O'ZBEKISTON RESPUBLIKASI
OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

NAMANGAN DAVLAT UNIVERSITETI

Matematika kafedrasi

# "MATEMATIK ANALIZ"

fanidan

# O' Q U V – U S L U B I Y M A J M U A

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Ta'lim sohasi: 130 000 – Matematika

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O‘quv uslubiy majmua _______ yil O‘zR OO‘MTV tomonidan № __________ raqami bilan _____ yil ___________dagi _____-sonli buyrug‘i bilan tasdiqlangan fan dasturi asosida ishlab chiqilgan.

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O‘quv uslubiy majmua Namangan davlat universiteti Kengashininig _____ yil "…" ________dagi "..." - son yig‘ilishida ko‘rib chiqilgan va foydalanishga tavsiya etilgan.

# Mundarija

# So`z boshi

So`nggi yillarda mamlakatimizda oliy ta'lim sifatini oshirishga qaratilgan bir qancha chora-tadbirlar amalga oshirilmoqda. Chunki, jahon talablari darajasidagi raqobatbardosh kadrlar tayyorlash maqsadida talabalarga dunyo standartlariga javob beradigan bilim va ko`nikmalar berish bugungi kunning eng dolzarb masalalaridan biri bo`lib qolmoqda.

Mazkur o`quv-uslubiy majmua "Matematik analiz" fani bo`yicha tayyorlangan bo`lib, u "5130100-Matematika" yo`nalishi talabalari uchun mo`ljallangan va Namangan davlat universiteti "Matematika" kafedrasi o`qituvchilari tomonidan tayyorlangan. Ushbu majmua mamlakatimizda "Matematik analiz" fanini o`qitish bo`yicha uzoq yillardan beri to`plangan boy tajriba hamda rivojlangan horijiy davlatlarning yetakchi Oliy ta'lim muassasalarining tajribalaridan foydalangan holda, shuningdek, ularning o`quv dasturlaridagi asosiy adabiyotlardan foydalangan holda yaratildi.

Matematik analiz fani oliy matematikaning fundamentak bo`limlaridan biri bo`lib, u matematikaning poydevori hisoblanadi. Matematik analiz kursi davomida ko`pgina tushuncha va tasdiqlar, shuningdek, ularning tatbiqlari keltiriladi.

Matematik analiz fanining asosiy vazifasi shu fanning tushuncha, tasdiqlar va boshqa matematik ma'lumotlar majmuasi bilan tanishtirishgina bo`lmasdan, balki talabalarda mantiqiy fikrlash, matematik usullarni amaliy masalalarni yechishga qo`llash ko`nikmalarini shakllantirishdan iborat.

Ushbu o`quv-uslubiy majmuada dastlab sillabus hamda o`qitishda foydalaniladigan interfaol ta'lim metodlari berilgan bo`lib, so`ngra har bir mavzu bo`yicha materiallar batartib berilgan. Bunda har bir mavzu bo`yicha ma'ruza matnlari, nazorat savollari, mashqlar, glossariy, amaliy mashg`ulot materiallari, test savollari va keyslar banki keltirilgan.

# Mavzu. Boshlang'ich funksiya va aniqmas integral tushunchalari

## 1-ma'ruza

**Reja**

**1⁰.** Boshlang'ich funksiya tushunchasi. Funksiyaning aniqmas integrali.

**2⁰.** Integralning xossalari.

**3⁰.** Asosiy aniqmas integrallar jadvali.

### 1⁰. Boshlang'ich funksiya tushunchasi. Funksiyaning aniqmas integrali.

Faraz qilaylik, $f(x)$ funksiya $(a,b)$ intervalda ( bu interval chekli yoki cheksiz bo'lishi mumkin ) aniqlangan bo'lib, $F(x)$ funksiya esa shu intervalda differensiallanuvchi bo'lsin.

**1—ta'rif.** **([1], *Definition 9.1, 300-bet*)** *Agar* $\forall x \in (a,b)$ *da*

$$F'(x)= f(x) \text{ yoki } dF(x)= f(x)dx$$

*bo'lsa,* $F(x)$ *funksiya* $(a,b)$ *da* $f(x)$ *ning boshlang'ich funksiyasi deyiladi.*

Masalan.

1). $f(x)=x^2$ funksiyaning $R=(-\infty,+\infty)$ dagi boshlang'ich funksiyasi $F(x)=\frac{1}{3}x^3$ bo'ladi, chunki

$$F'(x)=\left(\frac{1}{3}x^3\right)'=x^2=f(x).$$

2). $f(x)=-\frac{x}{\sqrt{1-x^2}}$ funksiyaning $(-1,1)$ intervaldagi boshlang'ich funksiyasi $F(x)=\sqrt{1-x^2}$ bo'ladi, chunki

$$F'(x)=\left(\sqrt{1-x^2}\right)'=\frac{-x}{\sqrt{1-x^2}}=f(x).$$

Aytaylik, $f(x)$ funksiya $[a,b]$ da aniqlangan bo'lib, $F(x)$ funksiya shu segmendga differensiallanuvchi bo'lsin.

**2—ta'rif.** Agar

$$F'(x)=f(x) \qquad \left(x\in(a,b)\right)$$

bo'lib,

$$F'(a+0)=f(a). \qquad F'(b-0)=f(b).$$

bo'lsa, $F(x)$ funksiya $[a,b]$ da $f(x)$ ning *boshlang'ich funksiyasi* deyiladi.

**8.1—misol.** Ushbu

$$f(x)=\begin{cases}1, & \text{agar} \quad x>0. \\ 0, & \text{agar} \quad x=0 \\ -1, & \text{agar} \quad x<0\end{cases}$$

funksiya $(-1,1)$ intervalda boshlang'ch funksiyaga ega emasligi ko'rsatilsin.

◀ Teskarisini faraz qilaylik. Biror $F(x)$ funksiya $(-1,1)$ da $f(x)$ ning boshlang'ich funksiyasi bo'lsin: $(-1,1)$ da

$$F'(x)=f(x).$$

Lagranj teoremasiga ko'ra $[0,x]$ da $(0<x<1)$

$$F(x)-F(0)=F'(c)x=f(c)x=x \qquad (0<c<x)$$

bo'ladi. Keyingi tenglikdan topamiz:

$$F'(+0)=\lim_{x\to+0}\frac{F(x)-F(0)}{x}=1.$$

Bu esa $F'(+0)=F'(0)=f(0)=0$ bo'lishiga zid. ▶

**1—eslatma.** Keyinchalik, $F(x)$ funksiya boshlang'ich funksiyasi bo'ladigan oraliqni ko'rsatib o'tirmaymiz. Oraliq sifatida $f(x)$ ning aniqlanish oralig'i $X$ ko'zda tutiladi va $X$ sifatida

$$[a,b], (a,b), (a,b], \ [a,b), (-\infty,a], \ (-\infty,a),$$
$$[b,+\infty), \ (b,+\infty), \ (-\infty,+\infty)$$

lar olinishi mumkin.

**1—teorema**. *Agar $f(x)$ funksiya $X$ oraliqda uzluksiz bo'lsa, $f(x)$ shu oraliqda har doim boshlang'ich funksiyaga ega bo'ladi.*

Bu teoremaning isboti 9—bobda keltiriladi.

***([1], Proposition 9.3, 300-bet)*** *$F(x)$ va $\Phi(x)$ funksiyalarning har biri $f(x)$ funksiya uchun boshlan-g'ich funksiya bo'lsin:*

$$F'(x)=f(x), \quad \Phi'(x)=f(x).$$

*Demak, $F'(x)=\Phi'(x)$. Bundan 7—bobdagi 1—natijaga ko'ra*

$$F(x)=\Phi(x)+C \qquad (C=const)$$

*tenglik kelib chiqadi.*

*Demak, $f(x)$ funksiyaning barcha boshlang'ich funksiyalari bir—biridan o'zgarmas songa farq qiladi va istalgan boshlang'ich funksiyasi ushbu ko'rinishda ifodalanadi: $F(x)+C \qquad (C=const)$.*

**3—ta'rif.** $f(x)$ funksiya boshlang'ich funksiyalarining umumiy ifodasi $F(x)+C$ $(C=const)$ shu $f(x)$ *funksiyaning aniqmas integrali* deb ataladi va

$$\int f(x)dx$$

kabi belgilanadi. Bunda $\int$ —integral belgisi, $f(x)$ integral ostidagi funksiya, $f(x)dx$ esa integral ostidagi ifoda deyiladi.

Demak,

$$\int f(x)dx=F(x)+C.$$

Masalan,

$$\int 2^x dx = \frac{2^x}{\ln 2} + C$$

bo'ladi, chunki hosila olish qoidalariga ko'ra

$$\left(\frac{2^x}{\ln 2} + C\right)' = 2^x .$$

## 2. Integralning xossalari.

1). $f(x)$ funksiya aniqmas integarali $\int f(x)dx$ ning differensiali $f(x)dx$ ga teng:

$$d\left(\int f(x)dx\right) = f(x)dx$$

◀ Haqiqatan ham, $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi bo'lsin: $F'(x) = f(x)$. U holda

$$\int f(x)dx = F(x) + C$$

bo'ladi. Keyingi tenglikdan topamiz:

$$d\left(\int f(x)dx\right) = d\left(F(x) + C\right) = dF(x) = F'(x)dx = f(x)dx .$$ ▶

Bu xossa avval differensial belgisi $d$, so'ngra integral belgisi $\int$ kelib, ular yonma—yon turganda o'zaro bir—birini yo'qotishni ko'rsatadi.

2). Funksiya differensialining aniqmas integrali shu funksiya bilan o'zgarmas son yig'indisiga teng:

$$\int dF(x) = F(x) + C$$

◀ $F(x)$ funksiya $f(x)$ ning biror boshlang'ich funksiyasi bo'lsin: $F'(x) = f(x)$. u holda

$$\int f(x)dx = F(x) + C$$

tenglik o'rinli bo'ladi. Ikkinchi tomondan,

$$\int f(x)dx = \int F'(x)dx = \int dF(x) .$$

Oxirgi ikki tenglik 2)—xossani isbot etadi. ▶

Yuqorida keltirilganlardan, differensiallash (funksiyaning hosilasini hisoblash) hamda integrallash (funksiyaning aniqmas integarlini hisoblash) amallari o'zaro teskari amallar ekanligi kelib chiqadi.

Ayni paytda funksiya hosilasi hisoblanganda natija bitta funksiya bo'lsa, uning aniqmas integrali hisoblanganda esa natija cheksiz ko'p funksiya (ular bir—biridan o'zgarmas songa farq qiladi ) bo'ladi. Aniqmas integral deb yuritilishining boisi ham shu.

**$3^0$. Integrallashning sodda qoidalari.** ***([1], Theorem 9.8 (Linearity of the integral), 304-bet )*** 1) *Agar $f(x)$ funksiya boshlan-g'ich funksiyaga ega bo'lsa, u holda $kf(x)$ ($k$ o'zgarmas son) funksiya ham boshlang'ich funksiyaga ega va $k \neq 0$ da*

$$\int kf(x)dx = k\int f(x)dx \qquad (8.2)$$

*formula o'rinli bo'ladi.*

◀ $f(x)$ funksiyaning boshlang'ich funksiyasi $F(x)$ bo'lsin. U holda $F'(x)=f(x)$ va $\int f(x)dx = F(x)+C$ bo'lib,

$$k\int f(x)dx = k\left(F(x)+C\right) = kF(x)+kC \qquad (8.3)$$

bo'ladi, bunda $C$—ixtiyoriy o'zgarmas son. Ushbu

$$\left(kF(x)\right)' = kF'(x) = kf(x)$$

tenglik o'rinli bo'lishidan $kf(x)$ funksiyaning boshlang'ich funksiyasi $kF(x)$ ekanini topamiz. Demak,

$$\int kf(x)dx = kF(x)+C_1 \qquad (8.4)$$

bunda $C_1$ ixtiyoriy o'zgarmas son. Endi (8.3) va (8.4) munosabatlardan $C$ va $C_1$ o'zgarmas sonlarning ixtiyoriligi hamda $k\neq 0$ bo'lishidan (8.2) formulaning o'rinli ekani kelib chiqadi. ▶

Shunga o'xshash integralning quyidagi xossasi isbotlanadi:

***([1], Theorem 9.8 (Linearity of the integral), 304-bet )*** *2). Agar $f(x)$ va $g(x)$ funksiyar boshlang'ich funksiyalarga ega bo'lsa, $f(x)+g(x)$ ham boshlang'ich funksiyaga ega va*

$$\int\left(f(x)+g(x)\right)dx = \int f(x)dx + \int g(x)dx \qquad (8.5)$$

*fo'rmula o'rinli bo'ladi.*

Odatda bu xossa integralning *additivlik xossasi* deyiladi.

**2—eslatma.** Yuqorida keltirilgan (8.2) va (8.5) tengliklarni hamda kelgusida uchraydigan shunga o'xshash o'ng va chap tomonlaridagi ifodalar orasidagi ayirma o'zgarmas songa barobarligi ma'nosidagi (o'zgarmas son aniqligidan) tengliklar deb qaraladi.

### 3. Asosiy aniqmas integrallar jadvali. *([1], (9.1) formulalar, 303-bet )*

$$\int 0dx = C = const; \qquad \int 1dx = \int dx = x+C; \qquad \int x^{\mu}dx = \frac{x^{\mu+1}}{\mu+1}+C \qquad (\mu\neq -1);$$

$$\int \frac{1}{x}dx = \int\frac{dx}{x} = \ln|x|+C \quad (x\neq 0); \qquad \int\frac{1}{1+x^2}dx = \int\frac{dx}{1+x^2} = arctgx + C;$$

$$\int\frac{1}{\sqrt{1-x^2}}dx = \int\frac{dx}{\sqrt{1-x^2}} = \arcsin x + C; \qquad \int a^x dx = \frac{a^x}{\ln a}+C, \qquad (a>0);$$

$$\int \sin x dx = -\cos x + C; \qquad \int\cos x dx = \sin x + C; \qquad \int\frac{1}{\sin^2 x}dx = \int\frac{dx}{\sin^2 x} = -ctgx + C;$$

$$\int\frac{1}{\cos^2 x}dx = \int\frac{dx}{\cos^2 x} = tgx + C; \qquad \int shxdx = chx + C; \qquad \int chxdx = shx + C;$$

$$\int\frac{1}{sh^2 x}dx = -cthx + C; \qquad \int\frac{1}{ch^2 x}dx = thx + C.$$

## Adabiyotlar

1. **Claudio Canute, Anita Tabacco,** *Mathematical Analysis I*, Springer-Verlag Italia, Milan 2008

2. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

3. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, I q.* T. "Voris-nashriyot", 2010.

4. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.

# Glossariy

**1. Boshlang'ich funksiya** – Bir funksiyaning hosilasi ikkinchi funksiyaga teng bo'lsa, ikkinchi funksiya uchun birinchi funksiya boshlang'ich funksiya deyiladi.

**2. Aniqmas integral** – bir funksiyaning barcha boshlang'ich funksiyalar oilasiga shu funksiyaning aniqmas integrali deyiladi.

# Keys banki

**28-keys.** Masala o`rtaga tashlanadi: Ushbu

$$I = \int \frac{\sqrt{4+x^2} - 3\sqrt{4-x^2}}{\sqrt{16-x^4}} dx$$

integral hisoblansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 1-amaliy mashg'ulot

**Asosiy aniqmas integrallar jadvali.**

$$\int 0dx = C = const;\ \int 1dx = \int dx = x + C;\qquad \int x^{\mu}dx = \frac{x^{\mu+1}}{\mu+1} + C \qquad (\mu \neq -1);$$

$$\int \frac{1}{x}dx = \int \frac{dx}{x} = \ln|x| + C \qquad (x \neq 0) \qquad \int \frac{1}{1+x^2}dx = \int \frac{dx}{1+x^2} = arctgx + C;$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C; \qquad \int a^x dx = \frac{a^x}{\ln a} + C, \qquad (a > 0);$$

$$\int \sin xdx = -\cos x + C; \qquad \int \cos xdx = \sin x + C;\ \int \frac{1}{\sin^2 x}dx = \int \frac{dx}{\sin^2 x} = -ctgx + C;$$

$$\int \frac{1}{\cos^2 x}dx = \int \frac{dx}{\cos^2 x} = tgx + C; \qquad \int shxdx = chx + C; \qquad \int chxdx = shx + C;$$

$$\int \frac{1}{sh^2 x}dx = -cthx + C; \qquad \int \frac{1}{ch^2 x}dx = thx + C.$$

Sodda aniqmas integrallar asosan bevosita xossalardan hamda jadvaldan foydalanib hisoblanadi.

**1 – m i s o l .** ***Ushbu***

$$\int \frac{x^2 + x + 1}{\sqrt{x}}dx$$

***integral hisoblansin.***

◀Integral ostidagi funksiyani quydagicha $\frac{x^2 + x + 1}{\sqrt{x}} = x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ yozib olamiz. So'ng integralni xossalar va jadvaldan foydalanib topamiz:

$$\int \frac{x^2 + x + 1}{\sqrt{x}}dx = \int \left( x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right)dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} +$$

$$+ C = 2\sqrt{x}\left( \frac{x^2}{5} + \frac{x}{3} + 1 \right) + C. ▶$$

**2 – m i s o l .** ***Ushbu***

$$I = \int \frac{\sqrt{4 + x^2} - 3\sqrt{4 - x^2}}{\sqrt{16 - x^4}}dx$$

***integral hisoblansin.***

◀Aniqmas integrallarning xossalari va jalvaldan foydalanib topamiz:

$$I = \int \frac{\sqrt{4 + x^2}}{\sqrt{16 - x^4}}dx - 3\int \frac{\sqrt{4 - x^2}}{\sqrt{16 - x^4}}dx = \int \frac{dx}{\sqrt{4 - x^2}} - 3\int \frac{dx}{\sqrt{4 + x^2}} =$$

$$= \arcsin\frac{x}{2} + 3\ln\left|x + \sqrt{x^2+4}\right| + C. \blacktriangleright$$

**3 – m i s o l .** ***Ushbu***

$$I = \int \frac{dx}{3\cos^2 x + 4\sin^2 x}$$

***integral hisoblansin.***

◀Integral ostidagi ifodani quyidagi ko‘rinishda yozib olamiz:

$$I = \int \frac{d(tgx)}{3 + 4tg^2 x}.$$

Natijada,

$$I = \int \frac{dtgx}{3+4tg^2x} = \frac{1}{4}\int \frac{dtgx}{\left(\frac{\sqrt{3}}{2}\right)^2 + tg^2x} = \frac{1}{2\sqrt{3}}\,\mathrm{arctg}\,\frac{2tgx}{\sqrt{3}} + C$$ bo‘ladi.▶

**Quyidagi integrallar hisoblansin.**

1. $\int \sqrt{x}dx$
2. $\int \frac{dx}{2\sqrt{x}}$
3. $\int \frac{dx}{x^2}$
4. $\int \sqrt[m]{x^n}dx$
5. $\int 10^x dx$
6. $\int a^x e^x dx$
7. $\int \left(3 - x^2\right)^3 dx$
8. $\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}}dx$
9. $\int \left(1 - \frac{1}{x^2}\right)\sqrt{x\sqrt{x}}dx$
10. $\int (\sqrt{x}+1)(x - \sqrt{x} + 1)dx$
11. $\int \frac{\sqrt{x} - x^3e^x + x^2}{x^3}dx$
12. $\int \left(\frac{1-x}{x}\right)^2 dx$
13. $\int \frac{(1-x)^2}{x\sqrt{x}}dx$
14. $\int \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}dx$
15. $\int \frac{3\cdot 2^x - 2\cdot 3^x}{2^x}dx$
16. $\int \frac{(1-x)^3}{x\sqrt[3]{x}}dx$

$17.\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx$

$18.\int \frac{dx}{\sqrt{x^2+13}}$

$19.\int 2^{2x}\cdot e^x dx$

$20.\int \frac{(1+2x^2)}{x^2(1+x^2)}dx$

$21.\int \frac{x^2dx}{1+x^2}$

$22.\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx$

$23.\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx$

$24.\int \frac{\sqrt{4+x^2}+2\sqrt{4-x^2}}{\sqrt{16-x^4}}dx$

$25.\int (2^x+3^x)^2 dx$

$26.\int (1+\sin x+\cos x)dx$

$27.\int \sqrt{1-\sin 2x}\,dx$

$28.\int \frac{\cos 2x}{\cos^2 x\cdot \sin^2 x}dx$

$29.\int \frac{dx}{\cos 2x+\sin^2 x}$

$30.\int tg^2 x dx$

$31.\int ctg^2 x dx$

$32.\int 2\sin^2\frac{x}{2}dx$

**Berilgan $f(x)$ funksiya uchun grafigi $(x_0,y_0)$ nuqtadan o'tuvchi boshlang'ich funksiya $F(x)$ topilsin.**

$33.f(x)=\frac{1}{2\sqrt{x}}+\sin(x+1),\quad x\in(0,+\infty),\quad (x_0,y_0)=(1;1)$

$34.f(x)=\frac{2}{x}-\frac{3}{x^2},\quad x\in(-\infty;0),\quad (x_0,y_0)=(-1;1)$

$35.f(x)=|x|,\quad x\in R,\quad (x_0,y_0)=(-2;4)$

**36.**Agar $F(x)$ funksiya sonlar o'qida $f(x)$ funksiyaning boshlang'ich funksiyasi bo'lsa, u holda quyidagi tasdiqlarning to'g'ri yoki noto'g'riligi tekshirilsin:

**a)** Agar $f(x)$ funksiya davriy bo'lsa, unda $F(x)$ ham davriy funksiya

bo‘ladi.

**б)** Agar $f(x)$ - toq funksiya bo‘lsa, unda $F(x)$ - juft funksiya bo‘ladi.

**в)** Agar $f(x)$ - juft funksiya bo‘lsa, unda $F(x)$ - toq funksiya bo‘ladi.

**37.** Agar $\int f(x)dx = F(x) + C$ bo‘lsa, unda ushbu

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C \qquad (a \neq 0)$$

tenglikning o‘rinli bo‘lishi isbotlansin.

**38.** Uzilishga ega bo‘lgan shunday funksiyaga misol keltiringki, uning boshlang‘ich funksiyasi butun sonlar o‘qida aniqlangan bo‘lsin.

**Quyidagi funksiyalarning boshlang‘ich funksiyalari topilsin.**

**39.** $f(x)=|x|, \quad x \in R$

**40.** $f(x)=x|x|, \quad x \in R$

**41.** $f(x)=|1+x|-|1-x|, \quad x \in R$

**42.** $f(x)=(2x-3)\cdot|x-2|, \quad x \in R$

**43.** $f(x)=e^{|x|}, \quad x \in R$

**44.** $f(x)=|\text{sh}x|, \quad x \in R$

**45.** $f(x)=\begin{cases} 1-x^2, & \text{агар } |x| \leq 1 \text{ бўлса,} \\ 1-|x|, & \text{агар } |x| > 1 \text{ бўлса.} \end{cases}$

**46.** $f(x)=\max(1; x^2), \quad x \in R$

**47.** $f(x)=[x]\cdot|\sin \pi x|, \quad x \in [0;+\infty)$

# Test

| | | | | | |
|---|---|---|---|---|---|
| 1. | $\int 2(x-1)^3 dx - ?$ | * $\frac{(x-1)^4}{2}+C$ | $\frac{x-1}{2}+C$ | $2(x-1)^4+C$ | $(2x+5)^4+C$ |
| 2. | $\int (-2\sin x+\cos x)dx$ | * $2\cos x+\sin x+C$ | $2tgx+C$ | $\ln\sin x+\cos x+C$ | $2\sin x-2\cos x+c$ |
| 3. | $\int (2\sin x-\cos x)dx$ | * $-2\cos x-\sin x+C$ | $2tgx+C$ | $\ln\sin x+\cos x+C$ | $2\sin x-2\cos x+c$ |
| 4. | $\int (\sin x+3\cos x)dx$ | * $-\cos x+3\sin x+C$ | $2tgx+C$ | $\ln\sin x+\cos x+C$ | $2\sin x-2\cos x+c$ |
| 5. | $\int (2\cos 2x-1)dx$ | * $\sin 2x-x+C$ | $2\cos 2x-x+C$ | $tgx+C$ | $ctgx+C$ |
| 6. | $\int (\sin 2x-x)dx$ | * $-\frac{\cos 2x}{2}-\frac{x^2}{2}+C$ | $2\cos 2x+C$ | $tgx+C$ | $ctgx+C$ |
| 7. | $\int 2\cos 2xdx$ | * $\sin 2x+C$ | $2\cos 2x+C$ | $tgx+C$ | $ctgx+C$ |
| 8. | $\int a^x dx$ | * $\frac{a^x}{\ln a}+C$ | $e^x+C$ | $\sin x+C$ | $\ln x^a+C$ |
| 9. | $\int 2^x dx$ | * $\frac{2^x}{\ln 2}+C$ | $e^x+C$ | $2^x+C$ | $\ln x^a+C$ |
| 10. | $\int 2e^{2x}dx$ | * $e^{2x}+C$ | $e^x+C$ | $-e^{2x}+C$ | $\ln x^a+C$ |
| 11. | $\int -a^{-x}dx$ | * $\frac{a^{-x}}{\ln a}+C$ | $a^{-x}+C$ | $\sin x+C$ | $\ln x^a+C$ |
| 12. | $\int (-e+2x)dx$ | * $-ex+x^2C$ | $e^x+C$ | $-ex+2x^2+C$ | $\ln e^{-x}+C$ |
| 13. | $\int (e^{ax}+2)dx$ | * $\frac{e^{ax}}{a}+2x+C$ | $e^{ax}+x^2+C$ | $\sin x+C$ | $\ln x^a+C$ |

| | | | | | |
|---|---|---|---|---|---|
| 14. | $\int (e^x+4)\,dx$ | * $e^x+4x+C$ | $e^x+C$ | $\ln x+C$ | $e^{x+4}+C$ |
| 15. | $\int (x+4)^2dx-?$ | * $(x+4)^3/3+C$ | $x+C$ | $\ln x+C$ | $e^x+C$ |
| 16. | $\int (3x^2+2x)dx-?$ | * $x^3+x^2+C$ | $x+C$ | $x^3+C$ | $e^x+C$ |
| 17. | $\int (x^4+3x^2)dx-?$ | * $\frac{x^5}{5}+x^3+C$ | $2x+C$ | $51x+C$ | $9x^2+C$ |
| 18. | $\int 3x^2dx-?$ | * $x^3+C$ | $x+C$ | $x^2+C$ | $e^x+C$ |
| 19. | $\int x^6dx-?$ | * $\frac{x^7}{7}+C$ | $2x+C$ | $51x+C$ | $9x^2+C$ |
| 20. | $\int (x^2+x)dx-?$ | * $\frac{x^3}{3}+\frac{x^2}{2}+C$ | $x+C$ | $x^3+C$ | $e^x+C$ |
| 21. | $\int (x^6+4x^3)dx-?$ | * $\frac{x^7}{7}+x^4+C$ | $2x+C$ | $x^2+x+C$ | $9x^2+15x+C$ |
| 22. | $\int (x+\cos x)dx-?$ | * $\frac{x^2}{2}+\sin x+C$ | $\sin x+C$ | $x^2+C$ | $\cos e^x+C$ |
| 23. | $\int (x+3)^6dx-?$ | * $\frac{(x+3)^7}{7}+C$ | $2x+5+C$ | $\cos x+51x+C$ | $9x^2+C$ |

# Mavzu. Integrallash usullari. Sodda kasrlarni integrallash

## 2-ma'ruza

**Reja**

$1^0$.O'zgaruvchini almashtirib integrallash usuli.

$2^0$.Bo'laklab integrallash usuli.

$3^0$.Sodda kasrlarni integrallash.

### $1^0$. O'zgaruvchilarni almashtirib integrallash usuli.

***([1], Theorem 9.12 (Integration by substitution))***

Ushbu $\int f(x)dx$ aniqmas integralni hisoblash talab etilgan bo'lsin. Bunda $f(x)$ funksiya biror $X=(a,b)$ intervalda aniqlangan va

$$f(x)=\varphi(g(x))\cdot g'(x) \qquad (8.6)$$

ko'rinishda yozilishi mumkin deylik.

Agar $\varphi(t)$ funksiya $T=(t_1,t_2)$ intervalda boshlang'ich funksiya $\Phi(t)$ ga ega bo'lib, $g(x)$ funksiya $X=(a,b)$ intervalda (bunda $g(x)\subset T$ ) different-siallanuvchi bo'lsa, u holda

$$\int f(x)dx=\int \varphi(g(x))g'(x)dx=\Phi(g(x))+C \qquad (8.7)$$

formula o'rinli .

◀Haqiqatan, $[\Phi(g(x))]'=\Phi'(g(x))\cdot g'(x)=\varphi(g(x))g'(x)$. ▶

Odatda integralni bunday usul bilan hisoblash *o'zgaruvchini almashtirish usuli* bilan integrallash deb ataladi.

O'zgaruvchilarni almashtirish usulining muhim tomoni o'zgaruvchilarni juda ko'p usul bilan almashtirish imkoniyati bo'lgan holda ular ichidan integralni sodda va hisoblash uchun qulay holga keltiradiganini tanlab olishdan iborat.

**8.3—misol.** $\int \frac{xdx}{x^2+a^2}$ $(a=\text{const})$ ni hisoblansin.

◀Berilgan integralda o'zgaruvchi $x$ ni $x^2+a^2=t$ kabi almashtiramiz. Bunda $2xdx=dt$ bo'lib ( (8.6) va (8.7) ) larga qarang)

$$\int \frac{xdx}{x^2+a^2}=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\ln|t|+C=\frac{1}{2}\ln(x^2+a^2)+C \text{ ▶}$$

**8.4—misol.** $\int e^{\cos x}\sin x dx$ ni hisoblansin.

◀ Bu integralda $\cos x = t$ almashtirishni bajaramiz. Natijada $-\sin x dx = dt$ bo'lib,

$$\int e^{\cos x}\sin x dx = -\int e^t dt = -e^t + C = -e^{\cos x + C}$$

bo'ladi. ▶

**2⁰. Bo'laklab integrallash usuli.** ***([1], Theorem 9.10 (Integration by parts) )*** Ikki $u = u(x)$ va $v = v(x)$ funksiya $(a,b)$ intervalda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin. Ma'lumki, (6—bob-ning 4—§ ga qarang )

$$d(u(x)\cdot v(x)) = u(x)dv(x) + v(x)du(x) .$$

Bu tenglikdan

$$u(x)\cdot dv(x) = d(u(x)v(x)) - v(x)du(x) \qquad (8.8)$$

bo'lishi kelib chiqadi.

Endi $(8.8)$ tenglikni integrallab topamiz:

$$\int u(x)dv(x) = \int d(u(x)v(x)) - \int v(x)du = u(x)v(x) - \int v(x)du$$

Sunday qilib, quyidagi

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du \qquad (8.9)$$

formulaga kelamiz. Bu $(8.9)$ formula *bo'laklab integrallash formulasi* deyiladi.

Bo'laklab integrallash formulasidan foydalanish uchun integral ostidagi ifodani $u(x)$ hamda $dv(x)$ lar ko'paytmasi ko'rinishida yozib olinadi, bunda albatta $dv(x)$ hamda $v(x)du$ ifodalarning integrallarini oson hisoblana olinishi lozimligini e'tiborda tutish kerak.

**8.5—misol.** $\int \ln x dx$ ni hisoblansin.

◀Integral ostidagi $\ln x dx$ ifodani $u = \ln x,\ dv = dx$ lar ko'paytmasi deb olamiz. U holda $du = \frac{1}{x}dx,\ \ v = x$ bo'ladi. Bo'laklab integrallash formulasidan foydalanib topamiz:

$$\int \ln x dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x + C = x\ln\frac{x}{e} + C$$ ▶

**8.6—misol.** $J_n = \int \frac{dx}{(x^2+a^2)^n}$ $\quad (n = 1,2,3...)$ ni hisoblansin.

◀Bu integralda

$$u = \frac{1}{(x^2+a^2)^n},\quad dv = dx \qquad du = -\frac{2nxdx}{(x^2+a^2)^{n+1}},\quad v = x$$

bo'ladi. $(8.9)$ formuladan foydalanib topamiz:

$$J_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx . \qquad (8.10)$$

Bu tenglikning o'ng tomonidagi $\int \frac{x^2}{(x^2+a^2)^{n+1}}dx$ ni

$$\int \frac{x^2}{(x^2+a^2)^{n+1}}dx = \int \frac{1}{(x^2+a^2)^n}dx - a^2\int \frac{dx}{(x^2+a^2)^{n+1}}$$

ko'rinishda yozsak, unda $(8.10)$ munosabat ushbu

$$J_n = \frac{x}{(x^2+a^2)^n} + 2n\cdot J_n - 2na^2\cdot J_{n+1}$$

ko'rinishni oladi. Keyingi tenglikdan esa quyidagi

$$J_{n+1} = \frac{1}{2na^2}\cdot\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n}\cdot\frac{1}{a^2}\cdot J_n \qquad (8.11)$$

rekurrent formula kelib chiqadi.

Ravshanki, $n=1$ bo'lganda

$$J_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a}arctg\frac{x}{a} + C$$

bo'ladi.

$n\geq 2$ bo'lganda, mos $J_n$ integrallar $(8.11)$ rekurrent formula yordamida topiladi. Masalan:

$$J_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}\cdot J_1 = \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^3}arctg\frac{x}{a} + C\,.$$ ▶

**3⁰. Sodda kasrlarni integrallash.** ***([1], Theorem 9.15, (9,7) (9,8) 9,9) formulalar.)*** Sodda kasrlarning aniqmas integrallarini hisoblaymiz.

1). $\frac{A}{x-a}$ sodda kasrning aniqmas integrali .

$$\int \frac{A}{x-a}dx = A\int \frac{d(x-a)}{x-a} = A\ln|x-a| + C.$$

2). $\frac{A}{(x-a)^m}$ $(m>1)$ sodda kasrning aniqmas integrali ham tez hisoblanadi:

$$\int \frac{Adx}{(x-a)^m} = A\int \frac{d(x-a)}{(x-a)^m} = A\int (x-a)^{-m}d(x-a) = \frac{A}{1-m}\cdot\frac{1}{(x-a)^{m-1}} + C.$$

3). $\frac{Bx+C}{x^2+px+q}$ sodda kasrning (bunda $x^2+px+q$ kvadrat uchhad haqiqiy ildizga ega emas) integrali $\int \frac{Bx+C}{x^2+px+q}dx$ ni hisoblash uchun avval kasrning mahrajida turgan $x^2+px+q$ kvadrat uchhadni ushbu

$$x^2+px+q = \left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}$$

ko'rinishda yozib olamiz. U holda

$$\int \frac{Bx+C}{x^2+px+q}dx = \int \frac{Bx+C}{\left(x+\frac{p}{2}\right)^2 + a^2}dx$$

bo'ladi, bunda $a^2 = q - \frac{p^2}{4}$. Bu integralda $x + \frac{p}{2} = t$ almashtirishni bajaramiz:

$$\int \frac{Bx + C}{x^2 + px + q} dx = B\int \frac{tdt}{t^2 + a^2} + (C - \frac{Bp}{2})\int \frac{dt}{t^2 + a^2} = \frac{B}{2}\int \frac{d(t^2 + a^2)}{t^2 + a^2} +$$

$$+\left(C - \frac{Bp}{2}\right)\frac{1}{a}\int \frac{d(\frac{t}{a})}{1 + (\frac{t}{a})^2} = \frac{B}{2}\ln(t^2 + a^2) + \left(C - \frac{Bp}{2}\right)\frac{1}{a} arctg\frac{t}{a} + C_1 =$$

$$= \frac{B}{2}\ln(x^2 + px + q) + \frac{2C - Bp}{2\sqrt{q - \frac{p^2}{4}}} arctg\frac{x + \frac{p}{2}}{\sqrt{q - \frac{p^2}{4}}} + C_1 \ .$$

Demak,

$$\int \frac{Bx + C}{x^2 + px + q} dx = \frac{B}{2}\ln(x^2 + px + q) + \frac{2C - Bp}{2\sqrt{q - \frac{p^2}{4}}} arctg\frac{x + \frac{p}{2}}{\sqrt{q - \frac{p^2}{4}}} + C_1$$

bunda $C_1$ ixtiyoriy o'zgarmas.

4). $\frac{Bx + C}{(x^2 + px + q)^m}$ $(m > 1)$ sodda kasrning integrali $J_m = \int \frac{Bx + C}{(x^2 + px + q)^m} dx$ ni hisoblash uchun 3)—holdagidek o'zgaruvchini almashtiramiz: $x + \frac{p}{2} = t$. Natijada quyidagiga ega bo'lamiz:

$$J_m = \int \frac{Bx + C}{(x^2 + px + q)^m} dx = \int \frac{Bx + C}{\left(\left(x + \frac{p}{2}\right)^2 + p - \frac{p^2}{4}\right)^m} dx = \int \frac{Bt + \left(C - \frac{Bp}{t}\right)}{(t^2 + a^2)^m} \mathrm{dt} =$$

$$= \frac{B}{2}\int \frac{d(t^2 + a^2)}{(t^2 + a^2)^m} + \left(C - \frac{Bp}{2}\right)\int \frac{dt}{(t^2 + a^2)^m} = \frac{B}{2}\cdot\frac{1}{1 - m}\cdot\frac{1}{(t^2 + a^2)^{m-1}} + \left(C - \frac{Bp}{2}\right)\int \frac{dt}{(t^2 + a^2)^m}.$$

Bu munosabatdagi $\int \frac{dt}{(t^2 + a^2)^m}$ integral ushbu bobning 2—§ ida keltirilgan integ-ral bo'lib, u rekurrent formula orqali hisoblanadi.

## Adabiyotlar.


1. **Claudio Canute, Anita Tabacco,** *Mathematical Analysis I*, Springer-Verlag Italia, Milan 2008.
2. **Xudoyberganov G., Vorisov A. K., Mansurov X. T.,**

**Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, I q.* T. "Voris-nashriyot", 2010.

3. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
4. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

# Glossariy

1. **Bo'laklab integrallash usuli** – integral ostida bir funksiya bilan ikkinchi funksiyaning differensiali ko'payib turganda ishlatiladigan usul.
2. **Sodda kasrlar** – $\frac{A}{x-a}$, $\frac{A}{(x-a)^m}$, $(m>1)$, $\frac{Bx+C}{x^2+px+q}$ ushbu ko'rinishdagi kasr sodda kasrlar deyiladi.

# Keys banki

**29-keys.** Masala o`rtaga tashlanadi: Ushbu

$$J_n=\int\frac{dx}{(x^2+a^2)^n},\quad (n=1,2,3...)$$

integral hisoblansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 2-amaliy mashg'ulot.

**Misol.** $\int \frac{xdx}{x^2+a^2}$ $(a=\text{const})$ ni hisoblansin.

Berilgan integralda o'zgaruvchi $x$ ni $x^2+a^2=t$ kabi almashtiramiz. Bunda $2xdx=dt$ bo'lib ((8.6) va (8.7) ) larga qarang)

$$\int \frac{xdx}{x^2+a^2} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(x^2+a^2)+C$$

**Misol.** $\int e^{\cos x}\sin xdx$ ni hisoblansin.

Bu integralda $\cos x = t$ almashtirishni bajaramiz. Natijada $-\sin xdx = dt$ bo'lib,

$$\int e^{\cos x}\sin xdx = -\int e^t dt = -e^t + C = -e^{\cos x + C}$$

bo'ladi.

**Misol.** $\int \ln xdx$ ni hisoblansin.

Integral ostidagi $\ln xdx$ ifodani $u=\ln x,\ dv=dx$ lar ko'paytmasi deb olamiz. U holda $du=\frac{1}{x}dx,\ \ v=x$ bo'ladi. Bo'laklab integrallash formulasidan foydalanib topamiz:

$$\int \ln xdx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x + C = x\ln\frac{x}{e}+C$$

**Misol.** $J_n = \int \frac{dx}{(x^2+a^2)^n}$ $(n=1,2,3...)$ ni hisoblansin.

Bu integralda

$$u=\frac{1}{(x^2+a^2)^n},\quad dv=dx \qquad du=-\frac{2nxdx}{(x^2+a^2)^{n+1}},\quad v=x$$

bo'ladi. (8.9) formuladan foydalanib topamiz:

$$J_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx\ . \qquad (8.10)$$

Bu tenglikning o'ng tomonidagi $\int \frac{x^2}{(x^2+a^2)^{n+1}}dx$ ni

$$\int \frac{x^2}{(x^2+a^2)^{n+1}}dx = \int \frac{1}{(x^2+a^2)^n}dx - a^2\int \frac{dx}{(x^2+a^2)^{n+1}}$$

ko'rinishda yozsak, unda (8.10) munosabat ushbu

$$J_n = \frac{x}{(x^2+a^2)^n} + 2n\cdot J_n - 2na^2\cdot J_{n+1}$$

ko'rinishni oladi. Keyingi tenglikdan esa quyidagi

$$J_{n+1} = \frac{1}{2na^2} \cdot \frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n} \cdot \frac{1}{a^2} \cdot J_n \qquad (8.11)$$

rekurrent formula kelib chiqadi.

Ravshanki, $n=1$ bo'lganda

$$J_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a} arctg \frac{x}{a} + C$$

bo'ladi.

$n \geq 2$ bo'lganda, mos $J_n$ integrallar $(8.11)$ rekurrent formula yordamida topiladi. Masalan:

$$J_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2} \cdot \frac{x}{x^2+a^2} + \frac{1}{2a^2} \cdot J_1 = \frac{1}{2a^2} \cdot \frac{x}{x^2+a^2} + \frac{1}{2a^3} arctg \frac{x}{a} + C .$$

O'zgaruvchini almashtirish usulidan foydalanib quyidagi integrallar hisoblansin.

1. $\int \frac{6x-7}{3x^2-7x+1}dx$

2. $\int \frac{x-1}{x^2-x-1}dx$

3. $\int \frac{x}{\sqrt{1-x^2}}dx$

4. $\int \frac{xdx}{(x^2-1)^{3/2}}dx$

5. $\int x\sqrt{1-x^2}dx$

6. $\int \frac{dx}{x\sqrt{x^2-1}}$

7. $\int \frac{arctgx}{1+x^2}dx$

8. $\int \frac{dx}{e^{x/2}+e^x}$

9. $\int \frac{dx}{x\ln x \ln(\ln x)}$

10. $\int \frac{dx}{(1+x)\sqrt{x}}$

11. $\int \frac{dx}{\sqrt{x(1+x)}}$

12. $\int \frac{\ln^{100} x}{x}dx$

13. $\int xe^{-x^2}dx$

14. $\int \frac{x^2-1}{x^4+1}dx$

15. $\int \frac{2^x \cdot 3^x}{9^x-4^x}dx$

16. $\int \frac{xdx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}dx$

17. $\int \sqrt{1-x^2}dx$

18. $\int \cos^5 x\sqrt{\sin x}dx$

19. $\int \frac{\sin x - \cos x}{\sqrt[3]{\sin x - \cos x}}dx$

20. $\int \frac{dx}{\sin^2 x + 2\cos^2 x}dx$

21. $\int \frac{dx}{\sin x}$

22. $\int \frac{dx}{\cos x}$

23. $\int \frac{dx}{\operatorname{ch} x}$

24. $\int \frac{\sin x \cos^3 x}{1+\cos^2 x} dx$

25. $\int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}} \frac{dx}{1+x}$

26. $\int \frac{\sin 2x}{\sqrt{25\sin^2 x + 9\cos^2 x}} dx$

27. $\int \frac{\sqrt[4]{\operatorname{tg} x}}{\sin^2 x} dx$

28. $\int \frac{\sin x}{\sqrt{1+4\cos x + \cos^2 x}} dx$

**Ushbu $x = a\sin t$, $x = a\operatorname{tg} t$, $x = a\sin^2 t$ va boshqa trigonometrik almashtirishlardan foydalanib quyidagi integrallar hisoblansin.**

29. $\int \frac{dx}{(1-x^2)^{3/2}}$

30. $\int \frac{x^2}{\sqrt{x^2-2}} dx$

31. $\int \sqrt{a^2-x^2}\,dx$

32. $\int \frac{dx}{(x^2+a^2)^{3/2}}$

33. $\int \sqrt{\frac{a+x}{a-x}}\,dx$

34. $\int x\sqrt{\frac{x}{2a-x}}\,dx$

35. $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$

**Ko'rsatma** $x - a = (b-a)\sin^2 t$ almashtirishdan foydalanilsin.

36. $\int \sqrt{(x-a)(b-x)}\,dx$

**Ushbu $x = a\operatorname{sh} t$, $x = a\operatorname{ch} t$ va boshqa giperbolik almashtirishlardan foydalanib quyidagi integrallar hisoblansin.**

37. $\int \sqrt{a^2+x^2}\,dx$

38. $\int \frac{x^2}{\sqrt{a^2+x^2}} dx$

39. $\int \sqrt{\frac{x-a}{x+a}}\,dx$

40. $\int \frac{dx}{\sqrt{(x+a)(x+b)}}$

**Ko'rsatma,** $x + a = (b-a)\sin^2 t$ almashtirishdan foydalanilsin.

41. $\int \sqrt{(x+a)(x+b)}dx$

Bo'laklab integrallash usulidan foydalanib quyidagi integrallar hisoblansin.

42. $\int x\sin x\,dx$

43. $\int xe^{-x}\,dx$

44. $\int x^n \ln x\,dx \quad (n \neq -1)$

45. $\int \arcsin x\,dx$

46. $\int x^3 e^{-x^2}\,dx$

47. $\int \frac{\arcsin x}{x^2}dx$

48. $\int \mathrm{arctg}x\,dx$

49. $\int \ln\left(x+\sqrt{1+x^2}\right)dx$

50. $\int \sin x \ln(\mathrm{tg}x)dx$

51. $\int \cos(\ln x)dx$

52. $\int e^{ax}\cos bx\,dx$

53. $\int e^{ax}\sin bx\,dx$

54. $\int \sqrt{x}\cdot \ln^2 x\,dx$

55. $\int e^{2x}\cdot \sin^2 x\,dx$

56. $\int x\cdot(\mathrm{arctg}x)^2\,dx$

57. $\int x\ln\frac{1+x}{1-x}dx$

58. $\int x^2 \mathrm{sh}x\,dx$

59. $\int \left(\frac{\ln x}{x}\right)^2 dx$

60. $\int \frac{x}{\cos^2 x}dx$

61. $\int \frac{\mathrm{arctg}(e^x)}{e^x}dx$

Quyidagi $I_n \quad (n \in N)$ integrallar uchun rekurrent formulalar topilsin.

62. $I_n = \int x^n e^{ax}\,dx, \quad a \neq 0$

63. $I_n = \int \ln^n x\,dx$

64. $I_n = \int x^\alpha \ln^n x\,dx, \quad \alpha \neq -1$

65. $I_n = \int \frac{x^n dx}{\sqrt{x^2+a}}, \quad n > 2$

66. $I_n = \int \sin^n x\,dx, \quad n > 2$

67. $I_n = \int \cos^n x\,dx, \quad n > 2$

68. $I_n = \int \frac{dx}{\sin^n x}, \quad n > 2$

Quyidagi tengliklar hisoblansin.

69. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\mathrm{arctg}\frac{x}{a} + C$, $\quad (a \neq 0)$

70. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$, $(a \neq 0)$

71. $\int \frac{xdx}{a^2 \pm x^2} = \pm\frac{1}{2}\ln\left|a^2 \pm x^2\right| + C$.

72. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$, $(a > 0)$

73. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$, $(a > 0)$

74. $\int \frac{xdx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2} + C$.

75. $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$, $(a > 0)$

76. $\int \sqrt{x^2 \pm a^2}dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$.

69–76 misollardagi tengliklardan foydalanib quyidagi integrallar hisoblansin.

77. $\int \frac{x+1}{x^2 + x + 1}dx$

78. $\int \frac{xdx}{x^4 - 2x^2 - 1}$

79. $\int \frac{x^3dx}{x^4 - x^2 + 2}$

80. $\int \frac{dx}{\sin x + 2\cos x + 3}$

81. $\int \frac{dx}{\sqrt{a + bx^2}}$ $(b \neq 0)$

82. $\int \frac{dx}{\sqrt{1 - 2x - x^2}}$

83. $\int \frac{dx}{\sqrt{2x^2 - x + 2}}$

84. $\int \frac{dx}{\sqrt{5 + x - x^2}}$

85. $\int \frac{xdx}{\sqrt{1 - 3x^2 - 2x^4}}$

86. $\int \frac{\cos x}{\sqrt{1 + \sin x + \cos^2 x}}dx$

87. $\int \frac{dx}{x\sqrt{x^2 + x + 1}}$

88. $\int \frac{dx}{(x+1)\sqrt{x^2 + 1}}$

89. $\int \frac{dx}{(x-1)\sqrt{x^2 - 2}}$

90. $\int \sqrt{2 + x - x^2}\,dx$

91. $\int \sqrt{2+x+x^2}\,dx$

92. $\int \frac{x^2+1}{x\sqrt{x^4+1}}dx$

**Noma'lum koyeffitsentlar usulidan foydalanib quyidagi integrallar hisoblansin.**

93. $\int \frac{dx}{(x+1)(x-2)}$

94. $\int \frac{xdx}{2x^2-3x-2}$

95. $\int \frac{2x+11}{x^2+6x+13}dx$

96. $\int \frac{x^2-5x+9}{x^2-5x+6}dx$

97. $\int \frac{3x^3-5x+8}{x^2-4}dx$

98. $\int \frac{x^2dx}{x^2-6x+10}$

99. $\int \frac{xdx}{(x+1)(x+2)(x+3)}$

100. $\int \frac{xdx}{x^3-3x+2}$

101. $\int \frac{dx}{(x+1)(x^2+1)}$

102. $\int \frac{xdx}{(x+1)(x^2+1)}$

103. $\int \left(\frac{x}{x^2-3x+2}\right)^2 dx$

104. $\int \frac{dx}{x^3+1}$

105. $\int \frac{xdx}{x^3-1}$

105. $\int \frac{dx}{(x+1)^2(x^2+1)}$

107. $\int \frac{(3x^2-2)xdx}{(x+2)^2(3x^2-2x+4)}$

108. $\int \frac{x^2dx}{(x+1)(x^3+1)}$

109. $\int \frac{x^3+2x^2+3x+4}{x^4+x^3+2x^2}dx$

110. $\int \frac{dx}{x^4-x^3-x+1}$

111. $\int \frac{3x^2+x+3}{(x-1)^3(x^2+1)}dx$

112. $\int \frac{(x^4+1)dx}{(x-1)(x^4-1)}$

113. $\int \frac{x^3}{(x-1)^{100}}dx$

114. $\int \frac{dx}{x^4-1}$

115. $\int \frac{x^2+1}{x^4+1}dx$

116. $\int \frac{dx}{(x+1)(1+x^2)(1+x^3)}$

117. $\int \frac{1-x^7}{x(1+x^7)}dx$

118. $\int \frac{x^{11}}{x^8+3x^4+2}dx$

119. $\int \frac{dx}{x^6+1}$

120. $\int \frac{dx}{x^6-1}$

# Test

| | | | | |
|---|---|---|---|---|
| $\int 3x^2dx-?$ | $x^3+C$ | $x+C$ | $x^2+C$ | $e^x+C$ |
| $\int x^6dx-?$ | $\frac{x^7}{7}+C$ | $2x+C$ | $51x+C$ | $9x^2+C$ |
| $\int (x^2+x)dx-?$ | $\frac{x^3}{3}+\frac{x^2}{2}+C$ | $x+C$ | $x^3+C$ | $e^x+C$ |
| $\int (x^6+4x^3)dx-?$ | $\frac{x^7}{7}+x^4+C$ | $2x+C$ | $x^2+x+C$ | $9x^2+15x+C$ |
| $\int (x+\cos x)dx-?$ | $\frac{x^2}{2}+\sin x+C$ | $\sin x+C$ | $x^2+C$ | $\cos e^x+C$ |
| $\int (x+3)^6dx-?$ | $\frac{(x+3)^7}{7}+C$ | $2x+5+C$ | $\cos x+51x+C$ | $9x^2+C$ |
| $\int \left(x^2+\frac{1}{\cos^2 x}\right)dx-?$ | $\frac{x^3}{3}+tgx+C$ | $x+C$ | $x^3+C$ | $e^x+C$ |
| $\int (\sin x+4x^3)dx-?$ | $\cos x+x^4+C$ | $2x+C$ | $x^2+x+C$ | $9x^2+15x+C$ |
| $\int \left(2x+\frac{1}{\sin^2 x}\right)dx-?$ | $x^2-ctgx+C$ | $\sin x+C$ | $x^2+C$ | $\cos e^x+C$ |
| $\int \left(x^3+\frac{1}{1+x^2}\right)dx-?$ | $\frac{x^4}{4}+arctgx+C$ | $2x+5+C$ | $\cos x+51x+C$ | $9x^2+C$ |

# Mavzu. Ratsional funksiyalarni integrallash

## 3-ma'ruza

**Reja**

**1⁰**. Algebraning ba'zi ma'lumotlari va tasdiqlari

**2⁰**. Ratsional funksiyalarni integrallash.

### 1⁰. Algebraning ba'zi ma'lumotlari va tasdiqlari.

Biror

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \qquad (8.12)$$

ko'phad berilgan bo'lsin, bunda $a_1, a_2, a_3, \ldots, a_n$ o'zgarmas haqiqiy sonlar, $a_n \neq 0, n \in N$ esa ko'phadning darajasi.

Ma'lumki, biror $a \in R$ son uchun $P(a) = 0$ bo'lsa $a$ son $P(x)$ ko'phad-ning ildizi deb ataladi. U holda Bezu teoremasiga ko'ra $P(x)$ ko'phad $x - a$ ga qoldiqsiz bo'linib, u quydagii

$$P(x) = (x - a)Q(x)$$

ko'rinishda ifodalanadi, bunda $Q(x) - (n-1)$— darajali ko'phad.

Agar $(8.12)$ ko'phad $(x-a)^k$ $(k \in N)$ ga qoldiqsiz bo'linsa, $a$ son $(8.12)$ ko'phadning $k$ karrali ildizi bo'ladi. Bu holda $P(x)$ ko'phadni ushbu

$$P(x) = (x - a)^k \cdot R(x)$$

ko'rinishda ifodalash mumkin, bunda $R(x) - (n-k)$—darajali ko'phad.

Agar $z = \alpha + i\beta$ kompleks son $P(x)$ ko'phadning ildizi bo'lsa, u holda $z = \alpha - i\beta$ kompleks son ham bu ko'phadning ildizi bo'ladi. Shuningdek, $z = \alpha + i\beta$ son $P(x)$ ning $k$ karrali ildizi bo'lsa, $z = \alpha - i\beta$ son ham bu kop'hadning $k$ karrali ildizi bo'ladi.

Demak, $P(x)$ ko'phad $z = \alpha + i\beta$ komplaks ildizga ega bo'lganda uning ifodasida $(x-z)$ ko'paytuvchi bilan birga $x-\bar{z}$ ko'paytuvchi ham qatnashadi. Bunday holda $P(x)$ ko'phadning ifodasida quyidagi

$$(x-z)(x-\bar{z}) = [x-(\alpha+i\beta)][x-(\alpha-i\beta)] = x^2-2\alpha x+\alpha^2+\beta^2 = x^2+px+q$$

$$(p=-2\alpha, q=\alpha^2+\beta^2)$$

kvadrat uchhad ko'paytuvchi bo'lib qoladi.

Faraz qilaylik,

$$P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$

ko'phad berilgan bo'lib, $\alpha_1, \alpha_2, ..., \alpha_k$ lar uning mos ravishda $\lambda_1, \lambda_2, ..., \lambda_k$ karrali haqiqiy ildizlari $z_1, z_2, ..., z_s$ $z=\delta_j+ir_j$ $(j=1,2,\ldots,s)$ lar esa ko'phadning mos ravishda $\gamma_1, \gamma_2, ..., \gamma_s$ karrali ko'mpleks ildizlari bo'lsin. Bu ko'phadni ining ildizlariga ko'ra ko'paytuvchilarga ajratish haqidagi ushbu teoremani isbotsiz keltiramiz.

**2—teorema.** ***([1], Theorem 9.15, 312-bet)*** *Har qanday $n$-darajali $P(x)=a_0+a_1x+a_2x^2+...+a_nx^n$ ko'phad ($a_0, a_1, a_2, ..., a_n$ – o'zgarmas haqiqiy sonlar, $a_n \neq 0$) ushbu*

$$P(x) = (x-a_1)^{\lambda_1}(x-a_2)^{\lambda_2}\ldots(x-a_k)^{\lambda_k}\cdot(x^2+p_1x+q_1)^{\gamma_1}(x^2+p_2x+q_2)^{\gamma_2}\ldots(x^2+p_kx+q_k)^{\gamma_s}$$

*ko'rinishda ifodalanadi, bunda*

$$\lambda_1+\lambda_2+...+\lambda_k+2(\gamma_1+\gamma_2+\ldots+\gamma_s)=n$$

*bo'lib,* $x^2+p_jx+q_j=0$ $(j=1,2...,s)$ *tenglamalar haqiqiy ildizga ega emas.*

**Sodda kasrlar. To'g'ri kasrlarni sodda kasrlar orqali ifodalash.**

Ushbu

$$\frac{A}{(x-\alpha)^m}, \frac{Bx+C}{(x^2+px+q)^m} \qquad (m=1,2,3...) \qquad (8.13)$$

ko'rinishidagi kasrlar *sodda kasrlar* deb ataladi, bunda $A, B, C$ hamda $a, p, q$ lar o'zgarmas sonlar, $x^2+px+q$ kvadrat uchhad esa haqiqiy ildizga ega emas.

Ma'lumki, qiyidagi

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

va

$$Q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_\nu x^\nu$$

ko'phadlarning ( $a_0, a_1, a_2, ..., a_n, b_0, b_1, b_2, ..., b_\nu$ o'zgarmas sonlar $n \in N, \nu \in N$ ) nisbati

$$\frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + a_2x^2 + ... + a_nx^n}{b_0 + b_1x + b_2x^2 + ... + b_nx^n}$$

*kasr ratsional funksiya* deyiladi, $n < \nu$ bo'lganda esa u *to'g'ri kasr* deb ataladi.

Har qanday to'g'ri kasr (8.13) sodda kasrlar orqali ifodalanadi. Buni isbotlashdan avval, ikkita lemma keltiramiz.

**1—lemma.** Agar $\frac{P(x)}{Q(x)}$ to'g'ri kasr mahrajidagi $Q(x)$ ko'phad ushbu

$$Q(x) = (x-\alpha)^m Q_1(x) \qquad (m \in N)$$

ko'rinishda bo'lib , $Q_1(x)$ ko'phad esa $x-\alpha$ ga bo'linmasa, u holda berilgan to'g'ri kasr quydagi

$$\frac{P(x)}{Q(x)} = \frac{A_m}{(x-\alpha)^m} + \frac{A_{m-1}}{(x-\alpha)^{m-1}} + \ldots + \frac{A_1}{x-\alpha} + \frac{P_1(x)}{Q_1(x)}$$

ko'rinishda ifodalanishi mumkin, bunda $A_1, A_2, ..., A_m$ o'zgarmas haqiqiy sonlar, $P_1(x)$ ko'phad.

◀ $\frac{P(x)}{Q(x)}$ to'g'ri kasrni quydagi ko'rinishda yozib olamiz.

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-\alpha)^m Q_1(x)} = \frac{A_m}{(x-\alpha)^m} + \frac{P(x) - A_mQ_1(x)}{(x-\alpha)^m Q_1(x)} \qquad (8.14)$$

Ravshanki, (8.14) munosabatlardagi $P(x) - A_mQ_1(x)$ ayirma $A_m$ songa bo'g'liq. Bu sonni shunday tanlab olamizki, natijada $P(x) - A_mQ_1(x)$ ko'phad $x-\alpha$ ga bo'linsin. Buning uchun

$$P(\alpha) - A_mQ_1(\alpha) = 0$$

tenglik o'rinli bo'lishi kerak. Demak,

$$A_m=\frac{P(\alpha)}{Q_1(\alpha)}$$

deb olinsa, u holda $P(x)-A_mQ_1(x)$ ko'phad $x-\alpha$ ga bo'linadi.

Shunday qilib,

$$P(x)-A_mQ_1(x)=(x-\alpha)\cdot P_m(x)$$

bo'ladi, bunda $P_m(x)$—ko'phad .

Natjada (8.14) munosabat quydagi

$$\frac{P(x)}{Q(x)}=\frac{A_m}{(x-\alpha)^m}+\frac{P_m(x)}{(x-\alpha)^{m-1}Q_1(x)} \qquad (8.15)$$

ko'rinishga keladi, bunda $A_m$ son yuqoridek aniqlanadi.

Endi

$$\frac{P_m(x)}{(x-\alpha)^{m-1}Q_1(x)}=\frac{A_{m-1}}{(x-\alpha)^{m-1}}+\frac{P_m(x)-A_{m-1}Q_1(x)}{(x-\alpha)^{m-1}Q_1(x)}$$

tenglikning o'ng tomonidagi $A_{m-1}$ sonni shunday tanlab olamizki, $P_m(x)-A_{m-1}Q_1(x)$ ko'phad $x-\alpha$ ga bo'linsin. Buning uchun

$$P_m(\alpha)-A_{m-1}Q_1(\alpha)=0$$

tenglik o'rinli bo'lishi kerak. Demak,

$$A_{m-1}=\frac{P_m(\alpha)}{Q_1(\alpha)}$$

deb olinsa, u holda $P_m(x)-A_{m-1}Q_1(x)$ ko'phad $x-\alpha$ ga bo'linadi. Shunday qilib,

$$P_m(x)-A_{m-1}Q_1(x)=(x-\alpha)\cdot P_{m-1}(x) \qquad (8.16)$$

bo'ladi, bunda $P_{m-1}(x)$ — ko'phad.

(8.15) va (8.14) munosabatlardan topamiz:

$$\frac{P(x)}{Q(x)}=\frac{A_m}{(x-\alpha)^m}+\frac{A_{m-1}}{(x-\alpha)^{m-1}}+\frac{P_{m-1}(x)}{(x-\alpha)^{m-2}Q_1(x)}\ . \qquad (8.17)$$

Xuddi shunga o'xshash har gal $\frac{P(x)}{Q(x)}$ kasrni ifodalovchi tenglikning o'ng tomonidagi oxirgi hadidan, yuqoridagidek $\frac{A_i}{(x-\alpha)^i}$ qismini ajratib topamiz:

$$\frac{P_{m-1}(x)}{(x-\alpha)^{m-2}Q_1(x)}=\frac{A_{m-2}}{(x-\alpha)^{m-2}}+\frac{P_{m-2}(x)}{(x-\alpha)^{m-3}Q_1(x)} \qquad (8.18)$$

va h.k.

$$\frac{P_2(x)}{(x-\alpha)\ Q_1(x)}=\frac{A_1}{x-\alpha}+\frac{P_1(x)}{Q_1(x)} \qquad (8.19)$$

$(8.17)$, $(8.18)$, $(8.19)$ tengliklardan

$$\frac{P(x)}{Q\ (x)}=\frac{A_m}{(x-a)^m}+\frac{A_{m-1}}{(x-a)^{m-1}}+\ldots+\frac{A_1}{x-a}+\frac{P_1(x)}{Q_1(x)}$$

bo'lishi kelib chiqadi.►

**2—lemma.** Agar $\frac{P(x)}{Q\ (x)}$ to'gri kasr maxrajidagi $Q\ (x)$ ko'phad

$$Q\ (x)=(x^2+px+q)^n Q_1(x)$$

ko'rinishga ega bo'lib ( $x^2+px+q$ kvadrat uchhad haqiqiy ildizga ega emas), $Q_1(x)$ ko'phad $x^2+px+q$ ga bo'linmasa, u holda berilgan to'gri kasr quydagi ko'rinishda ifodalanishi mumkin:

$$\frac{P(x)}{Q(x)}=\frac{B_n(x)+C_n}{(x^2+px+q)^n}+\frac{B_{n-1}(x)+C_{n-1}}{(x^2+px+q)^{n-1}}+\ldots+\frac{B_1(x)+C_1}{x^2+px+q}+\frac{P_1(x)}{Q_1(x)}$$

bunda $B_1,B_2,\ldots,B_n$, $C_1,C_2,\ldots,C_n$ o'zgarmas sonlar, $P_1(x)$ ko'phad.

◄ $\frac{P(x)}{Q(x)}$ to'g'ri kasrni quydagicha yozib olamiz:

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x^2+px+q)^n Q_1(x)}=\frac{B_n x+C_n}{(x^2+px+q)^n}+\frac{P(x)-(B_n x+C_n)Q_1(x)}{(x^2+px+q)^n Q_1(x)}.$$

Bu tenglikdagi

$$P(x)-(B_n x+C_n)\,Q_1(x) \qquad (8.20)$$

ko'phad $B_n$ va $C_n$ sonlarga bog'liq. Endi $B_n$ va $C_n$ sonlarni shunday tanlab olish mumkinligini ko'rsatamizki, natijada $(8.20)$ ko'phad $x^2+px+q$ ga bo'linsin. Avvalo $P(x)$ va $Q_1(x)$ ko'phaglarning har birini $x^2+px+q$ kvadrat uchhadga bo'lib topamiz:

$$\frac{P(x)}{x^2+px+q}=R(x)+\frac{a_1x+b_1}{x^2+px+q}$$
$$\frac{Q_1(x)}{x^2+px+q}=S(x)+\frac{a_2x+b_2}{x^2+px+q} \qquad (8.21)$$

bunda $R(x)$ va $S(x)$ – ko'phadlar. U holda

$$\frac{P(x)-(B_nx+C_n)Q_1(x)}{x^2+px+q}=\frac{P(x)}{x^2+px+q}-(B_nx+C_n)\frac{Q_1(x)}{x^2+px+q}=$$
$$=R(x)-(B_nx+C_n)S(x)+\frac{a_1x+b_1-(B_nx+C_n)(a_2x+b_2)}{x^2+px+q}=$$
$$=R(x)-(B_nx+C_n)S(x)+B_na_2+\frac{(a_1+B_npa_2+C_na_2-B_nb_2)x+B_nqa_2+b_1-C_nb_2}{x^2+px+q}$$

bo'ladi. Bu tenglikdan ko'rinadiki $P(x)-(B_nx+C_n)Q_1(x)$ had $x^2+px+q$ ga bo'linishi uchun $x$ ning barcha qiymatlarida

$$(a_1+B_npa_2+C_na_2-B_nb_2)x+B_nqa_2+b_1-C_nb_2=0,$$

yani

$$\begin{cases}B_n(a_2p-b_2)-C_na_2+a_1=0,\\ B_na_2q-C_nb_2+b_1=0\end{cases} \qquad (8.22)$$

bo'lishi kerak. $B_n$ va $C_n$ larga nisbatan $(8.22)$ sistemaning determinanti

$$D=\begin{vmatrix}a_2p-b_2 & -a_2\\ a_2q & -b_2\end{vmatrix}\neq 0$$

bo'ladi. Buni isbotlaymiz. Teskarisini faraz qilaylik, ya'ni

$$D=-b_2(a_2p-b_2)+a_2{}^2q=0 \qquad (8.23)$$

bo'lsin.

Agar $a_2=0$ bo'lsa, unda $b_2=0$ bo'lib, natijada $(8.21)$ dan $Q_1(x)$ ko'phad $x^2+px+q$ ga bo'linishi kelib chiqadi. Bu esa $Q_1(x)$ ko'phad $x^2+px+q$ ga bo'linmaydi deb olinishiga ziddir. Demak, $a_2\neq 0$. Bu holda $(8.23)$ tenglik ushbu

$$\left(-\frac{b_2}{a_2}\right)^2 + p\cdot\left(-\frac{b_2}{a_2}\right) + q = 0$$

ko'rinishni olib, $-\frac{b_2}{a_2}$ haqiqiy son $x^2 + px + q = 0$ tenglamaning ildizi bo'lishini ko'ramiz. Bu esa $x^2 + px + q$ kvadrat uchhad haqiqiy ildizga ega bo'lmasin deb olinishiga ziddir. Demak, $(8.22)$ sistemaning determinanti noldan farqli ekan. U holda, bu sistemadan yagona $B_n$ va $C_n$ sonlar topiladi. Bu sonlarni $(8.20)$ ga qo'ysak, natijada $P(x) - (B_n x + C_n)Q_1(x)$ ko'phad $x^2 + px + q$ ga bo'linib, $\frac{P(x)}{Q(x)}$ kasr esa ushbu

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x^2+px+q)^n Q_1(x)} = \frac{B_n x + C_n}{(x^2+px+q)^n} + \frac{P_n(x)}{(x^2+px+q)^{n-1} Q_1(x)} \qquad (8.24)$$

ko'rinishga keladi, bunda $P_n(x)$ —ko'phad.

Xuddi shu yo'l bilan

$$\frac{P_n(x)}{(x^2+px+q)^{n-1} Q_1(x)} = \frac{B_{n-1} x + C_{n-1}}{(x^2+px+q)^{n-1}} + \frac{P_{n-1}(x)}{(x^2+px+q)^{n-2} Q_1(x)}, \qquad (8.25)$$

$$\frac{P_{n-1}(x)}{(x^2+px+q)^{n-2} Q_1(x)} = \frac{B_{n-2} x + C_{n-2}}{(x^2+px+q)^{n-2}} + \frac{P_{n-2}(x)}{(x^2+px+q)^{n-3} Q_1(x)} \qquad (8.26)$$

va hokozo

$$\frac{P_2(x)}{(x^2+px+q)\ Q_1(x)} = \frac{B_1 x + C_1}{x^2+px+q} + \frac{P_1(x)}{Q_1(x)} \qquad (8.27)$$

bo'lishi topiladi. $(8.24), (8.25), (8.26), (8.27)$ tengliklardan

$$\frac{P\ (x)}{Q\ (x)} = \frac{B_n x + C_n}{(x^2+px+q)^n} + \frac{B_{n-1} x + C_{n-1}}{(x^2+px+q)^{n-1}} + \ldots + \frac{B_1 x + C_1}{x^2+px+q} + \frac{P_1(x)}{Q_1(x)}$$

bo'lishi kelib chiqadi.►

**3-teorema.** ***([1], Theorem 9.15, 312-bet)*** *Har qanday to'g'ri kasr soda kasrlar yig'indisi orqali ifodalanadi.*

◄ $\frac{P(x)}{Q(x)}$ to'g'ri kasr bo'lsin. $Q(x)$ esa $n-$ darajali ko'phad bo'lib,

$$Q(x)=(x-a_1)^{n_1}\cdot(x-a_2)^{n_2}\dots(x-a_k)^{n_k}(x^2+p_1x+$$
$$+q_1)^{m_1}\cdot(x^2+p_2x+q_2)^{m_2}\dots(x^2+p_ix+q_i)^{m_i}$$

bo'lsin, bunda

$$n_1+n_2+\dots+n_k+2(m_1+m_2+\dots+m_i)=n$$

bo'lib, $x^2+p_jx+q_j \quad (j=1,2,\dots,i)$ kvadrat uchhadlar haqiqiy ildizga ega emas.

