x}{y}\right)^{y/z}$

3. Agar $u=f(x, y, z)$ ikki karrali differensiallanuvchi n o`lchovli bir jinsli funksiya bo`lsa, quyidagi tenglikni isbot qiling.

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^2 u=n(n-1)u$$

4. $Au=x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$ o`rinliligidan Au va $A^2u=A(Au)$ topilsin:

a) $u=\dfrac{x}{x^2+y^2}$ b) $u=\ln\sqrt{x^2+y^2}$.

5. $u=\ln\sqrt{(x-a)^2+(y-b)^2}$ (a,b- o`zgarmas son)

funksiya

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Laplas tenglamasini qanoatlantirishini ko’rsating.

6. $$u=\frac{1}{r}=\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}, \quad (r \quad 0)$$

funksiya

$$\Delta u \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

Laplas tenglamasini qanoatlantirishini isbotlang.

7. $u=\dfrac{c_1e^{-ar}+c_2^{\;ar}}{r}$ funksiya, bu yerda $r=\sqrt{x^2+y^2+z^2}$ va C_1, C_2 - o`zgarmaslar.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=a^2u.$$

Gelsman tenglamasini qanoatlantirishini isbot qiling.

8. $z=yf\left(x^2-y^2\right)$ funksiya (f – differensiallanuvchi funksiya) quyidagi tenglamani qanoatlantirishini ko`rsating.

$$y^2\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=xz.$$

9. φ, ψ ixtiyoriy funksiyalar yetarli tartibda differensiallanuvchi bo`lsa, quyidagi tengliklarni tekshiring.

9.1. $y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=0, \quad z=\varphi\left(x^2+y^2\right)$

9.2. $x^2\dfrac{\partial z}{\partial x}-xy\dfrac{\partial z}{\partial y}+y^2=0, \quad z=\dfrac{y^2}{3x}+\varphi(xy)$.

9.3. $(x^2-y^2)\dfrac{\partial z}{\partial x}+xy\dfrac{\partial z}{\partial y}=xyz, \quad z=e^y\varphi \;\; ye^{\frac{x^2}{2y^2}}$

9.4. $x\dfrac{\partial u}{\partial x}+\alpha y\dfrac{\partial u}{\partial y}+\beta z\dfrac{\partial u}{\partial z}=nu, \quad u=x^n\varphi \;\; \dfrac{y}{x^\alpha},\dfrac{z}{x^\beta}$;

9.5. $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}, \quad u=\varphi(x-at)+\psi(x+at);$

9.6. $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u = x\varphi(x+y) + y\psi(x+y);$

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