$\dfrac{P(x)}{Q(x)}$ to'g'ri kasrni quydagi

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x-a_1)^{n_1}\dots(x-a_k)^{n_k}(x^2+p_1x+q_1)^{m_1}\dots(x^2+p_ix+q_i)^{m_i}}$$

ko'rinishda yozib, bu tenglikning o'ng tomoniga 1—lemmani bir necha marta ($n_1+n_2+\dots+n_k$ marta ) qo'llanib topamiz:

$$\frac{P(x)}{Q(x)}=\frac{A_{n_1}^{(1)}}{(x-\alpha_1)^{n_1}}+\frac{A_{n_1-1}^{(1)}}{(x-\alpha_2)^{n_1-1}}+\dots+\frac{A_1^{(1)}}{x-\alpha_1}+\frac{A_{n_2}^{(2)}}{(x-\alpha_2)^{n_2}}+\frac{A_{n_2-1}^{(2)}}{(x-\alpha_2)^{n_2-1}}+\dots+$$
$$+\frac{A_1^{(2)}}{x-\alpha_2}+\dots+\frac{A_{n_k}^{(k)}}{(x-\alpha_k)^{n_k}}+\frac{A_{n_k-1}^{(k)}}{(x-\alpha_k)^{n_k-1}}+\dots+\frac{A_1^{(k)}}{x-\alpha_k}+\frac{P_1(x)}{Q\ (x)},$$

bunda

$$Q_1(x)=(x^2+p_1x+q_1)^{m_1}(x^2+p_2x+q_2)^{m_2}\dots(x^2+p_ix+q_i)^{m_i}.$$

Endi $\dfrac{P_1(x)}{Q_1(x)}$ kasrga 2—lemmani bir necha marta qo'llab, topamiz:

$$\frac{P_1(x)}{Q_1(x)}=\frac{P_1(x)}{(x^2+p_1x+q_1)^{m_1}(x^2+p_2x+q_2)^{m_2}\dots(x^2+p_ix+q_i)^{m_i}}=$$
$$=\frac{B_{m_1}^{(1)}x+C_{m_1}^{(1)}}{(x^2+p_1x+q_1)^{m_1}}+\frac{B_{m_1-1}^{(1)}x+C_{m_1-1}^{(1)}}{(x^2+p_1x+q_1)^{m_1-1}}+\dots+\frac{B_1^{(1)}x+C_1^{(1)}}{x^2+p_1x+q_1}+ \qquad (8.29)$$
$$+\frac{B_{m_2}^{(2)}x+C_{m_2}^{(2)}}{(x^2+p_2x+q_2)^{m_2}}+\frac{B_{m_2-1}^{(2)}x+C_{m_2-1}^{(2)}}{(x^2+p_2x+q_2)^{m_2-1}}+\dots+\frac{B_1^{(2)}x+C_1^{(2)}}{x^2+p_2x+q_2}+\dots+$$
$$+\frac{B_{m_i}^{(i)}x+C_{m_i}^{(i)}}{(x^2+p_ix+q_i)^{m_i}}+\frac{B_{m_i-1}^{(i)}x+C_{m_i-1}^{(i)}}{(x^2+p_ix+q_i)^{m_i-1}}+\dots+\frac{B_1^{(i)}x+C_1^{(i)}}{x^2+p_ix+q_i}.$$

(8.28) va (8.29) munosabatlardan teoremaning isboti kelib chiqadi. ►

Yuqorida isbotlangan teoremadagi o'zgarmas sonlarni boshqacha – noma'lum koeffitsientlar usuli deb atalgan usul bilan ham topish mumkin. Bunda $\frac{P(x)}{Q(x)}$ to'g'ri kasr noma'lum koeffitsientlari bo'lgan sodda kasrlarga yoyilib, so'ng tenglikning o'ng tomonidagi sodda kasrlar yig'indisi umumiy maxrajga keltiriladi.

Natijada

$$\frac{P(x)}{Q(x)} = \frac{R(x)}{Q(x)}$$

tenglik hosil bo'ladi va undan barcha $x$ lar uchun o'rinli bo'lgan $P(x) = R(x)$ tenglik kelib chiqadi. Bu tenglikning har ikki tomonidagi $x$ ning bir hil daragalari oldida turgan koeffitsientlarini tenglashtirib, sistema hosil qilinadi.

**8.7—misol.** $\frac{2x-1}{x^2-5x+6}$ to'g'ri kasrni soda kasrlarga ajratilsin.

◄ Bu kasrning maxraji $x^2-5x+6=(x-3)(x-2)$ bo'lgani uchun teoremaga ko'ra

$$\frac{2x-1}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}$$

bo'ladi. Uni

$$\frac{2x-1}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{A(x-2)+B(x-3)}{(x-2)(x-3)}$$

ko'rinishda yozib, ushbu

$$2x-1 = A(x-2)+B(x-3) \quad \text{yoki} \quad 2x-1=(A+B)x-(2A+3B)$$

tenglikka kelamiz. Ikki ko'phadning tengligidan foydalanib, $A$ va $B$ larga nisbatan ushbu

$$\begin{cases} A+B=2 \\ 2A+3B=1 \end{cases} \qquad (8.30)$$

sistemaga kelamiz. $(8.30)$ dan $A=5,\ B=-3$ bo'ladi. Shunday qilib, berilgan to'g'ri kasr sodda kasrlar orqali quyidagicha ifodalanadi:

$$\frac{2x-1}{x^2-5x+6}=\frac{5}{x-3}+\frac{-3}{x-2} \quad . ►$$

**$2^0$. Ratsional funksiyalarni integrallash.** ***([1], 9.2.1 Integrating rational maps, 310-bet)*** Ma`lumki, ratsional funksiya ikkita $P(x)$ va $Q(x)$ butun ratsional funksiyalar nisbatidan iborat:

$$f(x)=\frac{P(x)}{Q(x)}$$

Agar $\frac{P(x)}{Q(x)}$ noto'g'ri kasr bo'lsa, uning butun qismini ajratib, butun ratsional funksiya hamda to'g'ri kasr yig'indisi ko'rinishida quyidagicha ifodalab olinadi

$$\frac{P(x)}{Q(x)}=R(x)+\frac{P_1(x)}{Q_1(x)}$$

U holda

$$\int f(x)dx=\int\frac{P(x)}{Q(x)}dx=\int R(x)dx+\int\frac{P_1(x)}{Q_1(x)}dx \qquad (8.31)$$

bo'ladi.

$(8.31)$ munosabatdagi $\int R(x)dx$ integral butun ratsional funksiya (ko'phad) ning integrali bo'lib , u oson hisoblanadi.

To'g'ri kasrni integrallash uchun avval bu kasrni 3—teoremadan foydalanib, sodda kasrlar orqali ifodalab olinadi, so'ngra ularni $3^0$—bandda ko'rsatilgandek integrallanadi.

**8.8—misol.** $\int\frac{dx}{x^4-1}$ ni hisoblansin.

◄Integral ostidagi $\frac{1}{x^4-1}$ kasrni sodda kasrlarga ajratamiz:

$$\frac{1}{x^4-1}=\frac{1}{(x-1)(x+1)(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

Bu tenglikni quyidagicha yozib olmiz:

$$\frac{1}{x^4-1}=\frac{A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)}{(x-1)(x+1)(x^2+1)}$$

U holda

$$1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

ya'ni

$$1=(A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+(A-B-D)$$

bo'ladi. Natijada $A,B,C,D$ larni topish uchun

$$\begin{aligned}A+B+C&=0,\\A-B+D&=0,\\A+B-C&=0,\\A-B-D&=1.\end{aligned}$$

sistemaga kelamiz. Bu sistemani echib,

$$A=\frac{1}{4},\ B=-\frac{1}{4},\ C=0,\ D=-\frac{1}{2}$$

bo'lishini topamiz. Demak,

$$\frac{1}{x^4-1}=\frac{1}{4}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{x+1}-\frac{1}{2}\cdot\frac{1}{x^2+1}$$

bo'lib,

$$\int\frac{dx}{x^4-1}=\frac{1}{4}\cdot\int\frac{dx}{x-1}-\frac{1}{4}\cdot\int\frac{dx}{x+1}-\frac{1}{2}\cdot\int\frac{dx}{x^2+1}=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}arctgx+C$$

bo'ladi. ►

**Adabiyotlar**


1. **Claudio Canuto, Anita Tabacco,** *Mathematical Analysis I,* Springer-Verlag Italia, Milan 2008.
2. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, I q.* T. "Voris-nashriyot", 2010.
3. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
4. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

# Glossariy

1. **n-darajali ko‘phad** – $P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ ushbu ifodaga n-darajali ko‘phad deyiladi.

2. **Ratsional funksiya** – ikta ko‘phadning nisbatidan tashkil topgan funksiyaga ratsional funksiya deyiladi.

# Keys banki

**30-keys.** Masala o`rtaga tashlanadi: Ushbu

$$I = \int \frac{3x+1}{x\left(1+x^2\right)^2} dx$$

integral hisoblansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 3-amaliy mashg'ulot

**1.Sodda kasrlar va ularning integrallari.** Ushbu

1) $\frac{A}{x-a}$, $\quad (A=const,\quad a=const)$,

2) $\frac{A}{(x-a)^n}$, $\quad (A=const,\quad a=const,\quad n=2,3,...)$,

3) $\frac{Mx+N}{x^2+px+q}$ $\quad (M,N,p,q=const,\quad p^2-4q<0)$,

4) $\frac{Mx+N}{(x^2+px+q)^n}$

$(M,N,p,q=const,\quad p^2-4q<0,\quad n=2,3...)$

kasrlar **sodda kasrlar** deyiladi. Ularning integrallari quyidagicha bo'ladi:

$$\int\frac{A}{x-a}dx = A\ln|x-a|+C;$$

$$\int\frac{A}{(x-a)^n}dx = -\frac{A}{(n-1)(x-a)^{n-1}}+C \qquad (n\neq 1);$$

$$\int\frac{Mx+N}{x^2+px+q}dx = \frac{M}{2}\ln(x^2+px+q)+\frac{N-\frac{Mp}{2}}{\sqrt{q-\frac{p^2}{4}}}\operatorname{arctg}\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C;$$

$$\int\frac{Mx+N}{(x^2+px+q)^n}dx = \frac{M}{2}\frac{(x^2+px+q)^{1-n}}{1-n}+$$

$$+\left(N-\frac{Mp}{2}\right)\int\frac{dx}{\left[\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}\right]^n} \qquad (n=2,3,...)$$

Keyingi integral 29- mavzuda keltirilgan rekkurent formula yordamida hisoblanadi.

**2. Ratsional funksiyalarni integrallash.** Ushbu

$$f(x)=\frac{a_0+a_1x+a_2x^2+...+a_nx^n}{b_0+b_1x+b_2x^2+...+b_mx^m}$$

**ratsional funksiyaning** integrali $\int f(x)dx$ quyidagicha hisoblanadi:

agar $n \geq m$ bo'lsa, kasrning butun qismini ajratib, uni **butun ratsional funksiya** va **to'g'ri kasr** yig'indisi ko'rinishida yozib olinadi. Ravshanki, butun ratsional funksiyaning integrali oson hisoblanadi.

Ma'lumki, har qanday to'g'ri kasr sodda kasrlar yig'indisi sifatida ifodalanadi. Demak, to'g'ri kasrning integrali sodda kasrlarning integrallariga keltirib hisoblanadi.

**M i s o l .** ***Ushbu***

$$I = \int \frac{2x^5 + 6x^3 + 1}{x^4 + 3x^2} dx$$

***integral hisoblansin.***

◀Integral ostidagi kasrning suratini maxrajiga bo'lib, butun qismini ajratamiz:

$$\frac{2x^5 + 6x^3 + 1}{x^4 + 3x^2} = 2x + \frac{1}{x^4 + 3x^2}$$

so'ng bu tenglikning o'ng tomonidagi to'g'ri kasrni sodda kasrlarga yoyamiz:

$$\frac{1}{x^4 + 3x^2} = \frac{1}{x^2(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 3} \qquad (*)$$

Bundan

$$I = Ax(x^2+3) + B(x^2+3) + (Cx+D)x^2 = (A+C)x^3 + (B+D)x^2 + 3Ax + 3B$$

bo'lishi kelib chiqadi. Demak,

$$A + C = 0, \quad B + D = 0, \quad 3A = 0, \quad 3B = 1.$$

Keyingi tenglikdan esa

$$A = 0, \quad B = \frac{1}{3}, \quad C = 0, \quad D = -\frac{1}{3}$$

bo'lishini topamiz. (*) tenglikka ko'ra

$$\frac{1}{x^2(x^2+3)} = \frac{1}{3x^2} - \frac{1}{3(x^2+3)}$$

bo'ladi. Endi berilgan integralni hisoblaymiz:

$$\int \frac{2x^5 + 6x^3 + 1}{x^4 + 3x^2} dx = \int \left(2x + \frac{1}{x^4 + 3x^2}\right) dx = \int \left[2x + \frac{1}{3x^2} - \frac{1}{3(x^2+3)}\right] dx =$$

$$= x^2 - \frac{1}{3x} - \frac{1}{3\sqrt{3}} \operatorname{arctg} \frac{x}{\sqrt{3}} + C. ▶$$

**M i s o l .** ***Ushbu***

$$I = \int \frac{3x + 1}{x(1 + x^2)^2} dx$$

***integral hisoblansin.***

◄Integral ostidagi kasrni sodda kasrlarga yoyamiz:

$$\frac{3x+1}{x(1+x^2)^2}=\frac{A}{x}+\frac{Bx+C}{1+x^2}+\frac{Dx+F}{(1+x^2)^2}$$

Umumiy maxrajga keltirib topamiz:

$$3x+1=A(1+x^2)^2+(Bx+C)x(1+x^2)+(Dx+F)x=$$
$$=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+F)x+A.$$

$A,B,C,D,F$ larni topish uchun quyidagi

$$\begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ C+F=0 \\ A=1 \end{cases}$$

sistema hosil bo‘ladi. Bu sistemani echsak,

$$A=1,\quad B=-1,\quad C=0,\quad D=-1,\quad F=3$$

bo‘lishi kelib chiqadi. Demak, integral ostidagi funksiya

$$\frac{3x+1}{x(1+x^2)^2}=\frac{1}{x}-\frac{x}{1+x^2}+\frac{-x+3}{(1+x^2)^2}$$

bo‘ladi. Uning integralini hisoblaymiz:

$$I=\int\frac{3x+1}{x(1+x^2)^2}dx=\int\frac{dx}{x}-\int\frac{xdx}{1+x^2}-\int\frac{xdx}{(1+x^2)^2}+3\int\frac{dx}{(1+x^2)^2}=$$
$$=\ln|x|-\frac{1}{2}\ln(1+x^2)+\frac{1}{2(1+x^2)}+3\int\frac{dx}{(1+x^2)^2}.$$

Keyingi tenglikning o‘ng tomonidagi integralni **2-§** da keltirilgan rekkurent formulaga ko‘ra topamiz:

$$I_2=\int\frac{dx}{(x^2+1)^2}=\frac{1}{2}\cdot\frac{x}{x^2+1}+\frac{1}{2}I_1=\frac{x}{2(x^2+1)}+\frac{1}{2}\int\frac{dx}{x^2+1}=$$
$$=\frac{x}{2(x^2+1)}+\frac{1}{2}\operatorname{arctg}x+C$$

Demak,

$$I=\int\frac{3x+1}{x(1+x^2)}dx=\ln|x|-\frac{1}{2}\ln(1+x^2)+\frac{3x+1}{2(x^2+1)}+\frac{3}{2}\operatorname{arctg}x+C.$$ ►

**Noma'lum koeffitsientlar usulidan foydalanib quyidagi integrallar hisoblansin.**

1. $\int \frac{dx}{(x+1)(x-2)}$

2. $\int \frac{xdx}{2x^2-3x-2}$

3. $\int \frac{2x+11}{x^2+6x+13}dx$

4. $\int \frac{x^2-5x+9}{x^2-5x+6}dx$

4. $\int \frac{3x^3-5x+8}{x^2-4}dx$

5. $\int \frac{x^2dx}{x^2-6x+10}$

6. $\int \frac{xdx}{(x+1)(x+2)(x+3)}$

7. $\int \frac{xdx}{x^3-3x+2}$

8. $\int \frac{dx}{(x+1)(x^2+1)}$

9. $\int \frac{xdx}{(x+1)(x^2+1)}$

10. $\int \left(\frac{x}{x^2-3x+2}\right)^2 dx$

11. $\int \frac{dx}{x^3+1}$

12. $\int \frac{xdx}{x^3-1}$

13. $\int \frac{dx}{(x+1)^2(x^2+1)}$

14. $\int \frac{(3x^2-2)xdx}{(x+2)^2(3x^2-2x+4)}$

15. $\int \frac{x^2dx}{(x+1)(x^3+1)}$

16. $\int \frac{x^3+2x^2+3x+4}{x^4+x^3+2x^2}dx$

17. $\int \frac{dx}{x^4-x^3-x+1}$

18. $\int \frac{3x^2+x+3}{(x-1)^3(x^2+1)}dx$

19. $\int \frac{(x^4+1)dx}{(x-1)(x^4-1)}$

20. $\int \frac{x^3}{(x-1)^{100}}dx$

21. $\int \frac{dx}{x^4-1}$

22. $\int \frac{x^2+1}{x^4+1}dx$

23. $\int \frac{dx}{(x+1)(1+x^2)(1+x^3)}$

24. $\int \frac{1-x^7}{x(1+x^7)}dx$

25. $\int \frac{x^{11}}{x^8+3x^4+2}dx$

26. $\int \frac{dx}{x^6+1}$

27. $\int \frac{dx}{x^6-1}$

**Quyidagi integrallar Ostrogradskiy usulidan foydalanib hisoblansin.**

28. $\int \frac{xdx}{(x-1)^2(x+1)^3}$

29. $\int \frac{dx}{(x^3+1)^2}$

30. $\int \frac{dx}{(x^2+1)^3}$

31. $\int \frac{x^2dx}{(x^2+2x+2)^2}$

32. $\int \frac{dx}{(x^4+1)^2}$

33. $\int \frac{x^2+3x-2}{(x-1)(x^2+x+1)^2}dx$

34. $\int \frac{dx}{(x^4-1)^3}$

# Test

| | | | | |
|---|---|---|---|---|
| $\int\frac{dx}{x+1}=?$ | $\ln\lvert x+1\rvert+C$ | $\ln\lvert x+1\rvert+C$ | $\ln(x+1)+C$ | $2\ln(x+1)+C$ |
| $\int\frac{dx}{x^2+1}=?$ | $arctgx+C$ | $\ln\lvert x^2+1\rvert+C$ | $\ln(x^2+1)+C$ | $2\ln(x^2+1)+C$ |
| $\int\frac{dx}{x^2+9}=?$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+9\rvert+C$ | $\ln\lvert x^2+3\rvert+C$ | $2\ln(x^2+3)+C$ |
| $\int\frac{2xdx}{x^2+9}=?$ | $\ln\lvert x^2+9\rvert+C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $2\ln(x^2+3)+C$ | $\ln\lvert x^2+3\rvert+C$ |
| $\int\frac{x^2dx}{x^3+9}=?$ | $\frac{1}{3}\ln\lvert x^3+9\rvert+C$ | $2\ln(x^2+3)+C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+3\rvert+C$ |
| $\int\frac{dx}{x^2-1}=?$ | $\frac{1}{2}\ln\frac{\lvert x+1\rvert}{\lvert x-1\rvert}+C$ | $\ln\lvert x^2-1\rvert+C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+9\rvert+C$ |
| $\int\frac{xdx}{x^2-1}=?$ | $\frac{1}{2}\ln\lvert x^2-1\rvert+C$ | $\ln\lvert x^2-1\rvert+C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+1\rvert+C$ |
| $\int\frac{dx}{x^2-a^2}=?$ | $\frac{1}{2a}\ln\frac{\lvert x+a\rvert}{\lvert x-a\rvert}+C$ | $\frac{1}{2a}\ln\frac{\lvert x+1\rvert}{\lvert x-1\rvert}+C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+9\rvert+C$ |
| $\int\frac{dx}{x^2+a^2}=?$ | $\frac{1}{a}arctg\frac{x}{a}+C$ | $arctgx+C$ | $\ln\lvert x+1\rvert+C$ | $\ln\lvert x^2+a^2\rvert+C$ |
| $\int\frac{dx}{x^2}=?$ | $-\frac{1}{x}+C$ | $\ln\lvert x^2\rvert+C$ | $arctgx+C$ | $\ln\lvert x^2+a^2\rvert+C$ |

# Mavzu. Ba'zi irratsional funksiyalarni integrallash. Trigonometrik funksiyalarni integrallash

## 4-5-ma'ruza

**Reja**

**1$^0$**. $R(x, y(x))$ ko'rinishidagi funksiyalarni integralash.

**2$^0$**. Binominal differensialni integrallash.

**3$^0$**. Trigonometrik funksiyalarni integrallash.

**1$^0$. *([I] Integral calculus I, 299-bet)* $R(x, y(x))$ ko'rinishidagi funksiyalarni integralash.**

Ushbu

$$\int R(x, y(x))\,dx \qquad (8.32)$$

integralni qaraylik, bunda $R(x, y(x))$ funksiya $x$ va $y(x)$ larning ratsional funksiyasidir .

Agar $y(x)$ funksiya $x$ ning ratsional funksiyasi bo'lsa, ushbu

$$\int R(x, y(x))dx$$

integral ratsional funksiyaning integrali bo'ladi. Bunday integralar 3—§ da batafsil o'rganildi .

Agar $y(x)$ funksiya $x$ o'zgaruvchining ratsional funksiyasi bo'lmasa, u holda ravshanki, $R(x, y(x))$ ham $x$ o'zgaruvchining ratsional funksiyasi bo'lmaydi. Bu holda $x$ o'zgaruvchini almashtirish yordamida $R(x, y(x))$ ni ratsional funksiyaga keltirish masalasi kelib chiqadi . Agar biz shunday $x = \varphi(t)$ amashtirish topsakki, natijada $x = \varphi(t)$, $y(x) = y(\varphi(t))$ lar $t$ ning ratsional funksiyalari bo'lsa, (bunda $x' = \varphi'(t)$ ham ratsional funksiya bo'ladi ), u holda

$$\int R(x, y(x))dx = \int R((\varphi(t), y(\varphi(t)) \cdot \varphi'(t)dt$$

bo'lib , $\int R(x, y(x))dx$ integralini hisoblash ushbu

$$\int R((\varphi(t), y(\varphi(t)) \cdot \varphi'(t)dt$$

ratsional funksiyaning integralini hisoblashga keltiriladi .

Endi $y(x)$ funksiyaning ba`zi bir muayyan ko'rinishga ega bo'lgan hollarini qaraymiz :

**1).** (8.32) **integralda**

$$y(x)=\sqrt[n]{\frac{ax+b}{cx+d}}$$

bo'lsin, bunda $a,b,c,d$ o'zgarmas sonlar, $n\in N$. Bu holda (8.32) integral quyidagi

$$\int R(x,\sqrt[n]{\frac{ax+b}{cx+d}})dx \qquad (8.33)$$

ko'rinishni oladi . Bunda $a,b,c,d$ sonlardan tuzilgan determinant noldan farqli, ya`ni

$$\Delta=\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}\neq 0$$

deb qaraymiz . Agar

$$\Delta=\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}=0$$

bo'lsa, $a$ va $b$ sonlar $c,d$ sonlarga proporsional bo'lib , $\frac{ax+b}{cx+d}$ nisbat $x$ ga bog'liq bo'lmaydi va $R(x,\sqrt[n]{\frac{ax+b}{cx+d}})$ funksiya $x$ o'zgaruvchining ratsional funksiyasi bo'lib qoladi. Bu holda (8.33) integral 3—§ da o'rganilgan integralga keladi. Shunday qilib, keyingi mulohazalarda $\Delta\neq 0$ deymiz .

(8.33) integralda

$$t=\sqrt[n]{\frac{ax+b}{cx+d}}$$

almashtirish bajaramiz. Natijada

$$t=\sqrt[n]{\frac{ax+b}{cx+d}}\ , \qquad x=\frac{dt^n-b}{a-ct^n}=\varphi(t)$$

$$dx=\varphi'(t)dt=\frac{(ad-bc)nt^{n-1}}{(a-ct^n)^2}dt$$

bo'lib, (8.33) integral ushbu

$$\int R(x,\sqrt[n]{\frac{ax+b}{cx+d}})dx=\int R(\varphi(t),t)\cdot\varphi'(t)dt=\int R(\frac{dt^n-b}{a-ct^n}\ ,\ t)\cdot\frac{(ad-bc)nt^{n-1}}{(a-ct^n)^2}dt$$

ko'rinishni oladi .

Demak, qaralayotgan

$$\int R(x,\sqrt[n]{\frac{ax+b}{cx+d}})dx$$

integralni hisoblash ushbu $R(\frac{dt^n-b}{a-ct^n}\ ,\ t)\frac{(ad-bc)nt^{n-1}}{(a-ct^n)^2}$ ratsional funksiyaning integralini hisoblashga keladi .

**8.9—misol .** Ushbu

$$\int\sqrt{\frac{1+x}{1-x}}\,\frac{dx}{1-x}$$

integral hisoblansin .

◀ Bu integralni hisoblash uchun

$$t=\sqrt{\frac{1+x}{1-x}}$$

deb olamiz. U holda

$$x=\frac{t^2-1}{t^2+1}\,,\quad dx=\frac{4tdt}{(t^2+1)^2}$$

bo'lib ,

$$\int\sqrt{\frac{1+x}{1-x}}\,\frac{dx}{1-x}=2\int\frac{t^2dt}{t^2+1}$$

bo'ladi . Natijada berilgan integral uchun topamiz :

$$\int\sqrt{\frac{1+x}{1-x}}\,\frac{dx}{1-x}=2\int\frac{t^2dt}{t^2+1}=2t-2arctgt+C=2\sqrt{\frac{1+x}{1-x}}-2arctg\sqrt{\frac{1+x}{1-x}}+C\,.$$ ▶

Quyidagi

$$\int R\left(x,(\frac{ax+b}{cx+d})^{r_1},\,(\frac{ax+b}{cx+d})^{r_2},\,\ldots(\frac{ax+b}{cx+d})^{r_n}\right)dx \qquad (8.34)$$

integralni qaraylik. Agar $r_1,r_2,\ldots r_n$ ratsional sonlarni umumiy $m$ maxrajga keltirib , (8.34) integralda

$$t=\sqrt[m]{\frac{ax+b}{cx+d}}$$

almashtirish bajarilsa, natijada (8.34) integralni hisoblash ratsional funksiyani integrallashga keladi.

**8.10—misol.** Ushbu

$$\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$$

integral hisoblansin .

◀ Bu integralda $t=\sqrt[6]{x}$ almashtirish bajaramiz . Natijada

$$x=t^6\,,\quad dx=6t^5\,dt$$

bo'lib, berilgan integral uchun topamiz:

$$\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}=6\int\frac{t^3dt}{t+1}=6\int\left((t^2-t+1)-\frac{1}{t+1}\right)dt=6\left(\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|t+1|\right)+C=$$

$$=6(\frac{\sqrt{x}}{3}-\frac{\sqrt[3]{x}}{2}+\sqrt[6]{x}-\ln|\sqrt[6]{x}+1|)+C=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln|\sqrt[6]{x}+1|+C$$ ▶

**2).** (8.32) **integralda** $y=y(x)=\sqrt{ax^2+bx+c}$ bo'lsin**,** bunda $a,b,c$ o'zgarmas sonlar bo'lib, $ax^2+bx+c$ kvadrat uchhad teng ildizlarga ega emas.

(8.32) integral quyidagi

$$\int R(x,\sqrt{ax^2+bx+c})dx \qquad (a\neq 0) \qquad (8.35)$$

ko'rinishda bo'ladi.

Quyida keltiriladigan uchta almashtirish yordamida (8.35) integral ratsional funksiya integraliga keltiriladi.

**a)** $a>0$ **bo'lsin.** Bu holda (8.35) integral

$$t=\sqrt{a}x+\sqrt{ax^2+bx+c} \qquad (8.36)$$

( yoki $t=-\sqrt{a}x+\sqrt{ax^2+bx+c}$ )

almashtirish natijasida ratsional funksiyani integrallashga keladi:

$$\int R(x,\sqrt{ax^2+bx+c})dx=\int R\left(\frac{t^2-c}{2\sqrt{a}t+b},\frac{\sqrt{a}t^2+bt+c\sqrt{a}}{2\sqrt{a}t+b}\right)\frac{2(\sqrt{a}t^2+bt+c\sqrt{a})}{(2\sqrt{a}t+b)^2}dt.$$

**b)** $c>0$ **bo'lsin**. Bu holda (8.35) integral

$$t=\frac{1}{x}(\sqrt{ax^2+bx+c}-\sqrt{c}) \qquad (8.37)$$

( yoki $t=\frac{1}{x}(\sqrt{ax^2+bx+c}+\sqrt{c})$ )

almashtirish yordamida ratsional funksiyani integrallashga keladi:

$$\int R(x,\sqrt{ax^2+bx+c})dx=\int R\left(\frac{2\sqrt{c}t-b}{a-t^2}\cdot\frac{\sqrt{c}t^2-bt+a\sqrt{c}}{a-t^2}\right)\frac{2(\sqrt{c}t^2+bt+\sqrt{c}a)}{(a-t^2)^2}dt.$$

v) $ax^2+bx+c$ **kvadrat uchhad har hil** $x_1$ **va** $x_2$ **haqiqiy ildizlarga ega bo'lsin.** Ma'lumki, $x_1$ va $x_2$ ildizlar orqali $ax^2+bx+c$ kvadrat uchhadni

$$ax^2+bx+c=a(x-x_1)(x-x_2)$$

ko'rinishda ifodalash mumkin. Bu holda (8.35) integral ushbu

$$t=\frac{1}{x-x_1}(\sqrt{ax^2+bx+c}) \qquad (8.38)$$

almashtirish bilan

$$\int R(x,\sqrt{ax^2+bx+c})dx=\int R\left(\frac{-ax_2+x_1t^2}{t^2-a},\frac{a(x_1-x_2)}{t^2-a}t\right)\cdot\frac{2a(x_1-x_2)}{t^2-a}dt$$

ko'rinishga keladi. Bu tenglikning o'ng tomonidagi integral ostidagi funksiya $t$ o'zgaruvchining ratsional funksiyasidir.

Odatda (8.36), (8.37) va (8.38) almashtirishlar *Eyler almashtirishlari* deyiladi.

**8.11—misol.** $\int\frac{1}{x+\sqrt{x^2+x+1}}dx$ integral hisoblansin.

◄. Bu integral uchun $(a=1)$ Eylerning birinchi almashtirishini ((8.36) ga qarang) bajaramiz:

$$t=x+\sqrt{x^2+x+1}.$$

U holda

$$x=\frac{t^2-1}{1+2t},\sqrt{x^2+x+1}=\frac{t^2+t+1}{1+2t},\ dx=2\frac{t^2+t+1}{(1+2t)^2}dt$$

bo'lib, berilgan integral uchun

$$\int\frac{1}{x+\sqrt{x^2+x+1}}dx=2\int\frac{t^2+t+1}{t(1+2t)^2}dt$$

bo'ladi. Endi

$$2\frac{t^2+t+1}{t(1+2t)^2}=\frac{2}{t}-\frac{3}{1+2t}-\frac{3}{(1+2t)^2}$$

bo'lishini e'tiborga olib, topamiz:

$$\int\frac{1}{x+\sqrt{x^2+x+1}}dx=2\ln|t|-\frac{3}{2}\ln|1+2t|+\frac{3}{2}\cdot\frac{1}{1+2t}+C=$$

$$=2\ln\left|x+\sqrt{x^2+x+1}\right|-\frac{3}{2}\ln\left|1+2x+2\sqrt{x^2+x+1}\right|+\frac{3}{2}\cdot\frac{1}{1+2x+2\sqrt{x^2+x+1}}+C.$$ ▶

## $2^0$. Binominal differensiallarni integrallash.

Ushbu

$$x^m(a+bx^n)^p dx$$

differensial ifoda *binomial differensial* deb ataladi, bunda $a, b$ o'zgarmasli sonlar, $m, n, p$ ratsional sonlar.

Ushbu

$$\int x^m(a+bx^n)^p dx \qquad (8.37)$$

integralni hisoblash $m, n, p$ ratsional sonlariga bog'liq. Mashhur rus matematigi P. L. Chebishev ko'rsatganki $(8,37)$ integral quyidagi uchta

1) $p$ butun son , yoki

2) $\frac{m+1}{n}$ butun son, yoki

3) $\frac{m+1}{n}+p$ butun son,

bo'lgan holdagina ratsional funksiyalarning integrali orqali ifodalanadi.

**1)** $p$ — **butun son bo'lsin** . Bu holda $m$ va $n$ ratsional sonlar ( ya'ni kasrlar ) maxrajining eng kichik umumiy bo'luvchisini $\delta$ orqali belgilab, $(8,37)$ integralda $x=t^\delta$ almashtirish bajarilsa, integral ostidagi funksiya ratsional funksiyaga aylanib, $(8,37)$ integral ratsional funksiyaning integraliga keltiriladi.

**2)** $\frac{m+1}{n}$ — **butun son bo'lsin.** Avval $(8,37)$ integralda

$$x=t^{\frac{1}{n}}$$

almashtirish bajaramiz. Natijada $(8,37)$ integral quyidagi

$$\int x^m(a+bx^n)^p dx=\frac{1}{n}\int(a+bt)^p t^{\frac{m+1}{n}-1}dt \qquad (8.38)$$

ko'rinishni oladi . Qisqalik uchun

$$q = \frac{m+1}{n} - 1$$

deb belgilaymiz. Bu holda $p$ kasr sonning maxrajini $s$ bilan belgilab, (8.38) integralda

$$z = (a+bt)^{\frac{1}{s}} = (a+bx^n)^{\frac{1}{s}}$$

almashtirish bajarilsa, natijada integral ostidagi ifoda ratsional funksiyaga aylanib, ya'na (8,37) integral ratsional funksiya integralini hisoblashga keltiriladi.

**3). $p+q$ butun son bo'lsin.** Yuqoridagi (8,38) integralni quyidagicha yozib olamiz:

$$\int (a+bt)^p t^q dt = \int \left(\frac{a+bt}{t}\right)^p t^{p+q} dt .$$

Agar keyingi integralda

$$z = \left(\frac{a+bt}{t}\right)^{\frac{1}{s}}$$

almashtirish bajarilsa, (8,37) integral ratsional funksiyaning integraliga keladi.

**8.12–misol.** Ushbu

$$\int \frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)^2} dx$$

integral hisoblansin.

◄ Bu integralni (8.37) integral bilan taqqoslab, $p=-2$ ( butun son) ekanligini aniqlaymiz. Yuqorida qaralgan 1) holga ko'ra $x=t^6$ $(t=\sqrt[6]{x}\,)$ almashtirish bajarib topamiz:

$$\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2} dx = 6\int \frac{t^8}{(1+t^2)^2} dt .$$

Bu tenglikning o'ng tomonidagi integral ostidagi funksiyani

$$\frac{t^8}{(1+t^2)^2} = t^4 - 2t^2 + 3 - 4\frac{1}{t^2+1} + \frac{1}{(t^2+1)^2}$$

ko'rinishda yozish mumkin ekanini e'tiborga olsak, u holda

$$\int \frac{t^8}{(1+t^2)^2} dt = \frac{t^5}{5} - 2\frac{t^3}{3} + 3t - 4arctgt + \int \frac{dt}{(t^2+1)^2}$$

bo'ladi. Oxirgi integral shu bobning 2—§ ida keltirilgan (8.17) rekurrent munosabat yordamida osongina hisoblanadi.

$$\int \frac{dt}{(t^2+1)^2} = \frac{1}{2}\cdot\frac{1}{t^2+1} + \frac{1}{2}arctgt + C.$$

Natijada quyidagiga ega bo'lamiz:

$$\int \frac{t^8}{(1+t^2)^2}dt = \frac{1}{5}t^5 - \frac{2}{3}t^3 + 3t - \frac{7}{2}arctgt + \frac{1}{2}\cdot\frac{1}{t^2+1} + C.$$

Demak, $t=\sqrt[6]{x}$ ekanini etiborga olib, uzul – kesil yozamiz:

$$\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx = \frac{6}{5}\sqrt[6]{x^5} - 4\sqrt{x} + 18\sqrt[6]{x} - 21arctg\sqrt[6]{x} + 3\cdot\frac{\sqrt[6]{x}}{\sqrt[3]{x}+1} + C \quad ►$$

**8.13— misol.** $\int \frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$ integralni hisoblang.

◄ Bu integralni $\int \frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}} = \int x(1+x^{\frac{2}{3}})^{-\frac{1}{2}}dx$ ko'rinishda yozib, $m=1,\ n=\frac{2}{3},\ p=-\frac{1}{2}$ bo'lishini topamiz. Bu holda $\frac{m+1}{n}=3$ bo'lib,

$$t=(1+x^{\frac{2}{3}})^{\frac{1}{2}}$$

almashtirishni bajaramiz. Unda

$$1+x^{\frac{2}{3}}=t^2, \quad x=(t^2-1)^{\frac{3}{2}} \quad \text{va} \quad dx=\frac{3}{2}(t^2-1)^{\frac{1}{2}}\cdot 2tdt$$

bo'lib, berilgan integral uchun ushbu

$$\int x(1+x^{\frac{2}{3}})^{-\frac{1}{2}}dx = 3\int (t^2-1)^2t^2dt = 3\frac{t^7}{7} - 6\frac{t^6}{5} + t^3 + C, \quad t=\sqrt{1+x^{\frac{2}{3}}}$$

ifoda topiladi. ►

**3⁰. Trigonometrik funksiyalarni integrallash**

Yuqoridagidek, $R(\sin x, \cos x)$ orqali $\sin x$ va $\cos x$ larning ratsional funksiyasini belgilaylik. Bunday ifodaning

$$\int R(\sin x, \cos x)dx \qquad (8.39)$$

integralini qaraylik.

Agar (8.39) integralda

$$t = tg\frac{x}{2} \qquad (-\pi < x < \pi)$$

almashtirish bajarilsa, u holda (8.39) integral ostidagi $R(\sin x, \cos x)$ ifoda $t$ o'zgaruvchining ratsional funksiyasiga aylanib (8.39) integralni hisoblash ratsional funksiya integralini hisoblashga keladi.

Darhaqiqat, quyidagi

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},$$

$$x = 2arctgt, \quad dx = \frac{2dt}{1+t^2}$$

munosabatlarni e`tiborga olsak, u holda (8.39) integral quyidagi

$$\int R(\sin x,\cos x)dx = \int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}$$

ko'rinishga keladi. Ravshanki,

$$R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}$$

funksiya $t$ o'zgaruvchining ratsional funksiyasi. Demak, (8.39) integralni hisoblash ratsional funksiya integralini hisoblashga keladi.

**8.14 – misol.** $\int\frac{dx}{3+\sin x}$ integral hisoblansin.

◄ Bu integralda $t = tg\frac{x}{2}$ almashtirish bajarib topamiz:

$$\int\frac{dx}{3+\sin x} = \int\frac{1}{3+\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2} = 2\int\frac{dt}{3t^2+2t+3},$$

$$2\int\frac{dt}{3t^2+2t+1} = \frac{2}{3}\int\frac{d(t+\frac{1}{3})}{(t+\frac{1}{3})^2+\frac{8}{9}} = \frac{2}{3}\int\frac{d(t+\frac{1}{3})}{(t+\frac{1}{3})^2+(\frac{2\sqrt{2}}{3})^2} = \frac{\sqrt{2}}{2}arctg3\frac{t+\frac{1}{3}}{2\sqrt{2}}+C.$$

Demak,

$$\int\frac{dx}{3+\sin x} = \frac{\sqrt{2}}{2}arctg\frac{3tg\frac{x}{2}+1}{2\sqrt{2}}+C. \ ►$$

Shuni ta`kidlash lozimki, $\int R(\sin x,\cos x)dx$ integralda $t = tg\frac{x}{2}$ almashtirish universal almashtirish bo'lib, u (8.39) integralni har doim ratsional funksiya integraliga keltirsada ko'pincha bu almashtirish murakab hisoblashlarga olib keladi.

Ayrim hollarda trigonometrik funksiyalarni integrallashda $t = tgx$, $t = \sin x$, $t = \cos x$ almashtirishlar qulay bo'ladi.

**8.15—misol.** $\int\frac{dx}{\cos^4 x}$ integral hisoblansin.

◀ Agar bu integralda $t=tg\frac{x}{2}$ universal almashtirish bajarilsa , u holda

$$\int\frac{dx}{\cos^4 x}=2\int\frac{(1+t^2)^3}{(1-t^2)^4}dt$$

bo'ladi . Biroq qaralayotgan integralda $t=tgx$ almashtirish bajarilsa , u holda

$$\int\frac{dx}{\cos^4 x}=\int(1+tg^3x)d(tgx)=\int(1+t^2)dt$$

bo'lib , undan

$$\int\frac{dx}{\cos^4 x}=t+\frac{t^3}{3}+C=tgx+\frac{tg^3x}{3}+C$$

bo'lishini topamiz. ▶

## Mashqlar

**8.16.** Ushbu

$$f(x)=x|x|\,,\quad \varphi(x)=e^{|x|}x \qquad (x\in R)$$

funksiyalarning $(-\infty\,,\,+\infty)$ dagi boshlang'ich funksiyalari topilsin .

**8.17.** Ushbu

$$\int\frac{dx}{\sqrt{(x-a)(b-x)}}$$

integral hisoblansin.

**8.18.** Ushbu

$$\int e^{ax}\cos bx dx$$

integral hisoblansin.

**8.19.** Quyidagi

1) $$\int\frac{dx}{\sqrt{a^2\pm x^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C \qquad (a>0),$$

2) $$\int\frac{dx}{\sqrt{a^2\pm x^2}}=\pm\sqrt{a^2\pm x^2}+C \qquad (a>0),$$

3) $$\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C,$$

4) $$\int\sqrt{x^2\pm a^2}\,dx=\frac{x}{2}\sqrt{x^2\pm a^2}\pm\frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$$

tengliklar isbotlansin .

**8.20.** Ushbu

$$\int\frac{\sin 4x}{\sin^8 x+\cos^8 x}dx$$

integral hisoblansin .

## Adabiyotlar

1. **Claudio Canuto, Anita Tabacco,** *Mathematical Analysis I*, Springer-Verlag Italia, Milan 2008.
2. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'rizalar, I q.* T. "Voris-nashriyot", 2010.
3. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
4. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

# Glossariy

1. **Irratsional funksiyalar –** $R\left(x,\left(\frac{ax+b}{cx+d}\right)^{r_1},\left(\frac{ax+b}{cx+d}\right)^{r_2},\ldots,\left(\frac{ax+b}{cx+d}\right)^{r_n}\right)$, $\mathbf{r_1, r_2, \ldots, r_n}$ – ratsional sonlar, $\mathbf{a, b, c, d}$ – haqiqiy sonlar, $\mathbf{ad - bc \neq 0}$ ushbu ko'rinishdagi funksiyalarirratsional funksiyalar deyiladi.
2. **Trigonometrik funksiyalar** – $F(\sin x, \cos x)$ ushbu ko'rinishdagi funksiyalar trigonometrik funksiyalar deb ataladi.

# Keys banki

**31-keys.** Masala o`rtaga tashlanadi: Ushbu

$$I = \int \frac{dx}{x\sqrt{4x^2 + 4x + 3}}$$

integral hisoblansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 4-5-amaliy mashg‘ulot

$1^0$. $$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{r_1}, \left(\frac{ax+b}{cx+d}\right)^{r_2}, \dots, \left(\frac{ax+b}{cx+d}\right)^{r_n}\right)dx \qquad (1)$$

**ko‘rinishidagi integrallarni hisoblash.** (1) integralda $r_1, r_2, \dots, r_n -$ ratsional sonlar, $a, b, c, d -$ haqiqiy sonlar bo‘lib, $ad - bc \neq 0$.

Agar $r_1, r_2, \dots, r_n -$ ratsional sonlarning umumiy maxraji $p$ bo‘lsa, (1) integralda

$$\frac{ax+b}{cx+d} = t^p$$

almashtirish bilan qaralayotgan integral ratsional funksiyaning integraliga keladi.

$2^0$. $$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx \qquad \left(a \neq 0,\ \ b^2 - 4ac \neq 0\right) \qquad (2)$$

**ko‘rinishidagi integralni hisoblash.**

Bu integral quyidagi uchta almashtirish (**Eyler almashtirishlari**) bilan ratsional funksiyaning integraliga keladi:

1) $a > 0,\ \ \sqrt{ax^2+bx+c} = \pm\ t \pm \sqrt{a}\cdot t$,

2) $c > 0,\ \ \sqrt{ax^2+bx+c} = \pm\ x\cdot t \pm \sqrt{c}$,

3) $b^2 - 4ac > 0,\ \ \sqrt{ax^2+bx+c} = \pm\ t\,(x - x_1)$, bunda $x_1 -$ soni $ax^2 + bx + c = 0$ tenglamaning ildizlaridan biri.

$3^0$. $\int x^m\left(a + bx^n\right)^p dx$ **ko‘rinishidagi integralni hisoblash, bunda $m, n, p -$ ratsional sonlar.** Bu integral:

1) $p -$ butun son bo‘lgan holda $x = t^s$ almashtirish bilan, bunda $s$ сони $m$ va $n$ ratsional sonlarning umumiy mahraji;

2) $\frac{m+1}{n} -$ butun son bo‘lgan holda $a + bx^n = t^s$ almashtirish bilan, bunda $s$ сони $p$ ratsional sonning mahraji;

3) $\frac{m+1}{n} + p -$ butun son bo‘lgan holda $ax^{-n} + b\ = t^s$ almashtirish

bilan, bunda $s$ **сони** $p$ ratsional sonning mahraji, ratsional funksiyaning integraliga keladi.

**9 – m i s o l .** ***Ushbu***

$$I=\int\frac{dx}{(1-x)\sqrt{1-x^2}}$$

***integral hisoblansin.***

◄Ravshanki,

$$(1-x)\sqrt{1-x^2}=(1-x)\sqrt{(1-x)(1+x)}=(1-x)(1+x)\cdot\sqrt{\frac{1-x}{1+x}}.$$

Demak, integral ostidagi funksiya $x$ va $\sqrt{\frac{1-x}{1+x}}$ larning ratsional funksiyasi bo‘ladi.

Bu integralda $\frac{1-x}{1+x}=t^2$ $\left(t=\sqrt{\frac{1-x}{1+x}}\right)$ almashtirish bajaramiz. Unda

$$x=\frac{1-t^2}{1+t^2},\quad dx=\frac{-4tdt}{(1+t^2)^2},\quad 1-x=\frac{2t^2}{1+t^2},\quad 1+x=\frac{2}{1+t^2}$$

bo‘lib,

$$I=-\int\frac{4tdt}{(1+t^2)^2\frac{2t^2}{1+t^2}\cdot\frac{2}{1+t^2}\cdot t}=-\int\frac{dt}{t^2}=\frac{1}{t}+C$$

bo‘ladi. Demak,

$$I=\sqrt{\frac{1+x}{1-x}}+C.$$ ►

**10 – m i s o l .** ***Ushbu***

$$I=\int\frac{dx}{x\sqrt{4x^2+4x+3}}$$

***integral hisoblansin.***

◄Bu integralda $a=4>0$ bo‘lgani uchun $\sqrt{4x^2+4x+3}=t-2x$ almashtirish bajaramiz. Unda

$$4x^2+4x+3=t^2-4tx+4x^2,\quad 4x+3=t^2-4xt,\quad x=\frac{t^3-3}{4(1+t)},\quad dx=\frac{t^2+2t+3}{4(1+t)^2}dt,$$

$$\sqrt{4x^2+4x+3}=t-\frac{2(t^2-3)}{4(1+t)}=\frac{t^2+2t+3}{2(1+t)}$$

bo‘lib,

$$I=\int\frac{\frac{t^2+2t+3}{4(1+t)^2}dt}{\frac{t^3-3}{4(1+t)}\cdot\frac{t^2+2t+3}{2(1+t)}}=2\int\frac{dt}{t^2-3}$$

bo‘ladi. Ravshanki,

$$2\int\frac{dt}{t^2-3}=\frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+C$$

Demak,

$$I=\frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+C=\frac{1}{\sqrt{3}}\ln\left|\frac{2x+\sqrt{4x^2+4x+3}-\sqrt{3}}{2x+\sqrt{4x^2+4x+3}+\sqrt{3}}\right|+C$$

.►

**11 – m i s o l .** ***Ushbu***

$$I=\int\frac{dx}{\sqrt[4]{1+x^4}}$$

***integral hisoblansin.***

◄Berilgan integralni quyidagicha

$$I=\int(1+x^4)^{-\frac{1}{4}}dx$$

yozib olamiz. Integral ostidagi ifoda uchun

$$a=b=1,\quad m=0,\quad n=4,\quad p=-\frac{1}{4}$$

bo‘lib, $\frac{m+1}{n}+p=\frac{1}{4}-\frac{1}{4}=0$ bo‘ladi. Bu integralda $1+x^{-4}=t^4$ deb topamiz:

$$x=(t^4-1)^{-\frac{1}{4}},\quad (1+x^4)^{-\frac{1}{4}}=\frac{(t^4-1)^{\frac{1}{4}}}{t},\quad dx=-t^3(t^4-1)^{-\frac{5}{4}}dt$$

.

Natijada,

$$I=\int\frac{(t^4-1)^{\frac{1}{4}}}{t}\cdot\left(-t^3(t^4-1)^{-\frac{5}{4}}\right)dt=-\int\frac{t^2dt}{t^4-1}=-\frac{1}{2}\left[\int\frac{dt}{t^2+1}+\int\frac{dt}{t^2-1}\right]$$

bo‘ladi. Keyingi integralni hisoblab,

$$I=\frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|-\frac{1}{2}\text{arctg}t+C=\frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+4}{\sqrt[4]{1+x^4}-4}-\frac{1}{2}\text{arctg}\frac{\sqrt[4]{1+x^4}}{x}+C$$

bo‘lishini topamiz.►

**Integral ostidagi funksiyani ratsional funksiyaga keltirish yo‘li bilan quyidagi integrallar hisoblansin.**

1. $\int\frac{dx}{1+\sqrt{x}}$
2. $\int\frac{\sqrt{x}dx}{1+\sqrt{x}}$
3. $\int\frac{1-2\sqrt{x}}{1+2\sqrt{x}}dx$
4. $\int\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx$
5. $\int\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}dx$
6. $\int\sqrt[3]{\frac{x+1}{x-1}}dx$
7. $\int\frac{xdx}{\sqrt[4]{x^3(4-x)}}$
8. $\int\frac{\sqrt{x+4}}{x}dx$
9. $\int\frac{dx}{3x+\sqrt[3]{x^2}}$
10. $\int\sqrt[4]{x-2x}\,dx$
11. $\int\frac{\sqrt[3]{x+2}}{x+\sqrt[3]{x+2}}x\,dx$
12. $\int x\sqrt{\frac{x-1}{x+1}}dx$
13. $\int\sqrt[5]{\frac{x}{x+1}}\cdot\frac{dx}{x^3}$
14. $\int\frac{dx}{\sqrt[6]{(x-7)^7(x-5)^5}}$
15. $\int\frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}}\quad(a\neq b)$
16. $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x^2}}$
17. $\int\frac{\sqrt[6]{x}dx}{1+\sqrt[3]{x}}$
18. $\int\frac{dx}{\sqrt[3]{4x^2+4x+1}-\sqrt{2x+1}}$
19. $\int\frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}dx$
20. $\int\frac{dx}{\left(1+\sqrt[4]{x}\right)^3\sqrt{x}}$
21. $\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$

**Oddiy kvadratik irratsionalliklar qatnashgan integrallar hisoblansin.**

22. $\int\sqrt{3-4x+4x^2}dx$
23. $\int x\sqrt{x^2+2x+2}dx$

24. $\int x^2\sqrt{x^2+4}\,dx$

25. $\int \frac{x^2}{\sqrt{1+x+x^2}}dx$

26. $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

27. $\int \frac{dx}{(1-x)^2\sqrt{1-x^2}}$

28. $\int \frac{\sqrt{x^2+2x+2}}{x}dx$

29. $\int \frac{xdx}{(1+x)\sqrt{1-x-x^2}}$

30. $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}dx$

**Eyler almashtirishlaridan foydalanib quyidagi integrallar hisoblansin.**

31. $\int \frac{dx}{x+\sqrt{x^2+x+1}}$

32. $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$

33. $\int x\sqrt{x^2-2x+2}dx$

34. $\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx$

35. $\int \frac{dx}{\left[1+\sqrt{x(1+x)}\right]^2}$

**Binomial differensiallarni integrallashdan foydalanib quyidagi integrallar hisoblansin.**

36. $\int \sqrt{x^3+x^4}dx$

37. $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx$

38. $\int \frac{x^5dx}{\sqrt{1-x^2}}$

39. $\int \frac{dx}{\sqrt[3]{1+x^3}}$

40. $\int \frac{dx}{\sqrt[4]{1+x^4}}$

41. $\int \frac{dx}{x^3\sqrt[5]{1+\frac{1}{x}}}$

42. $\int \sqrt[3]{3x-x^3}dx$

43. $\int x^2\sqrt[3]{(x+1)^2}dx$

44. $\int \sqrt[3]{1+\sqrt[4]{x}}dx$

45. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$

46. $\int \frac{\sqrt[3]{x}}{\sqrt{1+\sqrt[3]{x}}}dx$

47. $\int \frac{dx}{x\sqrt[6]{x^6+1}}$

48. $\int \frac{x\,dx}{\sqrt{1+\sqrt[3]{x^2}}}$

49. $\int \frac{dx}{x^2\sqrt[3]{\left(2+x^3\right)^5}}$

50. $\int \frac{dx}{x^3\sqrt[5]{2-x^3}}$

51. $\int \sqrt[3]{x-x^3}\,dx$

**Turli usullardan foydalanib quyidagi integrallar hisoblansin.**

52. $\int \frac{dx}{\sqrt{x^2+1}-\sqrt{x^2-1}}$

53. $\int \frac{dx}{\sqrt{2+\sqrt{1-x}+\sqrt{1+x}}}$

54. $\int \frac{x+\sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}}dx$

55. $\int \frac{\left(x^2-1\right)dx}{\left(x^2+1\right)\sqrt{x^4+1}}$

56. $\int \frac{\left(x^2+1\right)dx}{\left(x^2-1\right)\sqrt{x^4+1}}$

57. $\int \frac{dx}{x\sqrt{x^4+2x^2-1}}$

**$1^0$. $\int R(\sin x, \cos x)dx$ кўринишидаги интегралларни ҳисоблаш.**

Бундай интегралларда

$$tg\frac{x}{2}=t \qquad (-\pi < x < \pi)$$

алмаштиришларни бажарилса, у ҳолда

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad x = 2arctgt, \quad dx = \frac{2dt}{1+t^2}$$

бўлиб, қаралаётган интеграл **рационал функциянинг интегралига** келади.

**$2^0$. $\int \sin^m x \cos^n x\,dx$ кўринишидаги интегралларни ҳисоблаш.**

Бундай интеграллар $m$ тоқ бўлганда $\cos x = t$ алмаштириш ёрдамида, $n$ тоқ бўлганда эса $\sin x = t$ алмаштириш ёрдамида ҳисобланади.

Агар $m$ ва $n$ лар жуфт сонлар бўлса, унда $\sin^2 x$ ва $\cos^2 x$ ни мос равишда

$$\frac{1-\cos 2x}{2}, \qquad \frac{1+\cos 2x}{2}$$

га алмаштириш лозим.

$3^0$. $\int \sin mx \cdot \cos nx\, dx$, $\int \sin mx \cdot \sin nx\, dx$, $\int \cos mx \cdot \cos nx\, dx$ **кўринишидаги интегралларни ҳисоблаш.** Бундай интегралларни қуйидаги

$$\sin mx \cdot \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\cos mx \cdot \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

$$\sin mx \cdot \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

формулалардан фойдаланиш керак.

**12 – м и с о л .** ***Ушбу***

$$I = \int \frac{1+\sin x}{\sin x(1+\cos x)}dx$$

***интеграл ҳисоблансин.***

◄Бу интегралда $tg\frac{x}{2} = t$ алмаштириш бажарамиз. Унда

$$I = \int \frac{\left(1+\frac{2t}{1+t^2}\right)\frac{2t}{1+t^2}dt}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)} = \frac{1}{2}\int\left(\frac{1}{t}+t+2\right)dt$$

бўлиб,

$$I = \frac{1}{2}\left(\ln|t| + \frac{1}{2}t^2 + 2t\right) + C = \frac{1}{2}\left(\ln\left|tg\frac{x}{2}\right| + \frac{1}{2}tg^2\frac{x}{2} + 2tg\frac{x}{2}\right) + C$$

бўлади.►

**13 – м и с о л .** ***Ушбу***

$$I = \int \sin^3 x \cos^5 x\, dx$$

***интеграл ҳисоблансин.***

◄Бу интегралда $\cos x = t$ алмаштириш бажариб, уни ҳисоблаймиз:

$$I = \int (1-\cos^2 x)\cos^5 x \sin x dx = [\cos x = t, \quad -\sin x dx = dt] =$$

$$= -\int (1-t^2)t^5 dt = \frac{1}{8}t^8 - \frac{1}{6}t^6 + C = \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C.$$ ►

**14 – м и с о л .** ***Ушбу***

$$I = \int \sin^3 2x \cos^2 3x\, dx$$

***интеграл ҳисоблансин.***

◄Бу интеграл қуйидагича ҳисобланади:

$$I=\int \sin 2x\sin^2 2x\cos^2 3x\,dx=\int \sin 2x\frac{1-\cos 4x}{2}\cdot\frac{1+\cos 6x}{2}dx=$$

$$=\frac{1}{4}\int \sin 2x(1-\cos 4x)(1+\cos 6x)dx=\frac{1}{4}\int(\sin 2x-\sin 2x\cdot\cos 4x)(1+\cos 6x)dx=$$

$$=\frac{1}{4}\int\left[\sin 2x-\frac{1}{2}(\sin(-2x)+\sin 6x)\right](1+\cos 6x)dx=$$

$$=\frac{1}{8}\int(3\sin 2x-\sin 6x)(1+\cos 6x)dx=$$

$$=\frac{1}{8}\int\left[3\sin 2x-\frac{3}{2}\sin 4x+\frac{3}{2}\sin 8x-\sin 6x-\frac{1}{12}\sin 12x\right]dx=$$

$$=-\frac{3}{16}\cos 2x+\frac{3}{64}\cos 4x-\frac{3}{128}\cos 8x+\frac{1}{48}\cos 6x+\frac{1}{192}\cos 12x+C.$$

►

**Ушбу $\int \sin^m x\cos^n x\,dx$ кўринишидаги интегралларни ҳисоблашдан фойдаланиб қуйидаги интеграллар ҳисоблансин.**

58. $\int \sin^2 x\cos^4 x\,dx$

59. $\int \sin^5 x\cos^5 x\,dx$

60. $\int \sin^6 x\,dx$

61. $\int \frac{dx}{\cos^3 x}$

62. $\int \frac{dx}{\sin^3 x}$

63. $\int \frac{dx}{\cos^4 x}$

64. $\int \frac{dx}{\sin^4 x}$

65. $\int \frac{dx}{\sin^4 x\cos^4 x}$

66. $\int \frac{dx}{\sin^3 x\cos^5 x}$

67. $\int \frac{dx}{\sin x\cos^4 x}$

68. $\int \cos^3 x\cdot\cos 2x\,dx$

69. $\int \cos^5 2x\cdot\sin^7 2x\,dx$

70. $\int \frac{dx}{\sqrt{\sin^3 x\cdot\cos^5 x}}$

71. $\int \frac{dx}{\sqrt[3]{tgx}}$

72. $\int \frac{\cos^2 x}{\sin 4x}dx$

73. $\int \frac{\sin 3x}{\cos^4 x}dx$

**74.** $\int \frac{\cos 3x}{\sin^5 x}dx$

# Test

| | | | | |
|---|---|---|---|---|
| $\int \frac{dx}{\sqrt{x+1}} = ?$ | $2\sqrt{x+1}+C$ | $\ln\lvert x+1\rvert + C$ | $\ln(x+1)+C$ | $2\ln(x+1)+C$ |
| $\int \frac{dx}{\sqrt{1-x^2}} = ?$ | $\arcsin x + C$ | $\ln\lvert x^2+1\rvert + C$ | $\ln(x^2+1)+C$ | $2\ln(x^2+1)+C$ |
| $\int \frac{dx}{x^2+9} = ?$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+9\rvert + C$ | $\ln\lvert x^2+3\rvert + C$ | $2\ln(x^2+3)+C$ |
| $\int \frac{2xdx}{x^2+9} = ?$ | $\ln\lvert x^2+9\rvert + C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $2\ln(x^2+3)+C$ | $\ln\lvert x^2+3\rvert + C$ |
| $\int \frac{x^2dx}{x^3+9} = ?$ | $\frac{1}{3}\ln\lvert x^3+9\rvert + C$ | $2\ln(x^2+3)+C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+3\rvert + C$ |
| $\int \frac{dx}{x^2-1} = ?$ | $\frac{1}{2}\ln\frac{\lvert x+1\rvert}{\lvert x-1\rvert}+C$ | $\ln\lvert x^2-1\rvert + C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+9\rvert + C$ |
| $\int \frac{xdx}{x^2-1} = ?$ | $\frac{1}{2}\ln\lvert x^2-1\rvert + C$ | $\ln\lvert x^2-1\rvert + C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+1\rvert + C$ |
| $\int \frac{dx}{x^2-a^2} = ?$ | $\frac{1}{2a}\ln\frac{\lvert x+a\rvert}{\lvert x-a\rvert}+C$ | $\frac{1}{2a}\ln\frac{\lvert x+1\rvert}{\lvert x-1\rvert}+C$ | $\frac{1}{3}arctg\frac{x}{3}+C$ | $\ln\lvert x^2+9\rvert + C$ |
| $\int \frac{dx}{x^2+a^2} = ?$ | $\frac{1}{a}arctg\frac{x}{a}+C$ | $arctgx+C$ | $\ln\lvert x+1\rvert + C$ | $\ln\lvert x^2+a^2\rvert + C$ |
| $\int \frac{dx}{x^2} = ?$ | $-\frac{1}{x}+C$ | $\ln\lvert x^2\rvert + C$ | $arctgx+C$ | $\ln\lvert x^2+a^2\rvert + C$ |

# Mavzu. Aniq integral tushunchasi

## 6-ma'ruza

**Reja**

$1^0$. Aniq integral ta'rifi.

$2^0$. Darbu yig'indilari.

### $1^0$. Aniq integral ta'rifi.

Biror $[a\,,b]\subset R$ segment berilgan bo'lsin . Uning ushbu

$$a_0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

munosabatda bo'lgan chekli sondagi ixtiyoriy $x_0, x_1, x_2 \ldots, x_{n-1}, x_n$ nuqtalari sistemasini olaylik . Agar $A_i = [x_{i-1},\, x_i]$, $i = 1,2,3\ldots n$ deb belgilasak , u holda ravshanki ,

1) $A_1 \cup A_2 \cup \ldots \cup A_n = [a, b]$;

2) $A_k \cap A_j = \varnothing$, $(k\,, j = 1, 2, \ldots n)$

Mazkur kursning 1—bobidagi to'plamni bo'laklash tushinchasi ta`rifiga binoan $\{A_1, A_2, \ldots, A_n\}$ sistema $[a\,, b]$ da bo'laklash bajargan bo'ladi , va aksincha, agar bizga $[a\,, b]$ segmentning biror chekli $\{B_1, B_2, \ldots, B_m\}$ bo'laklashi berilgan bo'lsa , u ushbu

$$a = y_0 < y_1 < y_2 < \ldots < y_m = b$$

munosabatda bo'lgan chekli sondagi $y_0, y_1, y_2, \ldots, y_{m-1}, y_m$ nuqtalar sistemasini aniqlaydi . Binobarin , biz to'plamni bo'laklash ta`rifiga ekvivalent bo'lgan quyidagi ta`rifni kirita olamiz .

**1—ta`rif .** $[a\,, b]$ segmentning ushbu

$$a_0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

munosabatda bo'lgan ixtiyoriy chekli sondagi $x_0, x_1, x_2 \ldots, x_{n-1}, x_n$ nuqtalari sistemasi $[a\,, b]$ *segmentda bo'laklash bajaradi deyiladi* .
Uni

$$P = \{x_0, x_1, x_2 \ldots, x_{n-1}, x_n\}$$

kabi belgilanadi .

Har bir $x_k$ $(k = 0,1,\ldots, n)$ nuqta bo'laklashning bo'luvchi nuqtasi, $[x_k\,, x_{k+1}]$ $(k = 0,1,\ldots, n-1)$ segment esa $P$ bo'laklashning oralig'i deyiladi.

$P$ bo'laklash oraliqlari uzunligi $\Delta x_k = x_{k+1} - x_k$ $(k = 0,1,\ldots, n-1)$ larning eng kattasi , ya`ni ushbu

$$\lambda_P = \max\{\Delta x_k\} = \max\{\Delta x_0, \Delta x_1, \Delta x_2 \ldots, \Delta x_{n-1}\}$$

miqdor $P$ *bo'laklashning diametri* deb ataladi. $[a,b]$ segment berilgan holda bu segmentni turli usullar bilan istalgan sondagi bo'laklashlarni tuzish mumkin ekan. Bu bo'laklashlardan iborat to'plamni $F$ bilan belgilaymiz : $F = \{P\}$.

**$2^0$ .** ***([1] 9.4 The Cauchy integral)*** **Integral yig'indi .** $[a,b]$ segmentda $f(x)$ funksiya aniqlangan bo'lsin . Shu segmentni

$$P = \{x_0, x_1, x_2 \ldots, x_k, \ldots, x_n\} \in F$$

bo'laklashi va bu bo'laklashning har bir $[x_k, x_{k+1}]$ $(k = 0,1,\ldots,n-1)$ oralig'ida ixtiyoriy $\xi_k$ $(\xi_k \in [x_k, x_{k+1}])$ nuqta olamiz . Berilgan funksiyaning $\xi_k$ nuqta-dagi qiymati $f(\xi_k)$ ni $\Delta x_k = x_{k+1} - x_k$ ga ko'paytirib, quyidagi yig'indini tuzamiz:

$$\sigma = f(\xi_0)\Delta x_0 + f(\xi_1)\Delta x_1 + \ldots + f(\xi_k)\Delta x_k + \ldots + f(\xi_{n-1})\Delta x_{n-1} = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k .$$

**2—ta`rif .** Ushbu

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k . \qquad (9.1)$$

yig'indi $f(x)$ funksiyaning *integral yig'indisi* deb ataladi .

Masalan, 1) $f(x) = x$ funksiyaning $[a,b]$ segmentdagi integral yig'indisi

$$\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k = \sum_{k=0}^{n-1} \xi_k \Delta x_k$$

bo'ladi , bunda

$$x_k \le \xi_k \le x_{k-1} \qquad (k = 0,1,\ldots,n-1)\,.$$

**3)** Dirixle funksiyasi

$$D(x) = \begin{cases} 1, & \text{agar } x \in [a,b] \text{ ratsional son bo'lsa}, \\ 0, & \text{agar } x \in [a,b] \text{ irratsional son bo'lsa}. \end{cases}$$

ning integral yig'indisi, masalan, barcha $\xi_k$ lar faqatgina ratsional son, yoki irratsional son deb qarasak

$$\sum_{k=0}^{n=1} D(\xi_k)\Delta x_k = \begin{cases} b-a, & \text{agar barcha } \xi_k \text{ ratsional son bo'lsa}, \\ 0, & \text{agar barcha } \xi_k \text{ irratsional son bo'lsa}. \end{cases}$$

ko'rinishga ega bo'ladi.

Ravshanki, $f(x)$ funksiyaning integral yig'indisi $\sigma$ :a) $f(x)$ funksiyaga, b) $[a,b]$ segmentni bo'laklash usuliga, v) har bir $[x_k, x_{k+1}]$ segmentdan olingan $\xi_k$ nuqtalarga bog'liq bo'ladi.

**$3^0$. Aniq integral ta'rifi.** $f(x)$ funksiya $[a,b]$ segmentda aniqlangan bo'lsin. $[a,b]$ segmentning shunday

$$P_1, P_2, \ldots, P_m, \ldots \qquad (9.2)$$

$(P_m \in F,\ m=1,2,...)$ bo'laklashlarni qaraymizki ularni mos diametrlaridan tashkil topgan

$$\lambda_{p_1}, \lambda_{p_2}, \lambda_{p_3}, \dots, \lambda_{p_m}, \dots$$

ketma—ketlik nolga intilsin: $\lambda_{p_m} \to 0$.

Bunday $P_m(m=1,2,3,\dots)$ bo'laklashlarga nisbatan $f(x)$ funksiyaning integral yig'indilarini tuzamiz. Natijada $[a,b]$ segmentni $(9.2)$ bo'laklashlarga mos $f(x)$ funksiyaning integral yig'indilari qiymatlaridan iborat quydagi:

$$\sigma_1, \sigma_2, \dots, \sigma_m, \dots$$

ketma—ketlik hosil bo'ladi. Ravshanki bu ketma—ketlikning har bir hadi $\xi_k$ huqtalarga bog'liqdir.

**3—ta'rif.** Agar $[a,b]$ segmentni har qanday $(9.2)$ bo'laklashlar ketma-ketligi $\{P_m\}$ olinganda ham unga mos integral yig'indi qiymatlaridan iborat $\{\sigma_m\}$ ketma—ketlik $\xi_k$ nuqtalarning tanlab olinishiga bog'liq bo'lmagan ravishda hamma vaqt bitta $J$ songa intilsa, bu $J$ son $\sigma$ yig'indining $\lambda_p \to 0$ dagi limiti deb ataladi. U

$$\lim_{\lambda_P \to 0} \sigma = \lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k$$

kabi belgilanadi.

Yig'indi limitini quyidagicha ham ta'riflash mumkin.

**4–ta'rif.** Agar $\forall \varepsilon > 0$ son olingnda ham shunday $\delta > 0$ son topilsaki, $[a,b]$ segmentni diametri $\lambda_p < \delta$, bo'lgan har qanday $P$ bo'laklash uchun tuzilgan $\sigma$ yig'indi ixtiyoriy $\xi_k$ nuqtalarda

$$|\sigma - J| < \varepsilon$$

tengsizliklarni qanoatlantirsa, $J$ son $\sigma$ yig'indining $\lambda_p \to 0$ dagi limiti deb ataladi.

**5—ta'rif**. Agar $\lambda_p \to 0$ da $f(x)$ funksiyaning integral yig'indisi $(9.1)$ chekli limitga ega bo'lsa, u holda $f(x)$ funksiya $[a,b]$ segmentda integrallanuvchi deyiladi, $\sigma$ yig'indining chekli limiti $J$ esa $f(x)$ funksiyaning $[a,b]$ segmentdagi aniq integrali deb ataladi. Funksiyaning aniq integrali

$$\int_a^b f(x)dx$$

kabi belgilanadi.

Demak,

$$\int_a^b f(x)dx = \lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k \ .$$

Bunda $a$ son integralning quyi chegarasi, $b$ son esa integralning yuqori chegarasi, $[a,b]$ segment integrallash oralig'i deb ataladi.

Agar $\lambda_p \to 0$ da yig'indining limiti mavjud bo'lmasa yoki uning limiti cheksiz bo'lsa, u holda funksiya $[a,b]$ segmentda integrallanmaydi deyiladi.

**9.1–misol.** $f(x) = C = const$ funksiyaning $[a,b]$ segmentdagi integrali hisoblansin.

◀ $[a,b]$ segmentni ixtiyoriy

$$P = \{x_0, x_1, \ldots, x_n\} \qquad (a = x_0 < x_1 < \ldots < x_n = b)$$

bo'laklashni olib, $f(x) = C$ funksiyaning integral yig'indisini topamiz:

$$\sigma = \sum_{k=0}^{n-1} C \cdot \Delta x_k = C\sum_{k=0}^{n-1} \Delta x_k = C[(x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})] = C(x_n - x_0) = C(b-a)$$

Ravshanki,

$$\lim_{\lambda_P \to 0} \sigma = \lim_{\lambda_P \to 0} C(b-a) = C(b-a).$$

Demak,

$$\int_a^b C dx = C(b-a) . \ ►$$

Xususan, $f(x) = 1$ bo'lganda quyidagiga egamiz:

$$\int_a^b 1 \cdot dx = \int_a^b dx = b - a .$$

**9.2—misol.** Ushbu $f(x) = x$ funksiyaning $[a,b]$ segmentdagi integrali hisoblansin.

◀ Ma'lumki, $[a,b]$ segmentda $f(x) = x$ funksiyaning integral yig'indisi

$$\sigma = \sum_{k=0}^{n-1} \xi_k \Delta x_k$$

bo'lib, bunda $\Delta x_k = x_{k+1} - x_k$ va

$$x_k \le \xi_k \le x_{k+1} .$$

Bu tengsizlikni $\Delta x_k > 0$ ga ko'paytirib topamiz:

$$x_k \cdot \Delta x_k \le \xi_k \cdot \Delta x_k \le x_{k+1} \cdot \Delta x_k \qquad (k = 0, 1, \ldots, n-1) .$$

Keyingi tengsizliklardan esa

$$\sum_{k=0}^{n-1} x_k \Delta x_k \le \sum_{k=0}^{n-1} \xi_k \Delta x_k \le \sum_{k=0}^{n-1} x_{k+1} \Delta x_k$$

tengsizliklar kelib chiqadi.

Demak,

$$\sum_{k=0}^{n-1} x_k \Delta x_k \le \sigma \le \sum_{k=0}^{n-1} x_{k+1} \Delta x_k .$$

Endi $\sum_{k=0}^{n-1} x_k \Delta x_k$ va $\sum_{k=0}^{n-1} x_{k+1} \Delta x_k$ yig'indilarni quyidagicha o'zgartirib yozib olamiz:

$$\sum_{k=0}^{n-1} x_k \Delta x_k = \sum_{k=0}^{n-1} x_k (x_{k+1} - x_k) = \frac{1}{2}\sum_{k=0}^{n-1} (x_{k+1}^2 - x_k^2) - \frac{1}{2}\sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 =$$

$$= \frac{1}{2}(x_n^2 - x_0^2) - \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 = \frac{b^2 - a^2}{2} - \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 .$$

Agar $x_{k+1} = x_k + \Delta x_k$ ekanini e'tiborga olsak, u holda

$$\sum_{k=0}^{n-1} x_{k+1}\Delta x_k = \sum_{k=0}^{n-1}(x_k + \Delta x_k)\Delta x_k = \sum_{k=0}^{n-1} x_k\Delta x_k + \sum_{k=0}^{n-1}\Delta x_k^2 = \frac{b^2 - a^2}{2} + \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 .$$

Demak,

$$\frac{b^2 - a^2}{2} - \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 \le \sigma \le \frac{b^2 - a^2}{2} + \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 .$$

Bu munosabatdan

$$\left|\sigma - \frac{b^2 - a^2}{2}\right| \le \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2$$

tengsizlik kelib chiqadi. So'ngra $\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2$ uchun

$$\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 \le \lambda_p \frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k \le \frac{b-a}{2}\lambda_p$$

(bunda $\lambda_p = \max_k\{\Delta x_k\}$) bo'lishidan $\lambda_p \to 0$ da

$$\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 \to 0$$

bo'lishini topamiz.

Demak,

$$\lim_{\lambda_p \to 0}\sigma = \frac{b^2 - a^2}{2}.$$

Bu esa ta'rifga ko'ra

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}$$

ekanini bildiradi. ►

**9.3—misol.** $[a,b]$ segmentda Dirixle funksiyasi uchun aniq integral mavjud emasligi ko'rsatilsin.

◄ Dirixle funksiyasi $D(x)$ uchun integral yig'idini hususan quyidagicha bo'lishini ko'rgan edik:

$$\sigma = \begin{cases} b-a, & \text{agar barcha } \xi_k \text{ ratsional son bo'lsa,} \\ 0, & \text{agar barcha } \xi_k \text{ irratsional son bo'lsa.} \end{cases}$$

Ravshanki, $\lambda_p \to 0$ da $\sigma$ yig'indi limitga ega emas. Demak, Dirixle funksiyasi $[a,b]$ segmentda integrallanmaydi. ►

Odatda, yuqorida keltirilgan aniq integral *Riman integrali*, integral yig'indini Riman yig'indisi deyiladi.

**1—eslatma.** Agar $f(x)$ funksiya $[a,b]$ segmentda chegaralanmagan bo'lsa, u shu segmentda integrallanmaydi.

**2$^0$. Darbu yig'indilari.** $f(x)$ funksiya $[a,b]$ oraliqda aniqlangan bo'lib, shu oraliqda chegaralangan bo'lsin:

$$m \le f(x) \le M, \qquad x \in [a,b].$$

$[a,b]$ oraliqda biror

$$P = \{x_0, x_2, \ldots, x_n\} \in F$$

$(a = x_0 < x_1 < \ldots < x_n = b)$ bo'laklashni olaylik. Bu funksiyaning aniq chegaralari

$$m_k = \inf\{f(x)\}, \quad x \in [x_k, x_{k+1}],$$
$$M_k = \sup\{f(x)\}, \quad x \in [x_k, x_{k+1}]$$

mavjud (2—bob, 6—§).

Ravshanki, ixtiyoriy $\xi_k \in [x_k, x_{k+1}]$ uchun

$$m_k \le f(\xi_k) \le M_k \qquad (9.3)$$

bo'ladi. Endi $m_k$ va $M_k$ sonlarni $[x_k, x_{k+1}]$ oraliqning uzunligi $\Delta x_k = x_{k+1} - x_k$ $(k = 0,\ldots, n-1)$ ga ko'paytirib quyidagi

$$\sum_{k=0}^{n-1} m_k \Delta x_k = m_0 \Delta x_0 + m_1 \Delta x_1 + \ldots + m_k \Delta x_k + \ldots + m_{n-1} \Delta x_{n-1},$$

$$\sum_{k=0}^{n-1} M_k \Delta x_k = M_0 \Delta x_0 + M_1 \Delta x_1 + \ldots + M_k \Delta x_k + \ldots + M_{n-1} \Delta x_{n-1}$$

yig'inilarni tuzamiz.

**6—ta'rif.** ***([1], 9.4 The Cauchy integral, 318-bet)*** Ushbu

$$s = \sum_{k=0}^{n-1} m_k \Delta x_k, \qquad S = \sum_{k=0}^{n-1} M_k \Delta x_k$$

yig'indilar mos ravishda *Darbuning quyi* hamda *yuqori yig'indilari* deb ataladi.

Darbu yig'indilari, funksiyaga hamda $P$ bo'laklashga bog'liq:

$$s = s(P), \ S = S(P)$$

va har doim

$$s(P) \le S(P)$$

bo'ladi.

(9.3) tengsizliklarni $\Delta x_k$ ga ko'paytirib topamiz:

$$m_k \Delta x_k \le f(\xi_k) \Delta x_k \le M_k x_k \qquad (k = 0,1,2\ldots, n-1).$$

Keyingi tengsizliklardan esa

$$\sum_{k=0}^{n-1} m_k \Delta x_k \le \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k \le \sum_{k=0}^{n-1} M_k \Delta x_k$$

tengsizliklar kelib chiqadi. Demak,

$$s(P) \le \sigma \le S(P)$$

Shunday qilib, $f(x)$ funksiyaning integral yig'indisi har doim uning Darbu yig'indilari orasida bo'lar ekan.

(9.3) munosabatdan yana bitta xulosa chiqarish mumkin: $\xi_k$ nuqtani tanlab olish hisobiga $f(\xi_k)$ ni $m_k$ shuningdek, $M_k$ qiymatlarga har qancha yaqin keltirish mumkin. Bundan esa Darbuning quyi va yuqori yig'indilari berilgan bo'laklash uchun integral yig'indining mos ravishda aniq quyi hamda aniq yuqori chegaralari bo'lishi kelib chiqadi:

$$s = \inf_{\xi_k} \{\sigma\}, \quad S = \sup_{\xi_k} \{\sigma\}.$$

Aniq chegaralar hossalaridan foydalanib topamiz:

$$m \le m_k\ , \qquad M_k \le M \qquad\qquad (k = 0,1,2,\ldots,n-1)\ .$$

Ravshanki,

$$\sum_{k=0}^{n-1} \Delta x_k = b - a\ .$$

Natijada

$$s(P) = \sum_{k=0}^{n-1} m_k \Delta x_k \ge m \sum_{k=0}^{n-1} \Delta x_k = m(b-a),$$

$$S(P) = \sum_{k=0}^{n-1} M_k \Delta x_k \le M \sum_{k=0}^{n-1} \Delta x_k = M(b-a)$$

bo'ladi.

Demak, $\forall P \in F$ uchun quyidagi

$$m(b-a) \le s(P) \le S(P) \le M(b-a) \qquad (9.4)$$

tengsizliklar o'rinli bo'ladi. Bu esa Darbu yig'indilarining chegaralanganligini bildiradi.

## Adabiyotlar


1. **Claudio Canute, Anita Tabacco,** *Mathematical Analysis I,* Springer-Verlag Italia, Milan 2008.
2. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'rizalar, I q.* T. "Voris-nashriyot", 2010.
3. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
4. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

## Glossariy

1. **Aniq integral** – funksiyadan biror chekli segmentda olingan integral.
2. **Integral yig‘indi** – $\sum_{k=0}^{n-1} f(\xi_k)\cdot \Delta x_k$ ushbu yig‘indiga integral yg’indi deyiladi.

## Keys banki

**32-keys.** Masala o`rtaga tashlanadi: Ushbu $f(x)=e^x$ funksiyaning [0,1] segmentdagi integral yig‘indisi topilsin, bunda [0,1] segmentni $n$ ta teng bo‘lakka bo‘lib, har bir bo‘lakda $\xi_k$, $(k=0,1,...,n-1)$ nuqta sifatida bo‘lakning chap chekkasi olinsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 6-amaliy mashg'ulot

$1^0$. **Funksiyaning integral va Darbu yig'indilari.** Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan bo'lsin. $[a,b]$ segmentning $a = x_0, x_1, x_2, \cdots, x_k, x_{k+1}, \cdots, x_n = b$ $(x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_n)$ nuqtalari uni $[x_k, x_{k+1}]$ $(k = 0,1,\ldots,n-1)$ bo'laklarga ajratadi. Bu nuqtalar sistemasi $[a,b]$ ni bo'laklash deyiladi va u $P = \{x_0, x_1, \cdots x_n\}$ kabi belgilanadi. Ushbu

$$\lambda_p = \max_k \{\Delta x_k\}, \Delta x_k = x_{k+1} - x_k \quad (k = 0,1,\cdots,n-1)$$

miqdor $P$ bo'laklashning **diametri** deyiladi.

Har bir $[x_k, x_{k+1}]$ da ihtiyoriy $\xi_k$ nuqtani:

$$\xi_k \in [x_k, x_{k+1}] \quad (k = 0,1,\cdots,n-1)$$

olib, quyidagi

$$\sigma = f(\xi_0)\cdot \Delta x_0 + f(\xi_1)\cdot \Delta x_1 + \cdots + f(\xi_{n-1})\cdot \Delta x_{n-1} = \sum_{k=0}^{n-1} f(\xi_k)\cdot \Delta x_k \qquad (1)$$

yig'indini tuzamiz. **(1)** yig'indi $f(x)$ funksiyaning **integral (Riman) yig'indisi** deyiladi. Ravshanki,

$$\sigma = \sigma_p(f : \xi_k)$$

Faraz qilaylik, $f(x)$ funksiya $[a,b]$ da chegaralangan bo'lsin. Unda

$$\begin{aligned} m_k &= \inf\{f(x)\}, \quad x \in [x_k, x_{k+1}] \\ M_k &= \sup\{f(x)\}, \quad x \in [x_k, x_{k+1}] \end{aligned} \qquad (k = 0,1,\ldots,n-1)$$

miqdorlar mavjud bo'ladi. Ushbu

$$s = \sum_{k=0}^{n-1} m_k \cdot \Delta x_k, \qquad S = \sum_{k=0}^{n-1} M_k \cdot \Delta x_k$$

yig'indilar mos ravishda **Darbuning quyi** va **yuqori** yig'indilari deyiladi.

Ravshanki,

$$s = s_p(f), \quad S = S_p(f).$$

$2^0$. **Aniq integral ta'rifi.**

**1-ta'rif.** *Agar $\forall \varepsilon > 0$ olinganda ham shunday $\delta = \delta(\varepsilon) > 0$ son topilsaki, $[a,b]$ segmentning diametri $\lambda_p < \delta$ bo'lgan har qanday $P$ bo'laklashi hamda har bir $[x_k, x_{k+1}]$ dan olingan ihtiyoriy $\xi_k$ uchun tuzilgan $\sigma$ integral yig'indisi ushbu*

$$|\sigma - \Im| < \varepsilon$$

*tengsizlikni qanoatlantirsa, $\Im$ son $\sigma$ yig'indining $\lambda_p \to 0$ dagi limiti deyiladi. Bu holda $f(x)$ funksiya $[a,b]$ da integrallanuvchi, $\Im$ son esa $f(x)$ funksiyaning a n i q i n t e g r a l i deyiladi. Uni*

$$\int_a^b f(x)dx$$

***kabi belgilanadi.*** Demak,

$$\int_a^b f(x)dx = \lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

Aytaylik, $\mathcal{P}$ – $[a,b]$ segmentning bo‘laklashlari to‘plami bo‘lsin: $\mathcal{P} = \{P\}$. Har bir $P \in \mathcal{P}$ bo‘laklashga nisbatan Darbu yig‘indilari $s(p)$ va $S(p)$ ni tuzib, ushbu

$$\{s(p)\}, \{S(p)\}$$

to‘plamlarni hosil qilamiz.

**2-ta’rif.** *Agar*

$$\operatorname{Sup}\{s(p)\} = \inf\{S(p)\}$$

***bo‘lsa,*** $f(x)$ ***funksiya*** $[a,b]$ ***da i n t e g r a l l a n u v ch i , bu tenglikning umumiy qiymatiga esa*** $f(x)$ ***funksiyaning a n i q i n t e g r a l i deyiladi.*** Demak,

$$\int_a^b f(x)dx = \inf\{S(p)\} = \sup\{s(p)\}$$

**1 – m i s o l .** ***Ushbu*** $f(x) = e^x$ ***funksiyaning*** $[0,1]$ ***segmentdagi integral yig‘indisi topilsin, bunda*** $[0,1]$ ***segmentni*** $n$ ***ta teng bo‘lakka bo‘lib, har bir bo‘lakda*** $\xi_k$ $(k = 0,1,\cdots,n-1)$ ***nuqta sifatida bo‘lakning chap chekkasi olinsin.***

◄Bu holda $[0,1]$ segmentning bo‘laklashi

$$P = \{x_0, x_1, x_2, \cdots, x_n\} = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{k}{n}, \cdots, 1\right\}$$

bo‘lib,

$$\Delta x_k = \frac{1}{n}, \quad \xi_k = \frac{k}{n} \quad (k = 0,1,\cdots,n-1)$$

bo‘ladi. SHularni e’tiborga olib topamiz.

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k = \sum_{k=0}^{n-1} e^{\frac{k}{n}} \cdot \frac{1}{n} = \frac{1}{n}\left(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right)$$

Ravshanki,

$$1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}} = \frac{e-1}{e^{\frac{1}{n}} - 1}$$

(mahraji $q = e^{\frac{1}{n}}$, birinchi hadi 1 bo‘lgan geometrik progressiyaning $n$ ta hadi yig‘indisi). Demak, berilgan funksiyaning integral yig‘indisi

$$\sigma = \frac{1}{n} \cdot \frac{e-1}{e^{\frac{1}{n}} - 1}$$

bo‘ladi.►

**2 – m i s o l .** $[a,b]$ *segmentda Dirixle funksiyasi*

$$D(x)=\begin{cases}1, \text{агар } x-\text{рационал сон} \in [a,b],\\ 0, \text{агар } x-\text{иррационал сон} \in [a,b]\end{cases}$$

*ning Darbu yig'indilari topilsin.*

◄ $[a,b]$ segmentning ihtiyoriy

$$P=\{x_0,x_1,x_2,\cdots,x_{n-1},x_n\} \qquad (a=x_0<x_1<x_2<\cdots<x_k<x_{k+1}<\cdots<x_n=b)$$

bo'laklashini olamiz.

Bu funksiya uchun

$$m_k=\inf\{D(x)\}=0,\ x\in[x_k,x_{k+1}]$$
$$M_k=\sup\{D(x)\}=1,\ x\in[x_k,x_{k+1}]$$

bo'ladi. Demak,

$$s(p)=\sum_{k=0}^{n-1}m_k\Delta x_k=0, \qquad S(p)=\sum_{k=0}^{n-1}M_k\Delta x_k=\sum_{k=0}^{n-1}\Delta x_k=b-a$$

bo'ladi.►

**3 – m i s o l .** *YUqoridagi 2-ta'rifdan foydalanib,* $f(x)=x$ *funksiyaning*

$$\int_a^b xdx$$

*integrali topilsin.*

◄ $[a,b]$ segmentning ihtiyoriy

$$P=\{x_0,x_1,x_2,...,x_{n-1},x_n\} \qquad (a=x_0<x_1<x_2<...<x_k<x_{k+1}<...<x_n=b)$$

bo'laklashini olamiz. Ravshanki, har bir $[x_k,x_{k+1}]$ $(k=0,1,...,n-1)$ segmentda $f(x)=x$ funksiyaning aniq chegaralari quyidagicha bo'ladi:

$$m_k=\inf\{f(x)\}=x_k \qquad (k=0,1,2,...,n-1)$$
$$M_k=\sup\{f(x)\}=x_{k+1} \qquad (k=0,1,2,...,n-1)$$

Demak,

$$s(p)=\sum_{k=0}^{n-1}m_k\Delta x_k=\sum_{k=0}^{n-1}x_k\cdot\Delta x_k,$$

$$S(p)=\sum_{k=0}^{n-1}M_k\Delta x_k=\sum_{k=0}^{n-1}x_{k+1}\cdot\Delta x_k.$$

Endi bu yig'indilarni (ularning aniq chegaralarini topish maqsadida) qulay ko'rinishda yozamiz:

$$s(p)=\sum_{k=0}^{n-1}x_k\Delta x_k=\sum_{k=0}^{n-1}x_k(x_{k+1}-x_k)=\frac{1}{2}\sum_{k=0}^{n-1}(x_{k+1}^2-x_k^2)-\frac{1}{2}\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2=$$

$$=\frac{1}{2}(x_n^2-x_{0.}^2)-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2=\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2,$$

$$S(p)=\sum_{k=0}^{n-1}x_{k+1}\Delta x_k=\sum_{k=0}^{n-1}(x_k+\Delta x_k)\cdot\Delta x_k=\sum_{k=0}^{n-1}x_k\Delta x_k+\sum_{k=0}^{n-1}\Delta x_k^2=$$

$$=\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2+\sum_{k=0}^{n-1}\Delta x_k^2=\frac{b^2-a^2}{2}+\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 .$$

Ravshanki,

$$\sup_p\{s(p)\}=\sup_p\left\{\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2\right\}=\frac{b^2-a^2}{2},$$

$$\inf_p\{S(p)\}=\inf_p\left\{\frac{b^2-a^2}{2}+\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2\right\}=\frac{b^2-a^2}{2}.$$

Demak,

$$\sup\{s(p)\}=\inf\{S(p)\}=\frac{b^2-a^2}{2}$$

bo‘lib,

$$\int_a^b xdx=\frac{b^2-a^2}{2}$$

bo‘ladi.►

**1.**Berilgan kesmani $n$ ta teng bo‘lakka ajratib va $\xi_i$ $(i=1,\cdots,n)$ argumentning qiymati sifatida har bir bo‘lakning o‘rtasini olib, $f(x)=1+x$ funksiya uchun $[-1;2]$ kesmada $\sigma$ integral yig‘indi yozilsin.

**2.**Berilgan kesmani $n$ ta teng bo‘lakka ajratish orqali $f(x)$ funksiya uchun Darbuning quyi $s_\tau$ va yuqori $S_\tau$ yig‘indilari topilsin:

a) $f(x)=x^3$, $[-2;3]$; b) $f(x)=\sqrt{x}$, $[0;1]$; v) $f(x)=2^x$, $[0;10]$;

**3.** $\int_0^T (gt+\vartheta_0)dt$ integral ta’rif yordamida hisoblansin.

**4.**Ta’rif yordamida hisoblansin.

a) $\int_0^1 e^x dx$; b) $\int_0^{\frac{\pi}{2}}\sin xdx$; v) $\int_0^x \cos tdt$; g) $\int_1^2\frac{dx}{x^2}$;

**5.** $[1,2]$ kesmani geometrik progressiya tashkil qiluvchi nuqtalar yordamida bo‘laklarga ajratib, $\int_1^2 x^3dx$ integralni integral yig‘indining limiti sifatida

hisoblansin.

**6.** $\int\limits_a^b \frac{dx}{x}$, $(0<a<b)$, integralni integral yig‘indining limiti sifatida hisoblansin.

**7.** Kesmada monoton bo‘lgan har qanday funksiya ushbu kesmada integrallanuvchi bo‘lishi isbotlansin.

**8.** Agar **f** funksiya $[0;1]$ kesmada uzluksiz va musbat bo‘lsa, u holda

$$\lim_{n\to\infty}\sqrt[n]{f\left(\frac{1}{n}\right)f\left(\frac{2}{n}\right)\cdots f\left(\frac{n}{n}\right)}=\exp\left(\int\limits_0^1 \ln f(x)dx\right)$$

bo‘lishi isbotlansin.

**9.** $f(x)$ va $\varphi(x)$ funksiyalar $[a,b]$ kesmada uzluksiz bo‘lsin. Quyidagi munosabatning to‘g‘riligi isbotlansin:

$$\lim_{\max|\Delta x_i|\to 0}\sum_{i=0}^{n-1} f(\xi_i)\varphi(\theta_i)\Delta x_i=\int\limits_a^b f(x)\varphi(x)dx,$$

bu erda $x_i\le\xi_i\le x_{i+1},\ x_i\le\theta_i\le x_{i+1}$ $(i=0,1,\cdots,n-1)$ va $\Delta x_i=x_{i+1}-x_i$ $(x_0=a,\ x_n=b)$.

**10.** **f** funksiya $[a,b]$ kesmada uzluksiz differensiallanuvchi va

$$\Delta_n=\int\limits_a^b f(x)dx-\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{k(b-a)}{n}\right)$$

bo‘lsin. $\lim\limits_{n\to\infty} n\Delta_n$ hisoblansin.

**11.** Uzilishga ega bo‘lgan $f(x)=\mathrm{signsin}\frac{\pi}{x}$ funksiyaning $[0,1]$ kesmada integrallanuvchi ekanligi isbotlansin.

**12.** $f(x)=\begin{cases}\frac{1}{x}-\left[\frac{1}{x}\right], & \text{агар } x\neq 0,\\ 0, & \text{агар } x=0.\end{cases}$

funksiya $[0,1]$ kesmada integrallanuvchi bo‘lishi isbotlansin

**13.** $\varphi(x)=\begin{cases} 0, & \text{агар } x-\text{иррационал}, \\ \frac{1}{n}, & \text{агар } x=\frac{m}{n} \end{cases}$ $(m \text{ ва } n-\text{ўзаро туб}, n\geq 1)$.

Riman funksiyasi ixtiyoriy chekli oraliqda integrallanuvchi bo‘lishi ko‘rsatilsin.

## Test

| | | | | |
|---|---|---|---|---|
| $\int_0^2 x^2dx$ | $\frac{8}{3}$ | -4 | 4 | $\frac{16}{3}$ |
| $\int_{-1}^0 3(2x+1)^2\,dx$ | 1 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $\int_{-1}^0 e^{-x}dx$ | $-1+e$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int_1^2 \frac{dx}{x+1}$ | $\ln\frac{3}{2}$ | 0 | $\ln 3$ | $\ln 3-1$ |
| $\int_0^\pi \sin x dx$ | 2 | 1 | -1 | -2 |
| $\int_0^2 x^3dx$ | 4 | 2 | 5 | 1 |
| $\int_0^\pi \sin 2xdx$ | 0 | -1 | 1 | 2 |
| $\int_{\pi/2}^\pi \cos xdx$ | -1 | 0 | 1 | -2 |
| $\int_3^5 \frac{dx}{x-1}$ | $\ln 2$ | $\ln\frac{3}{2}$ | $\ln\frac{4}{3}$ | 1 |
| $\int_{-1}^0 (2x+1)^2\,dx$ | $\frac{1}{3}$ | 1 | -3 | -1 |
| $\int_{-1}^0 2e^{-x}dx$ | $-2+2e$ | $e$ | $\frac{1}{e}-1$ | 5 |

# Mavzu. Funksiyaning integrallanuvchanlik mezoni (kriteriysi)

## 7-ma’ruza

**Reja**

**1⁰.** Darbu yig’indilarini xossalari.

**2⁰.** Aniq integralning mavjudligi.

**1⁰. Darbu yig’indilarining xossalari.** ***([1], 11.3 Upper and lower Riemann integrals)***

Faraz qilaylik, $F=\{P\}$ to’plam $[a,b]$ oraliqning barcha bo’laklashlaridan iborat to’plam bo’lsin. Agar $P_1\in F$ bo’laklashning har bir bo’luvchi nuqtasi $P_2\in F$ bo’laklashning ham bo’luvchi nuqtasi bo’lsa, $P_2$ bo’laklash $P_1$ ni ergashtiradi deyiladi va $P_1\subset P_2$ kabi belgilanadi.

Aytaylik, $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan bo’lib, $P_1\in F$ va $P_2\in F$ bo’laklashlari uchun Darbu yig’indilari

$$s(P_1), S(P_1);\ s(P_2), S(P_2)$$

bo’lsin.

1). Agar $P_1\subset P_2$ bo’lsa, u holda

$$s(P_1)\le s(P_2),\ \ S(P_1)\ge S(P_2)$$

bo’ladi.

2). $\forall P_1\in F,\ \ \forall P_2\in F$ uchun

$$s(P_2)\le S(P_1)$$

bo’ladi.

3). Darbu yig’ndilaridan tuzilgan

$$\{s(P)\}, \{S(P)\} \qquad (P\in F)$$

to’plam uchun

$$\sup_{P\in F}\{s(P)\}\le \inf_{P\in F}\{S(P)\}$$

ya’ni

$$\underline{J}\le \overline{J}$$

bo’ladi.

4). Ixtiyoriy $\varepsilon>0$ olinganda ham shunday $\delta>0$ topiladiki, diometri $\lambda_P<\delta$ bo’lgan $[a,b]$ oraliqning $P$ bo’laklashlari uchun

$$S(P) < \inf\{S(P)\} + \varepsilon$$
$$s(P) > \sup\{s(P)\} - \varepsilon$$

ya'ni

$$S(P) < \overline{J} + \varepsilon, \quad s(P) > \underline{J} - \varepsilon$$

bo'ladi.

Bu hossalardan birining masalan 2)— ning isbotini keltiramiz.

◄ Aytaylik, $P_1$ va $P_2$ lar $[a,b]$ oraliqning ixtiyoriy bo'laklashlari bo'lsin. Bu bo'laklashlarning barcha bo'luvchi nuqtalari yordamida $[a,b]$ ning yangi $P$ bo'laklashini hosil qilamiz. Ravshanki,

$$P_1 \subset P, \; P_2 \subset P$$

bo'ladi. $P$ bo'laklash uchun tuzilgan Darbu yig'indilari $s(P)$ va $S(P)$ lar uchun 1)—xossaga ko'ra

$$s(P_1) \le s(P), \; S(P) \le S(P_1),$$
$$s(P_2) \le s(P), \; S(P) \le S(P_2)$$

bo'lib, ulardan

$$s(P_2) \le s(P) \le S(P) \le S(P_1)$$

ya'ni

$$s(P_2) \le S(P_1)$$

bo'lishi kelib chiqadi. ►

Bu hossa $[a,b]$ oraliqni bo'laklashlari uchun tuzilgan quyi yig'indilar to'plami $\{s(P)\}$ ning har bir elementi yuqori yig'ndilar to'plami $\{S(P)\}$ ning istalgan elementidan katta emasligini bildiradi (Qolgan xossalarning isboti [1] ning 9—bobidan qaralsin ) .

**$2^0$. Aniq integralning mavjudligi.** ***([1], 11.5 Riemann integrability of continuous functions, 326-bet)*** Aytaylik, $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan bo'lsin .

**1–teorema .** *$f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lishi uchun $\forall\varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topilib, $[a,b]$ oraliqni diametri $\lambda_P < \delta$ bo'lgan har qanday $P$ bo'laklashi uchun Darbu yig'ndilari*

$$S(P) - s(P) < \varepsilon$$

*tengsizlikni qanoatlantirishi zarur va yetarli.*

◄ **Zarurligi.** $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan bo'lsin. Ta'rifga ko'ra $J = \overline{J} = \underline{J}$ bo'ladi, bunda

$$\underline{J} = \sup\{s(P)\}, \; \overline{J} = \inf\{S(P)\}.$$

$\forall\varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topiladiki, $[a,b]$ oraliqning diametri $\lambda_P < \delta$ bo'lgan har qanday $P$ bo'laklashida Darbu yig'indilari uchun $1^0$— dagi 4)–xossaga ko'ra $S(P) - \overline{J} < \frac{\varepsilon}{2}$, $\underline{J} - s(P) < \frac{\varepsilon}{2}$ tengsizliklar o'rinli bo'lib, undan $S(P) - s(P) < \varepsilon$ tengsizlik kelib chiqadi.

**Yetarliligi.** $\forall\varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topilib , $[a,b]$

oraliqning diametri $\lambda_P < \delta$ bo'lgan har qanday $P$ bo'laklashida Darbu yig'ndi-lari uchun

$$S(P) - s(P) < \varepsilon$$

tengsizlik o'rinli bo'lsin. $f(x)$ funksiya $[a,b]$ oraliqda chegaralanganligi uchun uning quyi hamda yuqori integrallari

$$\underline{J} = \sup\{s(P)\}, \quad \overline{J} = \inf\{S(P)\}$$

mavjud va $1^0$—dagi 3)—xossaga ko'ra $\underline{J} \leq \overline{J}$ tengsizlik o'rinli bo'ladi. Ravshanki,

$$s(P) \leq \underline{J} \leq \overline{J} \leq S(P) .$$

Bu munosabatdan

$$0 \leq \overline{J} - \underline{J} \leq S(P) - s(P)$$

bo'lishini topamiz. Demak, $\forall \varepsilon > 0$ son uchun $0 \leq \overline{J} - \underline{J} < \varepsilon$ bo'lib, undan $\overline{J} = \underline{J}$ bo'lishi kelib chiqadi. Bu esa $f(x)$ funksiyaning $[a,b]$ oraliqda integrallanuvchi ekanligini bildiradi. ►

Agar avvalgidek $f(x)$ funksiyaning $[x_k, x_{k+1}]$ ($k = 0,1,\ldots,n-1$) oraliq-dagi tebranishini $\omega_k$ orqali belgilasak, u holda

$$S(P) - s(P) = \sum_{k=0}^{n-1}(M_k - m_k)\Delta x_k = \sum_{k=0}^{n-1}\omega_k \Delta x_k$$

bo'lib, yuqorida keltirilgan teorema quyidagicha ifodalanadi.

**2—teorema**. *$f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lishi uchun $\forall \varepsilon > 0$ olinganda ham shunday $\delta > 0$ son topilib, $[a,b]$ oraliqni diametri $\lambda_P < \delta$ bo'lgan har qanday $P$ bo'laklashda*

$$\sum_{k=0}^{n-1}\omega_k \Delta x_k < \varepsilon \qquad (9.5)$$

*tengsizlikning bajarilisi zarur va yetarli.*

Ravshanki, (9.5) munosabatni quyidagi

$$\lim_{\lambda_P \to 0} \sum_{k=0}^{n-1}\omega_k \Delta x_k = 0$$

ko'rinishda ham yozish mumkin.

## Adabiyotlar


1. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.
2. **Claudio Canute, Anita Tabacco,** *Mathematical Analysis I*, Springer-Verlag Italia, Milan 2008.
3. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'rizalar, I q.* T. "Voris-nashriyot", 2010.
4. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.

# Glossariy

1. **Darbu yig‘indilari** – Ushbu $s=\sum_{k=0}^{n-1} m_k \Delta x_k$, $S=\sum_{k=0}^{n-1} M_k \Delta x_k$ yig’indilar mos ravishda Darbuning quyi hamda yuqori yig’indilari deb ataladi.

# Keys banki

**33-keys.** Masala o`rtaga tashlanadi: Agar $f(x)$ funksiya $[0,1]$ kesmada uzluksiz va musbat bo‘lsa, u holda

$$\lim_{n\to\infty}\sqrt[n]{f\left(\frac{1}{n}\right)f\left(\frac{2}{n}\right)\dots f\left(\frac{n}{n}\right)}=\exp\left(\int_0^1 \ln f(x)dx\right)$$

bo‘lishi isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 7-amaliy mashg‘ulot

$1^0$. **Funksiyaning integral va Darbu yig‘indilari.** Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan bo‘lsin. $[a,b]$ segmentning $a = x_0, x_1, x_2, \cdots, x_k, x_{k+1}, \cdots, x_n = b \quad (x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_n)$ nuqtalari uni $[x_k, x_{k+1}]$ $(k = 0,1,\ldots,n-1)$ bo‘laklarga ajratadi. Bu nuqtalar sistemasi $[a,b]$ ni bo‘laklash deyiladi va u $P = \{x_0, x_1, \cdots x_n\}$ kabi belgilanadi. Ushbu

$$\lambda_p = \max_k \{\Delta x_k\}, \Delta x_k = x_{k+1} - x_k \quad (k = 0,1,\cdots,n-1)$$

miqdor $P$ bo‘laklashning **diametri** deyiladi.

Har bir $[x_k, x_{k+1}]$ da ihtiyoriy $\xi_k$ nuqtani:

$$\xi_k \in [x_k, x_{k+1}] \quad (k = 0,1,\cdots,n-1)$$

olib, quyidagi

$$\sigma = f(\xi_0)\cdot \Delta x_0 + f(\xi_1)\cdot \Delta x_1 + \cdots + f(\xi_{n-1})\cdot \Delta x_{n-1} = \sum_{k=0}^{n-1} f(\xi_k)\cdot \Delta x_k \qquad (1)$$

yig‘indini tuzamiz. **(1)** yig‘indi $f(x)$ funksiyaning **integral (Riman) yig‘indisi** deyiladi. Ravshanki,

$$\sigma = \sigma_p(f : \xi_k)$$

Faraz qilaylik, $f(x)$ funksiya $[a,b]$ da chegaralangan bo‘lsin. Unda

$$m_k = \inf\{f(x)\}, \quad x \in [x_k, x_{k+1}]$$
$$M_k = \sup\{f(x)\}, \quad x \in [x_k, x_{k+1}] \qquad (k = 0,1,\ldots,n-1)$$

miqdorlar mavjud bo‘ladi. Ushbu

$$s = \sum_{k=0}^{n-1} m_k \cdot \Delta x_k, \qquad S = \sum_{k=0}^{n-1} M_k \cdot \Delta x_k$$

yig‘indilar mos ravishda **Darbuning quyi** va **yuqori** yig‘indilari deyiladi.

Ravshanki,

$$s = s_p(f), \quad S = S_p(f).$$

$2^0$. **Aniq integral ta’rifi.**

**1-ta’rif.** *Agar $\forall \varepsilon > 0$ olinganda ham shunday $\delta = \delta(\varepsilon) > 0$ son topilsaki, $[a,b]$ segmentning diametri $\lambda_p < \delta$ bo‘lgan har qanday $P$ bo‘laklashi hamda har bir $[x_k, x_{k+1}]$ dan olingan ihtiyoriy $\xi_k$ uchun tuzilgan $\sigma$ integral yig‘indisi ushbu*

$$|\sigma - \Im| < \varepsilon$$

*tengsizlikni qanoatlantirsa, $\Im$ son $\sigma$ yig‘indining $\lambda_p \to 0$ dagi limiti deyiladi. Bu holda $f(x)$ funksiya $[a,b]$ da integrallanuvchi, $\Im$ son esa $f(x)$ funksiyaning a n i q i n t e g r a l i deyiladi. Uni*

$$\int_a^b f(x)dx$$

***kabi belgilanadi.*** Demak,

$$\int_a^b f(x)dx = \lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

Aytaylik, $\mathcal{P}$ – $[a,b]$ segmentning bo‘laklashlari to‘plami bo‘lsin: $\mathcal{P} = \{P\}$. Har bir $P \in \mathcal{P}$ bo‘laklashga nisbatan Darbu yig‘indilari $s(p)$ va $S(p)$ ni tuzib, ushbu

$$\{s(p)\},\ \{S(p)\}$$

to‘plamlarni hosil qilamiz.

**2-ta’rif.** *Agar*

$$\operatorname{Sup}\{s(p)\} = \inf\{S(p)\}$$

***bo‘lsa,*** $f(x)$ ***funksiya*** $[a,b]$ ***da  i n t e g r a l l a n u v ch i ,  bu tenglikning umumiy qiymatiga esa*** $f(x)$ ***funksiyaning  a n i q  i n t e g r a l i  deyiladi.*** Demak,

$$\int_a^b f(x)dx = \inf\{S(p)\} = \sup\{s(p)\}$$

**1 – m i s o l .** ***Ushbu*** $f(x) = e^x$ ***funksiyaning*** $[0,1]$ ***segmentdagi integral yig‘indisi topilsin, bunda*** $[0,1]$ ***segmentni*** $n$ ***ta teng bo‘lakka bo‘lib, har bir bo‘lakda*** $\xi_k$ $(k = 0,1,\cdots,n-1)$ ***nuqta sifatida bo‘lakning chap chekkasi olinsin.***

◀Bu holda $[0,1]$ segmentning bo‘laklashi

$$P = \{x_0, x_1, x_2, \cdots, x_n\} = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{k}{n}, \cdots, 1\right\}$$

bo‘lib,

$$\Delta x_k = \frac{1}{n},\ \ \xi_k = \frac{k}{n} \quad (k = 0,1,\cdots,n-1)$$

bo‘ladi. SHularni e’tiborga olib topamiz.

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k = \sum_{k=0}^{n-1} e^{\frac{k}{n}} \cdot \frac{1}{n} = \frac{1}{n}\left(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right)$$

Ravshanki,

$$1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}} = \frac{e-1}{e^{\frac{1}{n}} - 1}$$

(mahraji $q = e^{\frac{1}{n}}$, birinchi hadi 1 bo‘lgan geometrik progressiyaning $n$ ta hadi yig‘indisi). Demak, berilgan funksiyaning integral yig‘indisi

$$\sigma = \frac{1}{n} \cdot \frac{e-1}{e^{\frac{1}{n}} - 1}$$

bo‘ladi.►

**2 – m i s o l .** $[a,b]$ *segmentda Dirixle funksiyasi*

$$D(x)=\begin{cases}1, \text{агар } x-\text{рационал сон} \in [a,b],\\ 0, \text{агар } x-\text{иррационал сон} \in [a,b]\end{cases}$$

***ning Darbu yig‘indilari topilsin.***

◄ $[a,b]$ segmentning ihtiyoriy

$$P=\{x_0,x_1,x_2,\cdots,x_{n-1},x_n\} \qquad (a=x_0<x_1<x_2<\cdots<x_k<x_{k+1}<\cdots<x_n=b)$$

bo‘laklashini olamiz.

Bu funksiya uchun

$$m_k=\inf\{D(x)\}=0,\ x\in[x_k,x_{k+1}]$$
$$M_k=\sup\{D(x)\}=1,\ x\in[x_k,x_{k+1}]$$

bo‘ladi. Demak,

$$s(p)=\sum_{k=0}^{n-1} m_k\Delta x_k=0, \qquad S(p)=\sum_{k=0}^{n-1} M_k\Delta x_k=\sum_{k=0}^{n-1}\Delta x_k=b-a$$

bo‘ladi.►

**3 – m i s o l .** *YUqoridagi 2-ta’rifdan foydalanib,* $f(x)=x$ *funksiyaning*

$$\int_a^b x\,dx$$

***integrali topilsin.***

◄ $[a,b]$ segmentning ihtiyoriy

$$P=\{x_0,x_1,x_2,\ldots,x_{n-1},x_n\} \qquad (a=x_0<x_1<x_2<\ldots<x_k<x_{k+1}<\ldots<x_n=b)$$

bo‘laklashini olamiz. Ravshanki, har bir $[x_k,x_{k+1}]$ $(k=0,1,\ldots,n-1)$ segmentda $f(x)=x$ funksiyaning aniq chegaralari quyidagicha bo‘ladi:

$$m_k=\inf\{f(x)\}=x_k \qquad (k=0,1,2,\ldots,n-1)$$
$$M_k=\sup\{f(x)\}=x_{k+1} \qquad (k=0,1,2,\ldots,n-1)$$

Demak,

$$s(p)=\sum_{k=0}^{n-1} m_k\Delta x_k=\sum_{k=0}^{n-1} x_k\cdot\Delta x_k,$$

$$S(p)=\sum_{k=0}^{n-1} M_k\Delta x_k=\sum_{k=0}^{n-1} x_{k+1}\cdot\Delta x_k.$$

Endi bu yig‘indilarni (ularning aniq chegaralarini topish maqsadida) qulay ko‘rinishda yozamiz:

$$s(p)=\sum_{k=0}^{n-1} x_k\Delta x_k=\sum_{k=0}^{n-1} x_k(x_{k+1}-x_k)=\frac{1}{2}\sum_{k=0}^{n-1}(x_{k+1}^2-x_k^2)-\frac{1}{2}\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2=$$

$$=\frac{1}{2}(x_n^2-x_{0.}^2)-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2=\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2,$$

$$S(p)=\sum_{k=0}^{n-1}x_{k+1}\Delta x_k=\sum_{k=0}^{n-1}(x_k+\Delta x_k)\cdot\Delta x_k=\sum_{k=0}^{n-1}x_k\Delta x_k+\sum_{k=0}^{n-1}\Delta x_k^2=$$

$$=\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2+\sum_{k=0}^{n-1}\Delta x_k^2=\frac{b^2-a^2}{2}+\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 .$$

Ravshanki,

$$\sup_p\{s(p)\}=\sup_p\left\{\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2\right\}=\frac{b^2-a^2}{2},$$

$$\inf_p\{S(p)\}=\inf_p\left\{\frac{b^2-a^2}{2}+\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2\right\}=\frac{b^2-a^2}{2}.$$

Demak,

$$\sup\{s(p)\}=\inf\{S(p)\}=\frac{b^2-a^2}{2}$$

bo‘lib,

$$\int_a^b xdx=\frac{b^2-a^2}{2}$$

bo‘ladi.►

**1.**Berilgan kesmani $\mathbf{n}$ ta teng bo‘lakka ajratib va $\xi_i$ $(i=1,\cdots,n)$ argumentning qiymati sifatida har bir bo‘lakning o‘rtasini olib, $f(x)=1+x$ funksiya uchun $[-1;2]$ kesmada $\sigma$ integral yig‘indi yozilsin.

**2.**Berilgan kesmani $\mathbf{n}$ ta teng bo‘lakka ajratish orqali $f(x)$ funksiya uchun Darbuning quyi $s_\tau$ va yuqori $S_\tau$ yig‘indilari topilsin:

a) $f(x)=x^3$, $[-2;3]$; b) $f(x)=\sqrt{x}$, $[0;1]$; v) $f(x)=2^x$, $[0;10]$;

**3.** $\int_0^T(gt+\vartheta_0)dt$ integral ta’rif yordamida hisoblansin.

**4.**Ta’rif yordamida hisoblansin.

a) $\int_0^1 e^x dx$; b) $\int_0^{\frac{\pi}{2}}\sin x dx$; v) $\int_0^x \cos t dt$; g) $\int_1^2\frac{dx}{x^2}$;

**5.** $[1,2]$ kesmani geometrik progressiya tashkil qiluvchi nuqtalar yordamida bo‘laklarga ajratib, $\int_1^2 x^3dx$ integralni integral yig‘indining limiti sifatida

hisoblansin.

**6.** $\int_a^b \frac{dx}{x}$, $(0 < a < b)$, integralni integral yig‘indining limiti sifatida hisoblansin.

**7.** Kesmada monoton bo‘lgan har qanday funksiya ushbu kesmada integrallanuvchi bo‘lishi isbotlansin.

**8.** Agar $f$ funksiya $[0;1]$ kesmada uzluksiz va musbat bo‘lsa, u holda

$$\lim_{n\to\infty} \sqrt[n]{f\left(\frac{1}{n}\right)f\left(\frac{2}{n}\right)\cdots f\left(\frac{n}{n}\right)} = \exp\left(\int_0^1 \ln f(x)dx\right)$$

bo‘lishi isbotlansin.

**9.** $f(x)$ va $\varphi(x)$ funksiyalar $[a,b]$ kesmada uzluksiz bo‘lsin. Quyidagi munosabatning to‘g‘riligi isbotlansin:

$$\lim_{\max|\Delta x_i|\to 0} \sum_{i=0}^{n-1} f(\xi_i)\varphi(\theta_i)\Delta x_i = \int_a^b f(x)\varphi(x)dx,$$

bu erda $x_i \le \xi_i \le x_{i+1}, x_i \le \theta_i \le x_{i+1}$ $(i = 0,1,\cdots,n-1)$ va $\Delta x_i = x_{i+1} - x_i$ $(x_0 = a, x_n = b)$.

**10.** $f$ funksiya $[a,b]$ kesmada uzluksiz differensiallanuvchi va

$$\Delta_n = \int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right)$$

bo‘lsin. $\lim\limits_{n\to\infty} n\Delta_n$ hisoblansin.

**11.** Uzilishga ega bo‘lgan $f(x) = \text{signsin}\frac{\pi}{x}$ funksiyaning $[0,1]$ kesmada integrallanuvchi ekanligi isbotlansin.

**12.** $f(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right], & \text{агар } x \ne 0, \\ 0, & \text{агар } x = 0. \end{cases}$

funksiya $[0,1]$ kesmada integrallanuvchi bo‘lishi isbotlansin

**13.** $\varphi(x)=\begin{cases} 0, & \text{агар } x-\text{иррационал},\\ \dfrac{1}{n}, & \text{агар } x=\dfrac{m}{n} \quad (m \text{ ва } n-\text{ўзаро туб}, n\geq 1). \end{cases}$

Riman funksiyasi ixtiyoriy chekli oraliqda integrallanuvchi bo‘lishi ko‘rsatilsin.

## Test

| | | | | |
|---|---|---|---|---|
| $\int_0^2 x^2dx$ | $\frac{8}{3}$ | -4 | 4 | $\frac{16}{3}$ |
| $\int_{-1}^0 3(2x+1)^2\,dx$ | 1 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $\int_{-1}^0 e^{-x}dx$ | $-1+e$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int_1^2 \frac{dx}{x+1}$ | $\ln\frac{3}{2}$ | 0 | $\ln 3$ | $\ln 3-1$ |
| $\int_0^\pi \sin x dx$ | 2 | 1 | -1 | -2 |
| $\int_0^2 x^3dx$ | 4 | 2 | 5 | 1 |
| $\int_0^\pi \sin 2xdx$ | 0 | -1 | 1 | 2 |
| $\int_{\pi/2}^\pi \cos xdx$ | -1 | 0 | 1 | -2 |
| $\int_3^5 \frac{dx}{x-1}$ | $\ln 2$ | $\ln\frac{3}{2}$ | $\ln\frac{4}{3}$ | 1 |
| $\int_{-1}^0 (2x+1)^2\,dx$ | $\frac{1}{3}$ | 1 | -3 | -1 |
| $\int_{-1}^0 2e^{-x}dx$ | $-2+2e$ | $e$ | $\frac{1}{e}-1$ | 5 |

# Mavzu. Integrallanuvchi funksiyalar sinfi

## 8-ma'ruza

**Reja**

**1$^0$.** Uzluksiz funksiyalarning integrallanuvchanligi.

**2$^0$.** Monoton funksiyalarning integrallanuvchanligi.

**3$^0$.** Uziladigan funksiyalarning integrallanuvchanligi

**1. Uzluksiz funksiyalarning integrallanuvchanligi.** ***([1], 11.5 Riemann integrability of continuous functions, 326-bet)***

Ushbu paragrafda aniq integralning mavjudligi haqidagi teoremadan foydalanib, bazi funksiyalarning sinfi integrallanuvchi bo'lishini ko'ramiz. $f(x)$ funksiya $[a,b]$ oraliqda aniqlangan bo'lsin .

**3 – teorema**. *Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsa, u shu oraliqda integrallanuvchi bo'ladi.*

◄ $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsin. Veyeshtrasning birinchi teoremasiga ( 5—bobdagi 7–teoremaga qarang ) ko'ra funksiya $[a,b]$ da chegaralangan. Ikkinchi tomondan, Kantor teoremasining ( 5— bobdagi 10—teoremaga qarang ) 3—natijaga ko'ra $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ son topilib, $[a,b]$ oraliqni uzluklari $\delta$ dan kichik bo'lgan bo'laklarga ajratilganda funksiyaning har bir bo'lakdagi tebranishi uchun $\omega_k<\varepsilon$ tengsizlik o'rinli bo'ladi. Demak, $[a,b]$ oraliqni diametri $\lambda_P<\delta$ bo'lgan har qanday $P$ bo'laklashda

$$S(P)-s(P)=\sum_{k=0}^{n-1}\omega_k\Delta x_k<\varepsilon\sum_{k=0}^{n-1}\Delta x_k=\varepsilon(b-a)$$

bo'lib, undan

$$\lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} \varpi_k \Delta x_k = 0$$

kelib chiqadi. Demak, $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi .►

**2. Monoton funksiyalarning integrallanuvchanligi.** ***([1], 11.6 Riemann integrability of monotone functions, 330-bet)***

**4—teorema**. *Agar $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan va monoton bo'lsa, funksiya shu oraliqda integrallanuvchi bo'ladi.*

◄ $f(x)$ funksiya $[a,b]$ da chegaralangan va shu oraliqda, aytaylik, o'suvchi bo'lsin. $\forall \varepsilon > 0$ sonni olib, unga ko'ra $\delta > 0$ sonni quyidagicha tanlaylik:

$$\delta = \frac{\varepsilon}{f(b) - f(a)} > 0 .$$

So'ngra $[a,b]$ oraliqni diametri $\lambda_P < \delta$ bo'lgan $P$ bo'laklashi uchun Darbu yig'indilari $S(P)$ va $s(P)$ ni tuzamiz. U holda

$$S(P) - s(P) = \sum_{k=0}^{n-1} \omega_k \Delta x_k = \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)] \Delta x_k \le \frac{\varepsilon}{f(b) - f(a)} \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)] =$$

$$= \frac{\varepsilon}{f(b) - f(a)} [f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots + f(x_n) - f(x_{n-1})] =$$

$$= \frac{\varepsilon}{f(b) - f(a)} (f(x_n) - f(x_0)) = \frac{\varepsilon}{f(b) - f(a)} (f(b) - f(a)) = \varepsilon$$

Demak, $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi .►

Chegaralangan hamda kamayuvchi funksiyaning integrallanuvchi bo'li-shi ham xuddi shunga o'xshash isbotlanadi.

## Uziladigan funksiyalarning integrallanuvchanligi

**5–teorema**. *Agar $f(x)$ funksiya $[a,b]$ oraliqda chegaralangan va bu oraliqning chekli sondagi nuqtalarida uzulishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa , funksiya shu oraliqda integrallanuvchi bo'ladi.*

◄ $f(x)$ funksiya $[a,b]$ da chegaralangan bo'lib, shu oraliqning faqat bitta

$x^*$ $(x^* \in [a,b])$ nuqtasida uzilishga ega, qolgan barcha nuqtalarida uzluksiz bo'lsin.

$\forall \varepsilon > 0$ son olib, $x^*$ nuqtaning

$$U_\varepsilon(x^*) = \{x : x \in R, x^* - \varepsilon < x < x^* + \varepsilon\}$$

atrofini tuzamiz. Bu atrof $[a,b]$ oraliqni

$$U_\varepsilon(x^*),\ [a,b] \setminus U_\varepsilon(x^*) = [a, x^* - \varepsilon] \cup [x^* + \varepsilon, b]$$

qismlarga ajratadi.

Shartga ko'ra, $f(x)$ funksiya $[a, x^* - \varepsilon]$ va $[x^* + \varepsilon, b]$ oraliqning har birida uzluksiz. Bu oraliqlarning har biriga alohida Kantor teoremasining natijasini (5— bobdagi 3— natijani qarang) qo'llaymiz. U holda olingan $\forall \varepsilon > 0$ son uchun shunday $\delta_1 > 0$ va $\delta_2 > 0$ sonlar topiladiki,

$$[a, x^* - \varepsilon] \text{ da } \Delta x_k < \delta_1 \text{ dan } \omega_k < \varepsilon,$$

$$[x^* + \varepsilon, b] \text{ da } \Delta x_k < \delta_2 \text{ dan } \omega_k < \varepsilon$$

tengsizliklar o'rinli ekani kelib chiqadi. Agar $\delta = \min\{\delta_1, \delta_2\}$ deb olsak , u holda ikkala oraliq uchun bir vaqtda

$$\Delta x_k < \delta \text{ dan } \omega_k < \varepsilon$$

tengsizliklar o'rinli ekani kelib chiqadi.

Endi yuqoridagi $\forall \varepsilon > 0$ songa ko'ra $\delta > 0$ sonni $\delta < \varepsilon$ deb olaylik. $[a,b]$ oraliqni diametri $\lambda_P < \delta$ bo'lgan bo'laklashlari uchun $f(x)$ funksiyaning Darbu yig'indilarini tuzib, quyidagi

$$S(P) - s(P) = \sum_{k=0}^{n-1} \omega_k \Delta x_k \qquad (9.6)$$

ayirmani qaraymiz. (9.6) yig'indining har bir hadida $[x_k, x_{k+1}]$ $(k = 0,1,\ldots,n-1)$ oraliqning uzunligi $\Delta x_k$ qatnashadi. Bu $[x_k, x_{k+1}]$ oraliqlarni $x^*$ nuqtaning $U_\varepsilon(x^*)$ atrofidan tashqarida joylashganiga, ya'ni $[x_k, x_{k+1}] \cap U_\varepsilon(x^*) = 0$ munosabat o'rinli bo'ladiganiga mos keladigan (9.6) yig'indining hadlaridan tuzilgan yig'indi

$$\sum_k{}' \omega_k \Delta x_k$$

bo'lsin. $(9.6)$ yig'indining qolgan barcha hadlaridan tashkil topgan yig'indi

$$\sum_k{}'' \omega_k \Delta x_k$$

bo'lsin, bunda $[x_k, x_{k+1}] \subset U_\varepsilon(x*)$ yoki $[x_k, x_{k+1}] \cap \{x^* - \varepsilon\} \neq \varnothing$ yoki $[x_k, x_{k+1}] \cap \{x^* + \varepsilon\} \neq \varnothing$ bo'ladi.

Natijada $(9.6)$ yig'indi ikki qismga ajraladi:

$$\sum_{k=0}^{n-1} \omega_k \Delta x_k = \sum_k{}' \omega_k \Delta x_k + \sum_k{}'' \omega_k \Delta x_k \qquad (9.7)$$

Endi bu yig'indilarni baholaymiz. Yuqoridagi $(9.6)$ munosabatdan foydalanib, topamiz:

$$\sum_k{}' \omega_k \Delta x_k < \sum_k{}' \varepsilon \cdot \Delta x_k \leq \varepsilon \sum_{k=0}^{n-1} \Delta x_k = \varepsilon(b-a). \qquad (9.8)$$

Ikkinchi yig'indi uchun

$$\sum_k{}'' \omega_k \Delta x_k \leq \sum_k{}'' \Omega \cdot \Delta x_k = \Omega \cdot \sum_k{}'' \Delta x_k$$

bo'lishini topamiz, bunda $\Omega -$ $f(x)$ funksiyaning $[a,b]$ oraliqdagi tebranishi.

Agar $U_\varepsilon(x^*)$ atrofida butunlay joylashgan $[x_k, x_{k+1}]$ oraliqlari uzunliklari- ning yig'indisi $2\varepsilon$ dan kichikligini hamda $x^* - \varepsilon$ va $x^* + \varepsilon$ nuqtalarni o'z ichi- ga olgan $[x_k, x_{k+1}]$ oraliqlar ikkita bo'lib, ularning uzunliklari yig'indisi ham $2\varepsilon$ (chunki $\delta < \varepsilon$) dan kichik bo'lishini etiborga olsak, u holda

$$\sum_k{}'' \Delta x_k < 4\varepsilon \qquad (9.9)$$

bo'ladi. Natijada $(9.7)$, $(9.8)$ va $(9.9)$ munosabatlardan

$$\sum_{k=0}^{n-1} \omega_k \Delta x_k < \varepsilon(b-a) + 4\varepsilon \cdot \Omega = \varepsilon[(b-a) + 4\Omega]$$

ekani kelib chiqadi. Demak, $\lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} \omega_k \Delta x_k = 0.$

Bu $f(x)$ funksiyaning $[a,b]$ da integrallanuvchi bo'lishini bildiradi.

$f(x)$ funksiya $[a,b]$ oraliqning chekli sondagi nuqtalarida uzulishga ega bo'lib, qolgan barcha nuqtalarida uzluksiz bo'lsa, uning $[a,b]$ da integrallanuvchi bo'lishi yuqoridagidek isbot etiladi.►

**2—eslatma.** $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsin. Biz

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

hamda $\int_a^a f(x)dx = 0$ tengliklar o'rinli deb kelishib olamiz.

### Adabiyotlar


1. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.
2. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, I q.* T. "Voris-nashriyot", 2010.
3. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.


## Glossariy

**Darbu yig'indilari** – Ushbu $s = \sum_{k=0}^{n-1} m_k \Delta x_k$, $S = \sum_{k=0}^{n-1} M_k \Delta x_k$ yig'indilar mos ravishda Darbuning quyi hamda yuqori yig'indilari deb ataladi.

## Keys banki

**34-keys.** Masala o`rtaga tashlanadi: Uzilishga ega bo'lgan $f(x) = \operatorname{sign}\sin\frac{\pi}{x}$ funksiyaning $[0,1]$ kesmada integrallanuvchi ekanligi isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 8-amaliy mashg'ulot.

$1^0$. **Funksiyaning integral va Darbu yig'indilari.** Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan bo'lsin. $[a,b]$ segmentning $a=x_0,x_1,x_2,\cdots,x_k,x_{k+1},\cdots,x_n=b$ $(x_0<x_1<\cdots<x_k<x_{k+1}<\cdots<x_n)$ nuqtalari uni $[x_k,x_{k+1}]$ $(k=0,1,...,n-1)$ bo'laklarga ajratadi. Bu nuqtalar sistemasi $[a,b]$ ni bo'laklash deyiladi va u $P=\{x_0,x_1,\cdots x_n\}$ kabi belgilanadi. Ushbu

$$\lambda_p=\max_k\{\Delta x_k\},\Delta x_k=x_{k+1}-x_k \quad (k=0,1,\cdots,n-1)$$

miqdor $P$ bo'laklashning **diametri** deyiladi.

Har bir $[x_k,x_{k+1}]$ da ihtiyoriy $\xi_k$ nuqtani:

$$\xi_k\in[x_k,x_{k+1}] \quad (k=0,1,\cdots,n-1)$$

olib, quyidagi

$$\sigma=f(\xi_0)\cdot\Delta x_0+f(\xi_1)\cdot\Delta x_1+\cdots+f(\xi_{n-1})\cdot\Delta x_{n-1}=\sum_{k=0}^{n-1}f(\xi_k)\cdot\Delta x_k \qquad (1)$$

yig'indini tuzamiz. **(1)** yig'indi $f(x)$ funksiyaning **integral (Riman) yig'indisi** deyiladi. Ravshanki,

$$\sigma=\sigma_p(f:\xi_k)$$

Faraz qilaylik, $f(x)$ funksiya $[a,b]$ da chegaralangan bo'lsin. Unda

$$\begin{aligned} m_k &=\inf\{f(x)\}, \quad x\in[x_k,x_{k+1}] \\ M_k &=\sup\{f(x)\}, \quad x\in[x_k,x_{k+1}] \end{aligned} \qquad (k=0,1,...,n-1)$$

miqdorlar mavjud bo'ladi. Ushbu

$$s=\sum_{k=0}^{n-1}m_k\cdot\Delta x_k, \qquad S=\sum_{k=0}^{n-1}M_k\cdot\Delta x_k$$

yig'indilar mos ravishda **Darbuning quyi** va **yuqori** yig'indilari deyiladi.

Ravshanki,

$$s=s_p(f), \quad S=S_p(f).$$

$2^0$. **Aniq integral ta'rifi.**

**1-ta'rif.** *Agar $\forall\varepsilon>0$ olinganda ham shunday $\delta=\delta(\varepsilon)>0$ son topilsaki, $[a,b]$ segmentning diametri $\lambda_p<\delta$ bo'lgan har qanday $P$ bo'laklashi hamda har bir $[x_k,x_{k+1}]$ dan olingan ihtiyoriy $\xi_k$ uchun tuzilgan $\sigma$ integral yig'indisi ushbu*

$$|\sigma-\Im|<\varepsilon$$

*tengsizlikni qanoatlantirsa, $\Im$ son $\sigma$ yig'indining $\lambda_p\to 0$ dagi limiti deyiladi. Bu holda $f(x)$ funksiya $[a,b]$ da integrallanuvchi, $\Im$ son esa $f(x)$ funksiyaning a n i q i n t e g r a l i deyiladi. Uni*

$$\int_a^b f(x)dx$$

***kabi belgilanadi.*** Demak,

$$\int_a^b f(x)dx = \lim_{\lambda_p \to 0} \sigma = \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k$$

Aytaylik, $\mathcal{P}$ – $[a,b]$ segmentning bo‘laklashlari to‘plami bo‘lsin: $\mathcal{P} = \{P\}$. Har bir $P \in \mathcal{P}$ bo‘laklashga nisbatan Darbu yig‘indilari $s(p)$ va $S(p)$ ni tuzib, ushbu

$$\{s(p)\}, \{S(p)\}$$

to‘plamlarni hosil qilamiz.

**2-ta’rif.** *Agar*

$$\sup\{s(p)\} = \inf\{S(p)\}$$

***bo‘lsa,*** $f(x)$ ***funksiya*** $[a,b]$ ***da integrallanuvchi, bu tenglikning umumiy qiymatiga esa*** $f(x)$ ***funksiyaning aniq integrali deyiladi.*** Demak,

$$\int_a^b f(x)dx = \inf\{S(p)\} = \sup\{s(p)\}$$

**1 – misol.** ***Ushbu*** $f(x) = e^x$ ***funksiyaning*** $[0,1]$ ***segmentdagi integral yig‘indisi topilsin, bunda*** $[0,1]$ ***segmentni*** $n$ ***ta teng bo‘lakka bo‘lib, har bir bo‘lakda*** $\xi_k$ $(k = 0,1,\cdots,n-1)$ ***nuqta sifatida bo‘lakning chap chekkasi olinsin.***

◄Bu holda $[0,1]$ segmentning bo‘laklashi

$$P = \{x_0, x_1, x_2, \cdots, x_n\} = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{k}{n}, \cdots, 1\right\}$$

bo‘lib,

$$\Delta x_k = \frac{1}{n}, \quad \xi_k = \frac{k}{n} \quad (k = 0,1,\cdots,n-1)$$

bo‘ladi. SHularni e’tiborga olib topamiz.

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k = \sum_{k=0}^{n-1} e^{\frac{k}{n}} \cdot \frac{1}{n} = \frac{1}{n}\left(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right)$$

Ravshanki,

$$1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}} = \frac{e-1}{e^{\frac{1}{n}} - 1}$$

(mahraji $q = e^{\frac{1}{n}}$, birinchi hadi 1 bo‘lgan geometrik progressiyaning $n$ ta hadi yig‘indisi). Demak, berilgan funksiyaning integral yig‘indisi

$$\sigma = \frac{1}{n} \cdot \frac{e-1}{e^{\frac{1}{n}} - 1}$$

bo‘ladi.►

**2 – m i s o l .** $[a,b]$ *segmentda Dirixle funksiyasi*

$$D(x)=\begin{cases}1, \text{агар } x-\text{рационал сон}\in[a,b],\\ 0, \text{агар } x-\text{иррационал сон}\in[a,b]\end{cases}$$

***ning Darbu yig'indilari topilsin.***

◄ $[a,b]$ segmentning ihtiyoriy

$$P=\{x_0,x_1,x_2,\cdots,x_{n-1},x_n\} \qquad (a=x_0<x_1<x_2<\cdots<x_k<x_{k+1}<\cdots<x_n=b)$$

bo'laklashini olamiz.

Bu funksiya uchun

$$m_k=\inf\{D(x)\}=0,\ x\in[x_k,x_{k+1}]$$
$$M_k=\sup\{D(x)\}=1,\ x\in[x_k,x_{k+1}]$$

bo'ladi. Demak,

$$s(p)=\sum_{k=0}^{n-1}m_k\Delta x_k=0, \qquad S(p)=\sum_{k=0}^{n-1}M_k\Delta x_k=\sum_{k=0}^{n-1}\Delta x_k=b-a$$

bo'ladi.►

**3 – m i s o l .** *YUqoridagi 2-ta'rifdan foydalanib,* $f(x)=x$ *funksiyaning*

$$\int_a^b x\,dx$$

***integrali topilsin.***

◄ $[a,b]$ segmentning ihtiyoriy

$$P=\{x_0,x_1,x_2,\dots,x_{n-1},x_n\} \qquad (a=x_0<x_1<x_2<\dots<x_k<x_{k+1}<\dots<x_n=b)$$

bo'laklashini olamiz. Ravshanki, har bir $[x_k,x_{k+1}]$ $(k=0,1,\dots,n-1)$ segmentda $f(x)=x$ funksiyaning aniq chegaralari quyidagicha bo'ladi:

$$m_k=\inf\{f(x)\}=x_k \qquad (k=0,1,2,\dots,n-1)$$
$$M_k=\sup\{f(x)\}=x_{k+1} \qquad (k=0,1,2,\dots,n-1)$$

Demak,

$$s(p)=\sum_{k=0}^{n-1}m_k\Delta x_k=\sum_{k=0}^{n-1}x_k\cdot\Delta x_k,$$

$$S(p)=\sum_{k=0}^{n-1}M_k\Delta x_k=\sum_{k=0}^{n-1}x_{k+1}\cdot\Delta x_k.$$

Endi bu yig'indilarni (ularning aniq chegaralarini topish maqsadida) qulay ko'rinishda yozamiz:

$$s(p)=\sum_{k=0}^{n-1}x_k\Delta x_k=\sum_{k=0}^{n-1}x_k(x_{k+1}-x_k)=\frac{1}{2}\sum_{k=0}^{n-1}(x_{k+1}^2-x_k^2)-\frac{1}{2}\sum_{k=0}^{n-1}(x_{k+1}-x_k)^2=$$

$$=\frac{1}{2}(x_n^2-x_{0.}^2)-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2=\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2,$$

$$S(p)=\sum_{k=0}^{n-1}x_{k+1}\Delta x_k=\sum_{k=0}^{n-1}(x_k+\Delta x_k)\cdot\Delta x_k=\sum_{k=0}^{n-1}x_k\Delta x_k+\sum_{k=0}^{n-1}\Delta x_k^2=$$

$$=\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2+\sum_{k=0}^{n-1}\Delta x_k^2=\frac{b^2-a^2}{2}+\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2 .$$

Ravshanki,

$$\sup_p\{s(p)\}=\sup_p\left\{\frac{b^2-a^2}{2}-\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2\right\}=\frac{b^2-a^2}{2},$$

$$\inf_p\{S(p)\}=\inf_p\left\{\frac{b^2-a^2}{2}+\frac{1}{2}\sum_{k=0}^{n-1}\Delta x_k^2\right\}=\frac{b^2-a^2}{2}.$$

Demak,

$$\sup\{s(p)\}=\inf\{S(p)\}=\frac{b^2-a^2}{2}$$

bo‘lib,

$$\int_a^b xdx=\frac{b^2-a^2}{2}$$

bo‘ladi.►

**1.**Berilgan kesmani $n$ ta teng bo‘lakka ajratib va $\xi_i$ $(i=1,\cdots,n)$ argumentning qiymati sifatida har bir bo‘lakning o‘rtasini olib, $f(x)=1+x$ funksiya uchun $[-1;2]$ kesmada $\sigma$ integral yig‘indi yozilsin.

**2.**Berilgan kesmani $n$ ta teng bo‘lakka ajratish orqali $f(x)$ funksiya uchun Darbuning quyi $s_\tau$ va yuqori $S_\tau$ yig‘indilari topilsin:

a) $f(x)=x^3$, $[-2;3]$;  b) $f(x)=\sqrt{x}$, $[0;1]$;  v) $f(x)=2^x$, $[0;10]$;

**3.** $\int_0^T(gt+\vartheta_0)dt$ integral ta’rif yordamida hisoblansin.

**4.**Ta’rif yordamida hisoblansin.

a) $\int_0^1 e^x dx$;  b) $\int_0^{\frac{\pi}{2}}\sin x dx$;  v) $\int_0^x \cos t dt$;  g) $\int_1^2\frac{dx}{x^2}$;

**5.** $[1,2]$ kesmani geometrik progressiya tashkil qiluvchi nuqtalar yordamida bo‘laklarga ajratib, $\int_1^2 x^3dx$ integralni integral yig‘indining limiti sifatida

hisoblansin.

**6.** $\int_a^b \frac{dx}{x}$, $(0 < a < b)$, integralni integral yig'indining limiti sifatida hisoblansin.

**7.** Kesmada monoton bo'lgan har qanday funksiya ushbu kesmada integrallanuvchi bo'lishi isbotlansin.

**8.** Agar $f$ funksiya $[0;1]$ kesmada uzluksiz va musbat bo'lsa, u holda

$$\lim_{n\to\infty} \sqrt[n]{f\left(\frac{1}{n}\right)f\left(\frac{2}{n}\right)\cdots f\left(\frac{n}{n}\right)} = \exp\left(\int_0^1 \ln f(x)dx\right)$$

bo'lishi isbotlansin.

**9.** $f(x)$ va $\varphi(x)$ funksiyalar $[a,b]$ kesmada uzluksiz bo'lsin. Quyidagi munosabatning to'g'riligi isbotlansin:

$$\lim_{\max|\Delta x_i|\to 0} \sum_{i=0}^{n-1} f(\xi_i)\varphi(\theta_i)\Delta x_i = \int_a^b f(x)\varphi(x)dx,$$

bu erda $x_i \le \xi_i \le x_{i+1}, x_i \le \theta_i \le x_{i+1}$ $(i = 0,1,\cdots,n-1)$ va $\Delta x_i = x_{i+1} - x_i$ $(x_0 = a, x_n = b)$.

**10.** $f$ funksiya $[a,b]$ kesmada uzluksiz differensiallanuvchi va

$$\Delta_n = \int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right)$$

bo'lsin. $\lim_{n\to\infty} n\Delta_n$ hisoblansin.

**11.** Uzilishga ega bo'lgan $f(x) = \operatorname{sign}\sin\frac{\pi}{x}$ funksiyaning $[0,1]$ kesmada integrallanuvchi ekanligi isbotlansin.

**12.** $f(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right], & \text{агар } x \neq 0, \\ 0, & \text{агар } x = 0. \end{cases}$

funksiya $[0,1]$ kesmada integrallanuvchi bo'lishi isbotlansin

**13.** $\varphi(x)=\begin{cases} 0, \text{ агар } x-\text{иррационал}, \\ \dfrac{1}{n}, \text{ агар } x=\dfrac{m}{n} \end{cases}$ $(m \text{ ва } n-\text{ўзаро туб}, n\geq 1)$.

Riman funksiyasi ixtiyoriy chekli oraliqda integrallanuvchi bo‘lishi ko‘rsatilsin.

## Test

| | | | | |
|---|---|---|---|---|
| $\int_0^2 x^2dx$ | $\frac{8}{3}$ | -4 | 4 | $\frac{16}{3}$ |
| $\int_{-1}^0 3(2x+1)^2\,dx$ | 1 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $\int_{-1}^0 e^{-x}dx$ | $-1+e$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int_1^2 \frac{dx}{x+1}$ | $\ln\frac{3}{2}$ | 0 | $\ln 3$ | $\ln 3-1$ |
| $\int_0^\pi \sin xdx$ | 2 | 1 | -1 | -2 |
| $\int_0^2 x^3dx$ | 4 | 2 | 5 | 1 |
| $\int_0^\pi \sin 2xdx$ | 0 | -1 | 1 | 2 |
| $\int_{\pi/2}^\pi \cos xdx$ | -1 | 0 | 1 | -2 |
| $\int_3^5 \frac{dx}{x-1}$ | $\ln 2$ | $\ln\frac{3}{2}$ | $\ln\frac{4}{3}$ | 1 |
| $\int_{-1}^0 (2x+1)^2\,dx$ | $\frac{1}{3}$ | 1 | -3 | -1 |
| $\int_{-1}^0 2e^{-x}dx$ | $-2+2e$ | $e$ | $\frac{1}{e}-1$ | 5 |

# Mavzu. Aniq integrallarning xossalari

## 9-ma’ruza

**Reja**

**1$^0$.** Integralning chiziqlilik hamda additivlik xossalari.

**2$^0$.** Integral tengsizliklar.

**3$^0$.** O’rta qiymat haqidagi teoremalar

### 1. Integralning chiziqlilik hamda additivlik xossalari.

Endi $f(x)$ funksiya aniq integralining xossalarini o’rganamiz.

**1—xossa**. Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo’lsa, u istalgan $[\alpha,\beta]\subset[a,b]$ oraliqda ham integrallanuvchi bo’ladi.

◄. $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo’lsin. U holda 1—teoremaga ko’ra $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ son topiladiki, $[a,b]$ oraliqni diametri $\lambda_P<\delta$ bo’lgan har qanday $P=\{x_0, x_1,\ldots,x_n\}$ bo’laklash uchun

$$S(P)-s(P)<\varepsilon \qquad (9.10)$$

tengsizlik bajariladi.

$P$ bo’laklashning bo’luvchi nuqtalari $x_0, x_1,\ldots,x_n$ qatoriga $\alpha$ hamda $\beta$ nuqtalarni qo’shib, $[a,b]$ oraliqni yangi $P_1$ bo’laklashni hosil qilamiz. Ravshanki, $P\subset P_1$ bo’ladi. U holda Darbu yig’ndilarining xossasiga ko’ra (ushbu bobning 4— **§**, 2— bandiga qarang)

$$s(P)\le s(P_1),\ S(P_1)\le S(P) \qquad (9.11)$$

tengsizliklar o’rinli bo’ladi. (9.10) va (9.11) munosabatlardan

$$S(P_1)-s(P_1)<\varepsilon \qquad (9.12)$$

bo’lishi kelib chiqadi.

$[\alpha,\beta]$ oraliqdagi $P_1$ bo’laklashning bo’luvchi nuqtalarini $[\alpha,\beta]$ oraliqning biror $P_2$ bo’laklashning bo’luvchi nuqtalari sifatida qaraymiz. Bu $P_2$ bo’laklash uchun $f(x)$ funksiyaning Darbu yig’ndilari $s(P_2), S(P_2)$ bo’lsin. U holda

$$S(P_1)-s(P_1)=\sum_{[a,b]}(M_k-m_k)\Delta x_k,$$

$$S(P_2)-s(P_2)=\sum_{[\alpha,\beta]}(M_k-m_k)\Delta x_k$$

yig’indilarni taqqoslab ,

$$S(P_2)-s(P_2)\le S(P_1)-s(P_1)$$

bo'lishini topamiz. Natijada (9.12) munosabatni e'tiborga olsak

$$S(P_2)-s(P_2)<\varepsilon$$

kelib chiqadi. Bundan 1— teoremaga ko'ra $f(x)$ funksiyaning $[\alpha,\beta]$ oraliqda integrallanuvchi ekani kelib chiqadi. ►

**2—xossa .** Agar $f(x)$ funksiya $[a,c]$ hamda $[c,b]$ oraliqlarda integrallanuvchi bo'lsa, u holda funksiya $[a,b]$ oraliqda ham integrallanuvchi va ushbu

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

formula o'rinli bo'ladi.

◄ $f(x)$ funksiya $[a,c]$ hamda $[c,b]$ oraliqlarda integrallanuvchi bo'lsin $(a<c<b)$. U holda $\forall\varepsilon>0$ olinganda ham $\frac{\varepsilon}{2}$ songa ko'ra shunday $\delta_1>0$ son topiladiki, $[a,c]$ oraliqni diametric $\lambda_{P_1}<\delta_1$ bo'lgan har qanday $P_1$ bo'laklashida Darbu yig'ndilari uchun

$$S(P_1)-s(P_1)<\frac{\varepsilon}{2}$$

tengsizlik o'rinli bo'ladi. Shuningdek, o'sha $\frac{\varepsilon}{2}$ songa ko'ra shunday $\delta_2>0$ son topiladiki, $[c,b]$ oraliqni diametri $\lambda_{P_2}<\delta_2$ bo'lgan har qanday $P_2$ bo'laklashida Darbu yig'ndilari uchun

$$S(P_2)-s(P_2)<\frac{\varepsilon}{2}$$

tengizlik o'rinli bo'ladi. Endi $\delta=\min\{\delta_1,\delta_2\}$ deb, $[a,b]$ oraliqni diametri $\lambda_{P_3}<\delta$ bo'lgan ixtiyoriy $P_3$ bo'laklashni olaylik. Bu bo'laklashning bo'luvchi nuqtalari qatoriga $c$ $(a<c<b)$ nuqtani ham qo'shib, $[a,b]$ oraliqni yangi $P$ bo'laklashni hosil qilamiz. Bu bo'laklash uchun Darbu yig'indilari $S(P),\ s(P)$ bo'lsin. $[a,c]$ oraliqdagi $P$ bo'laklashning bo'luvchi nuqtalarini shu $[a,c]$ oraliqni biror $P_1'$ bo'laklashning bo'luvchi nuqtalari $[a,c]$ oraliqni biror $P_2'$ bo'laklashning bo'luvchi nuqtalari sifatida qaraymiz. Bu bo'laklashlar uchun Darbu yig'indilarini tuzamiz:

$$S(P_1'),\ s(P_1'),\ S(P_2'),\ s(P_2')$$

Ravshanki, bu yig'indilar uchun mos ravishda yuqoridagi (9.13), (9.14) tengsizliklar o'rinli bo'ladi:

$$S(P_1')-s(P_1')<\frac{\varepsilon}{2},$$

$$S(P_2')-s(P_2')<\frac{\varepsilon}{2}.$$

Ikkinchi tomondan,

$$S(P)=S(P_1^{'})+S(P_2^{'}),$$
$$s(P)=s(P_1^{'})+s(P_2^{'})$$

bo'lib, natijada

$$S(P)-s(P)=\left[S(P_1^{'})-s(P_1^{'})\right]+\left[S(P_2^{'})-s(P_2^{'})\right]<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

bo'lishi kelib chiqadi. Demak , $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ son topiladiki, $[a,b]$ oraliqni diametri $\lambda_P<\delta$ bo'lgan har qanday $P$ bo'laklashda Darbu yig'indilari uchun

$$S(P)-s(P)<\varepsilon$$

bo'ladi. Bu esa 1-teoremaga ko'ra $f(x)$ funksiyaning $[a,b]$ oraliqda integral-lanuvchi ekanini ko'rsatadi.

Yuqoridagi $P$ bo'laklash uchun $f(x)$ funksiyaning $[a,b]$ oraliqdagi integral yig'indilarini tuzib, ularni mos ravishda quyidagi

$$\sum_{[a,b]} f(\xi_k)\Delta x_k\,,\quad \sum_{[a,c]} f(\xi_k)\Delta x_k\,,\quad \sum_{[c,b]} f(\xi_k)\Delta x_k\,,$$

ko'rinishda belgilasak, u holda

$$\sum_{[a,b]} f(\xi_k)\Delta x_k=\sum_{[a,c]} f(\xi_k)\Delta x_k+\sum_{[c,b]} f(\xi_k)\Delta x_k \qquad (9.15)$$

bo'ladi. $f(x)$ funksiya $[a,c]$, $[c,b]$ hamda $[a,b]$ oraliqlarda integrallanuvchi bo'lgani uchun

$$\lim_{\lambda_P\to 0}\sum_{[a,c]} f(\xi_k)\Delta x_k=\int_a^c f(x)dx,$$

$$\lim_{\lambda_P\to 0}\sum_{[c,b]} f(\xi_k)\Delta x_k=\int_c^b f(x)dx,$$

$$\lim_{\lambda_P\to 0}\sum_{[a,b]} f(\xi_k)\Delta x_k=\int_a^b f(x)dx,$$

tengliklarga egamiz. (9.15) tenglikdan $\lambda_P\to 0$ da izlangan formula kelib chiqadi.

Endi $c$ nuqta $[a,b]$ oraliqdan tashqarida yotsin, ya'ni $c$ nuqta $c<a<b$ yoki $a<b<c$ tengsizlikni qanoatlantirsin. Agar $c<a<b$ bo'lsa, u holda $[a,b]\subset[c,b]$ bo'lgani uchun 1—xossaga ko'ra $f(x)$ funksiya $[a,b]$ da integral-lanuvchi bo'lib, yuqorida isbot etilganiga asosan

$$\int_c^b f(x)dx=\int_c^a f(x)dx+\int_a^b f(x)dx$$

formula o'rinli bo'ladi. Bundan esa, 2—eslatmadan foydalanib

$$\int_a^b f(x)dx=-\int_c^a f(x)dx+\int_c^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

bo'lishini topamiz.

Xuddi shunga o'xshash $a<b<c$ bo'lganda ham $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lishi va tegishli formulaning o'rinli ekani

ko'rsatiladi. ►

**3—xossa.** Agar $f(x)$ funksiya $[a,b]$ da integrallanuvchi bo'lsa, u holda $cf(x)$ $(c = const)$ ham shu oraliqda integrallanuvchi bo'ladi va ushbu

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

formula o'rinli bo'ladi.

◄ $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsin. Demak,

$$\lim_{\lambda_P\to 0}\sigma = \lim_{\lambda_P\to 0}\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k = \int_a^b f(x)dx.$$

Endi $cf(x)$ funksiyaning mos integral yig'indisini tuzamiz:

$$\sigma_1 = \sum_{k=0}^{n-1} cf(\xi_k)\cdot\Delta x_k.$$

U holda

$$\sigma_1 = \sum_{k=0}^{n-1} cf(\xi_k)\cdot\Delta x_k = c\sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k = c\cdot\sigma.$$

Bundan $\lambda_P \to 0$ da quyidagi

$$\lim_{\lambda_P\to 0}\sigma_1 = \lim_{\lambda_P\to 0} c\sigma = c\lim_{\lambda_P\to 0}\sigma = c\int_a^b f(x)dx$$

tenglik kelib chiqadi. Bu 3—xossa va undagi formulaning o'rinli ekanini anglatadi. ►

**4—xossa.** Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi va $f(x)\ge d>0$ bo'lsa, u holda $\dfrac{1}{f(x)}$ funksiya ham shu oraliqda integrallanuvchi bo'ladi.

◄ $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsin. Demak, $\forall\varepsilon>0$ olinganda ham $\varepsilon d^2$ ga ko'ra shunday $\delta>0$ son topiladiki, $[a,b]$ oraliqni diametri $\lambda_P<\delta$ bo'lgan har qanday $P$ bo'laklash uchun

$$S(P)-s(P) = \sum_{k=0}^{n-1}(M_k - m_k)\Delta x_k < d^2\varepsilon$$

bo'ladi. Bunda

$$M_k = \sup\{f(x)\},\qquad x\in[x_k, x_{k+1}]$$
$$m_k = \inf\{f(x)\},\qquad x\in[x_k, x_{k+1}].$$

$f(x)\ge d>0$ bo'lganligini etiborga olib, $\dfrac{1}{f(x)}$ funksiya uchun

$$M_k^* = \sup\left\{\frac{1}{f(x)}\right\},\qquad x\in[x_k, x_{k+1}]$$

$$m_k^* = \inf\left\{\frac{1}{f(x)}\right\},\qquad x\in[x_k, x_{k+1}].$$

mavjud bo'lishini aniqlaymiz. Ravshanki,

$$M_k^* = \frac{1}{m_k}, \quad m_k^* = \frac{1}{M_k},$$

bo'ladi. Natijada

$$S(P) - s(P) = \sum_{k=0}^{n-1}(M_k * - m_k *)\Delta x_k = \sum_{k=0}^{n-1}(\frac{1}{m_k} - \frac{1}{M_k})\Delta x_k = \sum_{k=0}^{n-1}\frac{M_k - m_k}{m_k M_k}\Delta x_k \le$$

$$\le \frac{1}{d^2}\sum_{k=0}^{n-1}(M_k - m_k)\Delta x_k < \varepsilon$$

bo'ladi. Bu esa $\frac{1}{f(x)}$ funksiyaning $[a,b]$ oraliqda integrallanuvchi ekanini bildiradi. ►

**5—xossa.** Agar $f(x)$ va $g(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda $f(x) \pm g(x)$ funksiya ham shu oraliqda integrallanuvchi bo'ladi va ushbu

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

formula o'rinli bo'ladi .

◄ $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi bo'lsin. Demak,

$$\lim_{\lambda_\rho \to 0} \sigma_1 = \lim_{\lambda_\rho \to 0} \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k = \int_a^b f(x)dx,$$

$$\lim_{\lambda_\rho \to 0} \sigma_2 = \lim_{\lambda_\rho \to 0} \sum_{k=0}^{n-1} g(\xi_k)\Delta x_k = \int_a^b g(x)dx$$

Endi $f(x) \pm g(x)$ funksiyaning $[a,b]$ oraliqdagi mos integral yig'indisini yozamiz:

$$\sigma = \sum_{k=0}^{n-1}[f(\xi_k) \pm (\xi_k)]\Delta x_k = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k \pm \sum_{k=0}^{n-1} g(\xi_k)\Delta x_k = \sigma_1 \pm \sigma_2 .$$

Bundan $\lambda_\rho \to 0$ da quyidagiga egamiz:

$$\lim_{\lambda_\rho \to 0} \sigma = \lim_{\lambda_\rho \to 0} \sigma_1 \pm \lim_{\lambda_\rho \to 0} \sigma_2 = \int_a^b f(x)dx + \int_a^b g(x)dx .$$

Bu 5—xossa va undagi formulaning o'rinli ekanligini anglatadi.►

**6—xossa.** $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda $f(x)g(x)$ funksiya ham shu oraliqda integrallanuvchi bo'ladi.

◄ $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi bo'lsin. U holda integralning mavjudligi haqidagi teoremaga ko'ra

$$\lim_{\lambda_\rho \to 0}(S_f(P) - s_f(P)) = \lim_{\lambda_\rho \to 0} \sum_{k=0}^{n-1}(M_k - m_k)\Delta x_k = 0, \qquad (9.6)$$

$$\lim_{\lambda_\rho\to 0}(S_g(P)-s_g(P))=\sum_{k=0}^{n-1}(M'_k-m'_k)\Delta x_k=0. \qquad (9.7)$$

Avval barcha $x\in[a,b]$ lar uchun $f(x)\ge 0,\ g(x)\ge 0$ deb qaraylik. U holda $\forall x\in[x_k,x_{k+1}]$ uchun

$$0\le m_k\le f(x)\le M_k$$

$$0\le m'_k\le g(x)\le M'_k$$

tengsizliklar o'rinli bo'lib, undan quyidagi

$$0\le m_k m'_k\le f(x)g(x)\le M_k M'_k$$

tengsizlik kelib chiqadi.

Ravshanki, $[x_k,x_{k-1}]$ $(k=0,1,\dots.n-1)$ oraliqda $f(x)g(x)$ quyidagi aniq chegaralari

$$m_k^0=\inf\{f(x)g(x)\}$$

$$m_k^0=\sup\{f(x)g(x)\}$$

mavjud bo'lib, ular uchun

$$m_k m'_k\le m_k^0 \langle M_k^0\le M_k M'_k$$

tengsizliklar o'rinli bo'ladi. U holda quyidagi

$$M_k^0-m_k^0\le M_k M'_k-m_k m'_k=M'_k(M_k-m_k)+m_k(M'_k-m'_k)$$

$$M=\sup_{a\le x\le b}\{f(x)\}\ge M_k,\quad M=\sup_{a\le x\le b}\{g(x)\}\ge M'_k$$

tengsizliklarni e'tiborga olib ( $f(x)$ va $g(x)$ funksiyalar $[a,b]$ da chegaralanganligi uchun $M<\infty,\ M'<\infty$ bo'ladi ) topamiz:

$$S_{fg}(P)-s_{fg}(P)=\sum_{k=0}^{n-1}(M_k^0-m_k^0)\Delta x_k\le M'\sum_{k=0}^{n-1}(M_k-m_k)\Delta x_k+M\sum_{k=0}^{n-1}(M'_k-m'_k)\Delta x_k.$$

Endi (9.6) va (9.7) munosabatlardan foydalansak, u holda quyidagi

$$\lim_{\lambda_\rho\to 0}(S_{fg}(P)-s_{fg}(P)=\lim_{\lambda_\rho\to 0}\sum_{k=0}^{n-1}(M_k^0-m_k^0)\Delta x_k=0$$

tenglik kelib chiqadi. Demak, $f(x)g(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi.

Endi $f(x)$ va $g(x)$ funksiyalar ixtiyoriy integrallanuvchi funksiyalar bo'lsin. Bir tomondan $\forall x\in[a,b]$ lar uchun

$$f(x)-\inf\{f(x)\}=f(x)-m\ge 0,$$

$$g(x)-\inf\{g(x)\}=g(x)-m'\ge 0$$

tengsizliklar o'rinli. Ikkinchi tomonda,

$$f(x)g(x)=[f(x)-m][g(x)-m']+mg(x)+m'f(x)-mm'$$

deb yoza olamiz. Yuqorida isbot etilganiga hamda 4—xossaning natijasiga (1-natijaga qarang) ko'ra, $f(x)g(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'ladi. ►

4) – va 5) – xossalardan quyidagi natija kelib chiqadi.

**1—natija.** Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi va $g(x)\ge d>0$ bo'lsa, u holda $\dfrac{f(x)}{g(x)}$ funksiya ham $[a,b]$ oraliqda integrallanuvchi bo'ladi.

**7-xossa** Agar $f(x)$ $[a,b]$ oraliqda integrallanuvchi bo'lib, $\forall x\in[a,b]$ lar uchun $f(x)\geq 0$ bo'lsa, u holda

$$\int_a^b f(x)dx\geq 0 \qquad (a<b)$$

bo'ladi.

◀ Ta'rifga ko'ra ushbu

$$\lim_{\lambda_\rho\to 0}\sigma=\lim_{\lambda_\rho\to 0}\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k$$

limit mavjud. Modomiki, $\forall x\in[a,b]$ lar uchun $f(x)\geq 0$ ekan, unda

$$\sigma=\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k\geq 0$$

va

$$\lim_{\lambda_\rho\to 0}\sum_{k=0}^{n-1} f(\xi_k)\,\Delta x_k\geq 0$$

bo'ladi. Demak,

$$\int_a^b f(x)dx\geq 0. \ ▶$$

**2—natija.** Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda integrallanuvchi bo'lib, $\forall x\in[a,b]$ lar uchun $f(x)\leq g(x)$ tengsizlik o'rinli bo'lsa, u holda
ushbu

$$\int_a^b f(x)dx\leq\int_a^b g(x)dx$$

tengsizlik ham o'rinli bo'ladi.

◀Haqiqatdan ham, $f(x)$ va $g(x)$ funksiyalar $[a,b]$ oraliqda inegrallanuvchi bo'lganida $g(x)-f(x)\geq 0$ funksiyaning integrallanuvchiligi 4—xossadan kelib chiqadi. 6—xossaga ko'ra bu holda

$$\int_a^b [g(x)-f(x)]dx\geq 0$$

tengsizlik kelib chiqadi. Bundan izlangan tengsizlikka ega bo'lamiz. ▶

## 2. Integral tengsizliklar.

**3—natija (Koshi–Bunyakovskiy tengsizligi).** Agar $f(x)$ va $g(x)$ funksi-yalar $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda (yuqoridagi xossalarga ko'ra) ushbu $f(x)-\alpha\, g(x)$ ($\alpha-$ ixtiyoriy o'zgarmas) funksiya ham $[a,b]$ oraliqda integrallanuvchi va

$$\int_a^b [f(x)-\alpha\, g(x)]^2\, dx \geq 0$$

tengsizlik o'rinli bo'ladi.

Demak, ixtiyoriy o'zgarmas $\alpha$ son uchun

$$\alpha^2 \int_a^b g^2(x)dx - 2\alpha \int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \geq 0$$

tensizlik o'rinli. Bu tengsizlikning chap tomonidagi ifoda $\alpha$ ga nisbatan kvadrat uchhad bo'lib, u $\alpha$ ning barcha haqiqiy qiymatlarida manfiy emas. Demak, bu kvadrat uchhadning diskriminanti musbat emas, ya'ni

$$\left[\int_a^b f(x)g(x)dx\right]^2 - \int_a^b f^2(x)dx \int_a^b g^2(x)dx \leq 0 .$$

Natijada quyidagi

$$\left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b f^2(x)dx}\sqrt{\int_a^b g^2(x)dx}$$

tengsizlikka kelamiz. Bu tengsizlik Koshi–Bunyakovskiy tengsizligi deb ataladi.

**8—xossa.** Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsa, u holda $|f(x)|$ funksiya ham shu oraliqda integrallanuvchi bo'ladi va

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx$$

tengsizlik o'rinli bo'ladi.

◀ $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsin. U holda $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ son topiladiki, $[a,b]$ oraliqni diametri $\lambda_P<\delta$ bo'lgan har qanday $P=\{x_0, x_1, \ldots, x_n\}$ bo'laklash uchun

$$S(P)-s(P)=\sum_{k=0}^{n-1}\omega_k\,\Delta x_k < \varepsilon$$

bo'ladi, bunda $\omega_k - f(x)$ funksiyaning $[x_k, x_{k+1}]$ oraliqdagi tebranishi.

Ravshanki, $\forall x' \in [a,b]$, $\forall x'' \in [a,b]$ lar quyidagi

$$\big||f(x')|-|f(x'')|\big| \leq |f(x')-f(x'')|$$

tengsizlik o'rinli bo'lib, undan

$$\sup\big||f(x')|-|f(x'')|\big| \leq \sup|f(x')-f(x'')|$$

tengsizlik kelib chiqadi. Demak, $\overline{\omega}_k \leq \omega_k$, bunda $\overline{\omega}_k - |f(x)|$ funksiyaning $[x_k, x_{k+1}]$ dagi tebranishi. Natijada

$$S_{|f|}(P)-s_{|f|}(P)=\sum_{k=0}^{n-1}\overline{\omega}_k\,\Delta x_k \leq \sum_{k=0}^{n-1}\omega_k\,\Delta x_k < \varepsilon$$

bo'ladi. Bundan $|f(x)|$ funksiyaning $[a,b]$ oraliqda integrallanuvchi bo'lishi kelib chiqadi.

$f(x)$ hamda $|f(x)|$ funksiyalarning $[a,b]$ oraliqda integral yig'indi—larini yozamiz:

$$\sigma=\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k ,$$

$$\sigma_1=\sum_{k=0}^{n-1} |f(\xi_k)|\Delta x_k .$$

U holda

$$|\sigma|=\left|\sum_{k=0}^{n-1} f(\xi_k)\Delta x_k\right|\le\sum_{k=0}^{n-1}|f(\xi_k)|\Delta x_k=\sigma_1$$

bo'ladi va $\lambda_P\to 0$ da limitga o'tib 8—xossaga va unda ta'kidlangan tengsizlikning o'rinli ekaniga ishonch hosil qilamiz. ►

### 3. O'rta qiymat haqidagi teoremalar

$f(x)$ funksiya $[a,b]$ oraliqda aniqlangan va chegaralangan bo'lsin. U hol-da $[a,b]$ oraliqda

$$m=\inf\{f(x)\},\quad M=\sup\{f(x)\}$$

mavjud va $\forall x\in[a,b]$ uchun

$$m\le f(x)\le M \tag{9.18}$$

tengsizliklar o'rinli bo'ladi

**6—teorema** *Agar* $f(x)$ *funksiya* $[a,b]$ *oraliqda inegrallanuvchi bo'lsa, u holda shunday o'zgarmas* $\mu$ $(m\le\mu\le M)$ *son mavjudki, ushbu*

$$\int_a^b f(x)dx=\mu(b-a)$$

*tenglik o'rinli bo'ladi.*

◄ (9.18) tengsizliklardan foydalanib topamiz:

$$\int_a^b m\,dx\le\int_a^b f(x)\,dx\le\int_a^b M\,dx.$$

Bundan

$$m(b-a)\le\int_a^b f(x)dx\le M(b-a).$$

Bu tengsizliklarni $b-a>0$ songa bo'lamiz:

$$m\le\frac{\int_a^b f(x)dx}{b-a}\le M .$$

Agar

$$\mu = \frac{\int_a^b f(x)dx}{b-a}$$

deb olsak, u holda izlangan tenglik kelib chiqadi. ►

**4—natija** Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bolsa, u holda bu oraliqda shunday $c$ $(c \in [a,b])$ nuqta topiladiki,

$$\int_a^b f(x)dx = f(c)(b-a)$$

tenglik o'rinli bo'ladi.

**7—teorema.** *Agar* $f(x)$ *va* $g(x)$ *funksiyalar* $[a,b]$ *oraliqda integral-lanuvchi bo'lib,* $g(x)$ *funksiya shu oraliqda o'z ishorasini o'zgartirmasa, u holda shunday o'zgarmas* $\mu$ $(m \le \mu \le M)$ *son mavjudki,*

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx \qquad (9.19)$$

*tenglik o'rinli bo'ladi.*

◄ Aniq integrallning 5—xossasiga asosan $f(x)g(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'ladi. Endi $g(x)$ funksiya $[a,b]$ oraliqda manfiy bo'lmasin, ya'ni $\forall x \in [a,b]$ lar uchun $g(x) \ge 0$ bo'lsin deylik. U holda $m \le f(x) \le M$ tengsiz-liklarni $g(x)$ ga ko'paytirib, so'ngra hosil bo'lgan ushbu

$$mg(x) \le f(x)g(x) \le Mg(x)$$

tengsizliklarni $[a,b]$ oraliqda integrallab topamiz:

$$m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx. \qquad (9.20)$$

Ikki holni qaraylik:

a) $\int_a^b g(x)dx = 0$ bo'lsin. U holda

$$\int_a^b f(x)g(x)dx = 0$$

bo'lib, bunda $\mu$ deb $m \le \mu \le M$ tengsizliklarni qanoatlantiruvchi ixtiyoriy sonni olish mumkin.

b) $\int_a^b g(x)dx > 0$ bo'lsin. Bu holda (9.20) tengsizliklardan

$$m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M$$

bo'lishi kelib chiqadi.

$$\mu = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$$

deb olsak, unda

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$$

bo'ladi.

$[a,b]$ oraliqda $g(x) \le 0$ bo'lganda $(9.19)$ formula xuddi shunga o'xshash isbotlanadi

**5—natija:** Agar $[a,b]$ oraliqda $f(x)$ funksiya uzluksiz, $g(x)$ funksiya integrallanuvchi bo'lsa, hamda shu oraliqda $g(x)$ funksiya o'z ishorasini o'zgartirmasa, u holda shunday $c$ $(c \in [a,b])$ nuqta topiladiki,

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

tenglik o'rinli bo'ladi.

Bu natijaning isboti $(9.19)$ tenglikka asoslanadi.

## Adabiyotlar


1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'rizalar, I q.* T. "Voris-nashriyot", 2010.
2. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 1.* Hindustan Book Agency, India, 2014.

# Glossariy

**4. Darbu yig'indilari** – Ushbu $s = \sum_{k=0}^{n-1} m_k \Delta x_k$, $S = \sum_{k=0}^{n-1} M_k \Delta x_k$ yig'indilar mos ravishda Darbuning quyi hamda yuqori yig'indilari deb ataladi.

# Keys banki

**35-keys.** Masala o`rtaga tashlanadi: Ushbu

$$S_n = \frac{1^\alpha + 2^\alpha + \ldots + n^\alpha}{n^{\alpha+1}}, \ (\alpha > 0)$$

yig'indining limiti aniq integral yordamida topilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 9-amaliy mashg'ulot

Funksiyaning aniq integrallari asosan Nyuton-Leybnits formulasi, o'zgaruvchilarini almashtirish va bo'laklab integrallash usullari (formulalari) yordamida hisoblanadi.

**$1^0$. Nyuton-Leybnits formulasi.** Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda uzluksiz bo'lib, $F(x)$ funksiya esa uning boshlang'ich funksiyasi bo'lsin $(F'(x)=f(x),\quad x\in[a,b])$. U holda

$$\int_a^b f(x)dx = F(b)-F(a)=F(x)\Big|_a^b \qquad (2)$$

bo'ladi.

Aniq integralni hisoblash imkonini beradigan (2) formula **Nyuton-Leybnits formulasi** deyiladi.

**$2^0$. O'zgaruvchini almashtirish formulasi.** Faraz qilaylik:

1) $f(x)$ funksiya $[a,b]$ segmentda uzluksiz;
2) $\varphi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz va uzluksiz $\varphi'(t)$ hosilaga ega;
3) $\varphi(\alpha)=a,\ \varphi(\beta)=b$;
4) $f(\varphi(t))$ murakkab funksiya $[a,b]$ da aniqlangan va uzluksiz bo'lsin. U holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \qquad (3)$$

bo'ladi.

(3) formula **o'zgaruvchini almashtirish formulasi** deyiladi.

**$3^0$. Bo'laklab integrallash formulasi.** Aytaylik, $f(x)$ va $g(x)$ funksiyalar $[a,b]$ segmentda uzluksiz va uzluksiz $f'(x), g'(x)$ hosilalarga ega bo'lsin. U holda

$$\int_a^b f(x)\cdot g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx \qquad (4)$$

bo'ladi.

(4) formula **bo'laklab integrallash formulasi** deyiladi.

**Eslatma:** Ayrim hollarda yig'indining limiti aniq integralga keltirib hisoblanadi.

**1 – m i s o l .** *Ushbu integral*

$$\int_a^b f(x)dx = \int_a^b \frac{|x|}{x}dx$$

***hisoblansin.***

◄Aytaylik, $0 \le a \le b$ bo‘lsin. Bu holda $x > 0$ bo‘lib, $f(x) = \frac{|x|}{x} = 1$ bo‘ladi. Demak,

$$\int_a^b \frac{|x|}{x}dx = \int_a^b dx = x\Big|_a^b = b - a$$

Aytaylik, $a < b < 0$ bo‘lsin. Bu holda $x < 0$ bo‘lib, $f(x) = \frac{|x|}{x} = -1$ bo‘ladi. Demak,

$$\int_a^b \frac{|x|}{x}dx = \int_a^b (-1)dx = -x\Big|_a^b = a - b$$

Aytaylik, $a < 0 < b$ bo‘lsin. Bu holda

$$f(x) = \frac{|x|}{x} = \begin{cases} -1, & \text{агар } a < x < 0 \\ 1, & \text{агар } 0 < x < b \end{cases}$$

bo‘ladi. Demak,

$$\int_a^b \frac{|x|}{x}dx = \int_a^0 (-1)dx + \int_0^b 1dx = -x\Big|_a^0 + x\Big|_0^b = a + b$$

YUqoridagi uchta holni birlashtirib,

$$\int_a^b \frac{|x|}{x}dx = |b| - |a|$$

bo‘lishini topamiz.►

**2 – m i s o l .** ***Ushbu***

$$\int_0^{10\pi} \sqrt{1 - \cos 2x}dx$$

***hisoblansin.***

◄Ma’lumki,

$$\sqrt{1 - \cos 2x} = \sqrt{2\sin^2 x} = \sqrt{2} \cdot |\sin x|$$

Unda

$$\int_0^{10\pi} \sqrt{1 - \cos 2x}dx = \sqrt{2}\int_0^{10\pi} |\sin x|dx$$

bo‘ladi. Aniq integralning xossalaridan foydalanib topamiz:

$$\int_0^{10\pi} |\sin x|dx = \int_0^{\pi} \sin xdx - \int_{\pi}^{2\pi} \sin xdx + \int_{2\pi}^{3\pi} \sin dx - \int_{3\pi}^{4\pi} \sin xdx + \cdots + \int_{8\pi}^{9\pi} \sin xdx - \int_{9\pi}^{10\pi} \sin xdx =$$

$$= \underbrace{2 + 2 + \cdots + 2}_{10\text{та}} = 20.$$

Demak,

$$\int_0^{10\pi}\sqrt{1-\cos 2x}\,dx = 20\cdot\sqrt{2}\,.\blacktriangleright$$

**3 – m i s o l .** ***Ushbu***

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

***integral hisoblansin.***

◀Bu integralni o‘zgaruvchini almashtirish usulidan foydalanib hisoblaymiz:

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \left[x = a\sin t\,,\ dx = a\cos t dt\,,\ t\in\left[0,\frac{\pi}{2}\right]\right] = a^4\int_0^{\frac{\pi}{2}}\sin^2 t\cdot\cos^2 t dt =$$

$$= \frac{a^4}{4}\int_0^{\frac{\pi}{2}}\sin^2 2t dt = \frac{a^4}{4}\int_0^{\frac{\pi}{2}}\frac{1-\cos 4t}{2}dt = \frac{a^4}{8}\left(t-\frac{\sin 4t}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{a^4}{16}\pi\ \blacktriangleright$$

**4 – m i s o l .** ***Ushbu***

$$\Im = \int_1^e (x\ln x)^2 dx$$

***integral hisoblansin.***

◀Bu integralni bo‘laklab integrallash formulasidan foydalanib hisoblaymiz:

$$\Im = \int_1^e (x\ln x)^2 dx = \left[u = \ln^2 x\,,\ du = 2\ln x\cdot\frac{1}{x}dx\,,\ dv = x^2 dx\,,\ v = \frac{x^3}{3}\right] =$$

$$= \frac{x^3}{3}\ln^2 x\Big|_1^e - \frac{2}{3}\int_1^e x^2\ln x dx = \frac{e^3}{3} - \frac{2}{3}\int_1^e x^2\ln x dx$$

Keyingi integral ham bo‘laklab integrallash usuli yordamida hisoblanadi:

$$\int_1^e x^2\ln x dx = \left[u = \ln x\,,\ du = \frac{1}{x}dx\,,\ dv = x^2 dx\,,\ v = \frac{x^3}{3}\right] =$$

$$= \frac{x^3}{3}\ln x\Big|_1^e - \frac{1}{3}\int_1^e x^2 dx = \frac{e^3}{3} - \frac{x^3}{9}\Bigg|_1^e = \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{2}{9}e^3 + \frac{1}{9}$$

Demak,

$$\Im = \frac{e^3}{3} - \frac{2}{3}\left(\frac{2}{9}e^3 + \frac{1}{9}\right) = \frac{e^3}{3} - \frac{4}{27}e^3 - \frac{2}{27} = \frac{1}{27}\left(5e^3 - 2\right).\blacktriangleright$$

**5 – m i s o l .** ***Ushbu***

$$S_n = \frac{1^\alpha + 2^\alpha + \ldots + n^\alpha}{n^{\alpha+1}} \qquad (\alpha > 0)$$

***yig‘indining limiti aniq integral yordamida topilsin.***

◀Avvalo berilgan yig‘indini quyidagicha yozib olamiz:

$$S_n = \frac{1^\alpha + 2^\alpha + \ldots + n^\alpha}{n^{\alpha+1}} = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^\alpha \cdot \frac{1}{n}$$

Endi $f(x) = x^\alpha$ $(\alpha > 0)$ funksiyani $[0, 1]$ segmentda qaraymiz. Ravshanki, bu funksiya $[0, 1]$ segmentda integrallanuvchi bo‘ladi. $[0, 1]$ segmentni $n$ ta teng bo‘lakka bo‘lib, ushbu

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{k-1}{n}, \frac{k}{n}, \ldots \frac{n}{n} = 1\right\}$$

bo‘laklashni hosil qilamiz. Har bir

$$\left[\frac{k-1}{n}, \frac{k}{n}\right] \quad (k = 1, 2, 3, \ldots, n)$$

bo‘laklashda $\xi_k = \dfrac{k}{n}$ deb, $P$ bo‘laklashga nisbatan $f(x) = x^\alpha$ funksiyaning integral yig‘indisini tuzamiz:

$$\sigma = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n}\right) \cdot \frac{1}{n}$$

Demak, yuqoridagi $S_n$ yig‘indi $\sigma$ integral yig‘indidan iborat ekan:

$$S_n = \sigma$$

$f(x) = x^\alpha$ funksiya $[0, 1]$ da integrallanuvchi bo‘lganligi sababli

$$\lim_{n\to\infty} \sigma = \int_0^1 x^\alpha dx$$

bo‘ladi. SHuni e’tiborga olib topamiz:

$$\lim_{n\to\infty} S_n = \int_0^1 x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}\Bigg|_0^1 = \frac{1}{\alpha+1}. \blacktriangleright$$

**Integrallar hisoblansin.**

1. $\int_{-1}^{2} x^3 dx$
2. $\int_{1}^{4} \frac{dx}{\sqrt[3]{x}}$
3. $\int_{1}^{2} (x^2 - 2x + 3) dx$
4. $\int_{-1}^{1} (x^3 - 2x^2 + x - 1) dx$
5. $\int_{0}^{1} \left(\sqrt{x} + \sqrt[3]{x^2}\right) dx$
6. $\int_{0}^{\pi} \sin x dx$

7. $\int\limits_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

8. $\int\limits_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

9. $\int\limits_{-\pi/4}^{0} \frac{dx}{\cos^2 x}$

10. $\int\limits_{2}^{9} \sqrt[3]{x-1}dx$

11. $\int\limits_{1}^{2} e^x dx$

12. $\int\limits_{0}^{2} 2^x dx$

13. $\int\limits_{2}^{4} \frac{dx}{x}$

14. $\int\limits_{1}^{2} \frac{dx}{2x-1}$

15. $\int\limits_{0}^{1} \frac{x^2 dx}{1+x^6}$

16. $\int\limits_{e}^{e^2} \frac{dx}{x\ln x}$

17. $\int\limits_{e^2}^{e^3} \frac{dx}{x\ln x}$

18. $\int\limits_{-\pi}^{\pi} \sin^2 x dx$

19. $\int\limits_{-\pi}^{\pi} \cos^2 x dx$

20. $\int\limits_{\pi/6}^{\pi/3} tg^4 x dx$

21. $\int\limits_{0}^{2} sh^3 x dx$

22. $\int\limits_{0}^{1} \frac{dx}{4x^2+4x+5}$

23. $\int\limits_{3}^{4} \frac{x^2+3}{x-2} dx$

24. $\int\limits_{-2}^{-1} \frac{x+1}{x^2(x-1)} dx$

25. $\int\limits_{2}^{3} \frac{dx}{x^2-2x-8}$

26. $\int\limits_{0}^{2} \frac{2x-1}{2x+1} dx$

27. $\int\limits_{0}^{1} \frac{x^2+3x}{(x+1)(x^2+1)} dx$

28. $\int\limits_{0}^{1} \frac{dx}{\sqrt{x^2+2x+2}}$

29. $\int\limits_{3/4}^{2} \frac{dx}{\sqrt{2+3x-2x^2}}$

30. $\int\limits_{0}^{4} \frac{dx}{1+\sqrt{x}}$

31. $\int\limits_{0}^{\pi/4} \frac{dx}{1+2\sin^2 x}$

32. $\int\limits_{0}^{1} xe^{-x} dx$

33. $\int\limits_{\pi/4}^{\pi/3} \frac{xdx}{\sin^2 x}$

34. $\int\limits_{1}^{3} \ln xdx$

35. $\int\limits_{1}^{2} x\ln xdx$

36. $\int\limits_{0}^{1/2} \arcsin xdx$

37. $\int\limits_{0}^{\pi} x^3 \sin xdx$

38. $\int\limits_{0}^{\pi/2} e^{2x}\cos xdx$

39. $\int\limits_{0}^{e} \sin\ln xdx$

40. $\int\limits_{0}^{2} |1-x|dx$

**41.**Tengsizliklar isbotlansin:

a) $\frac{1}{10\sqrt{2}} < \int\limits_{0}^{1} \frac{x^9}{\sqrt{1+x}}dx < \frac{1}{10}$

b) $\frac{1}{20\sqrt[3]{2}} < \int\limits_{0}^{1} \frac{x^{19}}{\sqrt[3]{1+x^6}}dx < \frac{1}{20}$

v) $0 < \int\limits_{0}^{200} \frac{e^{-5x}dx}{x+20} < 0{,}01$

g) $1 < \int\limits_{0}^{1} \frac{1+x^{20}}{1+x^{40}}dx < 1+\frac{1}{42}$

d) $1-\frac{1}{n} < \int\limits_{0}^{1} e^{-x^n}dx < 1, \quad n>1$

e) $0 < \int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin x}{x}dx < \ln 3$

j) $\sin 1 < \int\limits_{-1}^{1} \frac{\cos x}{1+x^2}dx < 2\sin 1$

z) $\frac{2}{\pi}\ln\frac{\pi+2}{2} < \int\limits_{0}^{\frac{\pi}{2}} \frac{\sin x}{x(x+1)}dx < \ln\frac{\pi+2}{3}$

i) $\frac{1}{8} < \frac{\pi}{6}\int\limits_{0}^{2} \frac{\sin\frac{\pi}{6}(x+1)}{(x+1)(3-x)}dx < \frac{1}{6}$

k) $0{,}03 < \int\limits_{0}^{1} \frac{x^7}{(e^x+e^{-x})\sqrt{1+x^2}}dx < 0{,}05$

**42.**Integrallar hisoblansin:

a) $\int\limits_{0}^{2} f(x)dx, \quad f(x)=\begin{cases} x^2, \text{ агар } 0\le x\le 1, \\ 2-x, \text{ агар } 1<x\le 2. \end{cases}$

b) $\int\limits_{0}^{1} f(x)dx, \quad f(x)=\begin{cases} x, \text{ агар } 0\le x\le t, \\ t\frac{1-x}{1-t}, \text{ агар } t<x\le 1. \end{cases}$

**43.** Tengliklarning nima uchun noto‘g‘riligi tushuntirilsin.

a) $\int_{-1}^{1} \frac{d}{dx}\left(\operatorname{arctg}\frac{1}{x}\right)dx = \operatorname{arctg}\frac{1}{x}\Big|_{-1}^{1} = \frac{\pi}{2}$

b) $\int_{-1}^{1} \frac{dx}{x} = \ln|x|\Big|_{-1}^{1} = 0$

v) $\int_{0}^{2\pi} \frac{\sec^2 x dx}{2+\operatorname{tg}^2 x} = \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\operatorname{tg}^2 x}{\sqrt{2}}\Big|_{-1}^{1} = 0$

**Aniq integrallar yordamida quyidagi limitlar hisoblansin:**

**44.** $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}\right)$

**45.** $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$

**46.** $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)$

**47.** $\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)$

**48.** $\lim_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}} \quad (p>0)$

**49.** $\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\cdots+\sqrt{1+\frac{n}{n}}\right)$

**50.** Limitlar hisoblansin:

a) $\lim_{x\to 0}\frac{\int_0^x \cos t^2 dt}{x}$

b) $\lim_{x\to+\infty}\frac{\int_0^x (\operatorname{arctg} t)^2 dt}{\sqrt{x^2+1}}$

v) $\lim\limits_{x\to+\infty}\dfrac{\left(\int\limits_0^x e^{t^2}dt\right)^2}{\int\limits_0^x e^{2t^2}dt}$ g) $\lim\limits_{x\to+0}\dfrac{\int\limits_0^{\sin x}\sqrt{\operatorname{tg} t}\,dt}{\int\limits_0^{\operatorname{tg} x}\sqrt{\sin t}\,dt}$

**O'zgaruvchilarni almashtirish va bo'laklab integrallash usullari yordamida hisoblansin:**

51. $\int\limits_{-\pi}^{\pi}\sqrt[3]{\sin x}\,dx$

52. $\int\limits_{-\pi}^{\pi}e^{x^2}\sin x\,dx$

53. $\int\limits_{-\pi/2}^{\pi/2}\left(\cos^2 x + x^2\sin x\right)dx$

54. $\int\limits_{-1}^{1}\cos x\,\operatorname{th} x\,dx$

55. $\int\limits_{-1}^{1}\left(e^x + e^{-x}\right)\operatorname{tg} x\,dx$

56. $\int\limits_{-\pi/3}^{\pi/3}\left(x^2\sin 5x + \cos\dfrac{x}{3} + \operatorname{tg}^3 x\right)dx$

57. $\int\limits_{-\pi/3}^{\pi/3}\dfrac{2x^7 - x^5 + 2x^3 - x + 1}{\cos^2 x}dx$

58. $\int\limits_0^2 e^{x^2}x\,dx$

59. $\int\limits_0^1 x^2\sqrt{1-x^2}\,dx$

60. $\int\limits_0^{\ln 2}\sqrt{e^x - 1}\,dx$

61. $\int\limits_0^{\ln 2}xe^{-x}dx$

62. $\int\limits_0^{\pi}x\sin x\,dx$

63. $\int\limits_0^{\pi/4}x\sin 2x\,dx$

64. $\int\limits_0^{2\pi}x^2\cos x\,dx$

65. $\int\limits_0^1 \arccos x\,dx$

66. $\int\limits_1^3 \operatorname{arctg}\sqrt{x}\,dx$

67. $\int\limits_1^e \dfrac{dx}{x\sqrt{1+\ln x}}$

68. $\int\limits_0^1 \dfrac{e^{2x}+2e^x}{e^{2x}+1}dx$

69. $\int\limits_0^{\pi/2}\dfrac{dx}{2-\sin x}$

70. $\int\limits_{\pi/3}^{\pi/2}\dfrac{dx}{3+\cos x}$

71. $\int\limits_{0}^{3/4} \frac{dx}{(x+1)\sqrt{x^2+1}}$

72. $\int\limits_{1}^{9} x\cdot\sqrt[3]{1-x}dx$

73. $\int\limits_{-3}^{-2} \frac{dx}{x\sqrt{x^2-1}}$

74. $\int\limits_{0}^{1} x^{15}\sqrt{1+3x^8}dx$

75. $\int\limits_{0}^{\pi/2} \sin x\sin 2x\sin 3xdx$

76. $\int\limits_{0}^{\ln 2} sh^4 xdx$

77. $\int\limits_{1}^{2} x^2\ln xdx$

78. $\int\limits_{1}^{n} x^n\ln xdx$

79. $\int\limits_{0}^{\sqrt{3}} xarctgxdx$

80. $\int\limits_{0}^{1} x\left(2-x^2\right)^{12}dx$

81. $\int\limits_{0}^{1} \arcsin\sqrt{x}dx$

82. $\int\limits_{0}^{\pi} e^x\cos^2 xdx$

**83.**Tenglik isbotlansin.

$$\int\limits_{0}^{1}(1-x)^m x^n dx=\frac{m!n!}{(m+n+1)!},\quad m\in N,\, n\in N.$$

**84.**Integrallar hisoblansin:

a) $\int\limits_{0}^{\pi}(x\sin x)^2 dx$

b) $\int\limits_{0}^{\pi}(x\cos x)^2 dx$

85. $$J_n=\int\limits_{0}^{\pi/2}\sin^n xdx,\quad n\ge 2$$

integral uchun

$$J_n=\frac{n-1}{n}J_{n-2}$$

rekurrent formula isbotlansin.

**86.**Isbotlansin:

$$\int_0^{\pi/2}\sin^n x dx=\int_0^{\pi/2}\cos^n x dx=\begin{cases}\frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}, \text{ агар } n-\text{жуфт бўлса},\\ \frac{(n-1)!!}{n!!}, \text{ агар } n-\text{ток бўлса}, n\in N.\end{cases}$$

**87.**Isbotlansin:

$$\lim_{n\to\infty}\frac{1}{2n+1}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2=\frac{\pi}{2}$$

**Ko‘rsatma:** 86-misol natijasidan foydalaning.

**88.**Isbotlansin:

$$\int_0^{\frac{\pi}{2}}\sin^m x\cos^n x dx=\begin{cases}\frac{(m-1)!!(n-1)!!}{(m+n)!!}\cdot\frac{\pi}{2}, & \text{агар } m \text{ ва } n \text{ жуфт бўлса},\\ \frac{(m-1)!!(n-1)!!}{(m+n)!!} & \text{бошка барча холларда, } m\in N, n\in N.\end{cases}$$

**89.** $n\in N$ uchun quyidagi tengliklar isbotlansin:

a) $\int_0^a (a^2-x^2)^n dx=a^{2n+1}\frac{(2n)!!}{(2n+1)!!}$  b) $\int_0^a (a^2-x^2)^{\frac{2n-1}{2}} dx=a^{2n}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}$

**90.**Isbotlansin:

a) $\int_0^{\pi/2}\cos^n x\sin nx dx=\frac{1}{2^{n+1}}\sum_{k=1}^{n}\frac{2^k}{k}, n\in N$  b) $\int_0^{\pi/2}\cos^n x\cos nx dx=\frac{\pi}{2^{n+1}}, n\in N$

**91.** $m\in N$ uchun tengliklar isbotlansin:

a) $\int_0^{\pi/2}\cos^m x\cos(m+2)x dx=0$  b) $\int_0^{\pi/2}\cos^m x\sin(m+2)x dx=\frac{1}{m+1}$

v) $\int_0^{\pi/2}\sin^m x\cos(m+2)x dx=-\frac{\sin\frac{m\pi}{2}}{m+1}$  g) $\int_0^{\pi/2}\sin^m x\sin(m+2)x dx=\frac{1}{m+1}\cos\frac{m\pi}{2}$

**92.** $f$ funksiya $[a,b]$ kesmada integrallanuvchi, $F$ funksiya $[a,b]$ kesmada chekli sondagi $x_1, \dots, x_n$ birinchi tur uzilish nuqtalariga ega bo‘lsin. Agar $F$ funksiya $[a,b]$ kesmaning chekli sondagi ichki nuqtalaridan boshqa barcha nuqtalarida differensiallanuvchi va $F'(x)=f(x)$ bo‘lsa, u holda quyidagi formula o‘rinli ekanligi isbotlansin:

$$\int_a^b f(x)dx = F(b-0)-F(b+0)-\sum_{k=1}^{n}[F(x_k+0)-F(x_k-0)]$$

($F$ funksiya $f$ funksiyaning **umumlashgan boshlang‘ich funksiyasi** deyiladi.)

**93.**Quyidagi uzilishga ega funksiyalarning uzluksiz umumlashgan boshlang‘ich funksiyalari topilsin:

a) $\operatorname{sign}x$ b) $\operatorname{sign}(\sin x)$ v) $[x]$ g) $x[x]$ d) $(-1)^{[x]}$

**94.** $\int_0^x f(t)dt$ integral hisoblansin. Bu erda

$$f(t)=\begin{cases} 1, \text{агар } |t|<l, \\ 0, \text{агар } |t|>l, \quad l\ge 0 \end{cases}$$

**95.**Hisoblansin:

a) $\int_0^3 \operatorname{sign}(x-x^3)dx$ b) $\int_0^2 [e^x]dx$

v) $\int_0^6 [x]\sin\frac{\pi x}{6}dx$ d) $\int_1^{n+1} \ln[x]dx, \quad n\in N$

g) $\int_0^{\pi} x\operatorname{sign}(\cos x)dx$ e) $\int_0^1 \operatorname{sign}(\sin\ln x)dx$

## Aniq integralning xossalari. O‘rta qiymat haqidagi teoremalar

**$1^0$. Aniq integralning asosiy xossalari.** $[a,b]$ segmentda integrallanuvchi funksiyalar sinfini $R([a,b])$ kabi belgilaymiz.

**1-xossa.** Agar $f(x)\in R([a,b])$, $c\in R$ bo‘lsa, u holda $c\cdot f(x)\in R([a,b])$ bo‘lib,

$$\int_a^b (c\cdot f(x))dx = c\int_a^b f(x)dx$$

bo‘ladi. (integralning **chiziqlilik** xossasi)

**2-xossa.** Agar $f(x)\in R([a,b])$, $g(x)\in R([a,b])$ bo‘lsa, u holda $f(x)+g(x)\in R([a,b])$ bo‘lib,

$$\int_a^b (f(x)+g(x))dx=\int_a^b f(x)dx+\int_a^b g(x)dx$$

bo‘ladi. (integralning **additivlik** xossasi)

**3-xossa.** Agar $f(x)\in R([a,b])$, $a<c<b$ bo‘lsa, u holda $f(x)\in R([a,c])$, $f(x)\in R([c,b])$ bo‘lib,

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

bo‘ladi.

**4-xossa.** Agar $f(x)\in R([a,b])$, $g(x)\in R([a,b])$ bo‘lib, $\forall x\in[a,b]$ da $f(x)\le g(x)$ bo‘lsa, u holda

$$\int_a^b f(x)dx\le\int_a^b g(x)dx$$

bo‘ladi. (integralning **monotonlik** xossasi)

**Natija.** Agar $f(x)\in R([a,b])$ bo‘lib, $f(x)\ge 0$ bo‘lsa,

$$\int_a^b f(x)dx\ge 0$$

bo‘ladi.

**$2^0$.Funksiyaning o‘rta qiymati va o‘rta qiymat haqidagi teoremalar.** Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan bo‘lib, u shu segmentda integrallanuvchi bo‘lsin.

Ushbu

$$M[f]=\frac{1}{b-a}\int_a^b f(x)dx$$

miqdor $f(x)$ funksiyaning $[a,b]$ dagi **o‘rta qiymati** deyiladi.

Aytaylik, $f(x)$ funksiya $[a,b]$ da chegaralangan bo‘lib, $\forall x\in[a,b]$ da $m\le f(x)\le M$ bo‘lsin.

**1 – t e o r e m a .** ***Agar $f(x)\in R([a,b])$ bo‘lsa, u holda shunday $\mu$ son $(m\le\mu\le M)$ topiladiki,***

$$\int_a^b f(x)dx=\mu\cdot(b-a)$$

***bo‘ladi.***

**2 – t e o r e m a .** ***Agar $f(x)\in C([a,b])$, $g(x)\in C([a,b])$ bo‘lib, $\forall x\in[a,b]$ da $g(x)>0$ yoki $g(x)<0$ bo‘lsa, u holda shunday $\mu$ son $(m\le\mu\le M)$ topiladiki,***

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx$$

*bo'ladi.*

**1 – m i s o l .** *Agar* $f(x)\in C([a,b])$, $g(x)\in C([a,b])$ *bo'lsa, ushbu*

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx \qquad (*)$$

***tengsizlikning o'rinli bo'lishi isbotlansin.***

◄Aniq integralning xossalariga ko'ra

$$f(x)-\alpha\cdot g(x)\in R([a,b]) \qquad (\alpha\in R)$$

bo'lib,

$$\int_a^b [f(x)-\alpha\cdot g(x)]^2dx \ge 0$$

bo'ladi. Bu munosabatdan

$$\alpha^2\int_a^b g^2(x)dx - 2\alpha\int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \ge 0$$

bo'lishi kelib chiqadi.

Ma'lumki, kvadrat uchhad manfiy bo'lmasa, uning diskriminanti musbat bo'lmaydi. Demak,

$$\left[\int_a^b f(x)g(x)dx\right]^2 - \int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx \le 0$$

Bu tengsizlikdan topamiz:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx$$ ►

(*) tengsizlik **Koshi-Bunyakovskiy tengsizligi** deyiladi.

**2 – m i s o l .** *Ushbu*

$$\int_0^\pi x\cdot\sqrt{\sin x}dx \le \frac{2}{3}\pi^3$$

***tengsizlik isbotlansin.***

◄Agar (*) tengsizlikda $f(x)=x$, $g(x)=\sqrt{\sin x}$ deyilsa, u holda

$$\int_0^\pi x\cdot\sqrt{\sin x}dx \le \int_0^\pi x^2dx\cdot\int_0^\pi \sin xdx = \frac{2}{3}\pi^3$$

bo'ladi.►

**3 – m i s o l .** *Agar*

$$f(x)=e^{2x},\ a=0,\ b=1$$

*bo'lsa, u holda* $c$ *ning qanday qiymatida ushbu*

$$\int_a^b f(x)dx = f(c)\cdot(b-a)$$

***tenglik o'rinli bo'ladi?***

◄Ravshanki,

$$\int_0^1 e^{2x}dx = \frac{e^{2k}}{2}\Big|_0^1 = \frac{e^2-1}{2}$$

bo'ladi. Demak,

$$e^{2c} = \frac{e^2-1}{2}$$

Bu tenglikdan $c$ ni topamiz:

$$2c = \ln\frac{e^2-1}{2}, \quad c = \frac{1}{2}\ln\frac{e^2-1}{2}$$ ►

**4 – m i s o l.** ***O'rta qiymat haqidagi teoremadan foydalanib, ushbu***

$$\int_a^b \frac{\sin x}{\sqrt{x}}dx \qquad (0<a<b)$$

***integral baholansin.***

◄O'rta qiymat haqidagi 2-teoremada $f(x)$ funksiya $[a,b]$ da uzluksiz bo'lsa, u holda shunday $c$ nuqta $(a<c<b)$ topiladiki,

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

bo'ladi. SHu tenglikdan foydalanib topamiz:

$$\int_a^b \frac{1}{\sqrt{x}}\sin x dx = \frac{1}{\sqrt{c}}\int_a^b \sin x dx = \frac{1}{\sqrt{c}}(\cos a - \cos b)$$

Agar $0<a<b$ bo'lganda

$$\frac{1}{\sqrt{c}} < \frac{1}{\sqrt{a}} \qquad \text{va} \qquad |\cos a - \cos b| \le 2$$

bo'lishini e'tiborga olsak, u holda

$$\left|\int_a^b \frac{\sin x}{\sqrt{x}}dx\right| \le \frac{2}{\sqrt{a}}$$

bo'lishi topiladi.►

**96.** Qaysi biri katta ekanligi aniqlansin:

a) $\int_0^{\pi/2} \frac{\sin x}{x}dx$ ёки $\int_0^{\pi} \frac{\sin x}{x}dx$ b) $\int_{1/2}^1 \frac{dx}{\sqrt{x}}$ ёки $\int_{1/2}^1 \frac{dx}{\sqrt[3]{x}}$

v) $\int_0^1 e^{-x}\sin x dx$ ёки $\int_0^1 e^{-x^2}\sin x dx$ g) $\int_1^2 \frac{dx}{\sqrt{1+x^2}}$ ёки $\int_1^2 \frac{dx}{x}$

**97.** Tengsizliklar isbotlansin:

a) $0 < \int_0^{\pi} \frac{\sin x}{\sqrt[5]{x^2+2}} dx < \frac{\pi}{\sqrt[5]{2}}$

b) $\frac{1}{\sqrt[3]{9}} < \frac{1}{\pi}\int_{-1}^{1} \frac{\pi + \operatorname{arctg} x}{\sqrt[3]{x^2+8}} dx < \frac{3}{2}$

v) $\frac{\sqrt{2}}{3} < \int_{-1}^{1} \frac{\cos x}{2+x^2} dx < 1$

**98.** $f$ funksiya $[a,b]$ kesmada integrallanuvchi bo‘lsin.

$$\int_a^b f^2(x)dx = 0$$

tenglik bajarilishi uchun, $f$ funksiyaning barcha uzluksiz nuqtalarida $f(x)=0$ bo‘lishi zarur va etarli ekanligi isbotlansin.

**99.** Berilgan funksiyalarning ko‘rsatilgan oraliqlardagi o‘rta qiymatlari aniqlansin:

a) $f(x) = x^2, \quad [0, 1];$ b) $f(x) = \sqrt{x}, \quad [0, 100];$

v) $f(x) = 10 + 2\sin x + 3\cos x, \quad [0, 2\pi];$ g) $f(x) = \sin x \sin(x+\varphi), \quad [0, 2\pi]$

**O‘rta qiymat haqidagi birinchi teoremadan foydalanib, integrallar baholansin:**

**100.** $\int_0^{2\pi} \frac{dx}{1+0{,}5\cos x}$ **101.** $\int_0^1 \frac{x^9}{\sqrt{1+x}} dx$ **102.** $\int_0^{100} \frac{e^{-x}}{x+100} dx$

**O‘rta qiymat haqidagi ikkinchi teoremadan foydalanib, integrallar baholansin:**

**103.** $\int\limits_{100\pi}^{200\pi} \frac{\sin x}{x} dx$

**104.** $\int\limits_{a}^{b} \frac{e^{-\alpha x}}{x} \sin x dx \quad (\alpha \ge 0 ; 0 < a < b)$

# Test

| | | | | |
|---|---|---|---|---|
| $\int\limits_0^2 x^2 dx$ | $\frac{8}{3}$ | -4 | 4 | $\frac{16}{3}$ |
| $\int\limits_{-1}^0 3(2x+1)^2 dx$ | 1 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $\int\limits_{-1}^0 e^{-x} dx$ | $-1+e$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int\limits_1^2 \frac{dx}{x+1}$ | $\ln\frac{3}{2}$ | 0 | $\ln 3$ | $\ln 3-1$ |
| $\int\limits_0^\pi \sin x dx$ | 2 | 1 | -1 | -2 |
| $\int\limits_0^2 x^3 dx$ | 4 | 2 | 5 | 1 |
| $\int\limits_0^\pi \sin 2x dx$ | 0 | -1 | 1 | 2 |
| $\int\limits_{\pi/2}^\pi \cos x dx$ | -1 | 0 | 1 | -2 |
| $\int\limits_3^5 \frac{dx}{x-1}$ | $\ln 2$ | $\ln\frac{3}{2}$ | $\ln\frac{4}{3}$ | 1 |
| $\int\limits_{-1}^0 (2x+1)^2 dx$ | $\frac{1}{3}$ | 1 | -3 | -1 |
| $\int\limits_{-1}^0 2e^{-x} dx$ | $-2+2e$ | $e$ | $\frac{1}{e}-1$ | 5 |

# Mavzu. Chegaralari o'zgaruvchi bo'lgan aniq integrallar

## 10-ma'ruza

REJA

**$1^0$.** Uzluksizligi.

**$2^0$.** Differensiallanuvchanligi.

**$1^0$. Uzluksizligi.** ***([1], 11.9 The two fundamental theorems of calculus, 338-bet)*** $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsin. U holda aniq integrallning 1)– xossasiga ko'ra $f(x)$ funksiya istalgan $[a,x]\subset[a,b]$ $(a\le x\le b)$ oraliqda ham integrallanuvchi bo'ldi. Ravshanki,

$$\int_a^x f(t)dt$$

integral $x$ ga bog'liq. Uni $F(x)$ deb belgilaymiz:

$$F(x)=\int_a^x f(t)dt$$

Endi $f(x)$ funksiyaga ko'ra $F(x)$ funksiyaning xossalarini (uzluksizligi, differensiallanuvchi bo'lishini) o'rganamiz.

**8—teorema.** *Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lsa, $F(x)$ funksiya shu oraliqda uzluksiz bo'ladi.*

◄ $f(x)$ funksiya integrallanuvchi bo'lgani uchun $\sup\{f(x)\}=M<\infty$ bo'ladi. $\forall x\in[a,b]$ nuqta olib, unga shunday $\Delta x>0$ orttirma beraylikki, $(x+\Delta x)\in[a,b]$ bo'lsin. U holda $F(x)$ funksiyaning orttirmasi uchun quyida-giga ega bo'lamiz:

$$\Delta F(x) = F(x+\Delta x) - F(x) = \int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = \int_x^{x+\Delta x} f(t)dt .$$

Aniq integrallning 7)–xossasidan foydalanib, topamiz:

$$|\Delta F(x)| = \left| \int_x^{x+\Delta x} f(t)\,dt \right| \le \int_x^{x+\Delta x} |f(t)|\,dt \le M \int_x^{x+\Delta x} dt = M \cdot \Delta x$$

Demak,

$$|\Delta F(x)| \le M \cdot \Delta x.$$

Bundan esa

$$\lim_{\Delta x \to +0} \Delta F(x) = 0$$

limit kelib chiqadi. $\Delta x < 0$ bo'lganda ham xuddi yuqoridagiga o'xshash $\lim_{\Delta x \to -0} \Delta F(x) = 0$ bo'lishi ko'rsatiladi. Bu esa $F(x)$ funksiyaning $x \in [a,b]$ nuqtada uzluksizligini bildiradi. ►

**2⁰. Differensiallanuvchanligi.**

**9—teorema.** ***([1], Theorem 11.9.1 (First Fundamental Theorem of Calculus, 338-bet)*** *Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lib, $x_0 \in [a,b]$ nuqtada uzluksiz bo'lsa, u holda $F(x)$ funksiya $x_0$ nuqtada differensialanuvchi bo'ladi va*

$$F'(x_0) = f(x_0).$$

◄ $F(x)$ funksiyaning $x_0$ nuqtadagi orttirmasi:

$$\Delta F(x_0) = \int_{x_0}^{x_0+\Delta x} f(t)dt \qquad (\Delta x > 0)$$

ni olib, quyidagi

$$\frac{\Delta F(x_0)}{\Delta x} - f(x_0)$$

ayrimani qaraymiz. Aniq integrallning xossalaridan foydalanib topamiz:

$$\frac{\Delta F(x_0)}{\Delta x} - f(x_0) = \frac{1}{\Delta x}\left[ \int_{x_0}^{x_0+\Delta x} f(t)dt - f(x_0)\int_{x_0}^{x_0+\Delta x} dt \right] = \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} (f(t) - f(x_0))dt.$$

Bu munosabatdan

$$\left|\frac{\Delta F(x_0)}{\Delta x}-f(x_0)\right| \le \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}\left|f(t)-f(x_0)\right|dt. \qquad (9.21)$$

tengsizlik kelib chiqadi

Shartga ko'ra $f(x)$ funksiya $x_0$ nuqtada uzluksiz. Ta'rifga asosan, $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ son topiladiki $|x-x_0|<\delta$ bo'lganda $|f(x)-f(x_0)|<\varepsilon$ bo'ladi. Agar $\Delta x<\delta$ deb olsak, u holda $\forall t\in[x_0, x_0+\Delta x]$ uchun

$$|f(t)-f(x_0)|<\varepsilon$$

bo'ladi. Natijada (9.21) tengsizlik quyidagi

$$\left|\frac{\Delta F(x_0)}{\Delta x}-f(x_0)\right| < \frac{\varepsilon}{\Delta x}\int_{x_0}^{x_0+\Delta x}dt$$

ko'rinishga keladi. Demak,

$$\left|\frac{\Delta F(x_0)}{\Delta x}-f(x_0)\right| < \varepsilon .$$

Bundan

$$\lim_{\Delta x\to+0}\frac{\Delta F(x_0)}{\Delta x}=f(x_0)$$

ya'ni

$$F'(x_0+0)=f(x_0)$$

tenglik kelib chiqadi. Yuqordagidek $\Delta x<0$ bo'lganda

$$\lim_{\Delta x\to-0}\frac{\Delta F(x_0)}{\Delta x}=f(x_0)$$

ya'ni

$$F'(x_0-0)=f(x_0)$$

tenglik ham o'rinli bo'lishi ko'rsatiladi. ►

Agar $f(x)$ funksiya $[a,b]$ oraliqda integrallanuvchi bo'lib, $x=a$

va $x=b$ nuqtalarda uzluksiz (bunda funksiyaning $x=a$ da o'ngdan $x=b$ da esa chapdan uzluksizligi tushuniladi) bo'lsa, u holda

$$F'(a+0)=f(a+0),\ F'(b-0)=f(b-0)$$

bo'lishi yuqoridagiga o'xshash ko'rsatiladi.

**6–natija.** $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsa, u holda $\forall x\in[a,b]$ uchun

$$F'(x)=f(x)$$

bo'ladi.

Demak, $[a,b]$ oraliqda uzluksiz $f(x)$ funkisya shu oraliqda boshlang'ich funkisyaga ega, jumladan $F(x)$ funksiya $f(x)$ ning $[a,b]$ dagi boshlang'ich funksiyasi bo'ladi.

Endi quyi chegarasi o'zgaruvchi bo'lgan integralni qaraymiz. $f(x)$ funkisya $[a,b]$ oraliqda integrallanuvchi bo'lsin. U holda bu funksiya $[x,b]\subset[a,b]$ $(a\le x\le b)$ oraliqda ham integrallanuvchi va bu integral $x$ ga bog'liq bo'ladi. Uni

$$\phi(x)=\int_x^b f(t)dt$$

deb belgilaymiz. Aniq integral xossasidan foydalanib topamiz.

$$\int_a^b f(t)dt=\int_a^x f(t)dt+\int_x^b f(t)dt=F(x)+\phi(x) \qquad (a\le x\le b).$$

Bundan esa

$$\phi(x)=\int_a^b f(t)dt-F(x)$$

bo'lishi kelib chiqadi. Bu tenglik $\phi(x)$ funksiyaning xossalarini $f(x)$ hamda $F(x)$ funksiyalarning xossalari orqali o'rganish mumkinligini ko'rsatadi. Jumladan, agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz bo'lsa, u holda

$$\phi'(x)=-f(x)$$

bo'ladi. Haqiqatan ham, bu holda $\int_a^b f(t)dt$ mavjud va u chekli son, $F(x)$ funksiya esa yuqorida keltirilgan teoremaga ko'ra $[a,b]$ da $F(x)$ hosilaga ega bo'ladi.

**Adabiyotlar**


1. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014
2. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, I q.* T. "Voris-nashriyot", 2010.
3. **Фихтенгольц Г. М.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.


# Glossariy

**Darbu yig'indilari** – Ushbu $s=\sum_{k=0}^{n-1} m_k \Delta x_k$, $S=\sum_{k=0}^{n-1} M_k \Delta x_k$ yig'indilar mos ravishda Darbuning quyi hamda yuqori yig'indilari deb ataladi.

# Keys banki

**36-keys.** Masala o`rtaga tashlanadi: Ushbu integral

$$\int_a^b f(x)dx=\int_a^b \frac{|x|}{x}dx$$

hisoblansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 10-amaliy mashg'ulot

Funksiyaning aniq integrallari asosan Nyuton-Leybnits formulasi, o'zgaruvchilarini almashtirish va bo'laklab integrallash usullari (formulalari) yordamida hisoblanadi.

**$1^0$. Nyuton-Leybnits formulasi.** Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda uzluksiz bo'lib, $F(x)$ funksiya esa uning boshlang'ich funksiyasi bo'lsin $(F'(x)=f(x), \quad x\in[a,b])$. U holda

$$\int_a^b f(x)dx = F(b)-F(a) = F(x)\Big|_a^b \qquad (2)$$

bo'ladi.

Aniq integralni hisoblash imkonini beradigan (2) formula **Nyuton-Leybnits formulasi** deyiladi.

**$2^0$. O'zgaruvchini almashtirish formulasi.** Faraz qilaylik:

5) $f(x)$ funksiya $[a,b]$ segmentda uzluksiz;
6) $\varphi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz va uzluksiz $\varphi'(t)$ hosilaga ega;
7) $\varphi(\alpha)=a, \ \varphi(\beta)=b$;
8) $f(\varphi(t))$ murakkab funksiya $[a,b]$ da aniqlangan va uzluksiz bo'lsin. U holda

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)dt \qquad (3)$$

bo'ladi.

(3) formula **o'zgaruvchini almashtirish formulasi** deyiladi.

**$3^0$. Bo'laklab integrallash formulasi.** Aytaylik, $f(x)$ va $g(x)$ funksiyalar $[a,b]$ segmentda uzluksiz va uzluksiz $f'(x), g'(x)$ hosilalarga ega bo'lsin. U holda

$$\int_a^b f(x)\cdot g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx \qquad (4)$$

bo'ladi.

(4) formula **bo'laklab integrallash formulasi** deyiladi.

**Eslatma:** Ayrim hollarda yig'indining limiti aniq integralga keltirib hisoblanadi.

**1 – m i s o l .** ***Ushbu integral***

$$\int_a^b f(x)dx = \int_a^b \frac{|x|}{x}dx$$

***hisoblansin.***

◄Aytaylik, $0 \le a \le b$ bo‘lsin. Bu holda $x > 0$ bo‘lib, $f(x) = \frac{|x|}{x} = 1$ bo‘ladi. Demak,

$$\int_a^b \frac{|x|}{x} dx = \int_a^b dx = x\Big|_a^b = b - a$$

Aytaylik, $a < b < 0$ bo‘lsin. Bu holda $x < 0$ bo‘lib, $f(x) = \frac{|x|}{x} = -1$ bo‘ladi. Demak,

$$\int_a^b \frac{|x|}{x} dx = \int_a^b (-1) dx = -x\Big|_a^b = a - b$$

Aytaylik, $a < 0 < b$ bo‘lsin. Bu holda

$$f(x) = \frac{|x|}{x} = \begin{cases} -1, \text{агар} & a < x < 0 \\ 1, \text{агар} & 0 < x < b \end{cases}$$

bo‘ladi. Demak,

$$\int_a^b \frac{|x|}{x} dx = \int_a^0 (-1) dx + \int_0^b 1 dx = -x\Big|_a^0 + x\Big|_0^b = a + b$$

YUqoridagi uchta holni birlashtirib,

$$\int_a^b \frac{|x|}{x} dx = |b| - |a|$$

bo‘lishini topamiz.►

**2 – m i s o l .** ***Ushbu***

$$\int_0^{10\pi} \sqrt{1 - \cos 2x} dx$$

***hisoblansin.***

◄Ma’lumki,

$$\sqrt{1 - \cos 2x} = \sqrt{2 \sin^2 x} = \sqrt{2} \cdot |\sin x|$$

Unda

$$\int_0^{10\pi} \sqrt{1 - \cos 2x} dx = \sqrt{2} \int_0^{10\pi} |\sin x| dx$$

bo‘ladi. Aniq integralning xossalaridan foydalanib topamiz:

$$\int_0^{10\pi} |\sin x| dx = \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx + \int_{2\pi}^{3\pi} \sin dx - \int_{3\pi}^{4\pi} \sin x dx + \cdots + \int_{8\pi}^{9\pi} \sin x dx - \int_{9\pi}^{10\pi} \sin x dx =$$

$$= \underbrace{2 + 2 + \cdots + 2}_{10\text{та}} = 20.$$

Demak,

$$\int_0^{10\pi}\sqrt{1-\cos 2x}\,dx = 20\cdot\sqrt{2}\,.\blacktriangleright$$

**3 – m i s o l .** ***Ushbu***

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

***integral hisoblansin.***

◀Bu integralni o‘zgaruvchini almashtirish usulidan foydalanib hisoblaymiz:

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \left[x = a\sin t\,,\ dx = a\cos t dt\,,\ t\in\left[0,\frac{\pi}{2}\right]\right] = a^4\int_0^{\frac{\pi}{2}}\sin^2 t\cdot\cos^2 t dt =$$

$$= \frac{a^4}{4}\int_0^{\frac{\pi}{2}}\sin^2 2t dt = \frac{a^4}{4}\int_0^{\frac{\pi}{2}}\frac{1-\cos 4t}{2}dt = \frac{a^4}{8}\left(t-\frac{\sin 4t}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{a^4}{16}\pi\ \blacktriangleright$$

**4 – m i s o l .** ***Ushbu***

$$\Im = \int_1^e (x\ln x)^2 dx$$

***integral hisoblansin.***

◀Bu integralni bo‘laklab integrallash formulasidan foydalanib hisoblaymiz:

$$\Im = \int_1^e (x\ln x)^2 dx = \left[u = \ln^2 x\,,\ du = 2\ln x\cdot\frac{1}{x}dx\,,\ dv = x^2dx\,,\ v = \frac{x^3}{3}\right] =$$

$$= \frac{x^3}{3}\ln^2 x\Big|_1^e - \frac{2}{3}\int_1^e x^2\ln x dx = \frac{e^3}{3} - \frac{2}{3}\int_1^e x^2\ln x dx$$

Keyingi integral ham bo‘laklab integrallash usuli yordamida hisoblanadi:

$$\int_1^e x^2\ln x dx = \left[u = \ln x\,,\ du = \frac{1}{x}dx\,,\ dv = x^2dx\,,\ v = \frac{x^3}{3}\right] =$$

$$= \frac{x^3}{3}\ln x\Big|_1^e - \frac{1}{3}\int_1^e x^2dx = \frac{e^3}{3} - \frac{x^3}{9}\Bigg|_1^e = \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{2}{9}e^3 + \frac{1}{9}$$

Demak,

$$\Im = \frac{e^3}{3} - \frac{2}{3}\left(\frac{2}{9}e^3 + \frac{1}{9}\right) = \frac{e^3}{3} - \frac{4}{27}e^3 - \frac{2}{27} = \frac{1}{27}\left(5e^3-2\right).\blacktriangleright$$

**5 – m i s o l .** ***Ushbu***

$$S_n = \frac{1^\alpha + 2^\alpha + \ldots + n^\alpha}{n^{\alpha+1}} \qquad (\alpha > 0)$$

***yig‘indining limiti aniq integral yordamida topilsin.***

◀Avvalo berilgan yig‘indini quyidagicha yozib olamiz:

$$S_n=\frac{1^\alpha+2^\alpha+\ldots+n^\alpha}{n^{\alpha+1}}=\sum_{k=1}^{n}\left(\frac{k}{n}\right)^\alpha\cdot\frac{1}{n}$$

Endi $f(x)=x^\alpha$ $(\alpha>0)$ funksiyani $[0,1]$ segmentda qaraymiz. Ravshanki, bu funksiya $[0,1]$ segmentda integrallanuvchi bo‘ladi. $[0,1]$ segmentni $n$ ta teng bo‘lakka bo‘lib, ushbu

$$P=\left\{0,\frac{1}{n},\frac{2}{n},\ldots\frac{k-1}{n},\frac{k}{n},\ldots\frac{n}{n}=1\right\}$$

bo‘laklashni hosil qilamiz. Har bir

$$\left[\frac{k-1}{n},\frac{k}{n}\right]\quad(k=1,2,3,\ldots,n)$$

bo‘laklashda $\xi_k=\dfrac{k}{n}$ deb, $P$ bo‘laklashga nisbatan $f(x)=x^\alpha$ funksiyaning integral yig‘indisini tuzamiz:

$$\sigma=\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\cdot\Delta x_k=\sum_{k=1}^{n}\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$

Demak, yuqoridagi $S_n$ yig‘indi $\sigma$ integral yig‘indidan iborat ekan:

$$S_n=\sigma$$

$f(x)=x^\alpha$ funksiya $[0,1]$ da integrallanuvchi bo‘lganligi sababli

$$\lim_{n\to\infty}\sigma=\int_0^1 x^\alpha dx$$

bo‘ladi. SHuni e’tiborga olib topamiz:

$$\lim_{n\to\infty}S_n=\int_0^1 x^\alpha dx=\left.\frac{x^{\alpha+1}}{\alpha+1}\right|_0^1=\frac{1}{\alpha+1}.\blacktriangleright$$

**Integrallar hisoblansin.**

1. $\int_{-1}^{2} x^3dx$
2. $\int_1^4\frac{dx}{\sqrt[3]{x}}$
3. $\int_1^2\left(x^2-2x+3\right)dx$
4. $\int_{-1}^{1}\left(x^3-2x^2+x-1\right)dx$
5. $\int_0^1\left(\sqrt{x}+\sqrt[3]{x^2}\right)dx$
6. $\int_0^\pi \sin x dx$

7. $\int\limits_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

8. $\int\limits_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

9. $\int\limits_{-\pi/4}^{0} \frac{dx}{\cos^2 x}$

10. $\int\limits_{2}^{9} \sqrt[3]{x-1}dx$

11. $\int\limits_{1}^{2} e^x dx$

12. $\int\limits_{0}^{2} 2^x dx$

13. $\int\limits_{2}^{4} \frac{dx}{x}$

14. $\int\limits_{1}^{2} \frac{dx}{2x-1}$

15. $\int\limits_{0}^{1} \frac{x^2 dx}{1+x^6}$

16. $\int\limits_{e}^{e^2} \frac{dx}{x\ln x}$

17. $\int\limits_{e^2}^{e^3} \frac{dx}{x\ln x}$

18. $\int\limits_{-\pi}^{\pi} \sin^2 x dx$

19. $\int\limits_{-\pi}^{\pi} \cos^2 x dx$

20. $\int\limits_{\pi/6}^{\pi/3} tg^4 x dx$

21. $\int\limits_{0}^{2} sh^3 x dx$

22. $\int\limits_{0}^{1} \frac{dx}{4x^2+4x+5}$

23. $\int\limits_{3}^{4} \frac{x^2+3}{x-2}dx$

24. $\int\limits_{-2}^{-1} \frac{x+1}{x^2(x-1)}dx$

25. $\int\limits_{2}^{3} \frac{dx}{x^2-2x-8}$

26. $\int\limits_{0}^{2} \frac{2x-1}{2x+1}dx$

27. $\int\limits_{0}^{1} \frac{x^2+3x}{(x+1)(x^2+1)}dx$

28. $\int\limits_{0}^{1} \frac{dx}{\sqrt{x^2+2x+2}}$

29. $\int\limits_{3/4}^{2} \frac{dx}{\sqrt{2+3x-2x^2}}$

30. $\int\limits_{0}^{4} \frac{dx}{1+\sqrt{x}}$

31. $\int\limits_{0}^{\pi/4} \frac{dx}{1+2\sin^2 x}$

32. $\int\limits_{0}^{1} xe^{-x}dx$

**33.** $\int\limits_{\pi/4}^{\pi/3} \frac{xdx}{\sin^2 x}$

**34.** $\int\limits_1^3 \ln xdx$

**35.** $\int\limits_1^2 x\ln xdx$

**36.** $\int\limits_0^{1/2} \arcsin xdx$

**37.** $\int\limits_0^{\pi} x^3 \sin xdx$

**38.** $\int\limits_0^{\pi/2} e^{2x} \cos xdx$

**39.** $\int\limits_0^{e} \sin\ln xdx$

**40.** $\int\limits_0^2 |1-x|dx$

**41.** Tengsizliklar isbotlansin:

a) $\frac{1}{10\sqrt{2}} < \int\limits_0^1 \frac{x^9}{\sqrt{1+x}}dx < \frac{1}{10}$

b) $\frac{1}{20\sqrt[3]{2}} < \int\limits_0^1 \frac{x^{19}}{\sqrt[3]{1+x^6}}dx < \frac{1}{20}$

v) $0 < \int\limits_0^{200} \frac{e^{-5x}dx}{x+20} < 0{,}01$

g) $1 < \int\limits_0^1 \frac{1+x^{20}}{1+x^{40}}dx < 1 + \frac{1}{42}$

d) $1 - \frac{1}{n} < \int\limits_0^1 e^{-x^n}dx < 1, \quad n > 1$

e) $0 < \int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin x}{x}dx < \ln 3$

j) $\sin 1 < \int\limits_{-1}^1 \frac{\cos x}{1+x^2}dx < 2\sin 1$

z) $\frac{2}{\pi}\ln\frac{\pi+2}{2} < \int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{x(x+1)}dx < \ln\frac{\pi+2}{3}$

i) $\frac{1}{8} < \frac{\pi}{6}\int\limits_0^2 \frac{\sin\frac{\pi}{6}(x+1)}{(x+1)(3-x)}dx < \frac{1}{6}$

k) $0{,}03 < \int\limits_0^1 \frac{x^7}{(e^x + e^{-x})\sqrt{1+x^2}}dx < 0{,}05$

**42.** Integrallar hisoblansin:

a) $\int\limits_0^2 f(x)dx, \quad f(x) = \begin{cases} x^2, \text{agar } 0 \le x \le 1, \\ 2-x, \text{agar } 1 < x \le 2. \end{cases}$

b) $\int\limits_0^1 f(x)dx, \quad f(x) = \begin{cases} x, \text{agar } 0 \le x \le t, \\ t\frac{1-x}{1-t}, \text{agar } t < x \le 1. \end{cases}$

**43.** Tengliklarning nima uchun noto‘g‘riligi tushuntirilsin.

a) $\int_{-1}^{1}\frac{d}{dx}\left(\operatorname{arctg}\frac{1}{x}\right)dx=\operatorname{arctg}\frac{1}{x}\Big|_{-1}^{1}=\frac{\pi}{2}$

b) $\int_{-1}^{1}\frac{dx}{x}=\ln|x|\Big|_{-1}^{1}=0$

v) $\int_{0}^{2\pi}\frac{\sec^2 x dx}{2+tg^2x}=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{tg^2x}{\sqrt{2}}\Big|_{-1}^{1}=0$

**Aniq integrallar yordamida quyidagi limitlar hisoblansin:**

**44.** $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}\right)$

**45.** $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$

**46.** $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)$

**47.** $\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)$

**48.** $\lim_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}}\quad (p>0)$

**49.** $\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\cdots+\sqrt{1+\frac{n}{n}}\right)$

**50.** Limitlar hisoblansin:

a) $\lim_{x\to 0}\frac{\int_0^x \cos t^2 dt}{x}$

b) $\lim_{x\to+\infty}\frac{\int_0^x(\operatorname{arctg}t)^2dt}{\sqrt{x^2+1}}$

v) $\lim\limits_{x\to+\infty}\dfrac{\left(\int\limits_0^x e^{t^2}dt\right)^2}{\int\limits_0^x e^{2t^2}dt}$ g) $\lim\limits_{x\to+0}\dfrac{\int\limits_0^{\sin x}\sqrt{\mathrm{tg}t}\,dt}{\int\limits_0^{\mathrm{tg}x}\sqrt{\sin t}\,dt}$

**O'zgaruvchilarni almashtirish va bo'laklab integrallash usullari yordamida hisoblansin:**

51. $\int\limits_{-\pi}^{\pi}\sqrt[3]{\sin x}\,dx$
52. $\int\limits_{-\pi}^{\pi}e^{x^2}\sin x\,dx$
53. $\int\limits_{-\pi/2}^{\pi/2}(\cos^2 x+x^2\sin x)dx$
54. $\int\limits_{-1}^{1}\cos x\,\mathrm{th}x\,dx$
55. $\int\limits_{-1}^{1}(e^x+e^{-x})\mathrm{tg}x\,dx$
56. $\int\limits_{-\pi/3}^{\pi/3}\left(x^2\sin 5x+\cos\dfrac{x}{3}+\mathrm{tg}^3x\right)dx$
57. $\int\limits_{-\pi/3}^{\pi/3}\dfrac{2x^7-x^5+2x^3-x+1}{\cos^2 x}dx$
58. $\int\limits_0^2 e^{x^2}x\,dx$
59. $\int\limits_0^1 x^2\sqrt{1-x^2}\,dx$
60. $\int\limits_0^{\ln 2}\sqrt{e^x-1}\,dx$
61. $\int\limits_0^{\ln 2}xe^{-x}dx$
62. $\int\limits_0^{\pi}x\sin x\,dx$
63. $\int\limits_0^{\pi/4}x\sin 2x\,dx$
64. $\int\limits_0^{2\pi}x^2\cos x\,dx$
65. $\int\limits_0^1 \arccos x\,dx$
66. $\int\limits_1^3 \mathrm{arctg}\sqrt{x}\,dx$
67. $\int\limits_1^e \dfrac{dx}{x\sqrt{1+\ln x}}$
68. $\int\limits_0^1 \dfrac{e^{2x}+2e^x}{e^{2x}+1}dx$
69. $\int\limits_0^{\pi/2}\dfrac{dx}{2-\sin x}$
70. $\int\limits_{\pi/3}^{\pi/2}\dfrac{dx}{3+\cos x}$

**71.** $\int_0^{3/4} \frac{dx}{(x+1)\sqrt{x^2+1}}$

**72.** $\int_1^9 x\cdot\sqrt[3]{1-x}dx$

**73.** $\int_{-3}^{-2} \frac{dx}{x\sqrt{x^2-1}}$

**74.** $\int_0^1 x^{15}\sqrt{1+3x^8}dx$

**75.** $\int_0^{\pi/2} \sin x \sin 2x \sin 3x dx$

**76.** $\int_0^{\ln 2} sh^4 x dx$

**77.** $\int_1^2 x^2 \ln x dx$

**78.** $\int_1^n x^n \ln x dx$

**79.** $\int_0^{\sqrt{3}} x arctg x dx$

**80.** $\int_0^1 x(2-x^2)^{12} dx$

**81.** $\int_0^1 \arcsin\sqrt{x} dx$

**82.** $\int_0^{\pi} e^x \cos^2 x dx$

**83.**Tenglik isbotlansin.

$$\int_0^1 (1-x)^m x^n dx = \frac{m!n!}{(m+n+1)!}, \quad m\in N, n\in N.$$

**84.**Integrallar hisoblansin:

a) $\int_0^{\pi} (x\sin x)^2 dx$

b) $\int_0^{\pi} (x\cos x)^2 dx$

**85.**

$$J_n = \int_0^{\pi/2} \sin^n x dx, \quad n\geq 2$$

integral uchun

$$J_n = \frac{n-1}{n} J_{n-2}$$

rekurrent formula isbotlansin.

**86.**Isbotlansin:

$$\int_0^{\pi/2}\sin^n x dx=\int_0^{\pi/2}\cos^n x dx=\begin{cases}\frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}, \text{ агар } n-\text{жуфт бўлса,}\\ \frac{(n-1)!!}{n!!}, \text{ агар } n-\text{ток бўлса, } n\in N.\end{cases}$$

**87.**Isbotlansin:

$$\lim_{n\to\infty}\frac{1}{2n+1}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2=\frac{\pi}{2}$$

**Ko‘rsatma:** 86-misol natijasidan foydalaning.

**88.**Isbotlansin:

$$\int_0^{\frac{\pi}{2}}\sin^m x\cos^n xdx=\begin{cases}\frac{(m-1)!!(n-1)!!}{(m+n)!!}\cdot\frac{\pi}{2}, & \text{агар } m \text{ ва } n \text{ жуфт бўлса,}\\ \frac{(m-1)!!(n-1)!!}{(m+n)!!} & \text{бошка барча холларда, } m\in N, n\in N.\end{cases}$$

**89.** $n\in N$ uchun quyidagi tengliklar isbotlansin:

a) $\int_0^a (a^2-x^2)^n dx=a^{2n+1}\frac{(2n)!!}{(2n+1)!!}$ b) $\int_0^a (a^2-x^2)^{\frac{2n-1}{2}} dx=a^{2n}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}$

**90.**Isbotlansin:

a) $\int_0^{\pi/2}\cos^n x\sin nxdx=\frac{1}{2^{n+1}}\sum_{k=1}^{n}\frac{2^k}{k}, n\in N$ b) $\int_0^{\pi/2}\cos^n x\cos nxdx=\frac{\pi}{2^{n+1}}, n\in N$

**91.** $m\in N$ uchun tengliklar isbotlansin:

a) $\int_0^{\pi/2}\cos^m x\cos(m+2)xdx=0$ b) $\int_0^{\pi/2}\cos^m x\sin(m+2)xdx=\frac{1}{m+1}$

v) $\int_0^{\pi/2}\sin^m x\cos(m+2)xdx=-\frac{\sin\frac{m\pi}{2}}{m+1}$ g) $\int_0^{\pi/2}\sin^m x\sin(m+2)xdx=\frac{1}{m+1}\cos\frac{m\pi}{2}$

**92.f** funksiya $[a,b]$ kesmada integrallanuvchi, **F** funksiya $[a,b]$ kesmada chekli sondagi $x_1,\dots,x_n$ birinchi tur uzilish nuqtalariga ega bo‘lsin. Agar $F$ funksiya $[a,b]$ kesmaning chekli sondagi ichki nuqtalaridan boshqa barcha nuqtalarida differensiallanuvchi va $F'(x)=f(x)$ bo‘lsa, u holda quyidagi formula o‘rinli ekanligi isbotlansin:

$$\int_a^b f(x)dx = F(b-0)-F(b+0)-\sum_{k=1}^{n}[F(x_k+0)-F(x_k-0)]$$

($F$ funksiya $f$ funksiyaning **umumlashgan boshlang‘ich funksiyasi** deyiladi.)

**93.**Quyidagi uzilishga ega funksiyalarning uzluksiz umumlashgan boshlang‘ich funksiyalari topilsin:

a) $\operatorname{sign}x$ b) $\operatorname{sign}(\sin x)$ v) $[x]$ g) $x[x]$ d) $(-1)^{[x]}$

**94.** $\int_0^x f(t)dt$ integral hisoblansin. Bu erda

$$f(t)=\begin{cases}1, \text{агар } |t|<l,\\ 0, \text{агар } |t|>l, \quad l\ge 0\end{cases}$$

**95.**Hisoblansin:

a) $\int_0^3 \operatorname{sign}(x-x^3)dx$ b) $\int_0^2 [e^x]dx$

v) $\int_0^6 [x]\sin\frac{\pi x}{6}dx$ d) $\int_1^{n+1} \ln[x]dx, \quad n\in N$

g) $\int_0^\pi x\operatorname{sign}(\cos x)dx$ e) $\int_0^1 \operatorname{sign}(\sin\ln x)dx$

## Aniq integralning xossalari.
## O‘rta qiymat haqidagi teoremalar

**1$^0$. Aniq integralning asosiy xossalari.** $[a,b]$ segmentda integrallanuvchi funksiyalar sinfini $R([a,b])$ kabi belgilaymiz.

**1-xossa.** Agar $f(x)\in R([a,b])$, $c\in R$ bo'lsa, u holda $c\cdot f(x)\in R([a,b])$ bo'lib,

$$\int_a^b (c\cdot f(x))dx = c\int_a^b f(x)dx$$

bo'ladi. (integralning **chiziqlilik** xossasi)

**2-xossa.** Agar $f(x)\in R([a,b])$, $g(x)\in R([a,b])$ bo'lsa, u holda $f(x)+g(x)\in R([a,b])$ bo'lib,

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

bo'ladi. (integralning **additivlik** xossasi)

**3-xossa.** Agar $f(x)\in R([a,b])$, $a<c<b$ bo'lsa, u holda $f(x)\in R([a,c])$, $f(x)\in R([c,b])$ bo'lib,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

bo'ladi.

**4-xossa.** Agar $f(x)\in R([a,b])$, $g(x)\in R([a,b])$ bo'lib, $\forall x\in[a,b]$ da $f(x)\le g(x)$ bo'lsa, u holda

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

bo'ladi. (integralning **monotonlik** xossasi)

**Natija.** Agar $f(x)\in R([a,b])$ bo'lib, $f(x)\ge 0$ bo'lsa,

$$\int_a^b f(x)dx \ge 0$$

bo'ladi.

**$2^0$.Funksiyaning o'rta qiymati va o'rta qiymat haqidagi teoremalar.** Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan bo'lib, u shu segmentda integrallanuvchi bo'lsin.

Ushbu

$$M[f] = \frac{1}{b-a}\int_a^b f(x)dx$$

miqdor $f(x)$ funksiyaning $[a,b]$ dagi **o'rta qiymati** deyiladi.

Aytaylik, $f(x)$ funksiya $[a,b]$ da chegaralangan bo'lib, $\forall x\in[a,b]$ da $m\le f(x)\le M$ bo'lsin.

**1 – t e o r e m a .** *Agar* $f(x)\in R([a,b])$ *bo'lsa, u holda shunday* $\mu$ *son* $(m\le\mu\le M)$ *topiladiki,*

$$\int_a^b f(x)dx = \mu\cdot(b-a)$$

*bo'ladi.*

**2 – t e o r e m a .** *Agar* $f(x)\in C([a,b])$, $g(x)\in C([a,b])$ *bo'lib,* $\forall x\in[a,b]$ *da* $g(x)>0$ *yoki* $g(x)<0$ *bo'lsa, u holda shunday* $\mu$ *son* $(m\le\mu\le M)$ *topiladiki,*

$$\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx$$

*bo'ladi.*

**1 – m i s o l .** *Agar* $f(x)\in C([a,b])$, $g(x)\in C([a,b])$ *bo'lsa, ushbu*

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx \qquad (*)$$

*tengsizlikning o'rinli bo'lishi isbotlansin.*

◄Aniq integralning xossalariga ko'ra

$$f(x)-\alpha\cdot g(x)\in R([a,b]) \qquad (\alpha\in R)$$

bo'lib,

$$\int_a^b [f(x)-\alpha\cdot g(x)]^2dx \ge 0$$

bo'ladi. Bu munosabatdan

$$\alpha^2\int_a^b g^2(x)dx - 2\alpha\int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \ge 0$$

bo'lishi kelib chiqadi.

Ma'lumki, kvadrat uchhad manfiy bo'lmasa, uning diskriminanti musbat bo'lmaydi. Demak,

$$\left[\int_a^b f(x)g(x)dx\right]^2 - \int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx \le 0$$

Bu tengsizlikdan topamiz:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx$$ ►

(*) tengsizlik **Koshi-Bunyakovskiy tengsizligi** deyiladi.

**2 – m i s o l .** *Ushbu*

$$\int_0^\pi x\cdot\sqrt{\sin x}dx \le \frac{2}{3}\pi^3$$

*tengsizlik isbotlansin.*

◄Agar (*) tengsizlikda $f(x)=x$, $g(x)=\sqrt{\sin x}$ deyilsa, u holda

$$\int_0^{\pi} x\cdot\sqrt{\sin x}dx \le \int_0^{\pi} x^2dx\cdot\int_0^{\pi}\sin xdx = \frac{2}{3}\pi^3$$

bo‘ladi.►

**3 – m i s o l .** *Agar*

$$f(x) = e^{2x},\ a = 0,\ b = 1$$

***bo‘lsa, u holda*** $c$ ***ning qanday qiymatida ushbu***

$$\int_a^b f(x)dx = f(c)\cdot(b-a)$$

***tenglik o‘rinli bo‘ladi?***

◄Ravshanki,

$$\int_0^1 e^{2x}dx = \frac{e^{2k}}{2}\Big|_0^1 = \frac{e^2-1}{2}$$

bo‘ladi. Demak,

$$e^{2c} = \frac{e^2-1}{2}$$

Bu tenglikdan $c$ ni topamiz:

$$2c = \ln\frac{e^2-1}{2},\quad c = \frac{1}{2}\ln\frac{e^2-1}{2}$$ ►

**4 – m i s o l .** ***O‘rta qiymat haqidagi teoremadan foydalanib, ushbu***

$$\int_a^b \frac{\sin x}{\sqrt{x}}dx \qquad (0 < a < b)$$

***integral baholansin.***

◄O‘rta qiymat haqidagi 2-teoremada $f(x)$ funksiya $[a,b]$ da uzluksiz bo‘lsa, u holda shunday $c$ nuqta $(a < c < b)$ topiladiki,

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

bo‘ladi. SHu tenglikdan foydalanib topamiz:

$$\int_a^b \frac{1}{\sqrt{x}}\sin xdx = \frac{1}{\sqrt{c}}\int_a^b \sin xdx = \frac{1}{\sqrt{c}}(\cos a - \cos b)$$

Agar $0 < a < b$ bo‘lganda

$$\frac{1}{\sqrt{c}} < \frac{1}{\sqrt{a}} \qquad \text{va} \qquad |\cos a - \cos b| \le 2$$

bo‘lishini e’tiborga olsak, u holda

$$\left|\int_a^b \frac{\sin x}{\sqrt{x}}dx\right| \le \frac{2}{\sqrt{a}}$$

bo‘lishi topiladi.►

**96.** Qaysi biri katta ekanligi aniqlansin:

a) $\int\limits_0^{\pi/2} \frac{\sin x}{x}dx$ ёки $\int\limits_0^{\pi} \frac{\sin x}{x}dx$  b) $\int\limits_{1/2}^{1} \frac{dx}{\sqrt{x}}$ ёки $\int\limits_{1/2}^{1} \frac{dx}{\sqrt[3]{x}}$

v) $\int\limits_0^1 e^{-x}\sin x dx$ ёки $\int\limits_0^1 e^{-x^2}\sin x dx$  g) $\int\limits_1^2 \frac{dx}{\sqrt{1+x^2}}$ ёки $\int\limits_1^2 \frac{dx}{x}$

**97.**Tengsizliklar isbotlansin:

a) $0 < \int\limits_0^{\pi} \frac{\sin x}{\sqrt[5]{x^2+2}}dx < \frac{\pi}{\sqrt[5]{2}}$

b) $\frac{1}{\sqrt[3]{9}} < \frac{1}{\pi}\int\limits_{-1}^{1} \frac{\pi + \operatorname{arctg} x}{\sqrt[3]{x^2+8}}dx < \frac{3}{2}$

v) $\frac{\sqrt{2}}{3} < \int\limits_{-1}^{1} \frac{\cos x}{2+x^2}dx < 1$

**98.** $f$ funksiya $[a,b]$ kesmada integrallanuvchi bo‘lsin.

$$\int\limits_a^b f^2(x)dx = 0$$

tenglik bajarilishi uchun, $f$ funksiyaning barcha uzluksiz nuqtalarida $f(x)=0$ bo‘lishi zarur va etarli ekanligi isbotlansin.

**99.**Berilgan funksiyalarning ko‘rsatilgan oraliqlardagi o‘rta qiymatlari aniqlansin:

a) $f(x)=x^2,\quad [0,1];$  b) $f(x)=\sqrt{x},\quad [0,100];$

v) $f(x)=10+2\sin x+3\cos x,\quad [0,2\pi];$  g) $f(x)=\sin x\sin(x+\varphi),\quad [0,2\pi]$

**O‘rta qiymat haqidagi birinchi teoremadan foydalanib, integrallar baholansin:**

**100.** $\int\limits_0^{2\pi} \frac{dx}{1+0,5\cos x}$  **101.** $\int\limits_0^1 \frac{x^9}{\sqrt{1+x}}dx$  **102.** $\int\limits_0^{100} \frac{e^{-x}}{x+100}dx$

**O‘rta qiymat haqidagi ikkinchi teoremadan foydalanib, integrallar baholansin:**

**103.** $\int\limits_{100\pi}^{200\pi} \frac{\sin x}{x} dx$ **104.** $\int\limits_{a}^{b} \frac{e^{-\alpha x}}{x} \sin x dx \quad (\alpha \ge 0 ; 0 < a < b)$

**105.** $\int\limits_{a}^{b} \sin x^2 dx \quad (0 < a < b)$

## Test

| | | | | |
|---|---|---|---|---|
| $\int\limits_0^2 x^2 dx$ | $\frac{8}{3}$ | -4 | 4 | $\frac{16}{3}$ |
| $\int\limits_{-1}^0 3(2x+1)^2 dx$ | 1 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $\int\limits_{-1}^0 e^{-x} dx$ | $-1+e$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int\limits_1^2 \frac{dx}{x+1}$ | $\ln\frac{3}{2}$ | 0 | $\ln 3$ | $\ln 3-1$ |
| $\int\limits_0^\pi \sin xdx$ | 2 | 1 | -1 | -2 |
| $\int\limits_0^2 x^3 dx$ | 4 | 2 | 5 | 1 |
| $\int\limits_0^\pi \sin 2xdx$ | 0 | -1 | 1 | 2 |
| $\int\limits_{\pi/2}^\pi \cos xdx$ | -1 | 0 | 1 | -2 |
| $\int\limits_3^5 \frac{dx}{x-1}$ | $\ln 2$ | $\ln\frac{3}{2}$ | $\ln\frac{4}{3}$ | 1 |
| $\int\limits_{-1}^0 (2x+1)^2 dx$ | $\frac{1}{3}$ | 1 | -3 | -1 |
| $\int\limits_{-1}^0 2e^{-x} dx$ | $-2+2e$ | $e$ | $\frac{1}{e}-1$ | 5 |

# Mavzu. Aniq integrallarni hisoblash

## 11-12-ma'ruza

**Reja**

**1$^0$.** Nyuton-Leybnits formulasi.

**2$^0$.** O'zgaruvchilarni almashtirish formulasi.

**3$^0$.** Bo'laklab integrallash formulasi.

**1$^0$. Nyuton-Leybnits formulasi.**

Aytaylik, $f(x)$ funksiya $[a,b]$ segmentda berilgan va shu segmentda uzluksiz bo'lsin. U holda $f(x)$ boshlang'ich funksiya

$$F(x)=\int_a^x f(t)dt$$

ga ega bo'ladi.

Ravshanki, $\Phi(x)$ funksiya $f(x)$ ning ixtiyoriy boshlang'ich funksiyasi bo'lsa, u holda

$$\Phi(x)=F(x)+C \qquad (C=const)$$

bo'ladi.

Bu tenglikda, avval $x=a$ deb

$$\Phi(a)=C,$$

so'ngra $x=b$ deb

$$\Phi(b)=\int_a^b f(x)dx+C$$

bo'lishini topamiz.

Demak,

$$\int_a^b f(x)dx=\Phi(b)-\Phi(a). \qquad (1)$$

(1) formula Nyuton-Leybnits formulasi deyiladi.

Odatda, $\Phi(b)-\Phi(a)$ ayirma $\Phi(x)\Big|_a^b$ kabi yoziladi. Demak,

$$\int_a^b f(x)dx=\Phi(x)\Big|_a^b=\Phi(b)-\Phi(a).$$

Masalan,

$$\int_a^b \frac{1}{x}dx = \ln x\Big|_a^b = \ln b - \ln a = \ln\frac{b}{a}. \qquad (a>0, b>0)$$

**2[0]. O‘zgaruvchilarini almashtirish formulasi.**

Faraz qilaylik, $f(x)\in C[a,b]$ bo‘lsin. Ravshanki, bu holda

$$\int_a^b f(x)\,dx$$

integral mavjud bo‘ladi.

Ayni paytda, bu funksiya $[a,b]$ da boshlang‘ich $\Phi(x)$ funksiyaga ega bo‘lib,

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a)$$

bo‘ladi.

Aytaylik, aniq integralda $x$ o‘zgaruvchi ushbu

$$x = \phi(t)$$

formula bilan almashtirilgan bo‘lib, bunda $\phi(t)$ funksiya quyidagi shartlarni bajarsin:

1) $\phi(t)\in C[\alpha,\beta]$ bo‘lib, $\phi(t)$ funksiyaning barcha qiymat-lari $[a,b]$ ga tegishli;

2) $\phi(\alpha)=a, \quad \phi(\beta)=b$;

3) $\phi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz $\phi'(t)$ hosilaga ega bo‘lsin.

U holda

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(t))\cdot\phi'(t)\,dt \qquad (2)$$

bo‘ladi.

◄ Ravshanki, $\Phi(\varphi(t))$ murakkab funksiya $[\alpha,\beta]$ segmentda uzluksiz bo‘lib,

$$(\Phi(\varphi(t)))' = \Phi'(\varphi(t))\cdot\varphi'(t)$$

bo‘ladi.

Agar $\Phi'(x) = f(x)$ ekanini e’tiborga olsak, unda

$$(\Phi(\varphi(t)))' = f(\varphi(t))\cdot\varphi'(t)$$

bo‘lishini topamiz. Bu esa $\Phi(\varphi(t))$ funksiya $[\alpha,\beta]$ da $f(\varphi(t))\cdot\varphi'(t)$ funksiyaning boshlang‘ich funksiyasi ekanini bildiradi. Nyuton-Leybnits formulasiga ko‘ra

$$\int_{\alpha}^{\beta} f(\varphi(t))\cdot\varphi'(t)dt = \Phi(\varphi(\beta)) - \Phi(\varphi(\alpha)) = \Phi(b) - \Phi(a) \qquad (3)$$

bo‘ladi.

(2) va (3) munosabatlardan

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\cdot\varphi'(t)dt \qquad (4)$$

bo‘lishi kelib chiqadi. ►

(4) formula aniq integralda o‘zgaruvchini almashtirish formulasi deyiladi.

**Misol.** Ushbu

$$\int_{0}^{1} \sqrt{1-x^2}\,dx$$

integral hisoblansin.

**Yechilishi.** Berilgan integralda $x = \sin t$ almashtirishni bajara-miz. Unda

$$\int_{0}^{1} \sqrt{1-x^2}\,dx = \int_{0}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,\cos t dt = \int_{0}^{\frac{\pi}{2}} \cos^2 t dt =$$

$$= \int_{0}^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)dt = \left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right)\Bigg|_{0}^{\frac{\pi}{2}} = \frac{\pi}{4}$$

bo‘ladi.

**3⁰. Bo‘laklab integrallash formulasi.**

Aytaylik , $u(x)$ va $v(x)$ funksiyalarning har biri $[a,b]$ segmentda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega bo’lsin. U holda

$$\int_{a}^{b} u(x)\,d\,v(x) = (u(x)\cdot v(x))\Big|_{a}^{b} - \int_{a}^{b} v(x)\,d\,u(x) \qquad (3)$$

bo‘ladi.

(3) formula aniq integrallarda bo‘laklab integrallash formulasi deyiladi.

**Misol.** Ushbu

$$\int_{1}^{2} x \ln x\,d\,x$$

integral hisoblansin.

**Yechilishi.** Bu intervalda $u(x) = \ln x,\ d\,v(x) = x$ deb

$d\,u(x)=\frac{1}{x}\,d\,x,\;\; v(x)=\frac{x^2}{2}$ bo‘lishini topamiz. Unda (3) formulaga ko‘ra:

$$\int_1^2 x\ln x\,d\,x=\left(\frac{x^2}{2}\ln x\right)\Bigg|_1^2-\int_1^2\frac{x^2}{2}\cdot\frac{1}{x}\,d\,x=2\ln 2-\frac{1}{2}\int_1^2 x\,d\,x=2\ln 2-\frac{3}{4}$$

bo‘ladi.

**4-misol.** Ushbu

$$J_n=\int_0^{\frac{\pi}{2}}\sin^n xdx \qquad (n=0,1,2,...)$$

integral hisoblansin.

◀ Ravshanki,

$$J_0=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}\quad ,\quad J_1=\int_0^{\frac{\pi}{2}}\sin xdx=(-\cos x)\Bigg|_0^{\frac{\pi}{2}}=1.$$

$n\ge 2$ bo‘lganda berilgan integralni

$$J_n=\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,d(-\cos x)$$

ko‘rinishida yozib, unga bo‘laklab integrallash formulasini qo‘llaymiz. Natijada

$$J_n=(-\sin^{n-1}x\cdot\cos x)\Big|_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 xdx=$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)dx=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}xdx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n xdx=$$

$$=(n-1)J_{n-2}-(n-1)J_n$$

bo‘lib, undan ushbu

$$J_n=\frac{n-1}{n}J_{n-2}$$

rekurrent formula kelib chikadi.

Bu formula yordamida berilgan integralni $n=1,2,3,........$ bo‘lganda ketma-ket hisoblash mumkin.

Aytaylik, $n=2m$- juft son bo‘lsin. Unda

$$J_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}......\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot J_0=\frac{(2m-1)!!}{(2m)!!}\cdot\frac{\pi}{2}$$

bo‘ladi.

Aytaylik, $n=2m+1$- toq son bo‘lsin. Unda

$$J_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdot\ldots\ldots\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot J_1=\frac{(2m)!!}{(2m+1)!!}$$

bo‘ladi. ($m!!$ simvol $m$ dan katta bo‘lmagan va u bilan bir xil juftlikka ega bo‘lgan natural sonlarning ko‘paytmasini bildiradi.) ►

## Mashqlar

1. Agar $f(x)\in R([0,1])$ bo‘lsa,

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{1} f(x)dx=\int_0^1 f(x)dx$$

tenglik isbotlansin.

2. Ushbu integral

$$\int_0^{100\pi}\sqrt{1-\cos 2x}\,dx$$

hisoblansin.

3. Ushbu tenglik

$$\int_x^1\frac{dx}{1+x^2}=\int_1^{\frac{1}{x}}\frac{dx}{1+x^2}\qquad (x>0)$$

isbotlansin.

## Adabiyotlar


1. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’rizalar, I q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Kurs differensialnogo i integralnogo ischisleniya, 1 t.* M. «FIZMATLIT», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

# Glossariy

Nyuton-Leybnits formulasi

$$\int_a^b f(x)dx = \Phi(b) - \Phi(a). \qquad (1)$$

**O‘zgaruvchilarini almashtirish formulasi.**

$f(x) \in C[a,b]$ bo‘lsin. Ravshanki, bu holda

$$\int_a^b f(x)\,dx$$

integral mavjud bo‘ladi.

Ayni paytda, bu funksiya $[a,b]$ da boshlang‘ich $\Phi(x)$ funksiyaga ega bo‘lib,

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a)$$

bo‘ladi.

Aytaylik, aniq integralda $x$ o‘zgaruvchi ushbu

$$x = \phi(t)$$

formula bilan almashtirilgan bo‘lib, bunda $\phi(t)$ funksiya quyidagi shartlarni bajarsin:

1) $\phi(t) \in C[\alpha,\beta]$ bo‘lib, $\phi(t)$ funksiyaning barcha qiymat-lari $[a,b]$ ga tegishli;

2) $\phi(\alpha) = a, \quad \phi(\beta) = b$;

3) $\phi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz $\phi'(t)$ hosilaga ega bo‘lsin.

U holda

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(t))\cdot\phi'(t)\,dt \qquad (2)$$

bo‘ladi.

## Keys banki

**37-keys.** Masala o`rtaga tashlanadi: Agar $f(x)\in R([0,1])$ bo‘lsa, u holda quyidagi

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{1} f(x)dx = \int_0^1 f(x)dx$$

tenglik isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 11-12-amaliy mashg'ulot

**2 – м и с о л .** ***Ушбу***

$$\int_0^{10\pi} \sqrt{1-\cos 2x}dx$$

***ҳисоблансин.***

◀Маълумки,

$$\sqrt{1-\cos 2x} = \sqrt{2\sin^2 x} = \sqrt{2}\cdot|\sin x|$$

Унда

$$\int_0^{10\pi} \sqrt{1-\cos 2x}dx = \sqrt{2}\int_0^{10\pi} |\sin x| dx$$

бўлади. Аниқ интегралнинг хоссаларидан фойдаланиб топамиз:

$$\int_0^{10\pi} |\sin x| dx = \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx + \int_{2\pi}^{3\pi} \sin dx - \int_{3\pi}^{4\pi} \sin x dx + \cdots + \int_{8\pi}^{9\pi} \sin x dx - \int_{9\pi}^{10\pi} \sin x dx =$$
$$= \underbrace{2+2+\cdots+2}_{10\text{та}} = 20.$$

Демак,

$$\int_0^{10\pi} \sqrt{1-\cos 2x}dx = 20\cdot\sqrt{2}\,.\blacktriangleright$$

**3 – м и с о л .** ***Ушбу***

$$\int_0^{a} x^2\sqrt{a^2-x^2}dx$$

***интеграл ҳисоблансин.***

◀Бу интегрални ўзгарувчини алмаштириш усулидан фойдаланиб ҳисоблаймиз:

$$\int_0^a x^2\sqrt{a^2-x^2}dx=\left[x=a\sin t\,,\,dx=a\cos tdt\,,\,t\in\left[0,\frac{\pi}{2}\right]\right]=a^4\int_0^{\frac{\pi}{2}}\sin^2 t\cdot\cos^2 tdt=$$

$$=\frac{a^4}{4}\int_0^{\frac{\pi}{2}}\sin^2 2tdt=\frac{a^4}{4}\int_0^{\frac{\pi}{2}}\frac{1-\cos 4t}{2}dt=\frac{a^4}{8}\left(t-\frac{\sin 4t}{4}\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{a^4}{16}\pi \blacktriangleright$$

**Интеграллар ҳисоблансин.**

1914. $\int_{-1}^{2} x^3dx$

1915. $\int_1^4 \frac{dx}{\sqrt[3]{x}}$

1916. $\int_1^2 \left(x^2-2x+3\right)dx$

1917. $\int_{-1}^{1}\left(x^3-2x^2+x-1\right)dx$

1918. $\int_0^1\left(\sqrt{x}+\sqrt[3]{x^2}\right)dx$

1919. $\int_0^{\pi}\sin xdx$

1920. $\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{dx}{1+x^2}$

1921. $\int_{-1/2}^{1/2}\frac{dx}{\sqrt{1-x^2}}$

1922. $\int_{-\pi/4}^{0}\frac{dx}{\cos^2 x}$

1923. $\int_2^9\sqrt[3]{x-1}dx$

1924. $\int_1^2 e^x dx$

1925. $\int_0^2 2^x dx$

1926. $\int_2^4\frac{dx}{x}$

1927. $\int_1^2\frac{dx}{2x-1}$

1928. $\int_0^1\frac{x^2dx}{1+x^6}$

1929. $\int_e^{e^2}\frac{dx}{x\ln x}$

1930. $\int_{e^2}^{e^3}\frac{dx}{x\ln x}$

1931. $\int_{-\pi}^{\pi}\sin^2 xdx$

1932. $\int_{-\pi}^{\pi} \cos^2 x dx$

1933. $\int_{\pi/6}^{\pi/3} tg^4 x dx$

1934. $\int_{0}^{2} sh^3 x dx$

1935. $\int_{0}^{1} \frac{dx}{4x^2+4x+5}$

1936. $\int_{3}^{4} \frac{x^2+3}{x-2} dx$

1937. $\int_{-2}^{-1} \frac{x+1}{x^2(x-1)} dx$

1938. $\int_{2}^{3} \frac{dx}{x^2-2x-8}$

1939. $\int_{0}^{2} \frac{2x-1}{2x+1} dx$

1940. $\int_{0}^{1} \frac{x^2+3x}{(x+1)(x^2+1)} dx$

1941. $\int_{0}^{1} \frac{dx}{\sqrt{x^2+2x+2}}$

1942. $\int_{3/4}^{2} \frac{dx}{\sqrt{2+3x-2x^2}}$

1943. $\int_{0}^{4} \frac{dx}{1+\sqrt{x}}$

1944. $\int_{0}^{\pi/4} \frac{dx}{1+2\sin^2 x}$

1945. $\int_{0}^{1} xe^{-x} dx$

1946. $\int_{\pi/4}^{\pi/3} \frac{xdx}{\sin^2 x}$

1947. $\int_{1}^{3} \ln x dx$

1948. $\int_{1}^{2} x \ln x dx$

1949. $\int_{0}^{1/2} \arcsin x dx$

1950. $\int_{0}^{\pi} x^3 \sin x dx$

1951. $\int_{0}^{\pi/2} e^{2x} \cos x dx$

1952. $\int_{0}^{e} \sin \ln x dx$

1953. $\int_{0}^{2} |1-x| dx$

**1954.** Тенгсизликлар исботлансин:

а) $\frac{1}{10\sqrt{2}} < \int_{0}^{1} \frac{x^9}{\sqrt{1+x}} dx < \frac{1}{10}$

б) $\frac{1}{20\sqrt[3]{2}} < \int_{0}^{1} \frac{x^{19}}{\sqrt[3]{1+x^6}} dx < \frac{1}{20}$

в) $0 < \int_{0}^{200} \frac{e^{-5x} dx}{x+20} < 0,01$

г) $1 < \int_{0}^{1} \frac{1+x^{20}}{1+x^{40}} dx < 1 + \frac{1}{42}$

д) $1-\frac{1}{n}<\int_0^1 e^{-x^n}dx<1, \quad n>1$

е) $0<\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{\sin x}{x}dx<\ln 3$

ж) $\sin 1<\int_{-1}^{1}\frac{\cos x}{1+x^2}dx<2\sin 1$

з) $\frac{2}{\pi}\ln\frac{\pi+2}{2}<\int_0^{\frac{\pi}{2}}\frac{\sin x}{x(x+1)}dx<\ln\frac{\pi+2}{3}$

и) $\frac{1}{8}<\frac{\pi}{6}\int_0^2\frac{\sin\frac{\pi}{6}(x+1)}{(x+1)(3-x)}dx<\frac{1}{6}$

к) $0{,}03<\int_0^1\frac{x^7}{\left(e^x+e^{-x}\right)\sqrt{1+x^2}}dx<0{,}05$

**1955.** Интеграллар ҳисоблансин:

а) $\int_0^2 f(x)dx, \quad f(x)=\begin{cases} x^2, \text{ агар } 0\le x\le 1, \\ 2-x, \text{ агар } 1<x\le 2. \end{cases}$

б) $\int_0^1 f(x)dx, \quad f(x)=\begin{cases} x, \text{ агар } 0\le x\le t, \\ t\frac{1-x}{1-t}, \text{ агар } t<x\le 1. \end{cases}$

# Test

| | | | | |
|---|---|---|---|---|
| $\int_0^2 x^2 dx$ | $\frac{8}{3}$ | -4 | 4 | $\frac{16}{3}$ |
| $\int_{-1}^0 3(2x+1)^2 dx$ | 1 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $\int_{-1}^0 e^{-x} dx$ | $-1+e$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int_1^2 \frac{dx}{x+1}$ | $\ln\frac{3}{2}$ | 0 | $\ln 3$ | $\ln 3-1$ |
| $\int_0^\pi \sin x dx$ | 2 | 1 | -1 | -2 |
| $\int_0^2 x^3 dx$ | 4 | 2 | 5 | 1 |
| $\int_0^\pi \sin 2x dx$ | 0 | -1 | 1 | 2 |
| $\int_{\pi/2}^\pi \cos x dx$ | -1 | 0 | 1 | -2 |
| $\int_3^5 \frac{dx}{x-1}$ | $\ln 2$ | $\ln\frac{3}{2}$ | $\ln\frac{4}{3}$ | 1 |
| $\int_{-1}^0 (2x+1)^2 dx$ | $\frac{1}{3}$ | 1 | -3 | -1 |
| $\int_{-1}^0 2e^{-x} dx$ | $-2+2e$ | $e$ | $\frac{1}{e}-1$ | 5 |

# Mavzu. Tekis shaklning yuzi va uni hisoblash

## 13-ma’ruza

**Reja**

**1⁰.** Tekis shaklning yuzi tushunchasi.

**2⁰.** Egri chizikli trapetsiyaning yuzini hisoblash.

**3⁰.** Egri chiziqli sektorning yuzini hisoblash.

### 1⁰. Tekis shaklning yuzi tushunchasi.

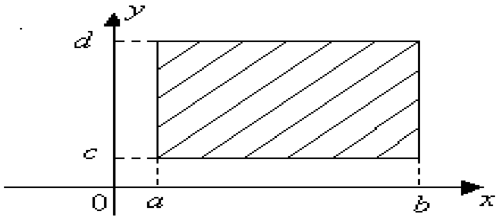
Ma’lumki, $(x, y)$ juftlik, $(x \in R, y \in R)$, tekislikda nuqtani ifodalaydi. Koordinatalari ushbu

$$a \le x \le b, \quad c \le y \le d \qquad (a \in R, b \in R, c \in R, d \in R)$$

tengsizliklarni qanoatlantiruvchi tekislik nuqtalaridan hosil bo‘lgan $D_0$ to‘plam :

$$D_0 = \{(x, y); x \in [a, b], y \in [c, d]\}$$

to‘g‘ri to‘rtburchak deyiladi (8-chizma)




8-чизма

Bu to‘g‘ri to‘rtburchakning tomonlari (chegaralari) mos ravishda koordinatalar o‘qiga parallel bo‘ladi.

$D_0$ to‘g‘ri to‘rtburchakning yuzi deb (uning chegarasining, ya’ni

$$x = a \quad , \quad x = b \qquad\qquad (c \le y \le d),$$

$$y = c \quad , \quad y = d \qquad\qquad (a \le x \le b)$$

to‘g‘ri chiziq kesmalarining $D_0$ ga tegishli bo‘lishi yoki tegishli bo‘lmasligidan qat’iy nazar) ushbu

$$\mu(D_0) = (b - a) \cdot (d - c)$$

miqdorga aytiladi.

Aytaylik, tekislik nuqtalaridan iborat biror $Q$ to‘plam berilgan bo‘lsin.

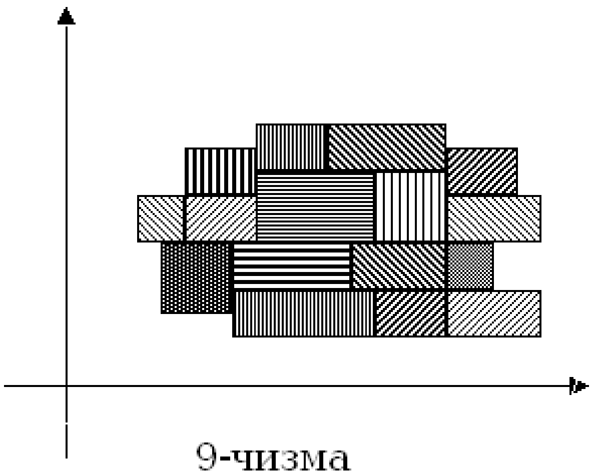
Agar shunday $D_0$ to‘g‘ri to‘rtburchak topilsaki,

$$Q \subset D_0$$

bo‘lsa, $Q$ chegaralangan to‘plam deyiladi.

Har qanday chegaralangan tekislik nuqtalaridan iborat to‘plam tekis shakl deyiladi.

Agar tekis shakl chekli sondagi kesishmaydigan to‘g‘ri to‘rtburchaklarning birlashmasi sifatida ifodalansa, uni to‘g‘ri ko‘pburchak deymiz.(9-chizma)



9-чизма

Bunday to‘g‘ri ko‘pburchakning yuzi deb, uni tashkil etgan to‘g‘ri to‘rtburchaklar yuzalari yig‘indisiga aytiladi.

To‘g‘ri ko‘pburchak yuzi quyidagi xossalarga ega:

1) To‘g‘ri ko‘pburchak yuzi har doim manfiy bo‘lmaydi: $\mu(D) \geq 0$;

2) Kesishmaydigan ikki $D_1$ va $D_2$ to‘g‘ri ko‘pburchaklar-dan tashkil topgan to‘g‘ri ko‘pburchak yuzi $D_1$ va $D_2$ larning yuzalari yig‘indisiga teng:

$$\mu(D_1 \cup D_2) = \mu(D_1) + \mu(D_2) \quad ;$$

3) Agar $D_1$ va $D_2$ to‘g‘ri ko‘pburchaklar uchun

$$D_1 \subset D_2$$

bo‘lsa, u holda

$$\mu(D_1) \leq \mu(D_2)$$

bo‘ladi.

Tekislikda biror chegaralangan $Q$ shakl berilgan bo‘lsin. Bu shaklning ichiga $A$ to‘g‘ri ko‘pburchak $(A \subset Q)$, so‘ngra $Q$ shaklni o‘z ichiga olgan $B$ to‘g‘ri ko‘pburchak $(Q \subset B)$ lar chizamiz. Ularning yuzlari mos ravishda $\mu(A)$ va $\mu(B)$ bo‘lsin.

Ravshanki, bunday to‘g‘ri ko‘pburchaklar ko‘p bo‘lib, ularning yuzalaridan iborat $\{\mu(A)\}$ va $\{\mu(B)\}$ to‘plamlar hosil bo‘ladi.

Ayni paytda, bu sonli to'plamlar chegaralangan bo'ladi. Binobarin, ularning aniq chegaralari

$$\sup\{\mu(A)\},\ \inf\{\mu(B)\}$$

lar mavjud.

**1-ta'rif.** Agar

$$\sup\{\mu(A)\} = \inf\{\mu(B)\}$$

bo'lsa, $Q$ shakl yuzaga ega deyiladi. Ularning umumiy qiymati $Q$ shaklning yuzi deyiladi va $\mu(Q)$ kabi belgilanadi:

$$\mu(Q) = \sup\{\mu(A)\} = \inf\{\mu(B)\}$$

**1-teorema.** Tekis shakl $Q$ yuzaga ega bo'lish uchun $\forall\varepsilon > 0$ son olinganda ham shunday $A$ $(A \subset Q)$ va $B$ $(Q \subset B)$ to'g'ri ko'pburchaklar topilib, ular uchun

$$\mu(B) - \mu(A) < \varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

◀ **Zarurligi.** Aytaylik, $Q$ shakl yuzaga ega bo'lsin. Unda ta'rifga binoan

$$\sup\{\mu(A)\} = \inf\{\mu(B)\} = \mu(Q)$$

bo'ladi.

Modomiki,

$$\sup\{\mu(A)\} = \mu(Q),$$

$$\inf\{\mu(B)\} = \mu(Q)$$

ekan, unda $\forall\varepsilon > 0$ olinganda ham shunday to'g'ri ko'pburchak $A(A \subset Q)$ hamda shunday to'g'ri ko'pburchak $B(Q \subset B)$ topiladiki,

$$\mu(Q) - \mu(A) < \frac{\varepsilon}{2},$$

$$\mu(B) - \mu(Q) < \frac{\varepsilon}{2}$$

bo'ladi. Bu tengsizliklardan

$$\mu(B) - \mu(A) < \varepsilon$$

bo'lishi kelib chiqadi.

**Etarliligi.** Aytaylik, $A$ $(A \subset Q)$ va $B$ $(Q \subset B)$ to'g'ri ko'pburchaklar uchun $\mu(B) - \mu(A) < \varepsilon$ tengsizligi bajarilsin.

Ravshanki,

$$\mu(A) \le \sup\{\mu(A)\},$$

$$\mu(B) \ge \inf\{\mu(B)\}.$$

Bu munosabatlardan

$$\inf\{\mu(B)\} - \sup\{\mu(A)\} \le \mu(B) - \mu(A) < \varepsilon$$

bo'lishini topamiz.

$\varepsilon$ -ixtiyoriy musbat son bo‘lganligidan

$$\sup\{\mu(A)\} = \inf\{\mu(B)\}$$

bo‘lishi kelib chikadi. Demak, $Q$ shakl yuzaga ega. ►

SHunga o‘xshash quyidagi teorema isbotlanadi.

**2-teorema.** Tekis shakl $Q$ yuzaga ega bo‘lishi uchun $\forall\varepsilon > 0$ son olinganda ham shunday yuzaga ega tekis shakllar $P$ va $S$ lar $(P \subset Q$ , $Q \subset S)$ topilib, ular uchun

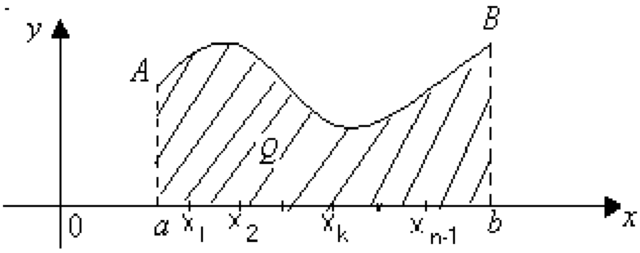
$$\mu(S) - \mu(P) < \varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

## $2^0$. Egri chizikli trapetsiyaning yuzini hisoblash.

Faraz qilaylik, $f(x) \in C[a,b]$ bo‘lib, $\forall x \in [a,b]$ da $f(x) \geq 0$ bo‘lsin.

YUqoridan $f(x)$ funksiya grafigi, yon tomonlardan $x = a$ , $x = b$ vertikal chiziqlar hamda pastdan absissa o‘qi bilan chegaralangan $Q$ shaklni qaraylik. (10-chizma)




10-chizma

Odatda, bu shakl egri chiziqli trapetsiya deyiladi. $[a,b]$ segmentni ixtiyoriy

$$P = \{x_0, x_1, x_2, ..., x_n\} \quad (a = x_0 < x_1 < x_2 < ... < x_n = b)$$

bo‘laklashni olamiz. Bu bo‘laklashning har bir $[x_k, x_{k+1}]$ oralig‘ida

$$\inf\{f(x)\} = m_k, \quad \sup\{f(x)\} = M_k \quad (k = 0,1,2,...,n-1)$$

mavjud bo‘ladi.

Endi asosi $\Delta x_k = x_{k+1} - x_k$, balandligi $m_k$ bo‘lgan $(k = 0,1,2,...,n-1)$ to‘g‘ri to‘rtburchaklarning birlashmasidan tash-kil topgan to‘g‘ri ko‘pburchakni $A$ deylik.

Shuningdek, asosi $\Delta x_k = x_{k+1} - x_k$, balandligi $M_k$ bo‘lgan $(k = 0,1,2,...,n-1)$ to‘g‘ri to‘rtburchaklarning birlashmasidan tashkil topgan to‘g‘ri ko‘pburchakni $B$ deylik. Ravshanki,

$$A \subset Q \ , \ \ Q \subset B$$

bo‘lib, ularning yuzalari

$$\mu(A)=\sum_{k=0}^{n-1} m_k\cdot\Delta x_k,\ \ \mu(B)=\sum_{k=0}^{n-1} M_k\cdot\Delta x_k$$

bo‘ladi.

Bu yig‘indilarni $f(x)$ funksiyaning $[a,b]$ segmentining $P$ bo‘laklashiga nisbatan Darbuning quyi hamda yuqori yig‘indilari ekanini payqash qiyin emas:

$$\mu(A)=s(f;P)\quad ,\quad \mu(B)=S(f;P).$$

$f(x)\in C[a,b]$ bo‘lgani uchun $f(x)$ funksiya $[a,b]$ da integralla-nuvchi bo‘ladi. Unda integrallanuvchilik mezoniga ko‘ra, $\forall\varepsilon>0$ olinganda ham $[a,b]$ segmentning shunday $P$ bo‘laklashi topiladiki,

$$S(f;P)-s(f;P)<\varepsilon$$

bo‘ladi. Birobarin, ushbu

$$\mu(B)-\mu(A)<\varepsilon$$

**tengsizlik bajariladi. Bu esa, 1-teoremaga muvofiq, qaralayotgan egri chiziqli trapetsiyaning yuzaga ega bo‘lishini bildiradi. Unda ta’rifga ko‘ra**

$$\sup\{\mu(A)\}=\inf\{\mu(B)\}$$

bo‘ladi.

Ayni paytda,

$$\sup\{\mu(A)\}=\underline{\int_a^b} f(x)dx,$$

$$\inf\{\mu(B)\}=\overline{\int_a^b} f(x)dx$$

bo‘lganligi sababli $Q$ egri chiziqli trapetsiyaning yuzi

$$\mu(Q)=\int_a^b f(x)dx \qquad (1)$$

ga teng bo‘ladi.

**1-misol.** Tekislikda ushbu

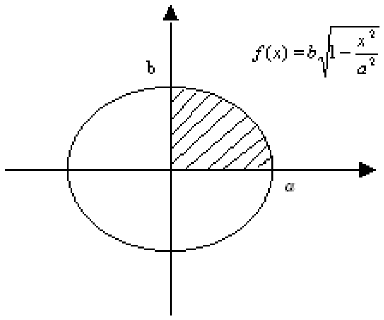
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ellips bilan chegaralangan $Q$ shaklning yuzi topilsin.

◄Ellips bilan chegaralangan $Q$ shaklning yuzi $OX$ va $OY$ koordinata o‘qlari hamda

$$f(x)=b\cdot\sqrt{1-\frac{x^2}{a^2}}\qquad ,\qquad 0\le x\le a$$

chiziqlar bilan chegaralangan egri chiziqli trapetsiya yuzi-ning 4 tasiga teng bo‘ladi. (11-chizma ).




11-чизма

Unda (1) formuladan foydalanib topamiz:

$$\mu(Q)=4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx=$$

$$=\left|\begin{matrix} x=a\sin t\ , & \\ & 0\le t\le\frac{\pi}{2} \\ dx=a\cos t dt\ , & \end{matrix}\right|=$$

$$=\frac{4b}{a}\cdot a^2\int_0^{\frac{\pi}{2}}\cos^2 t dt=4ab\cdot\frac{\pi}{4}=ab\pi.\ \blacktriangleright$$

Aytaylik, $f_1(x)\in C[a,b]$ , $f_2(x)\in C[a,b]$ bo‘lib, $\forall x\in[a,b]$ da

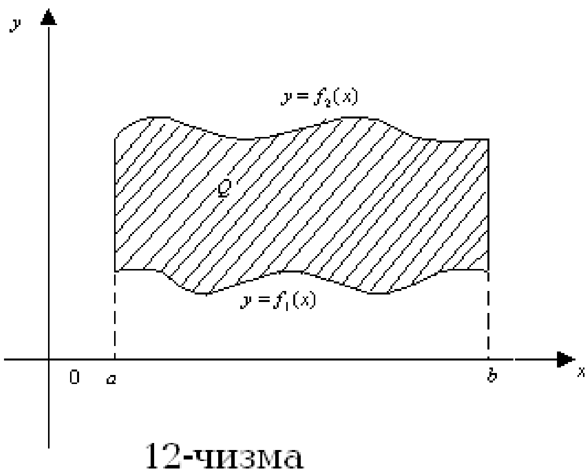
$$0\le f_1(x)\le f_2(x)$$

bo‘lsin.

Tekislikdagi $Q$ shakl quyidagi

$$y=f_1(x)\quad,\quad y=f_2(x)\quad,\quad x=a\quad,\quad x=b$$

chiziqlar bilan chegaralangan shaklni ifodalasin (12-chizma)




12-чизма

Bu shaklning yuzi

$$\mu(Q)=\int_a^b f_2(x)dx-\int_a^b f_1(x)dx=\int_a^b [f_2(x)-f_1(x)]dx \qquad (2)$$

bo‘ladi.

**2-misol.** Tekislikda ushbu

$$y=4-x^2 \quad , \qquad y=x^2-2x$$

chiziqlar (parabolalar) bilan chegaralangan $Q$ shaklning yuzi topilsin.

◀ Parabolalarning tenglamalari

$$y=4-x^2 \ ,$$

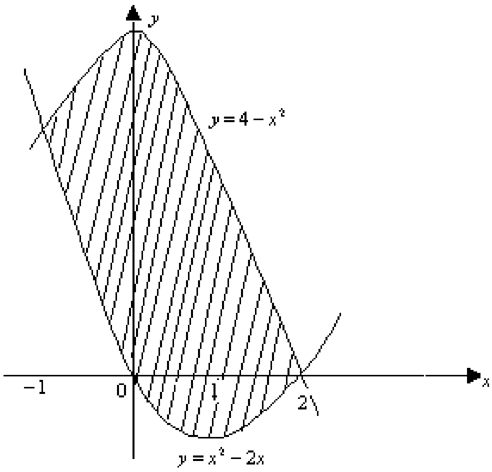
$$y=x^2-2x$$

ni birgalikda echib, ularning kesishish nuqtalarini topamiz:

$$4-x^2=x^2-2x \ ,$$

$$x_1=-1\,,\ x_2=2\,;\ y_1=3\,,\ y_2=0:\ \ A(-1;3)\,,\ B(2;0).$$

(13 -chizma).




13-чизма

Bu shaklning yuzini (2) formuladan foydalanib hisob-laymiz:

$$\mu(Q)=\int_{-1}^{2}\left[(4-x^2)-(x^2-2x)\right]dx=\int_{-1}^{2}\left(4+2x-2x^2\right)dx=(4x+x^2-\frac{2}{3}x^3)\Big|_{-1}^{2}=9. ▶$$

**Eslatma.** Agar $f(x)\in C[a,b]$ funksiya $[a,b]$ da ishora saqlamasa, (1) integral egri chiziqli trapetsiyalar yuzalari-ning yig‘indisidan iborat bo‘ladi. Bunda $OX$ o‘qining yuqori-sidagi yuza musbat ishora bilan, $OX$ o‘qining pastdagi yuza manfiy ishora bilan olinadi.

Masalan, $OX$ o‘qi hamda $f(x)=\sin x$ , $0\le x\le 2\pi$ funksiya grafigi bilan chegaralangan shaklning yuzi

$$\mu(Q)=\int_0^{\pi}\sin x+(-\int_{\pi}^{2\pi}\sin xdx)=(-\cos x)\Big|_0^{\pi}-(-\cos x)\Big|_{\pi}^{2\pi}=4$$

bo‘ladi.

### $3^0$. Egri chiziqli sektorning yuzini hisoblash.

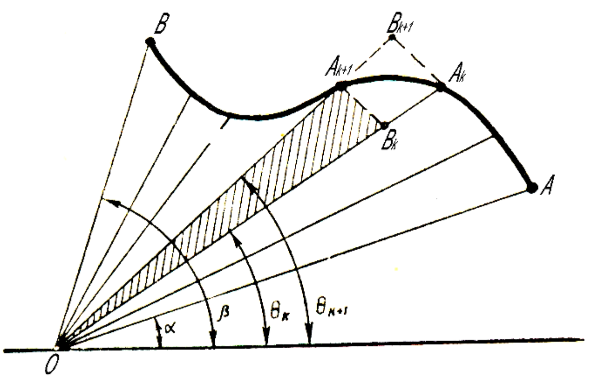
Aytaylik, $A\breve{B}$ egri chiziq qutb koordinatalar sistemasida ushbu

$$\rho=\rho(\theta)\quad,\quad\alpha\le\theta\le\beta\quad(\alpha\in R,\beta\in R)$$

tenglama bilan berilgan bo‘lsin. Bunda

$$\rho(\theta)\in C[\alpha,\beta]\ ,\qquad\forall\theta\in[\alpha,\beta]\quad\text{да}\qquad\rho(\theta)\ge 0\ .$$

Tekislikda $A\breve{B}$ egri chiziq hamda $OA$ va $OB$ radius-vektorlar bilan chegaralangan $Q$ shaklni qaraymiz. (14 -chizma).




14- chizma

$[\alpha,\beta]$ segmentni ixtiyoriy

$$P=\{\theta_0,\theta_1,\ldots,\theta_n\}\qquad(\alpha=\theta_0<\theta_1<\ldots<\theta_n=\beta)$$

bo‘laklashini olamiz. $O$ nuqtadan har bir qutb burchagi $\theta_k$ ga mos $OA_k$ radius-vektor o‘tkazamiz. Natijada $OAB$-egri chiziq-li sektor

$$OA_kA_{k+1}\ (k=0,1,2,\ldots,n-1\quad;\ A_0=A\ ,\ A_n=B)$$

egri chiziqli sektorchalarga ajraladi.

Ravshanki, $\rho=\rho(\theta)\in C[\alpha,\beta]$

bo‘lganligi uchun $[\theta_k\ ,\theta_{k+1}]$ da $(k=0,1,2,\ldots,n-1)$

$$m_k=\inf\{\rho(\theta)\}\quad,\quad M_k=\sup\{\rho(\theta)\}$$

lar mavjud.

Endi har bir $[\theta_k,\theta_{k+1}]$ segment uchun radius-vektorlari mos ravishda $m_k$ hamda $M_k$ bo‘lgan doiraviy sektorlarni hosil qilamiz. Bunday doiraviy sektorlar yuzaga ega bo‘lib, ularning yuzi mos ravishda

$$\frac{1}{2}m_k^2 \cdot \Delta\theta_k \quad , \quad \frac{1}{2}M_k^{2\cdot} \cdot \Delta\theta_k \qquad (\Delta\theta_k = \theta_{k+1} - \theta_k)$$

bo‘ladi.

Radius-vektorlari $m_k$ $(k = 0,1,2,...,n-1)$ bo‘lgan barcha doiraviy sektorlar birlashmasidan hosil bo‘lgan shaklni $Q_1$ desak, unda $Q_1 \subset Q$ bo‘lib, uning yuzi

$$\mu(Q_1) = \frac{1}{2}\sum_{k=0}^{n-1} m_k^2 \cdot \Delta\theta_k \qquad (3)$$

bo‘ladi.

SHuningdek, radius-vektorlari $M_k$ $(k = 0,1,2,...,n-1)$ bo‘l-gan barcha doiraviy sektorlar birlashmasidan hosil bo‘lgan shaklni $Q_2$ desak, unda $Q \subset Q_2$ bo‘lib, uning yuzi

$$\mu(Q_2) = \frac{1}{2}\sum_{k=0}^{n-1} M_k^2 \cdot \Delta\theta_k \qquad (4)$$

bo‘ladi.

(3) va (4) yig‘indilar $\frac{1}{2}\rho^2(\theta)$ funksiyaning Darbu yig‘indilari bo‘ladi. Ayni paytda, $\frac{1}{2}\rho^2(\theta)$ funksiya $[\alpha,\beta]$ da uzluksiz bo‘lgani uchun u integrallanuvchidir. Demak, $\forall\varepsilon > 0$ olinganda ham $[\alpha,\beta]$ segmentning shunday $P$ bo‘laklashi topiladiki,

$$S(\frac{1}{2}\rho^2(\theta); P) - s(\frac{1}{2}\rho^2(\theta); P) < \varepsilon$$

bo‘ladi. Binobarin, ushbu

$$\mu(Q_2) - \mu(Q_1) < \varepsilon$$

tengsizlik bajariladi. Bu esa, 2-teoremaga muvofiq, qaralayotgan egri chiziqli sektorning yuzaga ega bo‘lishini bildiradi. Unda ta’rifga ko‘ra

$$\sup\{\mu(Q_1)\} = \inf\{\mu(Q_2)\}$$

bo‘ladi.

Ayni paytda,

$$\sup\{\mu(Q_1)\} = \underline{\int_\alpha^\beta} \rho^2(\theta)d\theta,$$

$$\inf\{\mu(Q_2)\} = \overline{\int_\alpha^\beta} \rho^2(\theta)d\theta$$

bo‘lgani sababli $Q$ egri chiziqli sektorning yuzi

$$\mu(Q)=\frac{1}{2}\int_{\alpha}^{\beta}\rho^2(\theta)d\theta$$

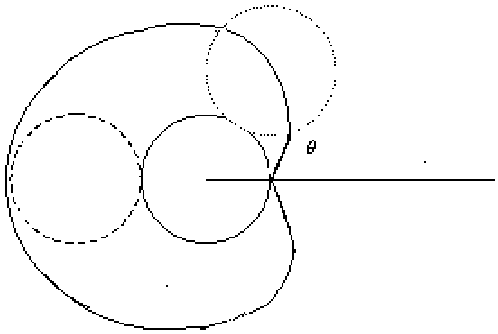
ga teng bo‘ladi.

**3-misol.** Ushbu

$$\rho=\rho(\theta)=a(1-\cos\theta)\quad (a\in R,\ 0\le\theta\le 2\pi)$$

funksiya grafigi bilan chegaralangan shaklning yuzi topilsin.

◄ Bu funksiya grafigi kardioidani ifodalaydi. Ma’lumki, kardioida radiusi $r$ ga teng bo‘lgan aylananing shu radiusli ikkinchi qo‘zg‘almas aylana bo‘ylab xarakati (sirpanmasdan dumalashi) natijasida birinchi aylana ixtiyoriy nuqtasining chizgan chizig‘idir. (15-chizma).




15-чизма

Kardioida qutb o‘qiga nisbatan simmetrik bo‘lganligi sababli yuqori yarim tekislikdagi shaklning yuzini topib, so‘ngra uni 2 ga ko‘paytirsak, izlanayotgan yuza kelib chiqadi.

$\theta$ o‘zgaruvchi $[0,\pi]$ da o‘zgarganda $\rho$ radius-vektor kardioidaning yuqori yarim tekislikdagi qismini chizadi. SHuning uchun

$$\mu(Q)=2\cdot\frac{1}{2}\int_0^{\pi}\rho^2(\theta)d\theta=\int_0^{\pi}a^2(1-\cos\theta)^2d\theta=$$

$$=a^2\int_0^{\pi}\left[\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right]d\theta=$$

$$=a^2\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi}=\frac{3}{2}\pi a^2$$

bo‘ladi. ►

## Mashqlar

1. Aytaylik, tekislikda $\breve{AB}$ egri chiziq $x=\varphi(t),\ y=\psi(t)\ (\alpha\le t\le\beta)$ tenglamalar bilan parametrik holda berilgan bo‘lsin, bunda $x=\varphi(t)$ funksiya

$[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hosilaga ega, $\varphi'(x)\geq 0$ va $\varphi(\alpha)=a$, $\varphi(\beta)=b$, $y=\psi(t)$ funksiya $[a,b]$ da uzluk-siz va $\psi(t)\geq 0$. U holda yuqoridan $\breve{AB}$ egri chiziq, yon tomon-laridagi $x=a$, $x=b$ vertikal chiziqlar, pastdan $[a,b]$ kesma bilan chegaralangan shaklning yuzi

$$S=\int_{\alpha}^{\beta}\psi(t)\varphi'(t)dt$$

bo‘lishini isbotlansin.

2. Ushbu

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$$

chiziq bilan chegaralangan shaklning yuzi topilsin.

## Adabiyotlar


1. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’rizalar, I q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Kurs differensialnogo i integralnogo ischisleniya, 1 t.* M. «FIZMATLIT», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

# Glossariy

**1-ta’rif.** Agar

$$\sup\{\mu(A)\} = \inf\{\mu(B)\}$$

bo‘lsa, $Q$ shakl yuzaga ega deyiladi. Ularning umumiy qiymati $Q$ shaklning yuzi deyiladi va $\mu(Q)$ kabi belgilanadi:

$$\mu(Q) = \sup\{\mu(A)\} = \inf\{\mu(B)\}$$

**1-teorema.** Tekis shakl $Q$ yuzaga ega bo‘lish uchun $\forall\varepsilon > 0$ son olinganda ham shunday $A$ $(A \subset Q)$ va $B$ $(Q \subset B)$ to‘g‘ri ko‘pburchaklar topilib, ular uchun

$$\mu(B) - \mu(A) < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

**Egri chiziqli sektorning yuzi**

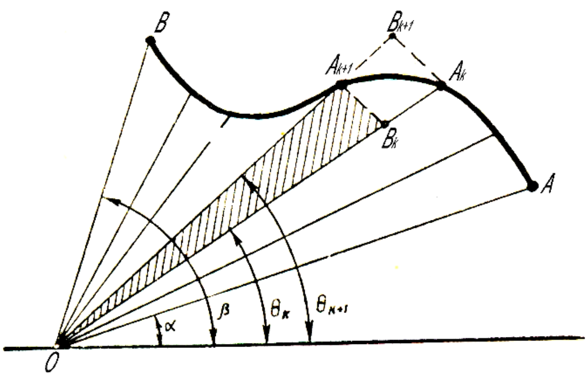
$\breve{AB}$ egri chiziq qutb koordinatalar sistemasida ushbu

$$\rho = \rho(\theta) \quad , \quad \alpha \le \theta \le \beta \quad (\alpha \in R, \beta \in R)$$

tenglama bilan berilgan bo‘lsin. Bunda

$$\rho(\theta) \in C[\alpha, \beta] \ , \quad \forall\theta \in [\alpha, \beta] \quad \text{да} \qquad \rho(\theta) \ge 0 \ .$$

Tekislikda $\breve{AB}$ egri chiziq hamda $OA$ va $OB$ radius-vektorlar bilan chegaralangan $Q$ shaklni qaraymiz. (14 -chizma).




14- chizma

$[\alpha,\beta]$ segmentni ixtiyoriy

$$P=\{\theta_0,\theta_1,\ldots,\theta_n\} \qquad (\alpha=\theta_0<\theta_1<\ldots<\theta_n=\beta)$$

bo‘laklashini olamiz. $O$ nuqtadan har bir qutb burchagi $\theta_k$ ga mos $OA_k$ radius-vektor o‘tkazamiz. Natijada $OAB$-egri chiziq-li sektor

$$OA_kA_{k+1}\ \left(k=0,1,2,\ldots,n-1\quad;\ A_0=A\,,\ A_n=B\right)$$

egri chiziqli sektorchalarga ajraladi.

Egri chiziqli sektor yuzi

$$\mu(Q)=\frac{1}{2}\int_{\alpha}^{\beta}\rho^2(\theta)d\theta$$

## Keys banki

**38-keys.** Masala o`rtaga tashlanadi: Aytaylik, tekislikda $\breve{AB}$ egri chiziq $x=\varphi(t)$, $y=\psi(t)$ $(\alpha\le t\le\beta)$ tenglamalar bilan parametrik holda berilgan bo‘lsin, bunda $x=\varphi(t)$ funksiya $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ hosilaga ega, $\varphi'(x)\ge 0$ va $\varphi(\alpha)=a$, $\varphi(\beta)=b$, $y=\psi(t)$ funksiya $[a,b]$ da uzluk-siz va $\psi(t)\ge 0$. U holda yuqoridan $\breve{AB}$ egri chiziq, yon tomon-laridagi $x=a$, $x=b$ vertikal chiziqlar, pastdan $[a,b]$ kesma bilan chegaralangan shaklning yuzi

$$S=\int_{\alpha}^{\beta}\psi(t)\varphi'(t)dt$$

bo‘lishini isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 13-amaliy mashg'ulot.

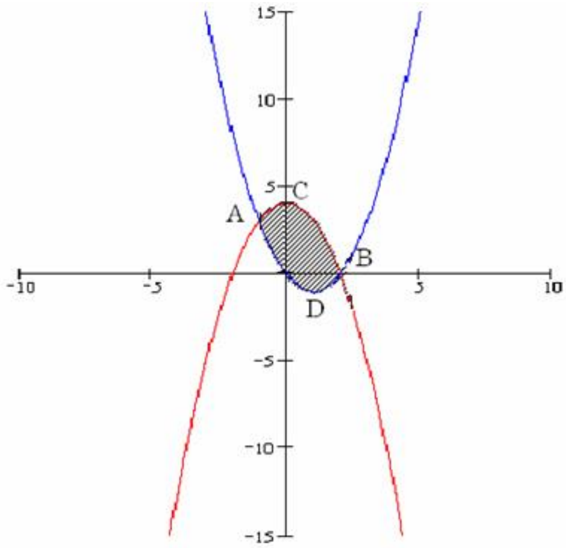
**3 – м и с о л .** ***Ушбу***

$$y = 4 - x^2 \qquad \text{ва} \qquad y = x^2 - 2x$$

***эгри чизиқлар (параболалар) билан чегараланган шаклнинг юзи топилсин.***

◄Параболалар $A(-1;3)$ ва $B(2;0)$ нуқталарда кесишади. Изланаётган шаклнинг юзаси

$$S = S_{A_1ACB} + S_{OBD} - S_{A_1AO}$$ бўлади. (12-чизма)




**12-чизма.**

Равшанки

$$S_{A_1ACB} = \int_{-1}^{2} (4 - x^2)dx = \left(4x - \frac{x^3}{3}\right)\Bigg|_{-1}^{2} = 9,$$

$$S_{OBD} = \int_{2}^{0} (x^2 - 2x)dx = \left(\frac{x^3}{3} - x^2\right)\Bigg|_{2}^{0} = \frac{4}{3},$$

$$S_{A_1AO} = \int_{-1}^{0}\left(x^2 - 2x\right)dx = \left(\frac{x^3}{3} - x^2\right)\Bigg|_{-1}^{0} = \frac{4}{3}.$$

Демак, $S = 9 + \frac{4}{3} - \frac{4}{3} = 9$. ►

**Тўғри бурчакли координаталар системасида берилган қуйидаги эгри чизиқлар билан чегараланган шаклларнинг юзаси топилсин.**

2097. $y = x^2 + 1, \quad x + y = 3$

2098. $y = 4x - x^2, \quad y = 0$

2099. $y^2 = 2px, \quad x^2 = 2py$

2100. $y^2 = 2x + 1, \quad x - y - 1 = 0$

2101. $y = \ln x, \quad y = 0, \quad x = e$

2102. $y^2 = x^3, \quad x = 0, \quad y = 4$

2103. $y = |\lg x|, \quad y = 0, \quad x = 0,1, \quad x = 10$

2104. $y = \frac{x^2}{2}, \quad y = 2 - \frac{3}{2}x$

2105. $y = x, \quad y = x + \sin^2 x, \quad 0 \le x \le \pi$

2106. $y = \operatorname{tg} x, \quad y = 0, \quad x = \frac{\pi}{3}$

2107. $y = \frac{1}{2} \cdot x^2, \quad y = \frac{1}{1 + x^2}$

2108. $y = \operatorname{tg} x, \quad y = \frac{2}{3} \cdot \cos x, \quad x = 0$

2109. $y = x^2, \quad y = \frac{x^2}{2}, \quad y = 2x$

2110. $y = e^x, \quad y = e^{-x}, \quad x = 1$

2111. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad x = 2a$

2112. $y = x - x^2, \quad y = x \cdot \sqrt{1 - x}$

**2113.** $y = \sin 2x, \quad y = \sin x, \quad \frac{\pi}{3} \le x \le \pi$

**2114.** $2y = x^2, \quad x^2 + y^2 = 8, \quad y \ge 0$

**2115.** $y = \ln(1 + x), \quad y = -xe^{-x}, \quad x = 1$

**2116.** $y = 6x^2 - 5x + 1, \quad y = \cos\pi x, \quad 0 \le x \le \frac{1}{2}$

**2117.** $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \quad x = 0, \quad x = a$

**2118.** $y = x^2 + 6x + 10, \quad x^2 + y^2 + 6x - 2y + 8 = 0$

**2119.** $6x = y^3 - 16y, \quad 24x = y^3 - 16y$

**2120.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

**2121.** $y^2 = x^2(a^2 - x^2)$

**2122.** $y = \ln\cos x, \quad 0 \le x \le a \quad \left(a < \frac{\pi}{2}\right).$

**2123.** $y = 2^{\sqrt{1+e^{\frac{x}{2}}}}, \quad \ln 9 \le x \le \ln 64$

**2124.** $x = \frac{2}{3} \cdot \sqrt{(y-1)^3}, \quad 0 \le x \le 2\sqrt{3}.$

**2125.** $y^2 = \frac{x^2}{2a - x}, \quad x = 2a$

**2126.** $x = a \cdot \ln\frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}, \quad y = 0$

**Параметрик кўринишда берилган эгри чизиқлар билан чегараланган шаклларнинг юзи топилсин:**

2127. $x = a\cos t, \quad y = a\sin t$

2128. $x = at - t^2, \quad y = at^2 - t^3 \qquad (a > 0)$

2129. $x = 1 + t - t^3, \quad y = 1 - 15t^2$

2130. $x = \frac{t(1 - t^2)}{1 + 3t^2}, \quad y = \frac{4t^2}{1 + 3t^2}$

2131. $x = \frac{1}{1 + t^2}, \quad y = \frac{t(1 - t^2)}{1 + t^2}$

2132. $x = \frac{3at}{1 + t^3}, \quad y = \frac{3at^2}{1 + t^3} \qquad (\text{бунда} \;\; 0 \le t < +\infty)$

2133. $x = a(t - \sin t), \quad y = a(1 - \cos t)$ ҳамда $y = 0 \quad (0 \le t \le 2\pi)$

2134. $x = a\sin^3 t, \quad y = b\cos^3 t, \quad 0 \le t \le 2\pi$

2135. $x = a\sin t \cdot \cos^2 t, \quad y = b\cos t \cdot \sin^2 t, \quad 0 \le t \le \frac{\pi}{2}$

2136. $x = a(1 - \cos t)\cos t, \quad y = a(1 - \cos t)\sin t$

2137. $x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t) \quad (0 \le t \le 2\pi)$ ва $(a,0), \quad (a,-2\pi a)$ нуқталарни бирлаштирувчи тўғри чизиқ кесмаси.

2138. $x = a(2\cos t - \cos 2t), \quad y = a(2\sin t - \sin 2t)$

2139. $x = a\sin 2t, \quad y = a\sin t \quad (a > 0)$

2140. $x = a\left(\frac{2}{\pi}t - \sin t\right), \quad y = a(1 - \cos t) \quad (a > 0)$

**2141.** $x = a\cos t, \quad y = \dfrac{a\sin^2 t}{2+\sin t}$

**Қутб координаталар системасида берилган қуйидаги эгри чизиқлар билан чегараланган шаклларнинг юзи топилсин.**

2142. $r = a\varphi \quad (0 \le \varphi \le 2\pi)$

2143. $r = a\cos\varphi$

2144. $r = r\sin 2\varphi$

2145. $r = a\sin 5\varphi$

2146. $r = a^2\cos 2\varphi$

2147. $r = a\cos 3\varphi$

2148. $r = a(1+\cos\varphi)$

2149. $r = 2 + \cos\varphi$

2150. $r = a\dfrac{\cos\varphi\cdot\sin\varphi}{\cos^3\varphi+\sin^3\varphi}, \quad 0 \le \varphi \le \dfrac{\pi}{2}$

2151. $r = 2\cdot\sqrt{3}\cdot a\cos\varphi, \quad r = 2a\sin\varphi$

2152. $r = 2 - \cos\varphi, \quad r = \cos\varphi$

2153. $r = a\cos\varphi, \quad r = a\cos\varphi + a\sin\varphi$

2154. $r = 2a\cos\varphi, \quad r = a\dfrac{\sin\varphi}{\cos^2\varphi}, \quad \varphi = 0$

2155. $r = \dfrac{1}{\varphi}; \quad r = \dfrac{1}{\sin\varphi} \quad \left(0 < \varphi \le \dfrac{\pi}{2}\right)$

2156. $r = a(1-\cos\varphi), \quad r = a$

2157. $\mathbf{r=\frac{p}{1+\frac{1}{2}\cos\varphi},\quad \varphi=\frac{\pi}{4},\quad \varphi=\frac{\pi}{2}}$

2158. $\mathbf{\varphi=r-\sin r,\quad \varphi=\pi}$

2159. $\mathbf{\varphi=r\cdot arctgr,\quad \varphi=0,\quad \varphi=\frac{\pi}{\sqrt{3}}}$

2160. $\mathbf{r=\frac{2at}{1+t^2},\quad \varphi=\frac{\pi t}{1+t}}$

# Test

| | | | | |
|---|---|---|---|---|
| $y=x$, $y=3x$ va $x=1$ to'g'ri chiziqlar bilan chegaralangan shaklning yuzasini hisoblang | 1 | 0 | 2 | -2 |
| $y=x$, $y=3x$ va $x=2$ to'g'ri chiziqlar bilan chegaralangan shaklning yuzasini hisoblang | 4 | 0 | -4 | 7 |
| $y=x$ to'g'ri chiziq va $y=x^2$ egri chiziq bilan chegaralangan shaklning yuzasini hisoblang | $\frac{1}{6}$ | 2 | 1 | 5 |
| $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips bilan chegaralangan shaklning yuzini hisoblang | $\pi ab$ | 2 | 3 | 4 |
| $x^2+y^2+z^2=1$ sfera bilan chegaralangan jismning hajmini hisoblang. | $\frac{4}{3}\pi$ | $\frac{4}{3}$ | -2 | 9 |
| $\int_0^1 x^2dx$ | $*\frac{1}{3}$ | -4 | 4 | $\frac{16}{3}$ |
| $\int_{-1}^0 6(2x+1)^2dx$ | *0 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $\int_{-1}^0 3e^{-x}dx$ | $*-3+3e$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int_6^7 \frac{dx}{x-5}$ | $*\ln 2$ | 0 | $\ln 3$ | $\ln 3-1$ |
| $\int_0^{\pi} 2\sin xdx$ | *4 | 1 | -1 | -2 |
| $\int_0^2 2xdx$ | * 4 | 2 | 5 | 1 |
| $\int_{\pi}^{2\pi} \sin 2xdx$ | * 0 | -1 | 1 | 2 |

| | | | | |
|---|---|---|---|---|
| $\int\limits_{\pi/2}^{\pi} 7\cos x dx$ | * -7 | 0 | 1 | -2 |
| $\int\limits_{4}^{7} \frac{dx}{x-1}$ | * $\ln 2$ | $\ln\frac{3}{2}$ | $\ln\frac{4}{3}$ | 1 |
| $\int\limits_{-1}^{0} (2x+1)^2 dx$ | * $\frac{1}{3}$ | 1 | -3 | -1 |
| $\int\limits_{-1}^{0} 2e^x dx$ | * $2-2e^{-1}$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int\left(x^3+\frac{1}{1+x^2}\right)dx - ?$ | * $\frac{x^4}{4}+arctgx+C$ | $2x+5+C$ | $\cos x+51x+C$ | $9x^2+C$ |
| $\int\limits_{0}^{\pi/2} 6\cos x dx$ | *6 | -1 | 0 | $\frac{\pi}{2}$ |
| $\int\limits_{0}^{\pi/4} \frac{5dx}{1+x^2}$ | *5 | -1 | 1 | 2 |
| $\int\limits_{0}^{1} \frac{3dx}{\sqrt{1-x^2}}$ | * $\frac{3\pi}{2}$ | 0 | 1 | 1 |
| Quyidagi tengliklardan qaysi biri o'rinli? | * $\int\limits_{a}^{b} f(x)dx = -\int\limits_{b}^{a} f(x)dx$ | $\int\limits_{a}^{-b} f(x)dx = \int\limits_{a}^{b} f(x)dx$ | $\int\limits_{-a}^{a} f(x)dx = 0$ | $\int\limits_{-a}^{-b} f(x)dx = -\int\limits_{a}^{b} f(x)dx$ |

# Mavzu. Yoy uzunligi va uni hisoblash

## 14-ma'ruza

**Reja**

**1⁰.** Yoy uzunligi tushunchasi.

**2⁰.** $y = f(x)$ tenglama bilan berilgan egri chiziq uzunligini hisoblash.

**3⁰.** Parametrik ko'rinishda berilgan egri chiziq uzunligini hisoblash.

### 1⁰. Yoy uzunligi tushunchasi.

Ma'lumki, tekislikdagi ikki $A(x_1, y_1)$ va $B(x_2, y_2)$ nuqtalarni birlashtiruvchi to'g'ri chiziq kesmasi $l_0$ uzunlikka ega va uning uzunligi

$$\mu(l_0) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

ga teng bo'ladi.

Aytaylik, tekislikdagi $l$ chiziq $A_0(x_0, y_0),\ A_1(x_1, y_1), ...,\ A_n(x_n, y_n)$ nuqtalarni $(n \in N)$ birin-ketin to'g'ri chiziq kesmalari bilan birlashtirishidan hosil bo'lgan bo'lsin. Odatda, bunday chiziq siniq chiziq deyiladi.

Siniq chiziq uzunligi (perimetri) deb, uni tashkil etgan to'g'ri chiziq kesmalari uzunliklarining yig'indisiga aytiladi:

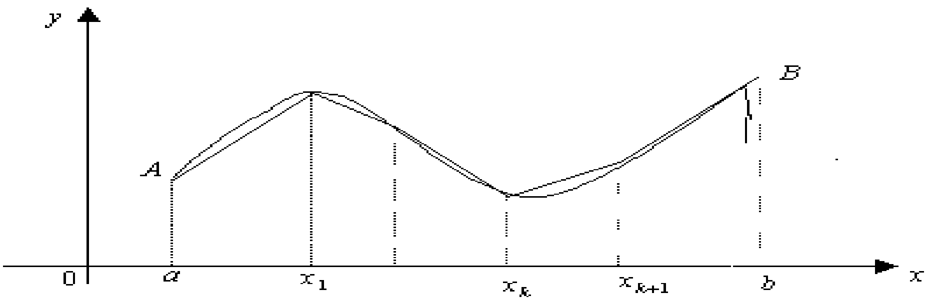
$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}.$$

Faraz qilaylik, tekislikdagi $\breve{AB}$ egri chizig'i (uni $\breve{AB}$ yoyi deb ham ataymiz) ushbu

$$y = f(x) \qquad (a \le x \le b)$$

tenglama bilan berilgan bo'lsin.

Bunda $f(x) \in C[a,b]$.




$[a,b]$ segmentning ixtiyoriy

$$P=\{x_0,x_1,\dots,x_n\} \qquad (a=x_0<x_1<\dots<x_n=b$$

bo'laklashni olib, bo'luvchi $x_k$ $(k=0,1,2,\dots,n)$ nuqtalar orqali $OY$ o'qiga parallel to'g'ri chiziqlar o'tkazamiz. Bu to'g'ri chiziqlarning $\breve{AB}$ yoyi bilan kesishgan nuqtalari

$$A_k(x_k, f(x_k)) \qquad (k=0,1,2,\dots,n\ ;\ A_0=A\ ,\ A_n=B)$$

bo'ladi.

$\breve{AB}$ yoyidagi bu $A_k(x_k, f(x_k))$ nuqtalarni bir-biri bilan to'g'ri chiziq kesmalari yordamida birlashtirib, $l$ siniq chiziqni hosil qilamiz. (16-chizma)

Odatda, $l$ siniq chiziq $\breve{AB}$ yoyiga chizilgan siniq chiziq deyiladi. U uzunlikka ega bo'lib, uzunligini (perimetrini) $\mu(l)$ deylik.

Agar $P_1$ va $P_2$ lar $[a,b]$ segmentning ikkita bo'laklashi bo'lib, $P_1\subset P_2$ bo'lsa, u holda bu bo'laklashlarga mos $\breve{AB}$ yoyiga chizilgan siniq chiziq $l_1$ , $l_2$ larning perimetrlari uchun

$$\mu(l_1)\le\mu(l_2)$$

bo'ladi.

Demak, $P$ bo'laklashning bo'luvchi nuqtalari sonini orttira borilsa, $\breve{AB}$ yoyiga chizilgan ularga mos siniq chiziqlar perimetrlari ham ortib boradi.

**1-ta'rif.** Agar $\lambda_p\to 0$ da $\breve{AB}$ yoyiga chizilgan siniq chiziq perimetri

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}$$

chekli limitga ega bo'lsa, $\breve{AB}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_p\to 0}\mu(l)=\mu(\breve{AB})$$

limit $\breve{AB}$ yoyining uzunligi deyiladi.

Masalan, agar

$$f(x)=kx+C \qquad (a\le x\le b)$$

bo‘lsa, unda $\breve{AB}$ ning uzunligi

$$\mu(\breve{AB}) = \lim_{\lambda_P \to 0} \sum_{k=0}^{n-1} \sqrt{(x_{k+1} - x_k)^2 + k^2 (x_{r+1} - x_k)^2} =$$

$$= \lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{1+k^2} \cdot (x_{k+1} - x_k) = \sqrt{1+k^2} \cdot (b-a)$$

bo‘ladi.

Aytaylik, $\breve{AB}$ egri chiziq ushbu

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}, \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan berilgan bo‘lsin.

Bunda:

1) $\varphi(t) \in C[\alpha, \beta]$ , $\psi(t) \in C[\alpha, \beta]$;

2) $\forall t_1, t_2 \in [\alpha, \beta]$ , $t_1 \ne t_2$ uchun (1)

$$A_1(x_1, y_1) = A_1(\varphi(t_1), \psi(t_1)) ,$$

$$A_2(x_2, y_2) = A_2(\varphi(t_2), \psi(t_2))$$

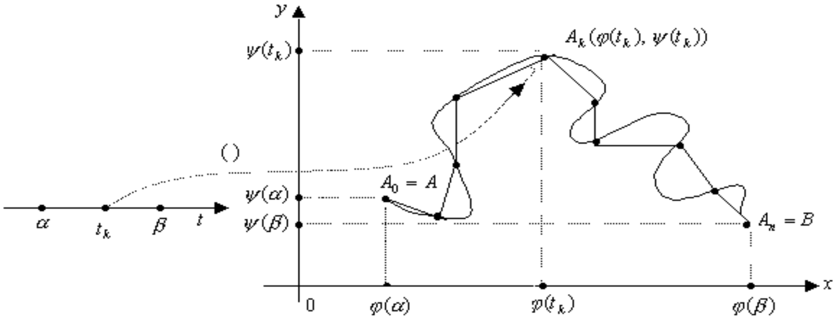
nuqtalar turlicha ;

3) $t = \alpha$ ga $A$ nuqta, $t = \beta$ ga $B$ nuqta mos kelsin.

$[\alpha, \beta]$ segmentning ixtiyoriy

$$P = \{t_0, t_1, ..., t_n\} \qquad (\alpha = t_0 < t_1 < ... < t_n = \beta$$

bo‘laklashni olib, bu bo‘laklashning bo‘luvchi $t_k$ $(k = 0,1,2,...,n)$ nuqtalariga mos kelgan $\breve{AB}$ yoydagi $A_k = A_k(x_k, y_k)$ $(x_k = \varphi(t_k)$ , $y_k = \psi(t_k)$ ; $k = 0,...,n)$ nuqtalarni bir-biri bilan to‘g‘ri chiziq kesmalari yordamida birlashtirib, $\breve{AB}$ yoyga chizilgan siniq chiziq $l$ ni hosil qilamiz.




Bu siniq chiziq perimetri

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{[\varphi(t_{k+1})-\varphi(t_k)]^2+[\psi(t_{k+1})-\psi(t_k)]^2}$$

bo‘ladi.

**2-ta’rif.** Agar $\lambda_p\to 0$ da $\breve{AB}$ yoyiga chizilgan siniq chiziq perimetri $\mu(l)$ chekli limitga ega bo‘lsa, $\breve{AB}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_p\to 0}\mu(l)=\mu(\breve{AB})$$

limit $\breve{AB}$ yoyining uzunligi deyiladi.

Yuqorida keltirilgan ta’riflardan yoy uzunligining ( agar u mavjud bo‘lsa ) musbat bo‘lishi kelib chiqadi.

Endi yoy uzunligining ikkita xossasini isbotsiz keltiramiz:

1) Agar $\breve{AB}$ yoyi uzunlikka ega bo‘lib, u $\breve{AB}$ yoydagi nuqtalar yordamida $n$ ta $A_k\breve{A}_{k+1}$ yoylarga $(k=0,1,2,...,n\ ;\ A_0=A, B=A_{n+1})$ ajralgan bo‘lsa, u holda har bir $A_k\breve{A}_{k+1}$ yoy uzunlikka ega va

$$\mu(\breve{AB})=\sum_{k=0}^{n}\mu(A_k\breve{A}_{k+1})$$

bo‘ladi.

2) Agar $\breve{AB}$ yoyi $n$ ta $A_k\breve{A}_{k+1}$ yoylarga ajralgan bo‘lib, har bir $A_k\breve{A}_{k+1}$ yoy uzunlikka ega bo‘lsa, u holda $\breve{AB}$ yoyi ham uzunlikka ega bo‘ladi.

**2⁰. $y=f(x)$ tenglama bilan berilgan egri chiziq uzunligini hisoblash.**

Faraz qilaylik, $\breve{AB}$ egri chiziq ushbu

$$y=f(x)\quad ,\quad a\le x\le b$$

tenglama bilan berilgan bo‘lsin. Bunda $f(x)$ funksiya $[a,b]$ segmentda uzluksiz va uzluksiz $f'(x)$ hosilaga ega.

$[a,b]$ segmentning ixtiyoriy

$$P=\{x_0,x_1,...,x_n\}\quad (a=x_0<x_1<...<x_n=b)$$

bo‘laklashini olib, unga mos $\breve{AB}$ yoyiga chizilgan $l$ siniq chiziqni hosil qilamiz. Bu siniq chiziqning perimetri

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}$$

bo‘ladi.

Har bir $[x_k,x_{k+1}]$ segmentda $f(x)$ funksiyaga Lagranj teoremasini qo‘llab topamiz:

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+[f'(\tau_k)\cdot(x_{k+1}-x_k)]^2}=$$

$$=\sum_{k=0}^{n-1}\sqrt{1+f'^2(\tau_k)}\cdot(x_{k+1}-x_k)=\sum_{k=0}^{n-1}\sqrt{1+f'^2(\tau_k)}\,\Delta x_k\quad ,$$

bunda $\tau_k\in[x_k,x_{r+1}]$.

Bu tenglikdagi yig‘indining $\sqrt{1+f'^2(x)}$ funksiyaning integral yig‘indisidan farqi shuki, integral yig‘indida $\xi_k\in[x_k,x_{k+1}]$ nuqta ixtiyoriy bo‘lgan holda yuqoridagi yig‘indida esa $\tau_k$ nuqta Lagranj teoremasiga muvofiq olingan tayin nuqta bo‘lishidadir. Ammo $\sqrt{1+f'^2(x)}$ funksiya integrallanuvchi bo‘lganligi sababli $\xi_k=\tau_k$ deb olinishi mumkin. Natijada

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k$$

bo‘lib, undan

$$\lim_{\lambda_p\to0}\mu(l)=\lim_{\lambda_p\to0}\sum_{k=0}^{n-1}\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k=\int_a^b\sqrt{1+f'^2(x)}dx$$

bo‘lishi kelib chiqadi.

Demak, $A\breve{B}$ yoyining uzunligi

$$\mu(A\breve{B})=\int_a^b\sqrt{1+f'^2(x)}dx \qquad (2)$$

bo‘ladi. Bu formula yordamida yoy uzunligi hisoblanadi.

**1-misol.** Ushbu

$$f(x)=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})\qquad (a>0\ ,\quad -a\le x\le a)$$

tenglama bilan berilgan $A\breve{B}$ egri chizig‘ining uzunligi topilsin.

Bu tenglama bilan aniqlanadigan chiziq zanjir chizig‘i deyiladi.

◄ Ravshanki,

$$f'(x)=\frac{1}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})\ ,$$

$$1+f'^2(x)=\frac{1}{4}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2\ ,$$

$$\sqrt{1+f'^2(x)}=\frac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$$

bo‘ladi. (2) formuladan foydalanib, zanjir chizig‘ining uzunligini topamiz:

$$\mu(\breve{AB}) = \int_{-a}^{a} \frac{1}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})dx = \frac{a}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})\Big|_{-a}^{a} = a(e - \frac{1}{e}). \blacktriangleright$$

**$3^0$. Parametrik ko‘rinishda berilgan egri chiziq uzun-ligini hisoblash.**

Faraz qilaylik, $\breve{AB}$ egri chiziq ushbu

$$\begin{cases} x = \varphi(t), \\ y = \psi(t) \end{cases} \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan berilgan bo‘lib, (1) shartlar-ning bajarilishi bilan birga $\varphi(t), \psi(t)$ funksiyalari $[\alpha, \beta]$ da uzluksiz $\varphi'(t)$ hamda $\psi'(t)$ hosilalarga ega bo‘lsin.

$[\alpha, \beta]$ segmentning ixtiyoriy

$$P = \{t_0, t_1, ..., t_n\} \qquad (\alpha = t_0 < t_1 < ... < t_n = \beta)$$

bo‘laklashini olib, ularga mos $\breve{AB}$ yoyiniig $A_k = A_k(x_k, y_k)$ $(x_k = \varphi(t_k),\ y_k = \varphi(t_k))$ nuqtalarini bir-biri bilan to‘g‘ri chiziq kesmasi yordamida birlashtirishdan hosil bo‘lgan $l$ siniq chiziq perimetri

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{[\varphi(t_{k+1}) - \varphi(t_k)]^2 + [\psi(t_{k+1}) - \psi(t_k))]^2}$$

ni qaraymiz.

Lagranj teoremasidan foydalanib topamiz:

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{\varphi'^2(\tau_k) \cdot (t_{k+1} - t_k)^2 + \psi'^2(\theta_k) \cdot (t_{k+1} - t_k)^2} =$$

$$= \sum_{k=0}^{n-1} \sqrt{\varphi'^2(\tau_k) + \psi'^2(\theta_k)} \cdot \Delta t_k \qquad (\Delta t_k = t_{k+1} - t_k)$$

bunda

$$\tau_k \in [t_k, t_{k+1}]. \qquad \theta_k \in [t_k, t_{k+1}].$$

Keyingi tenglikni quyidagicha yozib olamiz:

$$\mu(l) = \sum_{k=0}^{n-1} \sqrt{\varphi'^2(\xi_k) + \psi'^2(\xi_k)} \cdot \Delta t_k +$$

$$+ \sum_{k=0}^{n-1} [\sqrt{\varphi'^2(\tau_k) + \psi'^2(\theta_k)} - \sqrt{\varphi'^2(\xi_k) + \psi'^2(\xi_k)}] \cdot \Delta t_k \qquad (*)$$

bunda,

$$\xi_k \in [t_k, t_{k+1}].$$

Modomiki,

$$\sqrt{\varphi'^2(t) + \psi'^2(t)} \in C[\alpha, \beta]$$

ekan unda

$$\sqrt{\varphi'^2(t)+\psi'^2(t)} \in R[\alpha,\beta]$$

bo‘lib,

$$\lim_{\lambda_p \to 0} \sum_{k=0}^{n-1} \sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)} \cdot \Delta t_k = \int_\alpha^\beta \sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (3)$$

bo‘ladi.

Ixtiyoriy $a$, $b$, $c$, $d$ haqiqiy sonlar uchun ushbu

$$\left|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right| \le |a-c|+|b-d|$$

tengsizlik o‘rinli bo‘ladi.
Bu tengsizlikdan foydalanib topamiz:

$$\left|\sum_{k=0}^{n-1}[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)} - \sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}]\Delta t_k\right| \le$$

$$\le \sum_{k=0}^{n-1}|\varphi'(\tau_k)-\varphi'(\xi_k)|\Delta t_k + \sum_{k=0}^{n-1}|\psi'(\theta_k)-\psi'(\xi_k)|\Delta t_k \le$$

$$\le \sum_{k=0}^{n-1}\omega_k(\varphi')\cdot\Delta t + \sum_{k=0}^{n-1}\omega_k(\psi')\cdot\Delta t.$$

$$\varphi'(t) \in R[\alpha,\beta], \qquad \psi'(t) \in R[\alpha,\beta]$$

bo‘lganligi sababli

$$\lim_{\lambda_p \to 0} \sum_{k=0}^{n-1}[\sqrt{\varphi'^2(\tau_k)+\psi'^2(\theta_k)} - \sqrt{\varphi'^2(\xi_k)+\psi'^2(\xi_k)}]\Delta t_k = 0 \qquad (4)$$

bo‘ladi.

(3) va (4) munosabatlarni e’tiborga olib, $\lambda_p \to 0$ da (*) tenglikda limitga o‘tsak, u holda $A\breve{B}$ yoyining uzunligi uchun

$$\mu(A\breve{B}) = \int_\alpha^\beta \sqrt{\phi'^2(t)+\psi'^2(t)}\, dt \qquad (5)$$

bo‘lishi kelib chiqadi. Bu formula yordamida yoy uzunligi hisoblanadi.

## Mashqlar

1. Ushbu

$$x = \int_1^t \frac{\cos z}{z}dz, \qquad y = \int_1^t \frac{\sin z}{z}dz$$

$\left(1 \le t \le \frac{\pi}{2}\right)$ tenglamalar bilan berilgan egri chiziqning uzunligi topilsin.

2. Ushbu

$$x^2+y^2=2, \qquad y=\sqrt{|x|}$$

chiziqlar bilan chegaralangan egri chiziqli uchburchakning perimetri topilsin.

### Adabiyotlar


1. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'rizalar, I q.* T. "Voris-nashriyot", 2010.
2. **Fixtengols G. M.** *Kurs differensialnogo i integralnogo ischisleniya, 1 t.* M. «FIZMATLIT», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.


## Glossariy

**1-ta'rif.** Agar $\lambda_p \to 0$ da $\breve{AB}$ yoyiga chizilgan siniq chiziq perimetri

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+\left[f(x_{k+1})-f(x_k)\right]^2}$$

chekli limitga ega bo'lsa, $\breve{AB}$ yoy uzunlikka ega deyiladi.

Ushbu

$$\lim_{\lambda_p\to 0}\mu(l)=\mu(\breve{AB})$$

limit $\breve{AB}$ yoyining uzunligi deyiladi.

Masalan, agar

$$f(x)=kx+C \qquad (a\le x\le b)$$

bo'lsa, unda $\breve{AB}$ ning uzunligi

$$\mu(\breve{AB})=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\sqrt{(x_{k+1}-x_k)^2+k^2(x_{r+1}-x_k)^2}=$$

$$=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}\sqrt{1+k^2}\cdot(x_{k+1}-x_k)=\sqrt{1+k^2}\cdot(b-a)$$

bo'ladi.

Aytaylik, $\breve{AB}$ egri chiziq ushbu

$$\begin{cases} x=\varphi(t) \\ y=\psi(t)' \end{cases} \qquad (\alpha \le t \le \beta)$$

tenglamalar sistemasi bilan berilgan bo‘lsin.

Bunda:

1) $\varphi(t)\in C[\alpha,\beta]$ , $\psi(t)\in C[\alpha,\beta]$;

2) $\forall t_1,t_2\in[\alpha,\beta]$ , $t_1\ne t_2$ uchun (1)

$$A_1(x_1,y_1)=A_1(\varphi(t_1),\psi(t_1)) \ ,$$
$$A_2(x_2\ ,y_2)=A_2(\varphi(t_2),\psi(t_2))$$

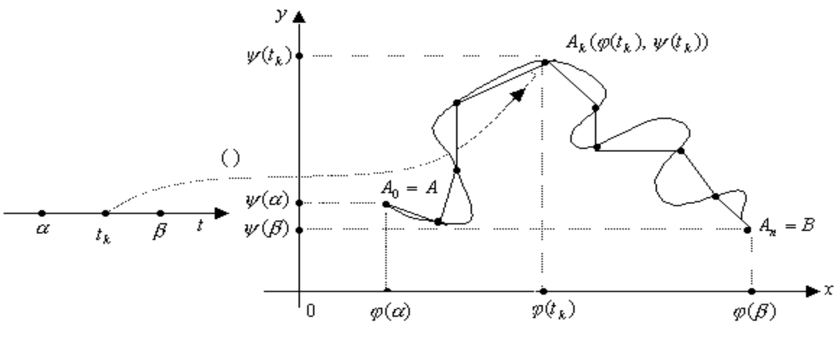
nuqtalar turlicha ;

3) $t=\alpha$ ga $A$ nuqta, $t=\beta$ ga $B$ nuqta mos kelsin.

$[\alpha,\beta]$ segmentning ixtiyoriy

$$P=\{t_0,t_1,...,t_n\} \qquad (\alpha=t_0<t_1<...<t_n=\beta$$

bo‘laklashni olib, bu bo‘laklashning bo‘luvchi $t_k$ $(k=0,1,2,...,n)$ nuqtalariga mos kelgan $\breve{AB}$ yoydagi $A_k=A_k(x_k,y_k)$
$(x_k=\varphi(t_k)$ , $y_k=\psi(t_k)$ ; $k=0,...,n)$ nuqtalarni bir-biri bilan to‘g‘ri chiziq kesmalari yordamida birlashtirib, $\breve{AB}$ yoyga chizilgan siniq chiziq $l$ ni hosil qilamiz.




Bu siniq chiziq perimetri

$$\mu(l)=\sum_{k=0}^{n-1}\sqrt{[\varphi(t_{k+1})-\varphi(t_k)]^2+[\psi(t_{k+1})-\psi(t_k)]^2}$$

bo‘ladi.

**2-ta'rif.** Agar $\lambda_p \to 0$ da $\breve{AB}$ yoyiga chizilgan siniq chiziq perimetri $\mu(l)$ chekli limitga ega bo'lsa, $\breve{AB}$ yoy uzunlikka ega deyiladi.

## Keys banki

**39-keys.** Masala o`rtaga tashlanadi: Ushbu

$$x^2 + y^2 = 2, \qquad y = \sqrt{|x|}$$

chiziqlar bilan chegaralangan egri chiziqli uchburchakning perimetri topilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 14-amaliy mashg'ulot

**6 – м и с о л .** *Ушбу*

$$f(x)=\frac{a}{2}\cdot\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)\quad (a>0)$$

***тенглама билан берилган эгри чизиқнинг (занжир чизиғининг)*** $[-a,a]$ ***оралиқдаги ёй узунлиги топилсин.***

◀Эгри чизиқнинг узунлигини (1) формуладан фойдаланиб топамиз. Равшанки,

$$f'(x)=\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right),\qquad 1+f'^2(x)=\frac{1}{4}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2$$

бўлиб,

$$\sqrt{1+f'^2(x)}=\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$$

бўлади. (1) формуладан фойдаланиб топамиз:

$$l=\int_{-a}^{a}\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)dx=\frac{a}{2}\left[e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right]\Bigg|_{-a}^{a}=a\left(e-\frac{1}{e}\right).$$ ▶

**7 – м и с о л .** *Ушбу*

$$\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}\qquad (0\le t\le 2\pi)$$

***тенгламалар системаси билан берилган эгри чизиқ ёйининг (циклоиданинг) узунлиги топилсин.***

◀Бу эгри чизиқнинг узунлигини топишда (2) формуладан фойдаланамиз.

Равшанки, $x'(t)=a(1-\cos t),\quad y'(t)=a\sin t$,

$$x'^2(t)+y'^2(t)=a^2(1-\cos t)^2+a^2\sin^2 t=2a^2(1-\cos t)$$

бўлиб,

$$\sqrt{x'^2(t)+y'^2(t)}=a\sqrt{2(1-\cos t)}$$

бўлади. Изланаётган эгри чизиқнинг узунлиги

$$l=\int_0^{2a} a\sqrt{2(1-\cos t)}dt$$

бўлади. Бу тенгликнинг ўнг томонидаги интегрални ҳисоблаймиз:

$$\int_0^{2\pi} a\sqrt{2(1-\cos t)}dt=a\int_0^{2\pi}\sqrt{4\cdot\sin^2\frac{t}{2}}=4a\int_0^{2\pi}\sin\frac{t}{2}d\left(\frac{t}{2}\right)=-4a\cdot\cos\frac{t}{2}\Bigg|_0^{2\pi}=8a.$$

Демак, $l=8a$.►

**Эгри чизиқ ёйининг узунлиги топилсин.**

**2161.** $y=\frac{4}{5}\cdot x^{\frac{5}{4}},\qquad 0\le x\le 9$

**2162.** $y=\frac{x}{6}\cdot\sqrt{x+12}\qquad (-11\le x\le -3)$

**2163.** $y=\frac{3}{2}\left(x^{\frac{1}{3}}-\frac{1}{5}x^{\frac{5}{3}}\right)\qquad 1\le x\le 8$

**2164.** $y=\ln x,\qquad 2\sqrt{2}\le x\le 2\sqrt{6}$

**2165.** $y=\frac{x^2}{2}-1,\qquad -\sqrt{2}\le x\le\sqrt{2}$

2166. $y=\ln(1-x^2), \quad -\frac{1}{2}\le x\le\frac{1}{2}$

2167. $y=1-\ln\cos x, \quad 0\le x\le\frac{\pi}{4}$

2168. $y=\frac{x}{4}\cdot\sqrt{2-x^2}, \quad 0\le x\le 1$

2169. $y=\sqrt{1-x^2}+\arcsin x, \quad 0\le x\le\frac{9}{16}$

2170. $y=a\cdot\ln\frac{a^2}{a^2-x^2}, \quad 0\le x\le b,\ \ b<a$

2171. $y=\frac{1}{4}\cdot y^2-\frac{1}{2}\ln y, \quad 1\le y\le e$

2172. $y^2=(x-1)^3, \quad 2\le x\le 5$

2173. $y=\arcsin e^{-x}, \quad 0\le x\le 1$

2174. $y=\ln\sin x, \quad \frac{\pi}{3}\le x\le\frac{\pi}{2}$

2175. $y=\sqrt{e^{2x}-1}-\arg\operatorname{tg}\sqrt{e^{2x}-1}, \quad 0\le x\le 1$

2176. $x=a\cdot\ln\frac{a+\sqrt{a^2-y^2}}{y}-\sqrt{a^2-y^2}, \quad 0<b\le y\le a$

2177. $x=t^2,\ \ y=t-\frac{1}{2}\cdot t^3, \quad 0\le t\le\sqrt{3}$

**2178.** $y = t^2, \quad y = \frac{t}{3}(t^2 - 3), \qquad 0 \le t \le \sqrt{3}$

**2179.** $y = a\cos^3 t, \quad y = a\sin^3 t, \qquad 0 \le t \le 2\pi$

**2180.** $x = e^t \sin t, \quad y = e^t \cos t, \qquad 0 \le t \le \frac{\pi}{2}$

**2181.** $x = t - \frac{1}{2}\mathrm{sh}2t, \quad y = 2\mathrm{ch}t, \qquad 0 \le t \le t_0$

**2182.** $x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t), \qquad 0 \le t \le 2\pi$

**2183.** $r = a\varphi, \qquad 0 \le \varphi \le 2\pi$

**2184.** $r = \frac{p}{1 + \cos\varphi}, \qquad -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

**2185.** $r = a(1 - \sin\varphi), \qquad -\frac{\pi}{2} \le \varphi \le -\frac{\pi}{6}$

**2186.** $r = a\varphi^2, \qquad 0 \le \varphi \le 4$

**2187.** $r = a \cdot \mathrm{th}\frac{\varphi}{2}, \qquad 0 \le t \le 2\pi$

# Test

| | | | | |
|---|---|---|---|---|
| $y=x$, $y=3x$ va $x=1$ to'g'ri chiziqlar bilan chegaralangan shaklning yuzasini hisoblang | 1 | 0 | 2 | -2 |
| $y=x$, $y=3x$ va $x=2$ to'g'ri chiziqlar bilan chegaralangan shaklning yuzasini hisoblang | 4 | 0 | -4 | 7 |
| $y=x$ to'g'ri chiziq va $y=x^2$ egri chiziq bilan chegaralangan shaklning yuzasini hisoblang | $\frac{1}{6}$ | 2 | 1 | 5 |
| $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips bilan chegaralangan shaklning yuzini hisoblang | $\pi ab$ | 2 | 3 | 4 |
| $x^2+y^2+z^2=1$ sfera bilan chegaralangan jismning hajmini hisoblang. | $\frac{4}{3}\pi$ | $\frac{4}{3}$ | -2 | 9 |
| $\int_0^1 x^2dx$ | *$\frac{1}{3}$ | -4 | 4 | $\frac{16}{3}$ |
| $\int_{-1}^0 6(2x+1)^2\,dx$ | *0 | 2 | $\frac{1}{2}$ | $\frac{1}{3}$ |
| $\int_{-1}^0 3e^{-x}dx$ | *$-3+3e$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int_6^7 \frac{dx}{x-5}$ | *$\ln 2$ | 0 | $\ln 3$ | $\ln 3-1$ |
| $\int_0^\pi 2\sin x dx$ | *4 | 1 | -1 | -2 |
| $\int_0^2 2xdx$ | * 4 | 2 | 5 | 1 |
| $\int_\pi^{2\pi} \sin 2xdx$ | * 0 | -1 | 1 | 2 |

| | | | | |
|---|---|---|---|---|
| $\int_{\pi/2}^{\pi} 7\cos x dx$ | * -7 | 0 | 1 | -2 |
| $\int_{4}^{7} \frac{dx}{x-1}$ | * $\ln 2$ | $\ln\frac{3}{2}$ | $\ln\frac{4}{3}$ | 1 |
| $\int_{-1}^{0} (2x+1)^2 dx$ | * $\frac{1}{3}$ | 1 | -3 | -1 |
| $\int_{-1}^{0} 2e^x dx$ | * $2-2e^{-1}$ | $e$ | $\frac{1}{e}-1$ | 5 |
| $\int\left(x^3+\frac{1}{1+x^2}\right)dx-?$ | * $\frac{x^4}{4}+arctgx+C$ | $2x+5+C$ | $\cos x+51x+C$ | $9x^2+C$ |
| $\int_{0}^{\pi/2} 6\cos x dx$ | *6 | -1 | 0 | $\frac{\pi}{2}$ |
| $\int_{0}^{\pi/4} \frac{5dx}{1+x^2}$ | *5 | -1 | 1 | 2 |
| $\int_{0}^{1} \frac{3dx}{\sqrt{1-x^2}}$ | * $\frac{3\pi}{2}$ | 0 | 1 | 1 |
| Quyidagi tengliklardan qaysi biri o‘rinli? | * $\int_{a}^{b} f(x)dx=-\int_{b}^{a} f(x)dx$ | $\int_{a}^{-b} f(x)dx=\int_{a}^{b} f(x)dx$ | $\int_{-a}^{a} f(x)dx=0$ | $\int_{-a}^{-b} f(x)dx=-\int_{a}^{b} f(x)dx$ |

# Mavzu. Aniq integralning tadbiqlari

## 15-16-ma’ruza

**Reja**

**1⁰.** Aylanma jism yuzi.

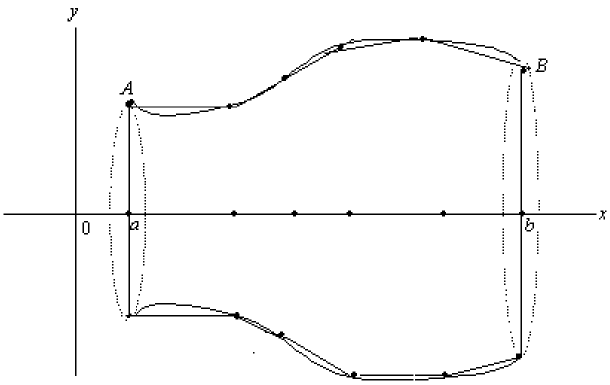
**2⁰.** Inersiya momenti.

**3⁰.** O‘zgaruvchi kuchning bajargan ishi.

### 1⁰. Aylanma sirtning yuzi va uni hisoblash

**1. Aylanma sirt va uning yuzi tushunchasi.** Ma’lumki, to‘g‘ri chiziq kesmasini biror o‘q atrofida aylantirishdan silindrik, konus (kesik konus) sirtlar hosil bo‘ladi. Bu sirtlar yuzaga ega va ular ma’lum formulalar yordamida topiladi.

Aytaylik, $f(x) \in C[a,b]$ bo‘lib, $\forall x \in [a,b]$ da $f(x) \geq 0$ bo‘lsin. Bu funksiya grafigi $\breve{AB}$ yoyini tasvirlasin (20-chizma )




20-чизма

$\breve{AB}$ yoyni $Ox$ o‘qi atrofida aylantirishdan hosil bo‘lgan sirt aylanma sirt deyiladi. Uni $\Pi$ deylik. $[a,b]$ segmentni ixtiyoriy

$$P = \{x_0, x_1, ..., x_n\} \qquad (a = x_0 < x_1 < ... < x_n = b)$$

bo‘laklashni olaylik. Bu bo‘laklashning har bir

$$x_k \, (k = 0,1,2,...,n)$$

bo‘luvchi nuqtalari orqali $Oy$ o‘qiga parallel to‘g‘ri chiziqlar o‘tkazib, ularning $\breve{AB}$ yoyi bilan kesishish nuqtalarini $A_k = A_k(x_k, f(x_k))$ bilan belgilaylik. $(A_0 = A,\ A_n = B; \quad k = 0,1,2,...,n)$ Bu nuqtalarni o‘zaro to‘g‘ri chiziq kesmalari bilan birlash-tirib, $\breve{AB}$ yoyiga $L$ siniq chiziq chizamiz.

$\breve{AB}$ yoyini $Ox$ o‘qi atrofida aylantirish bilan birga $L$ siniq chiziqni ham shu o‘q atrofida aylantiramiz. Natijada kesik konus sirtlarining birlashmasidan tashkil topgan $K$ sirt hosil bo‘ladi. Bu $K$ sirt yuzaga ega va uning yuzi

$$\mu(K)=2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}$$

**ga teng. (Bunda kesik konusning yon sirtining yuzini topish formulasidan foydalanildi).**

Ravshanki, $K$ sirt, binobarin uning yuzi $\mu(K)$ $[a,b]$ segmentning bo‘laklashlariga bog‘liq bo‘ladi.

**1-ta’rif**. Agar $\forall\varepsilon>0$ son olinganda ham shunday $\delta>0$ son topilsaki, $[a,b]$ segmentning diametri $\lambda_p<\delta$ bo‘lgan ixtiyoriy $P$ bo‘laklashi uchun

$$|\mu(K)-S|<\varepsilon \quad (S\in R)$$

tengsizlik bajarilsa, $S$ son $\mu(K)$ ning $\lambda_p\to 0$ dagi limiti deyiladi:

$$\lim_{\lambda_p\to 0}\mu(K)=S .$$

**2-ta’rif.** Agar $\lambda_p\to 0$ da $\mu(K)$ yig‘indi chekli $S$ limitga ega bo‘lsa, $\Pi$ aylanma sirt yuzaga ega deyiladi.

Bunda $S$ son $\Pi$ aylanma sirtning yuzi deyiladi:

$$S=\mu(\Pi).$$

Demak,

$$\mu(\Pi)=\lim_{\lambda_p\to 0}2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2} .$$

**2. Aylanma sirt yuzini hisoblash.** Faraz qilaylik, $f(x)\in C[a,b]$ bo‘lib, u $[a,b]$ segmentda uzluksiz $f'(x)$ hosilaga ega bo‘lsin.

Bu funksiya grafigi $\breve{AB}$ yoyini $Ox$ o‘qi atrofida aylantirishdan hosil bo‘lgan $\Pi$ aylanma sirtning yuzini topamiz.

◄ $[a,b]$ segmentning ixtiyoriy $P$ bo‘laklashini olib, yuqoridagidek

$$\mu(K)=2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}$$

yig‘indini tuzamiz.

Lagranj teoremasiga ko‘ra

$$f(x_{k+1})-f(x_k)=f'(\xi_k)(x_{k+1}-x_k)=f'(\xi_k)\cdot\Delta x_k$$

bo‘ladi, bunda $\xi_k\in[x_k,x_{k+1}]$. Natijada

$$\mu(K)=2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\sqrt{1+f'^2(\xi_k)}\Delta x_k$$

bo‘ladi.

Keyingi tenglikni quyidagicha yozib olamiz:

$$\mu(K)=2\pi\sum_{k=0}^{n-1}f(\xi_k)\sqrt{1+f'^2(\xi_k)}\Delta x_k+\pi\Big\{\sum_{k=0}^{n-1}[(f(x_k)-$$
$$-f(\xi_k))+(f(x_{k+1})-f(\xi_k))]\sqrt{1+f'^2(\xi_k)}\Delta x_k\Big\}. \quad (1)$$

$f'(x)\in C[a,b]$ bo‘lganligi sababli

$$f(x)\sqrt{1+f'^2(x)}\in R[a,b]$$

bo‘ladi. Demak, $\lambda_p\to 0$ da

$$2\pi\sum_{k=0}^{n-1}f(\xi_k)\sqrt{1+f'^2(\xi_k)}\Delta x_k\to 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx \quad (2)$$

Ravshanki,

$$\sqrt{1+f'^2(x)}\in C[a,b].$$

Demak, bu funksiya $[a,b]$ da o‘zining maksimum qiymatiga ega bo‘ladi. Uni $M$ deylik:

$$M=\max_{a\le x\le b}\sqrt{1+f'^2(x)}.$$

$f(x)$ funksiya $[a,b]$ segmentda tekis uzluksiz. Unda $\forall\varepsilon>0$ olinganda ham, $\frac{\varepsilon}{2M(b-a)}$ ga ko‘ra shunday $\delta>0$ son topiladiki, $\lambda_p<\delta$ bo‘lganda

$$|f(x_k)-f(\xi_k)|<\frac{\varepsilon}{2M(b-a)},\quad |f(x_{k+1})-f(\xi_k)|<\frac{\varepsilon}{2M(b-a)}$$

bo‘ladi. Shularni e’tiborga olib topamiz:

$$\left|\left\{\sum_{k=0}^{n-1}[(f(x_k)-f(\xi_k))+(f(x_{k+1})-f(\xi_k))]\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k\right\}\right|\le$$
$$\le\sum_{k=0}^{n-1}\big[|f(x_k)-f(\xi_k)|+|f(x_{k+1})-f(\xi_k)|\big]\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k<$$
$$<M\left[\frac{\varepsilon}{2M(b-a)}+\frac{\varepsilon}{2M(b-a)}\right]\cdot\sum_{k=0}^{n-1}\Delta x_k<\varepsilon\ .$$

Bundan $\lambda_p\to 0$ da

$$\sum_{k=0}^{n-1}[(f(x_k)-f(\xi_k))+(f(x_{k+1})-f(\xi_k))]\cdot\sqrt{1+f'^2(\xi_k)}\,\Delta x_k\to 0 \quad (3)$$

bo‘lishi kelib chiqadi.

$\lambda_p\to 0$ da (1) tenglikda limitga o‘tib, (bunda (2) va (3) munosabatlarni e’tiborga olib) aylanma sirtning yuzi uchun

$$\mu(\Pi)=2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx \quad (4)$$

bo‘lishini topamiz. ►

**1-misol.** Ushbu

$$f(x)=\frac{a}{2}(e^{\frac{x}{a}}+e^{\frac{-x}{a}}),\quad a>0,\ 0\le x\le a$$

zanjir chizig‘ini $Ox$ o‘q atrofida aylantirishdan hosil bo‘lgan aylanma sirtning yuzi topilsin.

◄ Ravshanki,

$$f'(x)=\frac{1}{2}(e^{\frac{x}{a}}-e^{\frac{-x}{a}})$$

(4) formuladan foydalanib, izlanayotgan aylanma sirtning yuzini topamiz:

$$\mu(\Pi)=2\pi\int_0^a\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})\sqrt{1+\frac{1}{4}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})^2}\,dx=$$

$$=\frac{\pi a}{2}\int_0^a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2dx=\frac{\pi a}{2}\int_0^a(e^{\frac{2x}{a}}+2+e^{-\frac{2x}{a}})dx=$$

$$=\frac{\pi a}{2}\left[\frac{a}{2}e^{\frac{2x}{a}}+2x-\frac{a}{2}e^{-\frac{2x}{a}}\right]\Bigg|_0^a=\frac{\pi a^2}{4}(e^2-e^{-2}+4)$$ ►

Aytaylik, $\breve{AB}$ egri chiziq yuqori yarim tekislikda joylashgan bo‘lib, u ushbu

$$\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}\quad(\alpha\le t\le\beta)$$

parametrik tenglamalar sistemasi bilan berilgan bo‘lsin. Bunda $\varphi(t),\ \psi(t)$ funksiyalari $[\alpha,\beta]$ da uzluksiz va uzluksiz $\varphi'(t),\psi'(t)$ hosilalarga ega. Bu egri chiziqni $Ox$ o‘qi atrofida aylantirishdan hosil bo‘lgan aylanma sirtning yuzi

$$\mu(\Pi)=2\pi\int_\alpha^\beta\psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt \qquad (5)$$

bo‘ladi.

**2-misol.** Ushbu

$$x^2+(y-2)^2=1$$

aylanani $Ox$ o‘qi atrofida aylantirishdan hosil bo‘lgan aylanma sirtning (torning) yuzi topilsin.

◄ Aylananing tenglamasini quyidagicha

$$\begin{aligned}x&=\varphi(t)=\cos t\\ y&=\psi(t)=2+\sin t\end{aligned}\qquad(0\le t\le 2\pi)$$

parametrik ko‘rinishda yozamiz.

Izlanayotgan aylanma sirtning yuzi, (5) formulaga ko‘ra

$$\mu(\Pi) = 2\pi \int_0^{2\pi} (2+\sin t)\sqrt{(\cos t)'^2 + (2+\sin t)'^2}\, dt =$$

$$= 2\pi \int_0^{2\pi} (2+\sin t)dt = 8\pi^2$$

bo'ladi. ►

**$2^0$. Inersiya momenti.**

Mexanikada moddiy nuqta harakati muhim tushunchalardan biri hisoblanadi.

Odatda, o'lchami etarli darajada kichik va massaga ega bo'lgan jism moddiy nuqta deb qaraladi.

Aytaylik, tekislikda $m$ massaga ega bo'lgan $A$ moddiy nuqta berilgan bo'lib, bu nuqtadan biror $l$ o'qqacha (yoki $O$ nuqtagacha) bo'lgan masofa $r$ ga teng bo'lsin.

Ushbu

$$J = mr^2$$

miqdor $A$ moddiy nuqtaning $l$ o'qqa ($O$ nuqtaga) nisbatan inersiya momenti deyiladi.

Masalan, $A = A(x,y)$ moddiy nuqtaning koordinata o'qlariga hamda koordinata boshiga nisbatan inersiya momentlari mos ravishda

$$J_x = my^2 \;,\; J_y = mx^2 \;,\; J_0 = m\sqrt{x^2+y^2}$$

bo'ladi.

Tekislikda, har biri mos ravishda

$$m_0, m_1, m_2, \ldots, m_{n-1}$$

massaga ega bo'lgan moddiy nuqtalar sistemasi

$\{A_0, A_1, A_2, \ldots, A_{n-1}\}$

ning biror $l$ o'qqa ($O$ nuqtaga) nisbatan inersiya momenti ushbu

$$J_n = \sum_{k=0}^{n-1} m_k r_k^{\,2}$$

yig'indi bilan ta'riflanadi, bunda $r_k - A_k$ nuqtadan $l$ o'qqacha ($O$ nuqtagacha) bo'lgan masofa $(k = 0,1,2,\ldots,n-1)$.

Faraz qilaylik, $y = f(x)$ egri chiziq yoyi $\breve{AB}$ bo'yicha zichligi $\rho = 1$ ga teng massa tarqatilgan bo'lib, bunda $f(x)$ funksiya $[a,b]$ segmentda uzluksiz hamda uzluksiz $f'(x)$ hosilaga ega bo'lsin.

Ravshanki, bu holda massa yoy uzunligiga teng bo'ladi:

$$m = \int_a^b \sqrt{1+f'^2(x)}\, dx.$$

$[a,b]$ segmentning ixtiyoriy

$$P=\{x_0,x_1,...,x_n\} \quad (a=x_0<x_1<...<x_n=b)$$

bo‘laklashini olamiz. Bu bo‘laklash $\breve{AB}$ yoyni

$$A_k=A_k(x_k, f(x_k)) \quad (k=0,1,2,...,n-1)$$

nuqtalar bilan $n$ ta $A_k\breve{A}_{k+1}$ $(A_0=A,\ A_{n-1}=B)$ bo‘lakka ajratadi. Bunda $A_k\breve{A}_{k+1}$ bo‘lakning massasi

$$m_k=\int_{x_k}^{x_{k+1}}\sqrt{1+f'^2(x)}\,dx \qquad (k=0,1,2,...,n-1)$$

bo‘ladi. O‘rta qiymat haqidagi teoremadan foydalanib topamiz:

$$m_k=\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

bunda,

$$\xi_k\in[x_k,x_{k+1}]\,,\Delta x_k=x_{k+1}-x_k.$$

Ma’lumki,

$$(\xi_k, f(\xi_k)) \quad (k=0,1,2,...,n-1)$$

moddiy nuqtaning koordinata o‘qlariga hamda kordinata boshiga nisbatan inersiya momentlari mos ravishda

$$J'_{x_k}=m_k\cdot f^2(\xi_k)=f^2(\xi_k)\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

$$J'_{y_k}=m_k\cdot\xi_k^2=\xi_k^2\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

$$J'_0=m_k(\xi_k^2+f^2(\xi_k))=(\xi_k^2+f^2(\xi_k))\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k$$

bo‘ladi. Unda ushbu

$$\{(\xi_0, f(\xi_0)),\ (\xi_1, f(\xi_1),...,(\xi_{n-1}, f(\xi_{n-1}))\}$$

moddiy nuqtalar sistemasining inersiya momentlari mos ravishda

$$J_x^{(n)}=\sum_{k=0}^{n-1}f^2(\xi_k)\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

$$J_y^{(n)}=\sum_{k=0}^{n-1}\xi_k^2\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

$$J_0^{(n)}=\sum_{k=0}^{n-1}(\xi_k^2+f^2(\xi_k))\cdot\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k,$$

tengliklar bilan ifodalanadi.

Agar $P$ bo‘laklashning diametri $\lambda_p$ nolga intila borsa, unda har bir $A_k\breve{A}_{k+1}$ yoyning uzunligi ham nolga intila borib, yuqoridagi

$$J_x^{(n)},\ J_y^{(n)},\quad J_0^{(n)},$$

yig‘indilarning limitini massaga ega bo‘lgan $\breve{AB}$ egri chiziqning mos ravishda koordinata boshi hamda koordinata o‘qlariga nisbatan inersiya momentlarini ifodalaydi deb qarash mumkin.

Ayni paytda,

$$\lim_{\lambda_p \to 0} J_x^{(n)} = \int_a^b f^2(x)\sqrt{1+f'^2(x)}\,dx\ ,$$

$$\lim_{\lambda_p \to 0} J_y^{(n)} = \int_a^b x^2\sqrt{1+f'^2(x)}\,dx\ ,$$

$$\lim_{\lambda_p \to 0} J_0^{(n)} = \int_a^b (x^2+f^2(x))\sqrt{1+f'^2(x)}\,dx$$

bo‘ladi.

Demak, massaga ega bo‘lgan $\breve{AB}$ egri chiziqning koordinata o‘qlariga hamda koordinata boshiga nisbatan inersiya momentlari aniq integrallar yordamida topiladi:

$$J_x = \int_a^b f^2(x)\sqrt{1+f'^2(x)}\,dx\ ,$$

$$J_y = \int_a^b x^2\sqrt{1+f'^2(x)}\,dx\ ,$$

$$J_0 = \int_a^b (x^2+f^2(x))\sqrt{1+f'^2(x)}\,dx.$$

**$3^0$. O‘zgaruvchi kuchning bajargan ishi.**

Biror jismni $Ox$ o‘qi bo‘ylab, shu o‘q yo‘nalishida bo‘lgan $F = F(x)$ kuch ta’siri ostida $a$ nuqtadan $b$ nuqtaga $(a < b)$ o‘tkazish uchun bajaril-gan ishni topish lozim bo‘lsin.

Ravshanki, jismga ta’sir etuvchi kuch o‘zgarmas, ya’ni

$$F(x) = C - const$$

bo‘lsa, unda jismni $a$ nuqtadan $b$ nuqtaga o‘tkazish uchun bajarilgan ish

$$A = C\cdot(b-a)$$

ga teng bo‘ladi.

Aytaylik, jismga ta’sir etuvchi kuch $x$ ga $(x\in[a,b])$ bog‘liq bo‘lib, u $[a,b]$ da uzluksiz bo‘lsin:

$$F = F(x)\in C[a,b].$$

$[a,b]$ segmentning ixtyoriy

$$P = \{x_0,\ x_1,\dots,\ x_n\}\quad (a = x_0 < x_1 < \dots < x_n = b)$$

bo‘laklashini olib, bu bo‘laklashning har bir

$$[x_k, x_{k+1}]\qquad\qquad (k = 0,1,2,\dots,n-1)$$

bo‘lakchasida ixtyoriy $\xi_k$ $\quad \xi_k \in [x_k, x_{k+1}];\ \ (k = 0,1,2\dots,n-1)$ nuqta olamiz.

Agar har bir $[x_k, x_{k+1}]$ da jismga ta’sir etuvchi kuchni o‘zgarmas va u $F(\xi_k)$ ga teng deyilsa, u holda $[x_k, x_{k+1}]$ oraliqda bajarilgan ish (kuch ta’sirida jismni $x_k$ nuqtadan $x_{k+1}$ nuqtaga o‘tkazish uchun bajarilgan ish) taxminan

$$F(\xi_k)\cdot(x_{k+1}-x_k)$$

formula bilan, $[a,b]$ oraliqda bajarilgan ish esa , taxminan

$$A\approx\sum_{k=0}^{n-1}F(\xi_k)\cdot(x_{k+1}-x_k)=\sum_{k=0}^{n-1}F(\xi_k)\cdot\Delta x_k \qquad (1)$$

formula bilan ifodalanadi.

$P$ bo‘laklashning diametri $\lambda_p$ nolga intila borganda yuqoridagi yig‘indining qiymati izlanayotgan ish miqdorini tobora aniqroq ifodalaydi. Bu hol $\lambda_p\to 0$ da (1) yig‘indi-ning chekli limitini bajarilgan ish deyilishi mumkinli-gini ko‘rsatadi.

Demak,

$$A=\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}F(\xi_k)\cdot\Delta x_k.$$

Modomiki, $F(x)\in C[a,b]$ ekan,

$$\lim_{\lambda_p\to 0}\sum_{k=0}^{n-1}F(\xi_k)\cdot\Delta x_k=\int_a^b F(x)dx$$

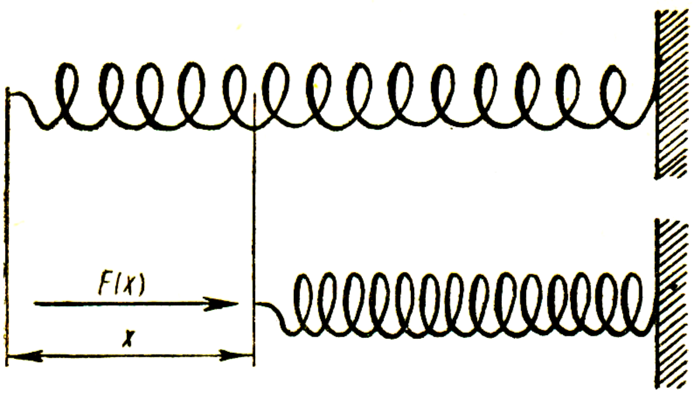
bo‘ladi.

SHunday qilib, o‘zgaruvchi $F(x)$ kuchning $[a,b]$ dagi bajargan ishi

$$A=\int_a^b F(x)dx \qquad (2)$$

formula bilan ifodalanadi.

**Misol.** Vintsimon prujinaning bir uchi mustahkamlan-gan, ikkinchi uchiga esa $F=F(x)$ kuch ta’sir etib, prujina qisilgan (21-chizma)




21-chizma

Agar prujinaning qisilishi unga ta’sir etayotgan $F(x)$ kuchga proporsional bo‘lsa, prujinani $a$ birlikka qisish uchun $F(x)$ kuchning bajargan ishi topilsin.

◄ Agar $F(x)$ kuch ta’sirida prujinaning qisilish miqdorini $x$ orqali belgilasak, u holda

$$F(x)=kx$$

bo‘ladi, bunda $k$-proporsionallik koeffitsienti (qisilish koeffitsienti). (2) formulaga ko‘ra bajarilgan ish

$$A=\int_0^a kxdx=\frac{ka^2}{2}$$

bo‘ladi. ►

## Mashqlar

1. Uchburchak asosiga nisbatan inersiya momentini topilsin.

2. Asosining radiusi $R$, balandligi $H$ bo‘lgan parabo-loid shaklidagi qozondan, undagi suvni chiqarishga sarflan-gan ish hisoblansin.

3. Aytaylik, $\breve{AB}$ egri chiziq $x=\varphi(t)$, $y=\psi(t)$ $(\alpha\le t\le\beta)$ tenglamalar bilan berilgan bo‘lib, $\varphi(t)$ va $\psi(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ va $\psi'(t)$ hosilalarga ega bo‘lsin. Bu egri chiziqni $OX$ o‘qi atrofida aylantirishdan hosil bo‘lgan aylana sirtining yuzi

$$P=2\pi\int_\alpha^\beta \psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (\psi(t)\ge 0)$$

bo‘lishi isbotlansin.

4. Ushbu

$$2ay=x^2-a^2 \qquad \left(0\le x\le 2\sqrt{2}a\right)$$

parabolani $OY$ o‘qi atrofida aylantirishdan hosil bo‘lgan aylana sirtining yuzi topilsin.

## Adabiyotlar


1. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’rizalar, I q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Kurs differensialnogo i integralnogo ischisleniya, 1 t.* M. «FIZMATLIT», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

# Glossariy

**1-ta'rif**. Agar $\forall\varepsilon>0$ son olinganda ham shunday $\delta>0$ son topilsaki, $[a,b]$ segmentning diametri $\lambda_p<\delta$ bo'lgan ixtiyoriy $P$ bo'laklashi uchun

$$|\mu(K)-S|<\varepsilon \quad (S\in R)$$

tengsizlik bajarilsa, $S$ son $\mu(K)$ ning $\lambda_p\to 0$ dagi limiti deyiladi:

$$\lim_{\lambda_p\to 0}\mu(K)=S .$$

**2-ta'rif.** Agar $\lambda_p\to 0$ da $\mu(K)$ yig'indi chekli $S$ limitga ega bo'lsa, $\Pi$ aylanma sirt yuzaga ega deyiladi.

Bunda $S$ son $\Pi$ aylanma sirtning yuzi deyiladi:

$$S=\mu(\Pi).$$

Demak,

$$\mu(\Pi)=\lim_{\lambda_p\to 0}2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2} .$$

**Aylanma sirt yuzini hisoblash.** Faraz qilaylik, $f(x)\in C[a,b]$ bo'lib, u $[a,b]$ segmentda uzluksiz $f'(x)$ hosilaga ega bo'lsin.

Bu funksiya grafigi $\breve{AB}$ yoyini $Ox$ o'qi atrofida aylantirishdan hosil bo'lgan $\Pi$ aylanma sirtning yuzini topamiz.

◄ $[a,b]$ segmentning ixtiyoriy $P$ bo'laklashini olib, yuqoridagidek

$$\mu(K)=2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\cdot\sqrt{(x_{k+1}-x_k)^2+[f(x_{k+1})-f(x_k)]^2}$$

yig'indini tuzamiz.

Lagranj teoremasiga ko'ra

$$f(x_{k+1})-f(x_k)=f'(\xi_k)(x_{k+1}-x_k)=f'(\xi_k)\cdot\Delta x_k$$

bo'ladi, bunda $\xi_k\in[x_k,x_{k+1}]$. Natijada

$$\mu(K) = 2\pi\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\sqrt{1+f'^2(\xi_k)}\Delta x_k$$

bo‘ladi.

Keyingi tenglikni quyidagicha yozib olamiz:

$$\mu(K) = 2\pi\sum_{k=0}^{n-1} f(\xi_k)\sqrt{1+f'^2(\xi_k)}\Delta x_k + \pi\left\{\sum_{k=0}^{n-1}[(f(x_k) - \right.$$
$$\left. - f(\xi_k)) + (f(x_{k+1}) - f(\xi_k))]\sqrt{1+f'^2(\xi_k)}\Delta x_k\right\}. \quad (1)$$

$f'(x) \in C[a,b]$ bo‘lganligi sababli

$$f(x)\sqrt{1+f'^2(x)} \in R[a,b]$$

bo‘ladi. Demak, $\lambda_p \to 0$ da

$$2\pi\sum_{k=0}^{n-1} f(\xi_k)\sqrt{1+f'^2(\xi_k)}\Delta x_k \to 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx \quad (2)$$

Ravshanki,

$$\sqrt{1+f'^2(x)} \in C[a,b].$$

Demak, bu funksiya $[a,b]$ da o‘zining maksimum qiymatiga ega bo‘ladi. Uni $M$ deylik:

$$M = \max_{a\le x\le b}\sqrt{1+f'^2(x)}.$$

$f(x)$ funksiya $[a,b]$ segmentda tekis uzluksiz. Unda $\forall\varepsilon > 0$ olinganda ham, $\frac{\varepsilon}{2M(b-a)}$ ga ko‘ra shunday $\delta > 0$ son topiladiki, $\lambda_p < \delta$ bo‘lganda

$$|f(x_k) - f(\xi_k)| < \frac{\varepsilon}{2M(b-a)}, \quad |f(x_{k+1}) - f(\xi_k)| < \frac{\varepsilon}{2M(b-a)}$$

bo‘ladi. Shularni e’tiborga olib topamiz:

$$\left|\left\{\sum_{k=0}^{n-1}[(f(x_k)-f(\xi_k))+(f(x_{k+1})-f(\xi_k))]\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k\right\}\right|\le$$

$$\le\sum_{k=0}^{n-1}\left[\left|f(x_k)-f(\xi_k)\right|+\left|f(x_{k+1})-f(\xi_k)\right|\right]\sqrt{1+f'^2(\xi_k)}\cdot\Delta x_k<$$

$$<M\left[\frac{\varepsilon}{2M(b-a)}+\frac{\varepsilon}{2M(b-a)}\right]\cdot\sum_{k=0}^{n-1}\Delta x_k<\varepsilon\ .$$

Bundan $\lambda_p\to 0$ da

$$\sum_{k=0}^{n-1}[(f(x_k)-f(\xi_k))+(f(x_{k+1})-f(\xi_k))]\cdot\sqrt{1+f'^2(\xi_k)}\,\Delta x_k\to 0 \qquad (3)$$

bo‘lishi kelib chiqadi.

$\lambda_p\to 0$ da (1) tenglikda limitga o‘tib, (bunda (2) va (3) munosabatlarni e’tiborga olib) aylanma sirtning yuzi uchun

$$\mu(\Pi)=2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx \qquad (4)$$

bo‘lishini topamiz. ►

## Keys banki

**40-keys.** Masala o`rtaga tashlanadi: Aytaylik, $A\breve{B}$ egri chiziq $x=\varphi(t),\ \ y=\psi(t)\ \ (\alpha\le t\le\beta)$ tenglamalar bilan berilgan bo‘lib, $\varphi(t)$ va $\psi(t)$ funksiyalar $[\alpha,\beta]$ da uzluksiz $\varphi'(t)$ va $\psi'(t)$ hosilalarga ega bo‘lsin. Bu egri chiziqni $OX$ o‘qi atrofida aylantirishdan hosil bo‘lgan aylana sirtining yuzi

$$P=2\pi\int_\alpha^\beta\psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}dt\ \ (\psi(t)\ge 0)$$

bo‘lishi isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 15-16-amaliy mashg'ulot

**10 – м и с о л .** *Ушбу*

$$f(x)=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)\qquad (0\le x\le a,\ \ a>0)$$

***занжир чизиғини* OX *ўқи атрофида айланишидан ҳосил бўлган сиркнинг юзи топилсин.***

◄Равшанки,

$$f'(x)=\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right).$$

Юқоридаги (1) формуладан фойдаланиб изланаётган сиртнинг юзасини топамиз:

$$S=2\pi\int_0^a\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)\sqrt{1+\frac{1}{4}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)^2}dx=\frac{\pi a}{2}\int_0^a\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2dx=$$

$$=\frac{\pi a}{2}\int_0^a\left(e^{\frac{2x}{a}}+2+e^{-\frac{2x}{a}}\right)dx=\frac{\pi a}{2}\left(\frac{a}{2}\cdot e^{\frac{2x}{a}}+2x-\frac{a}{2}e^{-\frac{2x}{a}}\right)\Bigg|_0^a=\frac{\pi a^2}{4}\left(e^2-e^{-2}+4\right)$$

►

**11 – м и с о л .** *Ушбу*

$$x^2+(y-2)^2=1$$

***айланани* OX *ўқи атрофида айланишдан ҳосил бўлган сиртнинг (торнинг) юзи топилсин.***

◄Берилган айлананинг тенгламасини қуйидагича

$$\begin{cases} x = \cos t, \\ y = 2 + \sin t \end{cases} \qquad (0 \le t \le 2\pi)$$

параметрик кўринишда ёзиб оламиз:

$$S = 2\pi\int_0^{2\pi}(2+\sin t)\sqrt{(\cos t)'^2 + (2+\sin t)'^2}\,dt =$$

$$= 2\pi\int_0^{2\pi}(2+\sin t)dt = 2\pi(2t - \cos t)\Big|_0^{2\pi} = 8\pi^2. \blacktriangleright$$

**Қуйидаги эгри чизиқларни айлантиришдан ҳосил бўлган айланма сиртларнинг юзлари топилсин.**

**2193.** $y = x^3;\ \ x = -\frac{2}{3},\ \ x = \frac{2}{3};$ **OX** ўқи атрофида

**2194.** $9y^2 = x(3-x)^2,\ \ 0 \le x \le 3;$ **OX** ўқи атрофида

**2195.** $y^2 = 2x,\ \ 0 \le x \le \frac{3}{2};$ **OX** ўқи атрофида

**2196.** $y = tgx,\ \ 0 \le x \le \frac{\pi}{4};$ **OX** ўқи атрофида

**2197.** $x = a\cos t,\ \ y = a\sin t;$ **OX** ўқи атрофида

**2198.** $x = e^t\sin t,\ \ y = e^t\cos t,\ \ 0 \le t \le \frac{\pi}{2};$ **OX** ўқи атрофида

**2199.** $y = x\cdot\sqrt{\frac{x}{a}},\ \ 0 \le x \le a;$ **OX** ўқи атрофида

**2200.** $3x^2 + 4y^2 = 12;$ **OX** ўқи атрофида

**2201.** $x = t(3 - t^2),\ \ y = t^2,\ \ -\sqrt{3} \le x \le \sqrt{3};$ **OX** ўқи атрофида

**2202.** $y = e^{-x},\ \ 0 \le x \le a;$ **OX** ўқи атрофида

**2203.** $y = \sin x,\ \ 0 \le x \le \pi;$ **OX** ўқи атрофида

**2204.** $y^2 = 4 + x,\ \ -4 \le x \le 2;$ **OX** ўқи атрофида

**2205.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}};$ **OX** ўқи атрофида

**2206.** $x = a(t - \sin t),\ \ y = a(1 - \cos t);$ **OX** ўқи атрофида

**2207.** $r = a(1 + \cos\varphi);$ кутб ўқи атрофида

**2208.** $r^2 = 2a^2\cos 2\varphi$ кутб ўқи атрофида.

# Test

1. Qachon aylanma sirt yuzaga ega bo'ladi?

A) Agar $\lambda_p \to 0$ da $\mu(K)$ yig'indi chekli $S$ limitga ega bo'lsa

B) Agar $\lambda_p \to 0$ da $\mu(K)$ yig'indi chekli $S$ limitga ega bo'lmasa

C) Agar $\lambda_p \to 1$ da $\mu(K)$ yig'indi chekli $S$ limitga ega bo'lsa

D) Agar $\lambda_p \to 1$ da $\mu(K)$ yig'indini limiti cheksiz bo'lsa

2. Berilgan $y = 3x$ funksiyani $[0,4]$ segmentdagi yoy uzunligi topilsin?

A) $4\sqrt{10}$

B) 10

C) 0

D) 7

3. Qaysi funksiyani $[1,9]$ kesmadagi yoy uzunligi eng katta bo'ladi?

A) $y = 8x$

B) $y = 4$

C) $y = 3x + 1$

D) $y = x$

4. $y = x$ funksiyani $[0,1]$ segmentda $Ox$ o'qi atrofida aylantirishdan hosil bo'lgan $\Pi$ aylanma sirtning yuzini toping?

A) $\pi\sqrt{2}$

B) $\sqrt{2}$

C) $\pi$

D) 2

5. Aylanma sirt yuzini topish formulasini toping?

A) $2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$

B) $2\pi\int_a^b \sqrt{1+f'^2(x)}\,dx$

C) $\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$

D) $2\pi\int_a^b \sqrt{1-f'^2(x)}\,dx$

6. Yoy uzunligi topish formulasi topilsin?

A) $\int_a^b \sqrt{1+f'^2(x)}\,dx$

B) $2\pi\int_a^b \sqrt{1-f'^2(x)}\,dx$

C) $2\pi\int_a^b \sqrt{1+f'^2(x)}\,dx$

D) $2\int_a^b \sqrt{1+f'^2(x)}\,dx$

7. O‘zgaruvchi $F(x)$ kuchning $[a,b]$ dagi bajargan ishi Qaysi formula yordamida topiladi?

A) $A=\int_a^b F(x)dx$

B) $A=\int_a^b F'(x)dx$

C) $A=2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$

D) $A=\int_a^b \sqrt{1+f'^2(x)}\,dx$

8. Agar $F(x)=C$ bo’lsa $[1,6]$ dagi bajarilgan ish nimaga teng?

A) $5C$

B) $6C$

C) $C$

D) $0$

9. Ushbu

$$f(x)=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{\frac{-x}{a}}\right),\quad a>0,\ 0\le x\le a$$

zanjir chizig‘ini $Ox$ o‘q atrofida aylantirishdan hosil bo‘lgan aylanma sirtning yuzi topilsin?

A) $\frac{\pi a^2}{4}(e^2-e^{-2}+4)$

B) $\frac{\pi a^2}{4}(e^2-e^{-2})$

C) $\frac{\pi a^2}{4}$

D) $\pi a^2$

10. Agar egri chiziq parametrik ko'rinishda berilgan bo'lsa , egri chiziqni o'qi atrofida aylantirishda hosil bo'lgan aylanma sirtning yuzini toppish formulasi?

A) $2\pi\int_{\alpha}^{\beta}\psi(t)\sqrt{\phi'^2(t)+\psi'^2(t)}\,dt$

B) $2\pi\int_{\alpha}^{\beta}\sqrt{\phi'^2(t)+\psi'^2(t)}\,dt$

C) $2\pi\int_{\alpha}^{\beta}\psi(t)\,dt$

D) $\int_{\alpha}^{\beta}\psi(t)\sqrt{\phi'^2(t)+\psi'^2(t)}\,dt$

# Mavzu. Chegaralari cheksiz xosmas integrallar

## 17-ma'ruza

**Reja**

**1$^0$.** Chegaralari cheksiz xosmas integral tushunchasi.

**2$^0$.** Yaqinlashuvchi xosmas integralning sodda xossalari.

**3$^0$.** Xosmas integralning yaqinlashuvchiligi.

### 1$^0$.Chegaralari cheksiz xosmas integral tushunchasi.

$f(x)$ funksiya $[a,+\infty)$ oraliqda $(a \in R)$ berilgan bo'lib, ixtiyoriy $[a,t]$ da $(a \le t < +\infty)$ integrallanuvchi bo'lsin: $f(x) \in R([a,t])$.

Ushbu

$$F(t) = \int_a^t f(x)dx$$

belgilashni kiritamiz.

**1-ta'rif.** Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti mavjud bo'lsa, bu limiti $f(x)$ funksiyaning $[a,+\infty)$ cheksiz oraliq bo'yicha xosmas integrali deyiladi va

$$\int_a^{+\infty} f(x)dx$$

kabi belgilanadi:

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} F(t) = \lim_{t \to +\infty} \int_a^t f(x)dx. \qquad (1)$$

(1) integralni chegarasi cheksiz xosmas integral ham deb yuritiladi.

Qulaylik uchun, bundan keyin "chegarasi cheksiz xosmas integral" deyish o'rniga "integral" deymiz.

**2-ta'rif.** Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti mavjud va chekli bo'lsa, (1) integral yaqinlashuvchi deyiladi.

Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti cheksiz yoki mavjud bo'lmasa,

(1) integral uzoqlashuvchi deyiladi.

**Misol.** Ushbu

$$\int_0^{+\infty} e^{-x}dx$$

integralni qaraylik. Bu holda

$$F(t)=\int_0^{t} e^{-x}dx=-e^{-t}+1$$

bo‘lib,

$$\lim_{t\to+\infty} F(t)=1$$

bo‘ladi.

Demak, berilgan integral yaqinlashuvchi va

$$\int_0^{+\infty} e^{-x}dx=1.$$

**2-misol.** Ushbu

$$\int_a^{+\infty}\frac{dx}{x^{\alpha}} \qquad (a>0,\quad \alpha>0)$$

integral uchun

$$F(t)=\int_a^{t}\frac{dx}{x^{\alpha}}=\begin{cases}\ln t-\ln a, & \text{агар} \quad \alpha=1 \text{ бўлса}\\ \dfrac{t^{-\alpha+1}}{-\alpha+1}-\dfrac{a^{-\alpha+1}}{-\alpha+1}, & \text{агар } \alpha\neq 1 \text{ бўлса}\end{cases},$$

bo‘lib, $t\to+\infty$ da

$$F(t)\to\frac{a^{1-\alpha}}{\alpha-1} \qquad (\alpha>1),$$

$$F(t)\to+\infty \qquad (\alpha\le 1)$$

bo‘ladi.

Demak,

$$\int_a^{+\infty}\frac{dx}{x^{\alpha}}$$

integral $\alpha>1$ bo‘lganda yaqinlashuvchi, $\alpha\le 1$ bo‘lganda uzoqla-shuvchi bo‘ladi.

**3-misol.** Ushbu

$$\int_0^{+\infty}\cos x dx$$

integral uzoqlashuvchi bo‘ladi, chunki $t\to+\infty$ da

$$F(t)=\int_0^{t}\cos x dx=\sin t$$

funksiyaning limiti mavjud emas.

YUqoridagidek,

$$\int_{-\infty}^{a} f(x)dx\,, \quad \int_{-\infty}^{+\infty} f(x)dx$$

xosmas integrallar va ularning yaqinlashuvchiligi, uzoqlashuvchiligi ta'riflanadi:

$$\int_{-\infty}^{a} f(x)dx = \lim_{t\to-\infty}\int_{t}^{a} f(x)dx \;,$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{u\to+\infty \\ v\to-\infty}}\int_{v}^{u} f(x)dx \;.$$

**$2^0$. Yaqinlashuvchi xosmas integralning sodda xossalari.**

Xosmas integralning turli xossalarini $f(x)$ funksiyaning $[a,+\infty)$ oraliq bo'yicha olingan

$$\int_{a}^{+\infty} f(x)dx$$

integrali uchun bayon etamiz. Bu xossalarni

$$\int_{-\infty}^{a} f(x)dx\,, \quad \int_{-\infty}^{+\infty} f(x)dx$$

integrallar uchun keltirishni o'quvchiga havola etamiz.

**1-xossa.** Agar $\int_{a}^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda

$$\int_{b}^{+\infty} f(x)dx \quad (a < b)$$

integral ham yaqinlashuvchi bo'ladi va aksincha . Bunda

$$\int_{a}^{+\infty} f(x)dx = \int_{a}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx \qquad (2)$$

tenglik bajariladi.

**2-xossa.** Agar $\int_{a}^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_{a}^{+\infty} C\cdot f(x)dx$ ham $(C = const)$ yaqinlashuvchi bo'lib,

$$\int_{a}^{+\infty} C\cdot f(x)dx = C\int_{a}^{+\infty} f(x)dx$$

bo'ladi.

**3-xossa.** Agar $\int_{a}^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lib, $\forall x\in[a,+\infty)$ da

$f(x)\geq 0$ bo‘lsa, u holda

$$\int_a^{+\infty} f(x)dx \geq 0$$

bo‘ladi.

**4-xossa.** Agar $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinla-shuvchi bo‘lsa, u holda $\int_a^{+\infty}(f(x)\pm g(x))dx$ integral ham yaqinla-shuvchi bo‘lib,

$$\int_a^{+\infty}(f(x)\pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$$

bo‘ladi.

**5-xossa.** Agar $\forall x\in[a,+\infty)$ da $f(x)\leq g(x)$ bo‘lib, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda

$$\int_a^{+\infty} f(x)dx \leq \int_a^{+\infty} g(x)dx$$

bo‘ladi.

2)- 5)- xossalarning isboti xosmas integral va uning yaqinlashuvchiligi ta’riflaridan bevosita kelib chiqadi.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ da berilgan bo‘lib, $f(x)$ funksiya chegaralangan ($m\leq f(x)\leq M$, $x\in[a,+\infty)$), $g(x)$ funksiya esa o‘z ishorasini o‘zgartirmasin ($\forall x\in[a,+\infty)$ da har doim $g(x)\geq 0$ yoki $g(x)\leq 0$).

**6-xossa.** Agar $\int_a^{+\infty} f(x)\cdot g(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqin-lashuvchi bo‘lsa, u holda shunday o‘zgarmas $\mu(m\leq\mu\leq M)$ topiladiki,

$$\int_a^{+\infty} f(x)\cdot g(x)dx = \mu\int_a^{+\infty} g(x)dx \qquad (3)$$

bo‘ladi.

Odatda, bu xossa o‘rta qiymat haqidagi teorema deyiladi.

**$3^0$. Xosmas integralning yaqinlashuvchiligi.**

Aytaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo‘lsin.

Ma’lumki,

$$\int_a^{+\infty} f(x)dx$$

xosmas integralning yaqinlashuvchiligi ushbu

$$F(t) = \int_a^t f(x)dx \qquad (t > a)$$

funksiyaning $t \to +\infty$ da chekli limitga ega bo‘lishidan iborat.

13-ma’ruzada funksiyaning chekli limitiga ega bo‘lishi haqidagi Koshi teoremasi, ya’ni $F(t)$ funksiyaning $t \to +\infty$ da chekli limitga ega bo‘lishi uchun

$$\forall \varepsilon > 0, \quad \exists t_0 > a, \quad \forall t' > t_0, \qquad \forall t'' > t_0 :$$

$$|F(t'') - F(t')| < \varepsilon$$

tengsizlikning bajarilishi zarur va etarli ekani keltirilgan edi.

Bu tushuncha va tasdiqdan

$$\int_a^{+\infty} f(x)dx \qquad (4)$$

xosmas integralning yaqinlashuvchiligini ifodalaydigan quyidagi teoremaga kelamiz.

**Teorema (Koshi teoremasi).** (4) integralning yaqinla-shuvchi bo‘lishi uchun $\forall \varepsilon > 0$ son olinganda ham shunday $t_0 \in R$ $(t_0 > a)$ topilib, ixtiyoriy $t' > t_0$, $t'' > t_0$ bo‘lganda

$$\left| \int_{t'}^{t''} f(x)dx \right| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

## Mashqlar

1. Ushbu

$$\int_0^{+\infty} x \sin x dx$$

xosmas integral yaqinlashuvchi bo‘ladimi?

## Adabiyotlar


1. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’rizalar, I q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Kurs differensialnogo i integralnogo ischisleniya, 1 t.* M. «FIZMATLIT», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

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**1-ta'rif.** Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud bo'lsa, bu limiti $f(x)$ funksiyaning $[a,+\infty)$ cheksiz oraliq bo'yicha xosmas integrali deyiladi va

$$\int_a^{+\infty} f(x)dx$$

kabi belgilanadi:

$$\int_a^{+\infty} f(x)dx=\lim_{t\to+\infty}F(t)=\lim_{t\to+\infty}\int_a^t f(x)dx. \qquad (1)$$

(1) integralni chegarasi cheksiz xosmas integral ham deb yuritiladi.

Qulaylik uchun, bundan keyin "chegarasi cheksiz xosmas integral" deyish o'rniga "integral" deymiz.

**2-ta'rif.** Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud va chekli bo'lsa, (1) integral yaqinlashuvchi deyiladi.

Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti cheksiz yoki mavjud bo'lmasa, (1) integral uzoqlashuvchi deyiladi.

**1-xossa.** Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda

$$\int_b^{+\infty} f(x)dx \quad (a<b)$$

integral ham yaqinlashuvchi bo'ladi va aksincha . Bunda

$$\int_a^{+\infty} f(x)dx=\int_a^b f(x)dx+\int_b^{+\infty} f(x)dx \qquad (2)$$

tenglik bajariladi.

**2-xossa.** Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda

$\int\limits_a^{+\infty} C\cdot f(x)dx$ ham $(C = const)$ yaqinlashuvchi bo‘lib,

$$\int\limits_a^{+\infty} C\cdot f(x)dx = C\int\limits_a^{+\infty} f(x)dx$$

bo‘ladi.

**3-xossa.** Agar $\int\limits_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo‘lib, $\forall x\in[a,+\infty)$ da $f(x)\geq 0$ bo‘lsa, u holda

$$\int\limits_a^{+\infty} f(x)dx \geq 0$$

bo‘ladi.

**4-xossa.** Agar $\int\limits_a^{+\infty} f(x)dx$ va $\int\limits_a^{+\infty} g(x)dx$ integrallar yaqinla-shuvchi bo‘lsa, u holda $\int\limits_a^{+\infty}(f(x)\pm g(x))dx$ integral ham yaqinla-shuvchi bo‘lib,

$$\int\limits_a^{+\infty}(f(x)\pm g(x))dx = \int\limits_a^{+\infty} f(x)dx \pm \int\limits_a^{+\infty} g(x)dx$$

bo‘ladi.

**5-xossa.** Agar $\forall x\in[a,+\infty)$ da $f(x)\leq g(x)$ bo‘lib, $\int\limits_a^{+\infty} f(x)dx$ va $\int\limits_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda

$$\int\limits_a^{+\infty} f(x)dx \leq \int\limits_a^{+\infty} g(x)dx$$

bo‘ladi.

2)- 5)- xossalarning isboti xosmas integral va uning yaqinlashuvchiligi ta’riflaridan bevosita kelib chiqadi.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ da berilgan bo‘lib, $f(x)$ funksiya chegaralangan ($m\leq f(x)\leq M$, $x\in[a,+\infty)$), $g(x)$ funksiya esa o‘z ishorasini o‘zgartirmasin ($\forall x \in [a,+\infty)$ da har doim $g(x) \geq 0$ yoki $g(x) \leq 0$).

**6-xossa.** Agar $\int_a^{+\infty} f(x) \cdot g(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqin-lashuvchi bo‘lsa, u holda shunday o‘zgarmas $\mu (m \leq \mu \leq M)$ topiladiki,

$$\int_a^{+\infty} f(x) \cdot g(x)dx = \mu \int_a^{+\infty} g(x)dx \qquad (3)$$

bo‘ladi.

Odatda, bu xossa o‘rta qiymat haqidagi teorema deyiladi.

## Keys banki

**41-keys.** Masala o`rtaga tashlanadi: Ushbu

$$\int_0^{+\infty} \frac{x \ln x}{\left(1+x^2\right)^3} dx = -\frac{1}{8}$$

tenglik isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 17-amaliy mashg'ulot

**1-misol.** $\int_a^{+\infty}\frac{dx}{x^\lambda}$ ($a>0$ va $\lambda - \forall$ haqiqiy son) xosmas integralni yaqinlashishga tekshiring.

$\triangleleft$ $\forall\ A>a$ olamiz

$$F(A)=\int_a^A\frac{dx}{x^\lambda}=\begin{cases}\left.\frac{x^{1-\lambda}}{1-\lambda}\right|_a^A=\frac{A^{1-\lambda}-a^{1-\lambda}}{1-\lambda}, & \lambda\neq 1,\\ \ln x|_a^A=\ln A-\ln a, & \lambda=1.\end{cases}$$

$\lambda>1$ bo\`lsa $\lim\limits_{A\to+\infty}F(A)=\frac{a^{1-\lambda}}{\lambda-1}$ bo\`lib, integral yaqinlashadi. $\lambda\le 1$ bo\`lsa $\lim\limits_{A\to+\infty}F(A)=\infty$ bo\`lib, integral uzoqlashadi.

Shunday qilib,

$$\int_a^{+\infty}\frac{dx}{x^\lambda}=\begin{cases}\text{yaqinlashadi, agar } \lambda>1 \text{ bo'lsa,}\\ \text{uzoqlashadi, agar } \lambda\le 1 \text{ bo'lsa.}\end{cases}\ \triangleright$$

**Quyidagi xosmas integrallar hisoblansin.**

**1.1** $\int_{-\infty}^{-2}\frac{dx}{x\sqrt{x^2-1}}$.

**1.2** $\int_0^{+\infty}\frac{dx}{e^x+\sqrt{e^x}}$.

**1.3** $\int_2^{+\infty}\frac{xdx}{x^3-1}$.

**1.4** $\int_1^{+\infty}\frac{x^2+1}{x^4+1}dx$.

**1.5** $\int_0^{+\infty}\frac{dx}{(x^2+9)\sqrt{x^2+9}}$.

**1.6** $\int_0^{+\infty}e^{-\sqrt{x}}dx$.

**1.7** $\int_1^{+\infty}\frac{x\cdot e^{arctgx}}{(1+x^2)\cdot\sqrt{1+x^2}}dx$.

**1.8** $\int_2^{+\infty}\frac{dx}{x\sqrt{x^2+x-1}}$.

**1.9** $\int_0^{+\infty} \frac{dx}{\left(\sqrt{x^2+1}+x\right)^2}$.

**1.10** $\int_0^{+\infty} \frac{dx}{\left(4x^2+1\right)\sqrt{x^2+1}}$.

**1.11** $\int_0^{+\infty} \frac{x^2+12}{\left(x^2+1\right)^2}dx$.

**1.12** $\int_0^{+\infty} e^{-ax}\cdot\sin^2 bxdx$.

**1.13** $\int_0^{+\infty} x^n e^{-x}dx, n\in N$.

**1.14** $\int_{\sqrt{2}}^{+\infty} \frac{dx}{(x-1)\sqrt{x^2-2}}$.

**1.15** $\int_1^{+\infty} \frac{dx}{(2x-1)\sqrt{x^2-1}}$.

**1.16** $\int_0^{+\infty} \frac{\ln x}{1+x^2}dx$.

**1.17** $\int_1^{+\infty} \frac{dx}{\left(4x^2-1\right)\sqrt{x^2-1}}$.

**1.18** $\int_0^{+\infty} \frac{x\ln x}{\left(1+x^2\right)^2}dx$.

**1.19** $\int_0^{+\infty} \frac{arctg(1-x)}{\sqrt[3]{(x-1)^4}}dx$.

**1.20** $\int_1^{+\infty} \frac{2-x}{x^3\cdot\sqrt{x^2-1}}dx$.

**1.21** $\int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+x+1\right)^3}$.

**Quyidagi II-tur xosmas integrallar hisoblansin.**

**2.1** $\int_0^{\pi/2} (\ln\cos x)\cdot\cos 2nxdx, n\in N$.

**2.2** $\int_0^{\pi} x\cdot(\ln\sin x)dx$.

**2.3** $\int_0^{\pi} \ln\cos xdx$..

**2.4** $\int_{-1}^{1} x^3\cdot\ln\frac{1+x}{1-x}\cdot\frac{dx}{\sqrt{1-x^2}}$.

**2.5** $\int_0^{1} \frac{x^3\arcsin x}{\sqrt{1-x^2}}dx$.

**2.6** $\int_0^{2} \left(x\sin\frac{\pi}{x^2}-\frac{\pi}{x}\cos\frac{\pi}{x^2}\right)dx$.

**2.7** $\int\limits_{-1}^{1}\frac{dx}{\sqrt{(1-x^2)\arccos x}}.$

**2.8** $\int\limits_{0}^{\pi/2}\sqrt{tg\,x}dx.$

**2.9** $\int\limits_{0}^{\pi/4}\sqrt{ctg\,x}dx.$

**2.10** $\int\limits_{a}^{b}x\cdot\sqrt{\frac{x-a}{b-x}}dx, b>a.$

**2.11** $\int\limits_{-a}^{a}\frac{dx}{\sqrt{a^2+b^2-2bx}}, a>0, b\geq 0.$

**2.12** $\int\limits_{-a}^{a}\frac{x^4dx}{(1+x^2)\cdot\sqrt{1-x^2}}$

**2.13** $\int\limits_{-1}^{1}\frac{dx}{(16-x^2)\cdot\sqrt{1-x^2}}.$

**2.14** $\int\limits_{-1}^{1}\frac{dx}{(4-x)\cdot\sqrt{1-x^2}}.$

**2.15** $\int\limits_{1}^{2}\frac{dx}{x\cdot\sqrt{3x^2-2x-1}}.$

**2.16** $\int\limits_{0}^{\pi/2}\ln\sin xdx.$

**2.17** $\int\limits_{\sqrt{2}}^{2}\frac{dx}{(x-1)\cdot\sqrt{x^2-2}}.$

**2.18** $\int\limits_{0}^{1}\frac{dx}{(2-x)\cdot\sqrt{1-x}}.$

**2.19** $\int\limits_{-0,5}^{-0,25}\frac{dx}{x\cdot\sqrt{2x+1}}.$

**2.20** $\int\limits_{0}^{1}x\ln^3 xdx.$

**2.21** $\int\limits_{a}^{b}\frac{dx}{\sqrt{(x-a)(b-x)}}, b>a.$

**3-masala. Quyidagi II-tur xosmas integrallarni yaqinlashishga tekshiring.**

**3.1** $\int\limits_{0}^{1}\frac{\ln\left(1+\sqrt[3]{x^2}\right)}{\sqrt{x}\cdot\sin\sqrt{x}}dx.$

**3.2** $\int\limits_{0}^{\pi}\sin\left(\frac{1}{\cos x}\right)\cdot\frac{dx}{\sqrt{x}}.$

**3.3** $\int_0^1 \frac{|\ln x|}{x^\alpha}dx.$

**3.4** $\int_{-1}^1 \frac{dx}{\ln(1+x)}.$

**3.5** $\int_0^1 \frac{dx}{\sqrt[5]{1-x^{10}}}.$

**3.6** $\int_0^2 \frac{\sqrt{x}dx}{e^{\sin x}-1}.$

**3.7** $\int_0^\pi \frac{\sin x}{x^2}dx.$

**3.8** $\int_0^1 \frac{dx}{\sqrt{x+arctgx}}.$

**3.9** $\int_1^2 \frac{(x-2)dx}{x^3-3x^2+4}.$

**3.10** $\int_0^\pi \frac{\ln x}{\sqrt{\sin x}}dx.$

**3.11** $\int_0^\pi \frac{\ln\sin x}{\sqrt[3]{x}}dx.$

**3.12** $\int_0^1 \frac{dx}{\arccos x}.$

**3.13** $\int_0^\pi \frac{1-\cos x}{x^\alpha}dx.$

**3.14** $\int_0^{\pi/2} \frac{e^{\alpha\cos x}-\sqrt{1+2\cos x}}{\sqrt{\cos^5 x}}dx.$

**3.15** $\int_0^1 \frac{\sqrt{e^2+x^2}-e^{\cos x}}{x^\alpha}dx$

**3.16** $\int_0^1 \frac{\ln(1+2x)-xe^{-x}}{1-\cos^\alpha x}dx.$

**3.17** $\int_0^{0.5} \frac{\ln^\alpha(1/x)}{tg^\beta x}dx.$

**3.18** $\int_0^1 x^\alpha\cdot(1-x)^\beta\cdot\ln x dx.$

**3.19** $\int_0^1 x^\alpha\cdot\ln^\beta\frac{1}{x}dx.$

**3.20** $\int_0^1 x^\alpha\cdot(1-x)^\beta dx.$

**3.21** $\int_0^{\pi/2} \sin^\alpha x\cdot\cos^\beta x dx.$

**4-masala. Quyidagi xosmas integrallarni yaqinlashishga tekshiring.**

**4.1** $\int\limits_{1}^{+\infty} \frac{\ln x dx}{x^{\alpha}}$.

**4.2** $\int\limits_{e}^{+\infty} \frac{dx}{x^{\alpha} \cdot \ln x}$.

**4.3** $\int\limits_{e}^{+\infty} \frac{dx}{x \cdot \ln^{\alpha} x}$.

**4.4** $\int\limits_{2}^{+\infty} \frac{e^{\alpha x} dx}{(x-1)^{\alpha} \cdot \ln x}$.

**4.5** $\int\limits_{0}^{+\infty} \frac{arctg 2x}{x^{\alpha}} dx$.

**4.6** $\int\limits_{0}^{+\infty} \frac{\ln\left(1+x^{-2\alpha}\right)}{\sqrt{x^{\alpha}+x^{-\alpha}}} dx$.

**4.7** $\int\limits_{0}^{+\infty} \frac{dx}{1+x^{\alpha} \cdot \sin^{2} x}$.

**4.8** $\int\limits_{0}^{+\infty} \frac{\ln\left(1+x^{2}\right)}{(x+\alpha)^{2}} dx$.

**4.9** $\int\limits_{0}^{+\infty} arctg \frac{x^{\alpha}}{1+x^{2}} \cdot \frac{dx}{x} (\alpha > 0)$.

**4.10** $\int\limits_{0}^{+\infty} \frac{\ln\left(1+x+x^{\alpha}\right)}{\sqrt{x^{3}}} dx (\alpha > 0)$.

**4.11** $\int\limits_{0}^{+\infty} \frac{\ln\left(x^{\alpha}+e^{x}\right)}{\sqrt{x^{3}+x^{5}}} dx (\alpha > 0)$.

**4.12** $\int\limits_{1}^{+\infty} \frac{dx}{x^{\alpha} \cdot \ln^{\beta} x}$.

**4.13** $\int\limits_{0}^{+\infty} \frac{x^{\alpha} dx}{x^{\beta}+1}, \beta \geq 0$.

**4.14** $\int\limits_{0}^{+\infty} \frac{dx}{x^{\alpha}+x^{\beta}}$.

**4.15** $\int\limits_{0}^{+\infty} \frac{\sin^{2} x}{x^{2}} dx$.

**4.16** $\int\limits_{2}^{+\infty} \left(\cos\frac{2}{x}-1\right) dx$.

**4.17** $\int\limits_{1}^{+\infty} \frac{\ln x dx}{x \cdot \sqrt{x^{2}-1}}$.

**4.18** $\int\limits_{0}^{+\infty} \frac{\sin\frac{1}{x}}{\left(x-\cos\frac{\pi}{x}\right)^{2}} dx$.

**4.19** $\int\limits_{0}^{+\infty} \frac{1}{\sqrt{x}} arctg \frac{x}{2+\sqrt{x}} dx$.

**4.20** $\int\limits_{0}^{+\infty} \frac{x dx}{1+x^{2} \cdot \sin^{2} x}$.

# Test

**1. Agar** $t \to +\infty$ **da** $F(t)$ **funksiyaning limiti mavjud bo‘lsa, bu limiti** $f(x)$ **funksiyaning** $[a,+\infty)$ **cheksiz oraliq bo‘yicha ...............deyiladi.**

A) xosmas integrali
B) Aniq integral
C) aniqmas integral
D) to’g’ri javob yo’q

**2.Agar** $t \to +\infty$ **da** $F(t)$ **funksiyaning limiti mavjud va chekli bo‘lsa,**

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty} \int_a^t f(x)dx.$ **integral .................. deyiladi.**

A) yaqinlashuvchi
B) uzoqlashuvchi
C) yaqinlashuvchi yoki uzoqlashuvchi
D) tog’ri javob A va B

**3. Agar** $t \to +\infty$ **da** $F(t)$ **funksiyaning limiti cheksiz yoki mavjud bo‘lmasa,**

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty} \int_a^t f(x)dx.$ **integral ....... deyiladi.**

A) uzoqlashuvchi
B) yaqinlashuvchi yoki uzoqlashuvchi
C) yaqinlashuvchi
D) tog’ri javob A va B

**4. Agar** $\int_a^{+\infty} f(x)dx$ **integral yaqinlashuvchi bo‘lsa, u holda** $\int_a^{+\infty} C\cdot f(x)dx$ **ham** $(C = const)$ **yaqinlashuvchi bo‘lib,...........bo‘ladi.**

A) $\int_a^{+\infty} C\cdot f(x)dx = C\int_a^{+\infty} f(x)dx$

B) $\int_a^{+\infty} f(x)dx \geq 0$

C) $\int_a^{+\infty} (f(x) \pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$

D) $\int_a^{+\infty} f(x)dx = \int_a^{b} f(x)dx + \int_b^{+\infty} f(x)dx$

5. Agar $\forall x \in [a,+\infty)$ da $f(x) \le g(x)$ bo‘lib, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda ………..bo‘ladi.

A) $\int_a^{+\infty} f(x)dx \le \int_a^{+\infty} g(x)dx$

B) $\int_a^{+\infty} f(x)dx = \int_a^{b} f(x)dx + \int_b^{+\infty} f(x)dx$

C) $\int_a^{+\infty} (f(x) \pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$

D) $\int_a^{+\infty} f(x)dx \ge 0$

**6. Hisoblang:** $\int_0^{\frac{1}{2}} \frac{dx}{x\ln^2 x}$

A) $\frac{1}{\ln 2}$

B) $\frac{1}{\ln 3}$

C) 2

D) 1

**7. Hisoblang:** $\int_0^{3} \frac{x^2 dx}{\sqrt{9-x^2}}$

A) $\frac{9\pi}{4}$

B) $\frac{9\pi}{2}$

C) $\frac{6\pi}{4}$

D) 1

**8. Hisoblang:** $\int_0^1 \ln x dx$

A) -1
B) -2
C) 2
D) 1

**9. Hisoblang:** $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}$

A) $\frac{\pi}{2}$
B) $\frac{3\pi}{2}$
C) $\frac{6\pi}{4}$
D) $\frac{\pi}{4}$

**10. Hisoblang:** $\int_2^3 \frac{2-x}{\sqrt{3-x}} dx$

A) $-\frac{4}{3}$
B) $\frac{4}{3}$
C) $-\frac{3}{2}$
D) 2

# Mavzu. Manfiy bo'lmagan funksiyaning xosmas integrallari. Integralning absolyut yaqinlashuvchililigi

## 18-ma'ruza

**Reja**

**$1^0$.** Manfiy bo'lmagan funksiya xosmas integralining yaqinlashuvchiligi.

**$2^0$.** Taqqoslash teoremalari.

**$3^0$.** Xosmas integralning absolyut yaqinlashuvchiligi.

**$1^0$. Manfiy bo'lmagan funksiya xosmas integralining yaqinlashuvchiligi.**

Aytaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo'lib, $\forall x \in [a,+\infty)$ da $f(x) \geq 0$ bo'lsin. Bu funksiyani $[a,t]$ da $(a < t < +\infty)$ integrallanuvchi deylik: $f(x) \in R([a,t])$. Bu holda

$$F(t) = \int_a^t f(x)dx$$

funksiya $(a,+\infty)$ oraliqda o'suvchi bo'ladi.

**1-teorema.** Manfiy bo'lmagan $f(x)$ funksiya xosmas integrali

$$\int_a^{+\infty} f(x)dx \qquad (f(x) \geq 0,\ x > a) \tag{1}$$

ning yaqinlashuvchi bo'lishi uchun $F(t)$ funksiyaning yuqoridan chegaralangan, ya'ni

$$\exists C \in R,\ \forall t > a:\ F(t) \leq C$$

bo'lishi zarur va yetarli.

◄**Zarurligi.** Aytaylik, (1) integral yaqinlashuvchi bo'lsin. Ta'rifga binoan

$$\lim_{t \to +\infty} F(t)$$

mavjud va chekli bo'ladi. Unda, $\exists\, C \in R$, $\forall t > a$ da $F(t) \leq C$ bo'ladi.

**Etarliligi.** Aytaylik, $F(t)$ funksiya $(a,+\infty)$ da yuqorida-gi chegaralangan bo‘lsin. Ayni paytda, $F(t)$ o‘suvchi funksiya. Demak, $t\to+\infty$ da $F(t)$ funksiya chekli limitga ega. Bu esa (1) integralni yaqinlashuvchi bo‘lishini bildiradi. ►

Bu teoremadan quyidagi natija kelib chiqadi.

**Natija.** Agar $F(t)$ funksiya $(t\in(a,+\infty))$ yuqoridan chegaralanmagan bo‘lsa, u holda

$$\int_a^{+\infty} f(x)dx$$

integral uzoqlashuvchi bo‘ladi.

**2$^0$. Taqqoslash teoremalari.**

Ikkita funksiya ma’lum munosabatda bo‘lganda birining xosmas integralining yaqinlashuvchi (uzoqlashuvchi) bo‘lishidan ikkinchisining ham yaqinlashuvchi (uzoqlashuvchi) bo‘lishini ifodalovchi teorema-larni keltiramiz. Odatda, ular taqqoslash teoremalari deyiladi.

**2-teorema.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo‘lib, $\forall x\in[a,+\infty)$ da

$$0\le f(x)\le g(x) \qquad (2)$$

bo‘lsin.

Agar $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi bo‘ladi.

Agar $\int_a^{+\infty} f(x)dx$ uzoqlashuvchi bo‘lsa, u holda $\int_a^{+\infty} g(x)dx$ ham uzoqlashuvchi bo‘ladi.

◄ Aytaylik, (2) munosabat o‘rinli bo‘lib, $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo‘lsin. Unda 1-teoremaga ko‘ra

$$G(t)=\int_a^{t} g(x)dx\le C$$

bo‘ladi. Ayni paytda,

$$F(t)=\int_a^{t} f(x)dx\le G(t)$$

bo‘lganligi sababli ya’ni 1-teoremaga binoan $\int_a^{+\infty} f(x)dx$ yaqinla-shuvchi bo‘ladi.

Aytaylik, (2) munosabat o‘rinli bo‘lib, $\int_a^{+\infty} f(x)dx$ uzoqla-shuvchi bo‘lsin.

Unda yuqorida keltirilgan natija va

$$F(t)\le G(t)$$

tengsizlikdan $\int_a^{+\infty} g(x)dx$ integralning uzoqlashuvchiligi kelib chiqadi.▶

**3-teorema.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ da $f(x)\ge 0$ $g(x)\ge 0$ bo‘lib,

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=k \qquad (0\le k\le +\infty)$$

bo‘lsin.

Agar $k<+\infty$ bo‘lib, $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi bo‘ladi.

Agar $k>0$ bo‘lib, $\int_a^{+\infty} g(x)dx$ uzoqlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ ham uzoqlashuvchi bo‘ladi.

**Natija.** Agar

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=k$$

bo‘lib, $0<k<+\infty$ bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar bir vaqtda yoki yaqinlashuvchi, yoki uzoqlashuvchi bo‘ladi.

Ko‘p hollarda biror xosmas integralning yaqinlashuvchiligini yoki uzoqlashuvchiligini aniqlashda avvaldan yaqinlashuvchiligi yoki uzoqlashuvchiligi ma’lum bo‘lgan integral bilan taqqoslab (yuqorida keltirilgan teoremalardan foydalanib) qaralayotgan integralning yaqinlashuvchi yoki uzoqlashuvchi bo‘lishi topiladi.

Masalan,

$$\int_a^{+\infty} f(x)dx$$

integralni

$$\int_a^{+\infty}\frac{dx}{x^{\alpha}} \qquad (a>0\ ,\alpha>0)$$

integral bilan taqqoslab, quyidagi natijaga kelamiz:

**Natija.** Aytaylik, biror $C$ $(0<C<+\infty)$ va $\alpha>0$ sonlar uchun $x\to+\infty$ da

$$f(x)\sim\frac{C}{x^{\alpha}},$$

ya’ni

$$\lim_{x\to+\infty} x^{\alpha}\cdot f(x)=C$$

bo‘lsin. Unda

$$\int_a^{+\infty} f(x)dx$$

integral $\alpha>1$ bo‘lganda yaqinlashuvchi, $\alpha\le 1$ bo‘lganda uzoqla-shuvchi bo‘ladi.

**Misol.** Ushbu

$$\int_0^{+\infty}\frac{\cos^2 x}{1+x^2}dx$$

integral yaqinlashuvchilikka tekshirilsin.

◄ Agar

$$f(x)=\frac{\cos^2 x}{1+x^2}\ ,\quad g(x)=\frac{1}{1+x^2}$$

deyilsa, unda $\forall x\in[0,+\infty)$

$$0\le f(x)\le g(x)$$

bo‘ladi.

Ravshanki, $\int_0^{+\infty}\frac{dx}{1+x^2}$

integral yaqinlashuvchi. 2-teoremaga ko‘ra berilgan xosmas integral yaqinlashuvchi bo‘ladi. ►

**2-misol.** Ushbu

$$\int_1^{+\infty} e^{-x^2}dx$$

integral yaqinlashuvchilikka tekshirilsin.

◄ $\forall x\ge 1$ da

$$f(x)=e^{-x^2}\ ,\quad g(x)=e^{-x}$$

funksiyalari uchun

$$0\le f(x)\le g(x)$$

bo‘ladi. Quyidagi

$$\int_1^{+\infty} e^{-x}dx$$

integralning yaqinlashuvchiligi ravshan. Demak,

$$\int_1^{+\infty} e^{-x^2}dx$$

integral yaqinlashuvchi bo‘ladi. ►

**3-misol.** Ushbu

$$\int_{1}^{+\infty} e^{-x}\ln x dx$$

integral yaqinlashuvchilikka tekshirilsin.

◀ $\forall x>1$ da

$$\ln x<x$$

bo‘lib, $f(x)=e^{-x}\ln x,\quad g(x)=xe^{-x}$ funksiyalar uchun

$$0\le f(x)\le g(x)$$

bo‘ladi. Endi

$$\int_{1}^{+\infty} g(x)dx=\int_{1}^{+\infty} xe^{-x}dx$$

integralning yaqinlashuvchiligini e’tiborga olib, 2-teorema-dan foydalanib, berilgan

$$\int_{1}^{+\infty} e^{-x}\ln x dx$$

integralning yaqinlashuvchiligini topamiz.▶

**4-misol.** Ushbu

$$\int_{1}^{+\infty}\frac{dx}{x\cdot\sqrt[3]{x^2+1}}$$

integral yaqinlashuvchilikka tekshirilsin.

◀Integral ostidagi

$$f(x)=\frac{1}{x\cdot\sqrt[3]{x^2+1}}$$

funksiya uchun

$$\lim_{x\to+\infty} x^{\frac{5}{3}}f(x)=\lim_{x\to+\infty} x^{\frac{5}{3}}\cdot\frac{1}{x\cdot\sqrt[3]{x^2+1}}=\lim_{x\to+\infty}\cdot\frac{1}{x\cdot\sqrt[3]{1+\frac{1}{x^2}}}=1$$

bo‘ladi.

Ravshanki,

$$\int_{1}^{+\infty}\frac{dx}{x^{\frac{5}{3}}}$$

integral yaqinlashuvchi. Demak, berilgan integral yaqinla-shuvchi bo‘ladi.▶

**4⁰. Xosmas integralning absolyut yaqinlashuvchiligi.**

Aytaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo‘lsin. Bunda, $\forall x\in[a,+\infty)$ uchun $f(x)\ge 0$ bo‘lishi shart emas

**Ta’rif.** Agar

$$\int_a^{+\infty} |f(x)|dx$$

integral yaqinlashuvchi bo‘lsa, $\int_a^{+\infty} f(x)dx$ integral absolyut yaqinlashuvchi deyiladi.

Agar $\int_a^{+\infty} f(x)dx$ yaqinlashuvchi bo‘lib, $\int_a^{+\infty} |f(x)|dx$ uzoqlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ shartli yaqinlashuvchi integral deyiladi.

**4-teorema.** Agar integral absolyut yaqinlashuvchi bo‘lsa, u yaqinlashuvchi bo‘ladi.

◀ Aytaylik,

$$\int_a^{+\infty} |f(x)|dx$$

integral yaqinlashuvchi bo‘lsin. Berilgan $f(x)$ va $|f(x)|$ funk-siyalar yordamida ushbu

$$\varphi(x) = \frac{1}{2}(f(x) + |f(x)|) \quad ,$$

$$\psi(x) = \frac{1}{2}(-f(x) + |f(x)|)$$

funksiyalarni tuzamiz.

Bu funksiyalar uchun, $\forall x \in [a,+\infty)$ da

1) $\varphi(x) \geq 0 \quad , \quad \psi(x) \geq 0$
2) $\varphi(x) \leq |f(x)| \quad , \psi(x) \leq |f(x)|$
3) $\varphi(x) - \psi(x) = f(x)$

bo‘ladi. YUqorida keltirilgan 2-teoremadan foydalanib, quyidagi

$$\int_a^{+\infty} \varphi(x)dx, \quad \int_a^{+\infty} \psi(x)dx$$

integral yaqinlashuvchiligini topamiz.

Unda

$$\int_a^{+\infty} (\varphi(x) - \psi(x))dx$$

integral ham yaqinlashuvchi bo‘ladi. Demak,

$$\int_a^{+\infty} f(x)dx$$

yaqinlashuvchi bo‘ladi. ▶

## Mashqlar

1. Ushbu integral

$$\int_0^{+\infty}\left(e^{-\frac{a^2}{x^2}}-e^{-\frac{b^2}{x^2}}\right)dx$$

integral yaqinlashuvchilikka tekshirilsin.

2. $k$ ning qanday qiymatlarida

$$\int_1^{+\infty} x^k \frac{x+\sin x}{x-\sin x}dx \qquad (k<1)$$

integral yaqinlashuvchi bo‘ladi?

## Adabiyotlar


1. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’rizalar, I q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Kurs differensialnogo i integralnogo ischisleniya, 1 t.* M. «FIZMATLIT», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.


# Glossariy

**1-teorema.** Manfiy bo‘lmagan $f(x)$ funksiya xosmas integrali

$$\int_a^{+\infty} f(x)dx \qquad (f(x)\ge 0,\ x>a) \tag{1}$$

ning yaqinlashuvchi bo‘lishi uchun $F(t)$ funksiyaning yuqoridan chegaralangan, ya’ni

$$\exists C\in R,\ \forall t>a:\ F(t)\le C$$

bo‘lishi zarur va yetarli.

**2-teorema.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo‘lib, $\forall x\in[a,+\infty)$ da

$$0\le f(x)\le g(x) \tag{2}$$

bo‘lsin.

Agar $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi bo‘ladi.

Agar $\int_a^{+\infty} f(x)dx$ uzoqlashuvchi bo‘lsa, u holda $\int_a^{+\infty} g(x)dx$ ham uzoqlashuvchi bo‘ladi.

**3-teorema.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ da $f(x)\geq 0$ $g(x)\geq 0$ bo‘lib,

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=k \qquad (0\leq k\leq+\infty)$$

bo‘lsin.

Agar $k<+\infty$ bo‘lib, $\int_a^{+\infty} g(x)dx$ yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ ham yaqinlashuvchi bo‘ladi.

Agar $k>0$ bo‘lib, $\int_a^{+\infty} g(x)dx$ uzoqlashuvchi bo‘lsa, u holda $\int_a^{+\infty} f(x)dx$ ham uzoqlashuvchi bo‘ladi.

## Keys banki

**42-keys.** Masala o`rtaga tashlanadi: $k$ ning qanday qiymatlarida

$$\int_1^{+\infty} x^k\frac{x+\sin x}{x-\sin x}dx, \ \ (k<1)$$

integral yaqinlashuvchi bo‘ladi?

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 18-amaliy mashg'ulot

Ushbu

$$\int_{0}^{+\infty} \frac{\cos^2 x}{1+x^2}dx$$

integral yaqinlashuvchilikka tekshirilsin.

◄ Agar

$$f(x)=\frac{\cos^2 x}{1+x^2}\ ,\quad g(x)=\frac{1}{1+x^2}$$

deyilsa, unda $\forall x\in[0,+\infty)$

$$0\le f(x)\le g(x)$$

bo‘ladi.

Ravshanki, $\int_{0}^{+\infty} \frac{dx}{1+x^2}$

integral yaqinlashuvchi. 2-teoremaga ko‘ra berilgan xosmas integral yaqinlashuvchi bo‘ladi. ►

**2-misol.** Ushbu

$$\int_{1}^{+\infty} e^{-x^2}dx$$

integral yaqinlashuvchilikka tekshirilsin.

◄ $\forall x\ge 1$ da

$$f(x)=e^{-x^2}, \quad g(x)=e^{-x}$$

funksiyalari uchun

$$0\le f(x)\le g(x)$$

bo‘ladi. Quyidagi

$$\int_1^{+\infty} e^{-x}dx$$

integralning yaqinlashuvchiligi ravshan. Demak,

$$\int_1^{+\infty} e^{-x^2}dx$$

integral yaqinlashuvchi bo‘ladi. ▶

**Quyidagi II-tur xosmas integrallarni yaqinlashishga tekshiring.**

**3.1** $\int_0^1 \frac{\ln\left(1+\sqrt[3]{x^2}\right)}{\sqrt{x}\cdot\sin\sqrt{x}}dx.$

**3.2** $\int_0^{\pi} \sin\left(\frac{1}{\cos x}\right)\cdot\frac{dx}{\sqrt{x}}.$

**3.3** $\int_0^1 \frac{|\ln x|}{x^{\alpha}}dx.$

**3.4** $\int_{-1}^1 \frac{dx}{\ln(1+x)}.$

**3.5** $\int_0^1 \frac{dx}{\sqrt[5]{1-x^{10}}}.$

**3.6** $\int_0^2 \frac{\sqrt{x}dx}{e^{\sin x}-1}.$

**3.7** $\int_0^{\pi} \frac{\sin x}{x^2}dx.$

**3.8** $\int_0^1 \frac{dx}{\sqrt{x+arctgx}}.$

**3.9** $\int_1^2 \frac{(x-2)dx}{x^3-3x^2+4}.$

**3.10** $\int_0^{\pi} \frac{\ln x}{\sqrt{\sin x}}dx.$

**3.11** $\int_0^{\pi} \frac{\ln\sin x}{\sqrt[3]{x}}dx.$

**3.12** $\int_0^1 \frac{dx}{\arccos x}.$

**3.13** $\int_0^{\pi} \frac{1-\cos x}{x^{\alpha}} dx.$

**3.14** $\int_0^{\pi/2} \frac{e^{\alpha \cos x} - \sqrt{1+2\cos x}}{\sqrt{\cos^5 x}} dx.$

**3.15** $\int_0^1 \frac{\sqrt{e^2+x^2} - e^{\cos x}}{x^{\alpha}} dx$

**3.16** $\int_0^1 \frac{\ln(1+2x) - xe^{-x}}{1-\cos^{\alpha} x} dx.$

**3.17** $\int_0^{0.5} \frac{\ln^{\alpha}(1/x)}{tg^{\beta} x} dx.$

**3.18** $\int_0^1 x^{\alpha} \cdot (1-x)^{\beta} \cdot \ln x dx.$

**3.19** $\int_0^1 x^{\alpha} \cdot \ln^{\beta} \frac{1}{x} dx.$

**3.20** $\int_0^1 x^{\alpha} \cdot (1-x)^{\beta} dx.$

**3.21** $\int_0^{\pi/2} \sin^{\alpha} x \cdot \cos^{\beta} x dx.$

**4-masala. Quyidagi xosmas integrallarni yaqinlashishga tekshiring.**

**4.1** $\int_1^{+\infty} \frac{\ln x dx}{x^{\alpha}}.$

**4.2** $\int_e^{+\infty} \frac{dx}{x^{\alpha} \cdot \ln x}.$

**4.3** $\int_e^{+\infty} \frac{dx}{x \cdot \ln^{\alpha} x}.$

**4.4** $\int_2^{+\infty} \frac{e^{\alpha x} dx}{(x-1)^{\alpha} \cdot \ln x}.$

**4.5** $\int_0^{+\infty} \frac{arctg 2x}{x^{\alpha}} dx.$

**4.6** $\int_0^{+\infty} \frac{\ln(1+x^{-2\alpha})}{\sqrt{x^{\alpha}+x^{-\alpha}}} dx.$

**4.7** $\int_0^{+\infty} \frac{dx}{1+x^{\alpha} \cdot \sin^2 x}.$

**4.8** $\int_0^{+\infty} \frac{\ln(1+x^2)}{(x+\alpha)^2} dx.$

**4.9** $\int_0^{+\infty} arctg \frac{x^{\alpha}}{1+x^2} \cdot \frac{dx}{x} (\alpha > 0).$

**4.10** $\int_0^{+\infty} \frac{\ln(1+x+x^{\alpha})}{\sqrt{x^3}} dx (\alpha > 0).$

**4.11** $\int\limits_0^{+\infty} \frac{\ln(x^\alpha + e^x)}{\sqrt{x^3 + x^5}} dx \, (\alpha > 0).$

**4.12** $\int\limits_1^{+\infty} \frac{dx}{x^\alpha \cdot \ln^\beta x}.$

**4.13** $\int\limits_0^{+\infty} \frac{x^\alpha dx}{x^\beta + 1}, \beta \geq 0.$

**4.14** $\int\limits_0^{+\infty} \frac{dx}{x^\alpha + x^\beta}.$

**4.15** $\int\limits_0^{+\infty} \frac{\sin^2 x}{x^2} dx.$

**4.16** $\int\limits_2^{+\infty} \left( \cos\frac{2}{x} - 1 \right) dx.$

**4.17** $\int\limits_1^{+\infty} \frac{\ln x dx}{x \cdot \sqrt{x^2 - 1}}.$

**4.18** $\int\limits_0^{+\infty} \frac{\sin\frac{1}{x}}{\left( x - \cos\frac{\pi}{x} \right)^2} dx.$

**4.19** $\int\limits_0^{+\infty} \frac{1}{\sqrt{x}} arctg \frac{x}{2 + \sqrt{x}} dx.$

**4.20** $\int\limits_0^{+\infty} \frac{xdx}{1 + x^2 \cdot \sin^2 x}.$

**4.21** $\int\limits_2^{+\infty} \frac{dx}{x^\alpha \cdot \ln^\beta x}.$

# Test

**1. Agar** $t \to +\infty$ **da** $F(t)$ **funksiyaning limiti mavjud bo‘lsa, bu limiti** $f(x)$ **funksiyaning** $[a,+\infty)$ **cheksiz oraliq bo‘yicha …………….deyiladi.**

A) xosmas integrali
B) Aniq integral
C) aniqmas integral
D) to’g’ri javob yo’q

**2.Agar** $t \to +\infty$ **da** $F(t)$ **funksiyaning limiti mavjud va chekli bo‘lsa,**

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty}\int_a^t f(x)dx$. **integral ………………. deyiladi.**

A) yaqinlashuvchi
B) uzoqlashuvchi
C) yaqinlashuvchi yoki uzoqlashuvchi
D) tog’ri javob A va B

**3. Agar** $t \to +\infty$ **da** $F(t)$ **funksiyaning limiti cheksiz yoki mavjud bo‘lmasa,**

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty}\int_a^t f(x)dx$. **integral ……. deyiladi.**

A) uzoqlashuvchi
B) yaqinlashuvchi yoki uzoqlashuvchi
C) yaqinlashuvchi
D) tog’ri javob A va B

**4. Agar** $\int_a^{+\infty} f(x)dx$ **integral yaqinlashuvchi bo‘lsa, u holda** $\int_a^{+\infty} C\cdot f(x)dx$ **ham** $(C=const)$ **yaqinlashuvchi bo‘lib,………..bo‘ladi.**

A) $\int_a^{+\infty} C\cdot f(x)dx = C\int_a^{+\infty} f(x)dx$

B) $\int_a^{+\infty} f(x)dx \ge 0$

C) $\int_a^{+\infty} (f(x)\pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$

D) $\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx$

5. Agar $\forall x \in [a,+\infty)$ da $f(x) \le g(x)$ bo‘lib, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda ………..bo‘ladi.

A) $\int_a^{+\infty} f(x)dx \le \int_a^{+\infty} g(x)dx$

B) $\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx$

C) $\int_a^{+\infty} (f(x) \pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$

D) $\int_a^{+\infty} f(x)dx \ge 0$

**6. Hisoblang:** $\int_0^{\frac{1}{2}} \frac{dx}{x\ln^2 x}$

A) $\frac{1}{\ln 2}$

B) $\frac{1}{\ln 3}$

C) 2

D) 1

**7. Hisoblang:** $\int_0^3 \frac{x^2 dx}{\sqrt{9-x^2}}$

A) $\frac{9\pi}{4}$

B) $\frac{9\pi}{2}$

C) $\frac{6\pi}{4}$

D) 1

**8. Hisoblang:** $\int_0^1 \ln x dx$

A) -1
B) -2
C) 2
D) 1

**9. Hisoblang:** $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}$

A) $\frac{\pi}{2}$
B) $\frac{3\pi}{2}$
C) $\frac{6\pi}{4}$
D) $\frac{\pi}{4}$

**10. Hisoblang:** $\int_2^3 \frac{2-x}{\sqrt{3-x}} dx$

A) $-\frac{4}{3}$
B) $\frac{4}{3}$
C) $-\frac{3}{2}$
D) 2

# Mavzu. Chegaralanmagan funksiyaning xosmas integrallari

## 19-20-ma'ruzalar

**Reja**

**$1^0$.** Chegaralanmagan funksiyaning xosmas integrali tushunchasi.

**$2^0$.** Xosmas integrallarni hisoblash.

### $1^0$. Chegaralanmagan funksiyaning xosmas integrali tushunchasi.

**1. Maxsus nuqta.** Aytaylik, $f(x)$ funksiya $X\subset R$ to'plam-da berilgan bo'lsin. $x_0\in R$ nuqtaning ushbu

$$\dot{U}_\delta(x_0)=\{x\in R;\ x_0-\delta<x<x_0+\delta;\ x\neq x_0\}$$

atrofida qaraymiz, bunda $\delta$ ixtiyoriy musbat son.

**1-ta'rif.** Agar $f(x)$ funksiya

$$X\bigcap \dot{U}_\delta(x_0)\neq\varnothing$$

to'plamda chegaralanmagan bo'lsa, $x_0$ nuqta $f(x)$ funksiyaning maxsus nuqtasi deyiladi.

Masalan, $[a,b)$ da berilgan $f(x)=\dfrac{1}{b-x}$ funksiya uchun $x_0=b$ maxsus nuqta; $R\setminus\{-1;0;1\}$ to'plamda berilgan $f(x)=\dfrac{1}{x(x^2-1)}$ funksiya uchun $x_0=-1,\ x_1=0,\ x_2=1$ nuqtalar maxsus nuqtalar bo'ladi.

**2.Chegaralanmagan funksiyaning xosmas integrali tushunchasi.** Faraz qilaylik, $f(x)$ funksiya $[a,b)$ da berilgan bo'lib, $b$ nuqta shu funksiyaning maxsus nuqtasi bo'lsin. Bu funksiya ixtiyoriy $[a,t]$ da $(a<t<b)$ integrallanuvchi bo'lsin. Ravshanki, bu integral $t$ ga bog'liq bo'ladi:

$$F(t)=\int_a^t f(x)dx \qquad (a<t<b).$$

**2-ta'rif.** Agar $t\to b-0$ da $F(t)$ funksiyaning limiti mavjud bo'lsa, bu limit chegaralanmagan $f(x)$ funksiyaning $[a,b)$ bo‘yicha xosmas integrali deyiladi va

$$\int_a^b f(x)dx$$

kabi belgilanadi:

$$\int_a^b f(x)dx = \lim_{t\to b-0} F(x) = \lim_{t\to b-0}\int_a^t f(x)dx. \qquad (1)$$

**3-ta’rif.** Agar $t\to b-0$ da $F(t)$ funksiyaning limiti mavjud va chekli bo‘lsa, (1) xosmas integral yaqinlashuvchi deyiladi.

Agar $t\to b-0$ da $F(t)$ funksiyaning limiti cheksiz yoki mavjud bo‘lmasa, (1) xosmas integral uzoqlashuvchi deyiladi.

$f(x)$ funksiya $(a,b]$ da berilgan bo‘lib, $x_0=a$ nuqta uning maxsus nuqtasi; $f(x)$ funksiya $(a,b)$ da berilgan bo‘lib, $x_0=a$, $x_1=b$ nuqtalar uning maxsus nuqtalari bo‘lgan holda shu funksiyaning $(a,b]$ hamda $(a,b)$ bo‘yicha xosmas integrallari, ularning yaqinlashuvchiligi hamda uzoqlashuvchiligi yuqorida-gidek ta’riflanadi:

$$\int_a^b f(x)dx = \lim_{t\to a+0} F(t) = \lim_{t\to a+0}\int_t^b f(x)dx;$$

$$\int_a^b f(x)dx = \lim_{\substack{t'\to a+0\\ t\to b-0}} F(t',t) = \lim_{\substack{t'\to a+0\\ t\to b-0}}\int_{t'}^t f(x)dx.$$

Aytaylik, $f(x)$ funksiya $(a,b)\setminus\{c\}$ to‘plamda $(a<c<b)$ berilgan bo‘lib, $x_0=a$ $,x_1=b, x_2=c$ nuqtalar uning maxsus nuqtalari bo‘lsin. Bu funksiyaning qo‘yidagi

$$\int_{t'}^t f(x)dx = \varphi(t',t)\ , \qquad (a<t'<t<c)$$

$$\int_{u'}^u f(x)dx = \psi(u',u)\ , \qquad (c<u'<u<b)$$

integrallari mavjud bo‘lsin.

**4-ta’rif.** Agar $t'\to a+0$, $t\to c-0$ hamda $u'\to c+0$, $u\to b-0$ da $\varphi(t',t)+\psi(u',u)$ funksiyaning limiti

$$\lim_{\substack{t'\to a+0\\ t\to c-0\\ u'\to c+0\\ u\to b-0}} [\varphi(t',t)+\psi(u',u)] = \lim_{\substack{t'\to a+0\\ t\to c-0\\ u'\to c+0\\ u\to b-0}} [\int_{t'}^t f(x)dx + \int_{u'}^u f(x)dx]$$

mavjud bo‘lsa, bu limit chegaralanmagan $f(x)$ funksiyaning $(a,b)$ bo‘yicha xosmas integrali deyiladi va

$$\int_a^b f(x)dx$$

kabi belgilanadi. Demak,

$$\int_a^b f(x)dx = \lim_{\substack{t'\to a+0\\ t\to c-0\\ u'\to c+0\\ u\to b-0}} [\int_{t'}^{t} f(x)dx + \int_{u'}^{u} f(x)dx] \qquad (2)$$

**5-ta'rif.** Agar $t' \to a+0$, $t \to c-0$ hamda $u' \to c+0$, $u \to b-0$ da $\varphi(t',t)+\psi(u',u)$ funksiyaning limiti mavjud va chekli bo'lsa, (2) integral yaqinlashuvchi deyiladi.

**1-misol.** Ushbu

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

integral yaqinlashuvchilikka tekshirilsin.

◄ Ravshanki, $x_0 = 0$ nuqta $f(x) = \frac{1}{\sqrt{x}}$ funksiyaning maxsus nuqtasi. Demak, qaralayotgan integral chegaralanmagan funk-siyaning xosmas integrali bo'ladi.

Ta'rifga binoan

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to+0} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to+0} 2(1-\sqrt{t}) = 2$$

bo'ladi. Demak, berilgan xosmas integral yaqinlashuvchi va u 2 ga teng. ►

**2-misol.** Ushbu

$$\int_0^1 \frac{dx}{x}$$

xosmas integral uzoqlashuvchi bo'ladi, chunki

$$\lim_{t\to+0} \int_t^1 \frac{dx}{x} = \lim_{t\to+0} (\ln x)_t^1 = +\infty .$$

**3-misol.** Ushbu

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

integral yaqinlashuvchilikka tekshirilsin.

◄ Integral ostidagi

$$f(x) = \frac{1}{\sqrt{x(1-x)}}$$

funksiya uchun $x_0 = 0$, $x_1 = 1$ nuqtalar maxsus nuqtalar bo'ladi. Xosmas integral ta'rifidan foydalanib topamiz:

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \lim_{\substack{t'\to+0\\ t\to1-0}} \int_{t'}^{t} \frac{dx}{\sqrt{x(1-x)}} = \lim_{\substack{t'\to+0\\ t\to1-0}} [\arcsin(2x-1)]_{t'}^{t} =$$

$$= \lim_{\substack{t'\to+0\\ t\to1-0}} [\arcsin(2t-1) - \arcsin(2t'-1)] = \frac{\pi}{2} + \frac{\pi}{2} = \pi \ .$$

Demak, integral yaqinlashuvchi va

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi . \ ▶$$

**4-misol.** Ushbu

$$J_1 = \int_a^b \frac{dx}{(x-a)^{\alpha}} \quad , \quad J_2 = \int_a^b \frac{dx}{(b-x)^{\alpha}} \qquad (\alpha > 0)$$

integrallar yaqinlashuvchilikka tekshirilsin.

◀ Ta'rifdan foydalanib topamiz:

$$\int_a^b \frac{dx}{(x-a)^{\alpha}} = \lim_{t\to a+0} \int_t^b \frac{dx}{(x-a)^{\alpha}} = \lim_{t\to a+0} [\frac{(x-a)^{1-\alpha}}{1-\alpha}]_t^b =$$

$$= \lim_{t\to a+0} \frac{1}{1-\alpha}[(b-a)^{1-\alpha} - (t-a)^{1-\alpha}], \quad (\alpha \neq 1) \quad .$$

Bu limit $\alpha$ ga bog'liq bo'lib, $\alpha < 1$ bo'lganda chekli, demak $J_1$ xosmas integral yaqinlashuvchi, $\alpha > 1$ bo'lganda esa cheksiz bo'lib, $J_1$ xosmas integral uzoqlashuvchi bo'ladi.

$\alpha = 1$ bo'lganda

$$\int_a^b \frac{dx}{x-a} = \lim_{t\to a+0} \int_t^b \frac{dx}{x-a} = \lim_{t\to a+0} (\ln(x-a))_t^b$$

bo'lib, $J_1$ integral uzoqlashuvchi.

Demak,

$$J_1 = \int_a^b \frac{dx}{(x-a)^{\alpha}} \qquad (\alpha > 0)$$

integral $\alpha < 1$ bo'lganda yaqinlashuvchi, $\alpha \geq 1$ bo'lganda uzoqla-shuvchi bo'ladi.

Xuddi shunga o'xshash ko'rsatish mumkin,

$$J_2 = \int_a^b \frac{dx}{(b-x)^{\alpha}} \qquad (\alpha > 0)$$

integral $\alpha < 1$ bo'lganda yaqinlashuvchi, $\alpha \geq 1$ bo'lganda uzoqla-shuvchi bo'ladi. ▶

Yuqorida keltirilgan ta'rif va misollardan chegara-lanmagan funksiyaning xosmas integrali ham 43-46- ma'ru-zalarda batafsil o'rganilgan chegaralari cheksiz (cheksiz oraliq bo'yicha) xosmas integral kabi ekanligini ko'ramiz.

Shuni e'tiborga olib, chegaralanmagan fuksiyaning xos-mas integrallari haqidagi tushuncha va tasdiqlarni kelti-rish bilangina kifoyalanamiz. Bunda $[a,b)$ da berilgan va $x=b$ uning maxsus nuqtasi bo'lgan $f(x)$ funksiyaning xosmas integrali $\int_a^b f(x)dx$ ni qaraymiz.

**3. Yaqinlashuvchi xosmas integralning sodda xossalari.**

1) Agar $\int_a^b f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda

$$\int_c^b f(x)dx \quad (a<c<b)$$

integral ham yaqinlashuvchi bo'ladi va aksincha. Bunda

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

tenglik o'rinli bo'ladi.

2) Agar $\int_a^b f(x)dx$ integral yaqinlashuvchi bo'lsa, u holda $\int_a^b cf(x)dx$ ham $(c-const)$ ham yaqinlashuvchi bo'lib,

$$\int_a^b c\cdot f(x)dx = c\cdot\int_a^b f(x)dx \qquad (c-const)$$

bo'ladi.

3) Agar $\int_a^b f(x)dx$ integral yaqinlashuvchi bo'lib, $\forall x\in[a,b)$ da $f(x)\geq 0$ bo'lsa, u holda

$$\int_a^b f(x)dx \geq 0$$

bo'ladi.

4) Agar $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ integrallar yaqinlashuvchi bo'lsa, u holda $\int_a^b (f(x)\pm g(x))dx$ integral ham yaqinlashuvchi bo'lib,

$$\int_a^b (f(x)\pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

bo'ladi.

5) Agar $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ integrallar yaqinlashuvchi bo'lib, $\forall x \in [a,b)$ da $f(x) \le g(x)$ bo'lsa, u holda

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

bo'ladi.

**4. Xosmas integralning yaqinlashuvchiligi**. Faraz qilaylik, $f(x)$ funksiya $[a,b)$ da berilgan bo'lib, $b$ nuqta shu funksiyaning maxsus nuqtasi bo'lsin.

**1-teorema (Koshi teoremasi).** Ushbu

$$\int_a^b f(x)dx$$

integralning yaqinlashuvchi bo'lishi uchun, $\forall \varepsilon > 0$ son olinganda ham, shunday $\delta > 0$ son topilib, $b-\delta < t' < b$, $b-\delta < t'' < b$ tengsizliklarni qanoatlantiruvchi ixtiyoriy $t'$ *ва* $t''$ lar uchun

$$\left| \int_{t'}^{t''} f(x)dx \right| < \varepsilon$$

tengsizlik bajarilishi zarur va yetarli.

**5. Manfiy bo'lmagan funksiyaning xosmas integral-lari.** Aytaylik, $f(x)$ funksiya $[a,b)$ da berilgan ($b$ nuqta shu funksiyaning maxsus nuqtasi) bo'lib, $\forall x \in [a,b)$ da $f(x) \ge 0$ bo'lsin.

**2-teorema.** Ushbu

$$\int_a^b f(x)dx$$

xosmas integral yaqinlashuvchi bo'lishi uchun $\forall t \in (a,b)$ da

$$F(t) = \int_a^t f(x)dx \le C \qquad (C = const)$$

tengsizlikning bajarilishi, ya'ni $F(t)$ funksiyaning yuqoridan chegaralangan bo'lishi zarur va yetarli.

**Natija.** Agar $F(t) = \int_a^t f(x)dx$ $(\forall t \in (a,b))$ yuqoridan chegaralanmagan bo'lsa, u holda $\int_a^b f(x)dx$ xosmas integral uzoqlashuvchi bo'ladi.

**6. Taqqoslash teoremalari.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,b)$ da berilgan bo'lib, $b$ nuqta shu funksiyalarning maxsus nuqtalari bo'lsin.

**3-teorema.** Agar $\forall x\in[a,b)$ da $0\le f(x)\le g(x)$ bo‘lib, $\int_a^b g(x)dx$ yaqinlashuvchi bo‘lsa, $\int_a^b f(x)dx$ ham yaqinlashuvchi bo‘ladi, $\int_a^b f(x)dx$ uzoqlashuvchi bo‘lsa, $\int_a^b g(x)dx$ ham uzoqlashuvchi bo‘ladi.

**4-teorema.** Aytaylik, $f(x)\ge 0$ , $g(x)\ge 0)$ $x\in[a,b)$ funksiyalari uchun

$$\lim_{x\to b-0}\frac{f(x)}{g(x)}=k$$

bo‘lsin.

Agar $k<+\infty$ bo‘lib $\int_a^b g(x)dx$ yaqinlashuvchi bo‘lsa, $\int_a^b f(x)dx$ ham yaqinlashuvchi bo‘ladi.

Agar $k>0$ bo‘lib $\int_a^b g(x)dx$ uzoqlashuvchi bo‘lsa, $\int_a^b f(x)dx$ ham uzoqlashuvchi bo‘ladi.

**Natija.** YUqoridagi 4-teoremaning shartida $0<k<+\infty$ bo‘lsa, u holda $\int_a^b f(x)dx$ va $\int_a^b g(x)dx$ integrallar bir vaqtda yoki yaqinlashuvchi yoki uzoqlashuvchi bo‘ladi.

**Natija.** Agar $x$ o‘zgaruvchining $b$ ga etarlicha yaqin qiymatlarida

$$f(x)=\frac{\varphi(x)}{(b-x)^{\alpha}} \qquad (\alpha>0)$$

bo‘lsa, u holda:

1) $\varphi(x)\le C<+\infty$ va $\alpha<1$ bo‘lganda $\int_a^b f(x)dx$ integral yaqinlashuvchi,

2) $\varphi(x)\ge C>0$ va $\alpha\ge 1$ bo‘lganda $\int_a^b f(x)dx$ integral uzoqlashuvchi bo‘ladi.

**5-misol.** Ushbu

$$\int_0^1\frac{\cos^2 x}{\sqrt[4]{1-x}}dx$$

integral yaqinlashuvchilikka tekshirilsin.

◄ Integral ostidagi funksiya

$$f(x)=\frac{\cos^2 x}{\sqrt[4]{1-x}}=\frac{\cos^2 x}{(1-x)^{\frac{1}{4}}}$$

bo‘lib, $\forall x\in[0,1)$ uchun

$$\varphi(x)=\cos^2 x\le 1\ , \qquad \alpha=\frac{1}{4}<1$$

bo‘ladi. YUqoridagi natijadan foydalanib berilgan integralning yaqinlashuvchi bo‘lishini topamiz. ►

**6-misol.** Ushbu

$$\int_0^1 \frac{xdx}{\sqrt{1-x^2}}$$

integral yaqinlashuvchilikka tekshirilsin.

◄ Ravshanki, quyidagi

$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$

xosmas integral yaqinlashuvchidir.

Endi ushbu

$$\lim_{x\to 1-0}\frac{\dfrac{x}{\sqrt{1-x^2}}}{\dfrac{1}{\sqrt{1-x}}}$$

limitni hisoblaymiz:

$$\lim_{x\to 1-0}\frac{\dfrac{x}{\sqrt{1-x^2}}}{\dfrac{1}{\sqrt{1-x}}}=\lim_{x\to 1-0} x\cdot\sqrt{\frac{1-x}{1-x^2}}=\frac{1}{\sqrt{2}}$$

Unda yuqoridagi natijaga ko‘ra berilgan xosmas integralning yaqinlashuvchi ekanini topamiz. ►

**7. Xosmas integralning absolyut yaqinlashuvchiligi.** Aytaylik, $f(x)$ funksiya $[a,b)$ da berilgan bo‘lib, $b$ nuqta shu funksiyaning maxsus nuqtasi bo‘lsin. (Bunda $\forall x\in[a,b)$ da $f(x)\ge 0$ bo‘lishi shart emas)

Ravshanki, ushbu

$$\int_a^b |f(x)|dx$$

integral manfiy bo‘lmagan funksiyaning xosmas integrali bo‘ladi.

**5-teorema.** Agar $\int_a^b |f(x)|dx$ integral yaqinlashuvchi bo‘lsa, u holda

$\int_a^b f(x)dx$ integral ham yaqinlashuvchi boʻladi.

**6-ta'rif.** Agar $\int_a^b |f(x)|dx$ integral yaqinlashuvchi boʻlsa, $\int_a^b f(x)dx$ absolyut yaqinlashuvchi integral deyiladi.

Agar $\int_a^b |f(x)|dx$ integral uzoqlashuvchi boʻlib, $\int_a^b f(x)dx$ yaqinlashuvchi boʻlsa, $\int_a^b f(x)dx$ shartli yaqinlashuvchi integral deyiladi.

**8. Xosmas integrallarni hisoblash**. Faraz qilaylik, $f(x)$ funksiya $[a,b)$ da uzluksiz boʻlib, uning boshlangʻich funksiyasi $F(x)$ $x\to b-0$ da chekli limitga ega boʻlsin:

$$\lim_{x\to b-0} F(x) = F(b).$$

U holda

$$\int_a^b f(x)dx = \lim_{x\to b-0}\int_a^t f(x)dx = \lim_{x\to b-0}[F(t)-F(a)] =$$

$$= F(b) - F(a) = F(x)\Big|_a^b$$

boʻladi.

Bu Nyuton-Leybnits formulasi deyiladi.

Aytaylik, $u(x)$ va $v(x)$ funksiyalari $[a,b)$ da berilgan va shu oraliqda uzluksiz $u'(x)$ va $v'(x)$ hosilalarga ega boʻlib, $b$ nuqta $v(x)\cdot u'(x)$ hamda $u(x)\cdot v'(x)$ funksiyalarning maxsus nuqtalari boʻlsin.

Agar:

1) $\int_a^b v(x)du(x)$ integral yaqinlashuvchi ;

2) Ushbu

$$\lim_{x\to b-0} u(t)\cdot v(t)$$

limiti mavjud va chekli boʻlsa, u holda $\int_a^b u(x)dv(x)$ integral yaqinlashuvchi va

$$\int_a^b u(x)dv(x) = u(x)v(x)\Big|_a^b - \int_a^b v(x)du(x) \qquad (3)$$

boʻladi, bunda

$$u(b)\cdot v(b)=\lim_{x\to b-0} u(t)\cdot v(t).$$

Quyidagi

$$\int_a^b f(x)dx$$

xosmas integralda ($b$-maxsus nuqta) $x=\varphi(z)$ almashtirish bajaramiz, bunda $\varphi(z)$ funksiya $[\alpha,\beta)$ oraliqda uzluksiz $\varphi'(z)>0$ hosilaga ega hamda

$$\varphi(\alpha)=a\ ,\quad \varphi(\beta)=\lim_{z\to\beta-0}\varphi(z)=b\,.$$

Agar

$$\int_\alpha^\beta f(\varphi(z))\varphi'(z)dz$$

integral yaqinlashuvchi bo‘lsa, u holda $\int_a^b f(x)dx$ integral ham yaqinlashuvchi bo‘lib,

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(z))\varphi'(z)dz$$

bo‘ladi.

**9. Chegaralanmagan funksiya xosmas integralining bosh qiymati.**

Faraz qilaylik, $f(x)$ funksiya $[a,b]\setminus\{c\}$ da berilgan bo‘lib, $c$ nuqta $(a<c<b)$ shu funksiyaning maxsus nuqtasi bo‘lsin.

Ma’lumki, ushbu

$$\lim_{\substack{\alpha\to 0\\ \alpha'\to 0}}\left[\int_a^{c-\alpha} f(x)dx+\int_{c+\alpha'}^{b} f(x)dx\right]$$

limit chegaralanmagan $f(x)$ funksiyaning xosmas integrali deyilib, u chekli bo‘lsa

$$\lim_{\substack{\alpha\to 0\\ \alpha'\to 0}}\left[\int_a^{c-\alpha} f(x)dx+\int_{c+\alpha'}^{b} f(x)dx\right]=\int_a^b f(x)dx$$

xosmas integral yaqinlashuvchi deyilar edi. Bunda $\alpha$ va $\alpha'$ o‘zgaruvchilar ixtiyoriy ravishda nolga intiladi deb qaraladi.

Xususan, $\int_a^b f(x)dx$ xosmas integral yaqinlashuvchi bo‘lsa,

$$\lim_{\alpha\to 0}\left[\int_a^{c-\alpha} f(x)dx+\int_{c+\alpha}^{b} f(x)dx\right]=\int_a^b f(x)dx$$

biroq,

$$F(\alpha,\alpha')=\int_a^{c-\alpha} f(x)dx+\int_{c+\alpha'}^{b} f(x)dx$$

funksiya, $\alpha=\alpha'$ bo‘lib, $\alpha\to 0$ da chekli limitga ega bo‘lishdan $\int_a^b f(x)dx$ xosmas integralning yaqinlashuvchi bo‘lishi kelib chiqavermaydi.

Masalan,

$$F(\alpha,\alpha')=\int_a^{c-\alpha}\frac{dx}{x-c}+\int_{c+\alpha'}^{b}\frac{dx}{x-c} \qquad (a<c<b)$$

uchun $\alpha=\alpha'$ bo‘lsa,

$$\lim_{\alpha\to 0}\left[\int_a^{c-\alpha}\frac{dx}{x-c}+\int_{c+\alpha}^{b}\frac{dx}{x-c}\right]=\ln\frac{b-c}{c-a}$$

bo‘ladi. Biroq,

$$F(\alpha,\alpha')=\int_a^{c-\alpha}\frac{dx}{x-c}+\int_{c+\alpha'}^{b}\frac{dx}{x-c}=\ln\frac{b-c}{c-a}+\ln\frac{\alpha}{\alpha'}$$

bo‘lib, $\alpha\to 0,\ \alpha'\to 0$ da uning limiti mavjud emas, ya’ni

$$\int_a^b\frac{dx}{x-c} \qquad (a<c<b)$$

xosmas integral yaqinlashuvchi emas.

**7-ta’rif.** Agar $\alpha=\alpha'$ bo‘lib, $\alpha\to 0$ da

$$F(\alpha,\alpha')=\int_a^{c-\alpha}f(x)dx+\int_{c+\alpha'}^{b}f(x)dx$$

funksiyaning limiti mavjud va chekli bo‘lsa, u holda $\int_a^b f(x)dx$ xosmas integral bosh qiymat ma’nosida yaqinlashuvchi deyilib,

$$\lim_{\alpha\to 0}\left[\int_a^{c-\alpha}f(x)dx+\int_{c+\alpha}^{b}f(x)dx\right]$$

limit esa $\int_a^b f(x)dx$ xosmas integralning bosh qiymati deyi-ladi. Uni

$$V.P.\int_a^b f(x)dx$$

kabi belgilanadi. Demak,

$$V.P.\int_a^b f(x)dx=\lim_{\alpha\to 0}\left[\int_a^{c-\alpha}f(x)dx+\int_{c+\alpha}^{b}f(x)dx\right]$$

**$2^0$. Xosmas integrallarni hisoblash**

**1. Nyuton-Leybnits formulasi.** Ushbu

$$\int_a^{+\infty}f(x)dx$$

xosmas integral yaqinlashuvchi bo‘lib, uni hisoblash talab etilsin.

Aytaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda boshlang‘ich $F(x)$ funksiyaga ega va $x \to +\infty$ da $F(x)$ funksiya chekli limiti mavjud bo‘lsin:

$$\lim_{x\to+\infty} F(x) = F(+\infty).$$

Unda

$$\int_a^{+\infty} f(x)dx = \lim_{x\to+\infty}\int_a^t f(x)dx = \\ = \lim_{t\to+\infty}(F(t)-F(a)) = F(+\infty)-F(a) = F(x)\Big|_a^{+\infty} \qquad (1)$$

bo‘ladi.

(1) formula Nyuton-Leybnits formulasi deyiladi.

**1-misol.** Ushbu,

$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^2}\sin\frac{1}{x}dx$$

integral hisoblansin.

◄ Ravshanki, $F(x)=\cos\frac{1}{x}$ funksiya $[\frac{2}{\pi},+\infty)$ oraliqda $f(x)=\frac{1}{x^2}\sin\frac{1}{x}$ funksiyaning boshlang‘ich funksiyasi bo‘ladi.

(1) formuladan foydalanib topamiz:

$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^2}\sin\frac{1}{x}dx = \cos\frac{1}{x}\Big|_{\frac{2}{\pi}}^{+\infty} = 1. \blacktriangleright$$

**2. Bo‘laklab integrallash.** Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda uzluksiz va uzluksiz, $f'(x)$ va $g'(x)$ hosilalarga ega bo‘lsin.

Agar

1) $\int_a^{+\infty} f(x)\cdot g'(x)dx$ $\left(\int_a^{+\infty} f'(x)g(x)dx\right)$ integral yaqinlashuvchi;

2) ushbu $\lim_{x\to+\infty}(f(x)g(x))$ limit mavjud va chekli bo‘lsa, u holda

$$\int_a^{+\infty} f'(x)\cdot g(x)dx \qquad \left(\int_a^{+\infty} f(x)g'(x)dx\right)$$

integral yaqinlashuvchi bo‘lib,

$$\int_a^{+\infty} f'(x)\cdot g(x)dx = \lim_{x\to+\infty}(f(x)g(x)) - f(a)\cdot g(a) - \int_a^{+\infty} f(x)g'(x)dx \qquad (2)$$

$$\left(\int_a^{+\infty} f(x)g'(x)dx = \lim_{x\to+\infty}(f(x)g(x)) - f(a)\cdot g(a) - \int_a^{+\infty} f'(x)g(x)dx\right)$$

bo‘ladi.

(2) formula bo‘laklab integrallash formulasi deyiladi.

**2-misol .** Ushbu

$$\int_0^{+\infty} xe^{-x}dx$$

integral hisoblansin.

◄ Agar $g(x)=x$, $f'(x)=e^{-x}$ deb olsak, unda

$$g'(x)=1, \quad f(x)=-e^{-x}$$

bo‘lib, (2) formulaga ko‘ra $(a=0)$

$$\int_0^{+\infty} xe^{-x}dx=\lim_{t+\infty}(-te^{-t})-0+\int_0^{+\infty} e^{-x}dx=1$$

bo‘ladi. ►

**3. O‘zgaruvchilarni almashtirib integrallash.**

Ushbu

$$\int_a^{+\infty} f(x)dx$$

xosmas integralni qaraymiz. Bu integralda $x=\varphi(z)$ almash-tirishni bajaramiz. Bunda $x=\varphi(z)$ funksiya quyidagi shart-larni qanoatlantirsin:

1) $\varphi(z)$ funksiya $[\alpha,+\infty)$ oraliqda uzluksiz va uzluksiz $\varphi'(z)$ hosilaga ega;

2) $\varphi(z)$ funksiya $[\alpha,+\infty)$ da qat’iy o‘suvchi;

3) $\varphi(\alpha)=a$, $\varphi(+\infty)=\lim\limits_{z\to+\infty}\varphi(z)=+\infty$.

Agar

$$\int_\alpha^{+\infty} f(\varphi(z))\cdot\varphi'(z)dz$$

xosmas integral yaqinlashuvchi bo‘lsa, u holda

$$\int_a^{+\infty} f(x)dx$$

integral ham yaqinlashuvchi bo‘lib,

$$\int_a^{+\infty} f(x)dx=\int_\alpha^{+\infty} f(\varphi(z))\cdot\varphi'(z)dz$$

bo‘ladi.

◄ Ixtiyoriy $z(\alpha<z<+\infty)$ ni olib, unga mos $\varphi(z)=t$ nuqta-ni topamiz.

Ravshanki, yuqoridagi shartlarda $[a,t)$ da 37-ma’ruzadagi (2) formulaga ko‘ra

$$\int_a^t f(x)dx = \int_\alpha^z f(\varphi(z))\cdot\varphi'(z)$$

bo‘ladi.

Keyingi tenglikda $t\to+\infty$ da (bunda $z=\varphi^{-1}(t)\to+\infty$) limit-ga o‘tib topamiz:

$$\int_a^{+\infty} f(x)dx = \int_\alpha^{+\infty} f(\varphi(z))\cdot\varphi'(z)dz$$

Bu esa keltirilgan tasdiqni isbotlaydi. ►

**3-misol.** Ushbu

$$J=\int_0^{+\infty}\frac{dx}{1+x^4}$$

integral hisoblansin.

◄ Bu integralda $x=\frac{1}{t}$ almashtirishni bajaramiz. Natijada

$$J=\int_{+\infty}^{0}\frac{1}{1+\frac{1}{t^4}}\left(-\frac{1}{t^2}\right)dt=\int_0^{+\infty}\frac{t^2dt}{1+t^4}$$

bo‘lib,

$$J=\frac{1}{2}\int_0^{+\infty}\frac{1+x^2}{1+x^4}dx$$

bo‘lishi kelib chiqadi.

Keyingi integralda $x-\frac{1}{x}=z$ deb, topamiz:

$$J=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{dz}{2+z^2}=\frac{1}{2\sqrt{2}}arctg\frac{z}{\sqrt{2}}\Big|_{-\infty}^{+\infty}=\frac{\pi}{2\sqrt{2}}.$$

Demak,

$$\int_0^{+\infty}\frac{dx}{1+x^4}=\frac{\pi}{2\sqrt{2}}.$$ ►

**4.Xosmas integrallarni taqribiy hisoblash.**

Aytaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda uzluksiz bo‘lib, ushbu

$$\int_a^{+\infty} f(x)dx$$

xosmas integral yaqinlashuvchi bo‘lsin. Ta’rifga binoan

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx \ ,$$

ya'ni

$$\forall \varepsilon > 0 \ , \ \exists t_0 > a \ , \ \forall t > t_0 \ :$$

$$\left| \int_a^{+\infty} f(x)dx - \int_a^t f(x)dx \right| < \varepsilon$$

bo'ladi.

Ravshanki,

$$\int_a^{+\infty} f(x)dx - \int_a^t f(x) = \int_t^{+\infty} f(x)dx \ .$$

Demak,

$$\left| \int_t^{+\infty} f(x)dx \right| < \varepsilon \ .$$

Natijada ushbu

$$\int_a^{+\infty} f(x)dx \approx \int_a^t f(x)dx \qquad (5)$$

taqribiy formulaga kelamiz. Uning xatoligi

$$\left| \int_t^{+\infty} f(x)dx \right| < \varepsilon$$

bo'ladi.

**4-misol.** Ushbu

$$\int_0^{+\infty} e^{-x^2} dx$$

xosmas integral taqribiy hisoblansin.

◀ (5) formulaga ko'ra, berilgan integralni taqribiy hisoblash uchun ushbu

$$\int_0^{+\infty} e^{-x^2} dx \approx \int_0^a e^{-x^2} dx \qquad (a > 0)$$

formulani hosil qilamiz. Uning hatoligi

$$\int_a^{+\infty} e^{-x^2} dx$$

ga teng bo'ladi. Bu hatolikni yuqoridan baholaymiz:

$$\int_a^{+\infty} e^{-x^2} dx \le \frac{1}{a} \int_a^{+\infty} x e^{-x^2} dx = \frac{1}{2a} \int_a^{+\infty} e^{-x^2} d(x^2) = \frac{1}{2a} (-e^{-x^2})\Big|_a^{+\infty} = \frac{1}{2a} e^{-a^2} .$$

Aytaylik, $a = 1$ bo'lsin. Bu holda

$$\int_a^{+\infty} e^{-x^2}dx \approx \int_0^1 e^{-x^2}dx$$

bo‘lib, bu taqribiy formulaning xatoligi uchun

$$\int_1^{+\infty} e^{-x^2}dx \le 0{,}1839$$

bo‘ladi.

Aytaylik, $a=2$ bo‘lsin. Bu holda

$$\int_a^{+\infty} e^{-x^2}dx \approx \int_0^2 e^{-x^2}dx$$

bo‘lib, bu taqribiy formulaning xatoligi uchun

$$\int_2^{+\infty} e^{-x^2}dx \le 0{,}00458$$

bo‘ladi.

Aytaylik, $a=3$ bo‘lsin . Bu holda

$$\int_a^{+\infty} e^{-x^2}dx \approx \int_0^3 e^{-x^2}dx$$

bo‘lib, bu taqribiy formulaning xatoligi uchun

$$\int_3^{+\infty} e^{-x^2}dx \le 0{,}00002$$

bo‘ladi. ►

**Mashqlar**

1. Ushbu

$$\int_0^{+\infty} \frac{\ln x}{1+x^2}dx$$

integral hisoblansin.

2. Ushbu

$$\int_0^{+\infty} \frac{dx}{(x^2-a^2)\sqrt{x^2-1}} = \frac{\arcsin a}{a\sqrt{1-a^2}} \qquad (a>0)$$

tenglik isbotlansin.

3. Ushbu

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

integral hisoblansin.

4. Ushbu

$$\int_0^{\frac{\pi}{2}} \ln \sin x dx$$

integralning yaqinlashuvchiligi isbotlansin, qiymati topilsin.

5. Ushbu integral

$$\int_0^1 \frac{dx}{\sqrt{x} + arctgx}$$

yaqinlashuvchilikka tekshirilsin.

**Adabiyotlar**


1. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'rizalar, I q.* T. "Voris-nashriyot", 2010.
2. **Fixtengols G. M.** *Kurs differensialnogo i integralnogo ischisleniya, 1 t.* M. «FIZMATLIT», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.


# Glossariy

**Maxsus nuqta -** Agar $f(x)$ funksiya

$$X \bigcap \dot{U}_{\delta}(x_0) \neq \varnothing$$

to'plamda chegaralanmagan bo'lsa, $x_0$ nuqta $f(x)$ funksiyaning maxsus nuqtasi deyiladi.

**Chegaralanmagan funksiyaning xosmas integrali -** Agar $t \to b-0$ da $F(t)$ funksiyaning limiti mavjud bo'lsa, bu limit chegaralanmagan $f(x)$ funksiyaning $[a,b)$ bo'yicha xosmas integrali deyiladi va

$$\int_a^b f(x)dx$$

kabi belgilanadi.

Chegaralanmagan $f(x)$ funksiyaning $(a,b)$ bo'yicha xosmas integrali - Agar $t' \to a+0$, $t \to c-0$ hamda $u' \to c+0$, $u \to b-0$ da $\varphi(t',t)+\psi(u',u)$

funksiyaning limiti

$$\lim_{\substack{t'\to a+0\\ t\to c-0\\ u'\to c+0\\ u\to b-0}} [\varphi(t',t)+\psi(u',u)] = \lim_{\substack{t'\to a+0\\ t\to c-0\\ u'\to c+0\\ u\to b-0}} [\int_{t'}^{t} f(x)dx + \int_{u'}^{u} f(x)dx]$$

mavjud bo‘lsa, bu limit chegaralanmagan $f(x)$ funksiyaning $(a,b)$ bo‘yicha xosmas integrali deyiladi va

$$\int_a^b f(x)dx$$

kabi belgilanadi.

## Keys banki

**43-keys.** Masala o`rtaga tashlanadi: Ushbu

$$\int_0^{+\infty} \frac{dx}{(x^2-a^2)\sqrt{x^2-1}} = \frac{\arcsin a}{a\sqrt{1-a^2}}, \quad (a>0)$$

tenglik isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 19-20-amaliy mashg'ulot

**1-misol.** Ushbu

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

integral yaqinlashuvchilikka tekshirilsin.

◀ Ravshanki, $x_0 = 0$ nuqta $f(x) = \frac{1}{\sqrt{x}}$ funksiyaning maxsus nuqtasi. Demak, qaralayotgan integral chegaralanmagan funksiyaning xosmas integrali bo'ladi.

Ta'rifga binoan

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to+0} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to+0} 2(1-\sqrt{t}) = 2$$

bo'ladi. Demak, berilgan xosmas integral yaqinlashuvchi va u 2 ga teng. ►

**2-misol.** Ushbu

$$\int_0^1 \frac{dx}{x}$$

xosmas integral uzoqlashuvchi bo'ladi, chunki

$$\lim_{t\to+0} \int_t^1 \frac{dx}{x} = \lim_{t\to+0} (\ln x)_t^1 = +\infty .$$

**5-masala. Quyidagi xosmas integrallar absolut va shartli yaqinlashishga tekshirilsin.**

**5.1** $\int_0^1 \frac{1}{x\sqrt{x}} \cos\frac{\sqrt{x}-1}{\sqrt{x}} dx.$

**5.2** $\int_0^{\pi/4} \sin\left(\frac{1}{\sin x}\right) \cdot \frac{dx}{\sin^{\alpha} x}.$

**5.3** $\int_0^{0.5} \frac{\cos^3(\ln x)}{x \ln x} dx.$

**5.4** $\int_{-1}^{1} \sin\frac{1+x}{1-x} \cdot \frac{dx}{(1-x^2)^\alpha}.$

**5.5** $\int_0^1 (1-x)^\alpha \sin\frac{\pi}{1-x} dx.$

**5.6** $\int_0^1 \frac{x^\alpha}{e^x-1} \sin\frac{1}{x} dx.$

**5.7** $\int_0^1 \frac{x^\alpha}{x^2+1} \sin\frac{1}{x} dx.$

**5.8** $\int_0^1 \frac{\sin x^\alpha}{x^2} dx.$

**5.9** $\int_0^1 \cos\left(\frac{1}{\sqrt{x}}-1\right) \frac{dx}{x^\alpha}.$

**5.10** $\int_0^1 \frac{\sin\frac{1}{x}}{(\sqrt{x}-x)^\alpha} dx.$

**5.11** $\int_0^{0.5} \left(\frac{x}{1-x}\right)^\alpha \cdot \cos\frac{1}{x^2} dx.$

**5.12** $\int_0^1 \frac{(1-x)^\alpha}{x} \sin\frac{1}{x} dx.$

**5.13** $\int_1^{+\infty} \frac{\sin x}{x^\alpha} dx.$

**5.14** $\int_1^{+\infty} \frac{x^\alpha \cdot \sin x}{x^3+1} dx.$

**5.15** $\int_2^{+\infty} \frac{(x+1)^\alpha \cdot \sin x}{\ln x} dx.$

**5.16** $\int_2^{+\infty} \frac{\cos x dx}{x^\alpha + \ln x}.$

**5.17** $\int_1^{+\infty} \frac{\sin(\ln x)}{x^\alpha} \cdot \sin x dx.$

**5.18** $\int_2^{+\infty} \frac{\cos\sqrt{x}}{x^\alpha \cdot \ln x} dx.$

**5.19** $\int_0^{+\infty} \frac{x^\alpha \cdot \sin x}{1+x^\beta} dx, \beta \geq 0.$

**5.20** $\int_0^{+\infty} x^\alpha \cdot \sin x^\beta dx.$

**5.21** $\int_0^1 \sin\left(\frac{1}{1-x}\right) \cdot \frac{dx}{1-x}.$

# Test

1. Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti mavjud bo‘lsa, bu limiti $f(x)$ funksiyaning $[a,+\infty)$ cheksiz oraliq bo‘yicha ……………deyiladi.

A) xosmas integrali
B) Aniq integral
C) aniqmas integral
D) to’g’ri javob yo’q

2.Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti mavjud va chekli bo‘lsa,

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty} \int_a^t f(x)dx$. integral ………………. deyiladi.

A) yaqinlashuvchi
B) uzoqlashuvchi
C) yaqinlashuvchi yoki uzoqlashuvchi
D) tog’ri javob A va B

3. Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti cheksiz yoki mavjud bo‘lmasa,

$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty} \int_a^t f(x)dx$. integral ……. deyiladi.

A) uzoqlashuvchi
B) yaqinlashuvchi yoki uzoqlashuvchi
C) yaqinlashuvchi
D) tog’ri javob A va B

4. Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} C \cdot f(x)dx$ ham $(C = const)$ yaqinlashuvchi bo‘lib,………..bo‘ladi.

A) $\int_a^{+\infty} C \cdot f(x)dx = C\int_a^{+\infty} f(x)dx$

B) $\int_a^{+\infty} f(x)dx \geq 0$

C) $\int_a^{+\infty} (f(x) \pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$

D) $\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx$

5. Agar $\forall x \in [a,+\infty)$ da $f(x) \le g(x)$ bo‘lib, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda ………..bo‘ladi.

A) $\int_a^{+\infty} f(x)dx \le \int_a^{+\infty} g(x)dx$

B) $\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx$

C) $\int_a^{+\infty} (f(x) \pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$

D) $\int_a^{+\infty} f(x)dx \ge 0$

6. Hisoblang: $\int_0^{\frac{1}{2}} \frac{dx}{x \ln^2 x}$

A) $\frac{1}{\ln 2}$

B) $\frac{1}{\ln 3}$

C) 2

D) 1

7. Hisoblang: $\int_0^3 \frac{x^2 dx}{\sqrt{9 - x^2}}$

A) $\frac{9\pi}{4}$

B) $\frac{9\pi}{2}$

C) $\frac{6\pi}{4}$

D) 1

8. Hisoblang: $\int_0^1 \ln x dx$

A) -1
B) -2
C) 2
D) 1

9. Hisoblang: $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}$

A) $\frac{\pi}{2}$
B) $\frac{3\pi}{2}$
C) $\frac{6\pi}{4}$
D) $\frac{\pi}{4}$

10. Hisoblang: $\int_2^3 \frac{2-x}{\sqrt{3-x}}dx$

A) $-\frac{4}{3}$
B) $\frac{4}{3}$
C) $-\frac{3}{2}$
D) 2

# Mavzu. Xosmas integralining yaqinlashish a`lomatlari va bosh qiymati

## 21-22-ma’ruzalar

**Reja**

**1⁰.** Dirixle alomati.

**2⁰.** Abel alomati.

**3⁰.** Xosmas integralning bosh qiymati.

### 1⁰. Dirixle alomati.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo‘lsin.

**1-teorema** (**Dirixle alomati**). $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a,+\infty)$ da uzluksiz va uning shu oraliqdagi boshlang‘ich $F(x)$ $(F'(x)=f(x))$ funksiyasi chegara-langan;

2) $g(x)$ funksiya $[a,+\infty)$ da uzluksiz $g'(x)$ hosilaga ega ;

3) $g(x)$ funksiya $[a,+\infty)$ da kamayuvchi;

4) $\lim\limits_{x\to+\infty} g(x)=0$.

U holda

$$\int_a^{+\infty} f(x)g(x)dx$$

integral yaqinlashuvchi bo‘ladi.

**Misol.** Ushbu

$$J=\int_1^{+\infty}\frac{\sin x}{x^\alpha}dx \qquad (\alpha>0)$$

integralni yaqinlashuvchilikka tekshirilsin.

**Yechilishi.** Berilgan integralni quyidagicha

$$J=\int_1^{+\infty}\sin x\frac{1}{x^\alpha}dx \qquad (\alpha>0)$$

yozib, $f(x)=\sin x$, $g(x)=\frac{1}{x^{\alpha}}$ deymiz. Bu funksiyalar yuqorida keltirilgan teoremaning barcha shartlarini qanoatlantiradi.

1) $f(x)=\sin x$ funksiya $[1,+\infty)$ oraliqda uzluksiz va uning boshlang‘ich funksiyasi $F(x)=-\cos x$ funksiya $[1,+\infty)$ da chegaralangan;

2) $g(x)=\frac{1}{x^{\alpha}}$ $(\alpha>0)$ funksiya $[1,+\infty)$ da

$$g'(x)=-\frac{\alpha}{x^{\alpha+1}}$$

hosilaga ega va u uzluksiz;

3) $g(x)=\frac{1}{x^{\alpha}}$ $(\alpha>0)$ funksiya $[1,+\infty)$ da kamayuvchi;

4) $\lim\limits_{x\to+\infty} g(x)=\lim\limits_{x\to+\infty}\frac{1}{x^{\alpha}}=0$. $(\alpha>0)$

Unda Dirixle alomatiga ko‘ra

$$\int_1^{+\infty}\frac{\sin x}{x^{\alpha}}dx \qquad (\alpha>0)$$

integral yaqinlashuvchi bo‘ladi. ►

## $2^0$. Abel alomati.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo‘lsin.

**2-teorema (Abel alomati).** $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a,+\infty)$ da uzluksiz bo‘lib, $\int_a^{+\infty} f(x)\,dx$ integral yaqinlashuvchi;

2) $g(x)$ funksiya $[a,+\infty)$ da uzluksiz $g'(x)$ hosilaga ega va bu hosila $[a,+\infty)$ da o‘z ishorasini saqlasin;

3) $g(x)$ funksiya $[a,+\infty)$ da chegaralangan.

U holda

$$\int_a^{+\infty} f(x)g(x)\,dx$$

integral yaqinlashuvchi bo‘ladi.

◄ Ravshanki, $\int_a^{+\infty} f(x)\,dx$ integralning yaqinlashuvchi bo‘li-shidan $f(x)$

funksiyaning $[a,+\infty)$ oraliqda chegaralangan $F(x)$ boshlang‘ich funksiyaga ega bo‘lishi kelib chiqadi.

Teoremaning 2)- va 3)- shartlaridan hamda monoton funksiyaning limiti haqidagi teoremadan foydalanib ushbu

$$\lim_{x\to+\infty} g(x)$$

limitning mavjud va chekli bo‘lishini topamiz:

$$\lim_{x\to+\infty} g(x)=b.$$

Unda

$$g_1(x)=g(x)-b$$

funksiya $x\to+\infty$ da monoton ravishda nolga intiladi:

$$\lim_{x\to+\infty} g_1(x)=0.$$

SHunday qilib $f(x)$ va $g_1(x)$ funksiyalari Dirixle alomati keltirilgan barcha shartlarni qanoatlantiradi. Dirixle alomatiga ko‘ra

$$\int_a^{+\infty} f(x)g_1(x)dx$$

integral yaqinlashuvchi bo‘ladi.

Ayni paytda,

$$f(x)g(x)=f(x)b+f(x)g_1(x)$$

bo‘lganligi sababli,

$$\int_a^{+\infty} f(x)g(x)dx$$

integral ham yaqinlashuvchi bo‘ladi. ►

**$3^0$.Xosmas integralning bosh qiymati.**

Faraz qilaylik, $f(x)$ funksiya $(-\infty,+\infty)$ da berilgan bo‘lib, bu oraliqning istalgan $[t',t]$ $(-\infty<t'<t<+\infty)$ qismida integrallanuvchi bo‘lsin:

$$F(t',t)=\int_{t'}^{t} f(x)dx.$$

Ma’lumki, ushbu

$$\lim_{\substack{t'\to-\infty,\\ t\to+\infty}} F(t',t)=\lim_{\substack{t'\to-\infty\\ t\to+\infty}}\int_{t'}^{t} f(x)dx$$

limit $f(x)$ funksiyaning $(-\infty,+\infty)$ oraliq bo‘yicha xosmas integrali deyilib, u chekli bo‘lsa,

$$\lim_{\substack{t'\to-\infty\\ t\to+\infty}}\int_{t'}^{t} f(x)dx=\int_{-\infty}^{+\infty} f(x)dx$$

xosmas integral yaqinlashuvchi deyilar edi.

Bunda $t'$ va $t$ o‘zgaruvchilarning ixtiyoriy ravishda

$$t'\to-\infty,\quad t\to+\infty$$

ga intilishi ko‘zda tutiladi.

Xususan, $\int\limits_{-\infty}^{+\infty} f(x)dx$ xosmas integral yaqinlashuvchi bo‘lsa,

$$\lim_{t\to+\infty}\int\limits_{-t}^{t} f(x)\,dx=\int\limits_{-\infty}^{\infty} f(x)dx$$

bo‘ladi.

Biroq

$$F(t',t)=\int\limits_{t'}^{t} f(x)dx$$

funksiya, $t'=-t$ bo‘lib, $t\to+\infty$ da chekli limitga ega bo‘lishidan $\int\limits_{-\infty}^{+\infty} f(x)dx$ xosmas integralning yaqinlashuvchi bo‘lishi kelib chiqavermaydi.

Masalan, ushbu

$$F(t',t)=\int\limits_{t'}^{t}\sin x\,dx$$

integrel uchun $t'=-t$ bo‘lsa,

$$\int\limits_{-t}^{t}\sin x\,dx=0\qquad(\forall t>0)$$

bo‘lib,

$$\lim_{t\to+\infty}\int\limits_{-t}^{t}\sin x\,dx=0$$

bo‘ladi. Biroq

$$\int\limits_{-\infty}^{+\infty}\sin x\,dx$$

xosmas integral yaqinlashuvchi emas.

**Ta’rif**. Agar $t'=-t$ bo‘lib, $t\to+\infty$ da

$$F(t',t)=\int\limits_{-t'}^{t} f(x)dx$$

funksiyaning limiti mavjud va chekli bo‘lsa, $\int\limits_{-\infty}^{+\infty} f(x)dx$ xosmas integral bosh qiymat ma’nosida yaqinlashuvchi deyilib,

$$\lim_{t\to+\infty}\int\limits_{-t}^{t} f(x)\,dx$$

limit esa $\int_{-\infty}^{+\infty} f(x)\,dx$ xosmas integralning bosh qiymati deb ataladi. Odatda, $\int_{-\infty}^{+\infty} f(x)\,dx$ xosmas integralning bosh qiymati

$$v.p.\int_{-\infty}^{+\infty} f(x)\,dx$$

kabi belgilanadi.

Demak,

$$v.p.\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_{-t}^{t} f(x)\,dx.$$

Bunda $v.p$ belgi fransuzcha "*valeur principiale*"- "bosh qiymat" so‘zlarining dastlabki harflarini ifodalaydi.

Shunday qilib, $\int_{-\infty}^{+\infty} f(x)\,dx$ xosmas integral yaqinlashuvchi bo‘lsa, u bosh qiymat ma’nosida ham yaqinlashuvchi bo‘ladi. Biroq, $\int_{-\infty}^{+\infty} f(x)\,dx$ xosmas integralning bosh qiymat ma’nosida yaqinlashuvchi bo‘lishidan uning yaqinlashuvchi bo‘lishi har doim ham kelib chiqavermaydi.

## Xosmas integrallarning umumiy holi

**$1^0$. Chegaralanmagan funksiyaning cheksiz oraliq bo`yicha xosmas integrali tushunchasi.** Faraz qilaylik, $f(x)$ funksiya $(a,+\infty)$ oraliqda berilgan bo`lib, $a$ nuqta uning maxsus nuqta-si bo`lsin.

Ayni paytda, bu funksiya istalgan chekli $[t,\tau]$ $(a<t<\tau<+\infty)$ oraliqda integrallanuvchi, ya`ni

$$\int_{t}^{\tau} f(x)dx = F(t,\tau)$$

integral mavjud bo`lsin.

Ma`lumki, $t\to a+0$ da $F(t,\tau)$ funkiyaning limiti mavjud bo`lsa, uni chegaralanmagan funksiyaning xosmas integrali deyilib,

$$\int_a^{\tau} f(x)dx$$

kabi belgilanar edi:

$$\int_a^{\tau} f(x)dx = \lim_{t\to a+0} F(t,\tau) = \lim_{t\to a+0}\int_t^{\tau} f(x)dx. \qquad (1)$$

Aytaylik, $f(x)$ funksiyaning $(a,\tau]$ oraliq bo`yicha xosmas integrali $\int_a^{\tau} f(x)dx$ mavjud bo`lsin. Ravshanki, bu integral $\tau$ ga bog`liq bo`ladi.

Agar $\tau \to +\infty$ da $\int_a^{\tau} f(x)dx$ ning limiti mavjud bo`lsa, bu limit $f(x)$ funksiyaning $(a,+\infty)$ oraliq bo`yicha xosmas integ-rali deyilib, uni $\int_a^{+\infty} f(x)dx$ kabi belgilanar edi:

$$\int_a^{+\infty} f(x)dx = \lim_{\tau\to+\infty}\int_a^{\tau} f(x)dx. \qquad (2)$$

(1) va (2) munosabatlardan topamiz:

$$\int_a^{+\infty} f(x)dx = \lim_{\tau\to+\infty}\lim_{t\to a+0}\int_t^{\tau} f(x)dx. \qquad (3)$$

Bu (3) munosabat chegaralanmagan funksiyaning cheksiz oraliq bo`yicha xosmas integralini ifodalaydi.

**$2^0$. $\int_0^{+\infty} x^{a-1}e^{-x}dx$ integral va uning yaqinlashuvchanligi**. Ravshanki, bu integral $a$ ga bog`liq. $a<1$ bo`lganda, $x=0$ nuqta integral ostidagi funksiyaning maxsus nuqtasi bo`ladi. Demak,

$$\int_0^{+\infty} x^{a-1}e^{-x}\,dx$$

integral chegaralanmagan funksiyaning cheksiz oraliq bo`yicha xosmas integrali.

Qaralayotgan integralni quyidagicha

$$\int_0^{+\infty} x^{a-1}e^{-x}dx = \int_0^1 x^{a-1}e^{-x}dx + \int_1^{+\infty} x^{a-1}e^{-x}dx$$

yozib, tenglikning o`ng tomonidagi integrallarning har birini alohida-alohida yaqinlashuvchilikka tekshiramiz.

Ushbu

$$\int_0^1 x^{a-1}e^{-x}dx$$

integralda, integral ostidagi funksiya uchun

$$\frac{1}{e}\frac{1}{x^{1-a}} \le x^{a-1}e^{-x} \le \frac{1}{x^{1-a}} \qquad (0 < x \le 1)$$

tengsizliklar o`rinli bo`ladi.

Ma`lumki,

$$\int_0^1 \frac{dx}{x^{1-a}}$$

integral $1-a<1$, ya`ni $a>0$ bo`lganda yaqinlashuvchi, $1-a \ge 1$, ya`ni $a \le 0$ bo`lganda uzoqlashuvchi.

Demak,

$$\int_0^1 x^{a-1}e^{-x}dx \quad \text{va} \quad \int_0^1 \frac{dx}{x^{1-a}}$$

integrallarda

$$x^{a-1}e^{-x} \le \frac{1}{x^{1-a}}$$

bo`lib, $a>0$ bo`lganda $\int_0^1 \frac{dx}{x^{1-a}}$ integral yaqinlashuvchi. Unda taqqoslash haqidagi teoremaga ko`ra $a>0$ bo`lganda

$$\int_0^1 x^{a-1}e^{-x}\,dx$$

integral yaqinlashuvchi bo`ladi.

Endi

$$\int_{1}^{+\infty} x^{a-1} e^{-x}\, dx$$

integralni yaqinlashuvchilikka tekshiramiz.

Ushbu

$$\int_{1}^{+\infty} x^{a-1} e^{-x} dx \quad \text{va} \quad \int_{1}^{+\infty} \frac{1}{x^2} dx$$

integrallarni qaraylik. Ravshanki, $\int_{1}^{+\infty} \frac{1}{x^2} dx$ integral yaqinla-shuvchi. Ayni paytda,

$$\lim_{x \to +\infty} \frac{x^{a-1} e^{-x}}{\frac{1}{x^2}} = \lim_{x \to +\infty} \frac{x^{a+1}}{e^x} = 0$$

bo`lganligi sababli, 44-ma`ruzada keltirilgan tasdiqqa ko`ra

$$\int_{0}^{+\infty} x^{a-1} e^{-x} dx$$

integral $a$ ning ixtiyoriy qiymatlarida yaqinlashuvchi bo`ladi.

Demak, berilgan integral

$$\int_{0}^{+\infty} x^{a-1} e^{-x} dx$$

$a > 0$ bo`lganda yakinlashuvchi bo`ladi.

**$3^0$. $\int_{0}^{1} x^{a-1} (1-x)^{b-1} dx$ integral va uning yaqinlashuvchiligi**.

Bu integraldagi integral ostidagi funksiya uchun:

а) $a < 1, \quad b \geq 1$ bo`lganda $x = 0$ maxsus nuqta,

б) $a \geq 1, \quad b < 1$ bo`lganda $x = 1$ maxsus nuqta,

в) $a < 1, \quad b < 1$ bo`lganda $x = 0$, $x = 1$ nuqtalar maxsus nuqtalar bo`ladi.

Berilgan integralni quyidagicha yozib olamiz:

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = \int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx + \int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}dx.$$

Ravshanki,

$$\lim_{x\to 0}\frac{x^{a-1}(1-x)^{b-1}}{x^{a-1}} = 1,$$

$$\lim_{x\to 1}\frac{x^{a-1}(1-x)^{b-1}}{(1-x)^{a-1}} = 1.$$

Unda 47- ma`ruzada keltirilgan tasdiqga ko`ra

$$\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx \ \text{ bilan } \ \int_0^{\frac{1}{2}} x^{a-1}dx,$$

$$\int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{b-1}dx \ \text{ bilan } \ \int_{\frac{1}{2}}^1 (1-x)^{b-1}dx$$

integrallar bir vaqtda yoki yaqinlashadi, yoki uzoqlashadi.

Ma`lumki, $a > 0$ bo`lganda

$$\int_0^{\frac{1}{2}} x^{a-1}dx$$

integral yaqinlashuvchi, $b > 0$ bo`lganda

$$\int_{\frac{1}{2}}^1 (1-x)^{b-1}dx$$

integral yaqinlashuvchi bo`ladi.

Demak, $a > 0$ bo`lganda

$$\int_0^{\frac{1}{2}} x^{a-1}(1-x)^{b-1}dx$$

integral yaqinlashuvchi bo`ladi, $b > 0$ bo`lganda

$$\int_{\frac{1}{2}}^{1} x^{a-1}(1-x)^{b-1}dx$$

integral yaqinlashuvchi bo`ladi.

Shunday qilib berilgan

$$\int_{0}^{1} x^{a-1}(1-x)^{b-1}dx$$

xosmas integral $a > 0$ va $b > 0$ bo`lganda yaqinlashuvchi bo`ladi.

## Mashqlar

1. Dirixle alomatidan foydalanib

$$\int_{0}^{+\infty} \frac{\sin\left(x + x^2\right)}{x^{\alpha}}dx \qquad (\alpha > 0)$$

integralni yaqinlashuvchi bo‘lishi isbotlansin.

2. Ushbu integral

$$\int_{0}^{+\infty} \frac{\sin x}{x^{\alpha}} arctgxdx \qquad (\alpha > 0)$$

yaqinlashuvchilikka tekshirilsin.

## Adabiyotlar


1. **Xudayberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’rizalar, I q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Kurs differensialnogo i integralnogo ischisleniya, 1 t.* M. «FIZMATLIT», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.

# Glossariy

**1-teorema (Dirixle alomati).** $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a,+\infty)$ da uzluksiz va uning shu oraliqdagi boshlang‘ich $F(x)$ $(F'(x)=f(x))$ funksiyasi chegara-langan;
2) $g(x)$ funksiya $[a,+\infty)$ da uzluksiz $g'(x)$ hosilaga ega ;
3) $g(x)$ funksiya $[a,+\infty)$ da kamayuvchi;
4) $\lim\limits_{x\to+\infty} g(x)=0$.

U holda

$$\int_a^{+\infty} f(x)g(x)dx$$

integral yaqinlashuvchi bo‘ladi.

**2-teorema (Abel alomati).** $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a,+\infty)$ da uzluksiz bo‘lib, $\int_a^{+\infty} f(x)\,dx$ integral yaqinlashuvchi;

2) $g(x)$ funksiya $[a,+\infty)$ da uzluksiz $g'(x)$ hosilaga ega va bu hosila $[a,+\infty)$ da o‘z ishorasini saqlasin;

3) $g(x)$ funksiya $[a,+\infty)$ da chegaralangan.

U holda

$$\int_a^{+\infty} f(x)g(x)\,dx$$

integral yaqinlashuvchi bo‘ladi.

# Keys banki

**44-keys.** Masala o`rtaga tashlanadi: Ushbu

$$\int_0^{+\infty} \frac{\sin x}{x^{\alpha}} arctgx dx, \ \ (\alpha > 0)$$

integral yaqinlashuvchilikka tekshirilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 21-22-amaliy mashg'ulot

**Xosmas integralning Koshi ma`nosidagi bosh qiymati topilsin:**

$$V.p.\int_0^{+\infty}\frac{dx}{x^2-3x+2}.$$

$$\triangleleft\ V.p\int_0^{+\infty}\frac{dx}{x^2-3x+2}=V.p\int_0^{+\infty}\frac{dx}{(x-1)(x-2)}=V.p\int_0^{1,5}\frac{dx}{(x-1)(x-2)}+V.p\int_{1,5}^{3}\frac{dx}{(x-1)(x-2)}+$$

$$+\int_3^{+\infty}\frac{dx}{(x-1)(x-2)}=\lim_{\alpha\to+0}\left[\int_0^{1-\alpha}\frac{dx}{(x-1)(x-2)}+\int_{1+\varepsilon}^{1,5}\frac{dx}{(x-1)(x-2)}\right]+\lim_{\beta\to+0}\left[\int_{1,5}^{2-\beta}\frac{dx}{(x-1)(x-2)}+\int_{2+\beta}^{3}\frac{dx}{(x-1)(x-2)}\right]+$$

$$+\lim_{A\to\infty}\int_3^{A}\frac{dx}{(x-1)(x-2)}$$

Agar $\int\frac{dx}{(x-1)(x-2)}=\int\left[\frac{1}{x-2}-\frac{1}{x-1}\right]dx=\ln|x-2|-\ln|x-1|$ ekanligidan foydalanib, yuqoridagi limitlarni hisoblasak, $V.p\int_0^{+\infty}\frac{dx}{x^2-3x+2}=-\ln 2$ tenglikni hosil qilamiz. $\triangleright$

Quyidagi xosmas integrallarning Koshi ma`nosidagi bosh qiymati topilsin.

**6.1** $V.p.\int_{-\infty}^{+\infty}\sin xdx.$

**6.2** $V.p.\int_2^{5}\frac{dx}{(x-3)^2}.$

**6.3** $V.p.\int_{0.5}^{4}\frac{dx}{x\ln x}.$

**6.4** $V.p.\int_{-\infty}^{+\infty}\cos xdx.$

**6.5** $V.p.\int_{-\infty}^{+\infty}arctgxdx.$

**6.6** $V.p.\int_0^{\pi}xtgxdx.$

**6.7** $V.p.\int_0^{\pi/2}\frac{dx}{3-5\sin x}.$

**6.8** $V.p.\int_{-\infty}^{+\infty}\left(arctgx+\frac{1}{1+x^2}-\frac{\pi}{2}\right)dx.$

**6.9** $V.p.\int_0^{\pi/2}\frac{dx}{\frac{1}{2}-\sin x}.$

**6.10** $V.p.\int_1^7\frac{dx}{5-x}.$

**6.11** $V.p.\int_{-\infty}^{+\infty}\frac{13+x}{17+x^2}dx.$

**6.12** $V.p.\int_0^{+\infty}\frac{dx}{x^2+x-6}.$

**6.13** $V.p.\int_0^{\pi/2}\frac{dx}{1-2\sin x}.$

**6.14** $V.p.\int_0^{\pi/2}\frac{dx}{\frac{1}{2}-\cos x}.$

**6.15** $V.p.\int_0^{+\infty}\frac{dx}{x^2+x-2}.$

**6.16** $V.p.\int_{-\infty}^{+\infty}\frac{dx}{x}.$

**6.17** $V.p.\int_0^{+\infty}\frac{dx}{1-x^2}.$

**6.18** $V.p.\int_0^{10}\frac{dx}{7-x}.$

**6.19** $V.p.\int_{-1}^{1}\frac{dx}{x}.$

**6.20** $V.p.\int_{-1}^{7}\frac{dx}{(x-1)^3}.$

**6.21** $V.p.\int_0^{+\infty}\frac{dx}{x^2-3x+2}.$

# Test

1. Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti mavjud bo‘lsa, bu limiti $f(x)$ funksiyaning $[a,+\infty)$ cheksiz oraliq bo‘yicha ……………deyiladi.

A) xosmas integrali
B) Aniq integral
C) aniqmas integral
D) to’g’ri javob yo’q

2.Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti mavjud va chekli bo‘lsa,
$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty}\int_a^t f(x)dx$. integral ………………. deyiladi.

A) yaqinlashuvchi
B) uzoqlashuvchi
C) yaqinlashuvchi yoki uzoqlashuvchi
D) tog’ri javob A va B

3. Agar $t \to +\infty$ da $F(t)$ funksiyaning limiti cheksiz yoki mavjud bo‘lmasa,
$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} F(t) = \lim_{t\to+\infty}\int_a^t f(x)dx$. integral ……. deyiladi.

A) uzoqlashuvchi
B) yaqinlashuvchi yoki uzoqlashuvchi
C) yaqinlashuvchi
D) tog’ri javob A va B

4. Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} C\cdot f(x)dx$ ham $(C = const)$ yaqinlashuvchi bo‘lib,………..bo‘ladi.

A) $\int_a^{+\infty} C\cdot f(x)dx = C\int_a^{+\infty} f(x)dx$

B) $\int_a^{+\infty} f(x)dx \ge 0$

C) $\int_a^{+\infty}(f(x)\pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$

D) $\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx$

5. Agar $\forall x \in [a,+\infty)$ da $f(x) \le g(x)$ bo‘lib, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda ...........bo‘ladi.

A) $\int_a^{+\infty} f(x)dx \le \int_a^{+\infty} g(x)dx$

B) $\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx$

C) $\int_a^{+\infty} (f(x) \pm g(x))dx = \int_a^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$

D) $\int_a^{+\infty} f(x)dx \ge 0$

6. Hisoblang: $\int_0^{\frac{1}{2}} \frac{dx}{x\ln^2 x}$

A) $\frac{1}{\ln 2}$

B) $\frac{1}{\ln 3}$

C) 2

D) 1

7. Hisoblang: $\int_0^3 \frac{x^2dx}{\sqrt{9-x^2}}$

A) $\frac{9\pi}{4}$

B) $\frac{9\pi}{2}$

C) $\frac{6\pi}{4}$

D) 1

8. Hisoblang: $\int_0^1 \ln x dx$

A) -1
B) -2
C) 2
D) 1

9. Hisoblang: $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}$

A) $\frac{\pi}{2}$
B) $\frac{3\pi}{2}$
C) $\frac{6\pi}{4}$
D) $\frac{\pi}{4}$

10. Hisoblang: $\int_2^3 \frac{2-x}{\sqrt{3-x}} dx$

A) $-\frac{4}{3}$
B) $\frac{4}{3}$
C) $-\frac{3}{2}$
D) 2

# Mavzu. $R^m$ fazo. $R^m$ fazoda ochiq va yopiq to‘plamlar

## 23-ma’ruza

**Reja**

$1^0$. $R^m$ fazo tushunchasi.

$2^0$. $R^m$ fazoda nuqtaning atrofi.

$3^0$. $R^m$ fazoda ochiq va yopiq to‘plamlar.

### $1^0$. $R^m$ fazo tushunchasi.

Haqiqiy sonlar to‘plami $R$ yordamida ushbu

$$\underbrace{R\times R\times\cdots\times R}_{m\ ma}=\{(x_1,x_2,...,x_m): x_1\in R,\ x_2\in R,...,x_m\in R\} \qquad (1)$$

to‘plamni ($R$ ning dekart ko‘paytmalaridan tuzilgan to‘plamni) hosil qilaylik. Ravshanki, (1) to‘plamning har bir elementi $m$ ta $x_1, x_2,..., x_m$ haqiqiy sonlardan tashkil topgan tartiblangan $m$ lik

$$(x_1, x_2,..., x_m)$$

dan iborat bo‘ladi. Uni (1) to‘plamning nuqtasi deyilib, bitta harf bilan belgilanadi:

$$x=(x_1, x_2,..., x_m).$$

Bunda $x_1, x_2, ..., x_m$ sonlar $x$ nuqtaning mos ravishda birinchi, ikkinchi, ... , $m$-koordinatalari deyiladi.

Agar $x=(x_1, x_2, ..., x_m)$, $y=(y_1, y_2, ..., y_m)$ nuqtalar uchun $x_1 = y_1$, $x_2 = y_2, ..., x_m = y_m$ bo‘lsa, $x = y$ deyiladi.

Faraz qilaylik,

$$x=(x_1, x_2, ..., x_m),\ y=(y_1, y_2, ..., y_m)$$

lar (1) to‘plamning ixtiyoriy ikki nuqtasi bo‘lsin. Ushbu

$$\sqrt{\sum_{k=1}^{m}(y_k - x_k)^2}$$

miqdor $x$ va $y$ nuqtalar orasidagi masofa deyiladi va $\rho(x, y)$ kabi belgilanadi:

$$\rho(x, y)=\sqrt{\sum_{k=1}^{m}(y_k - x_k)^2}\ . \tag{2}$$

Endi masofaning xossalarini keltiramiz:

1) Har doim $\rho(x, y) \geq 0$ va $\rho(x, y)=0 \Leftrightarrow x = y$ bo‘ladi.

◄ (2) munosabatga ko‘ra, har doim $\rho(x, y) \geq 0$ bo‘ladi. Agar $\rho(x, y)=0$ bo‘lsa, unda

$$(y_1 - x_1)^2 + (y_2 - x_2)^2 + ... + (y_m - x_m)^2 = 0$$

bo‘lib, natijada $x_1 = y_1$, $x_2 = y_2, ..., x_m = y_m$, ya’ni $x = y$ bo‘lishi kelib chiqadi. Aksincha, agar $x_1 = y_1$, $x_2 = y_2, ..., x_m = y_m$ bo‘lsa, unda (2) munosabatdan foydalanib $\rho(x, y)=0$ bo‘lishini topamiz.►

2) $\rho(x, y)$ masofa $x$ va $y$ ularga nisbatan simmetrik bo‘ladi: $\rho(x, y)=\rho(y, x)$.

◄ Bu xossanig isboti (2) munosabatdan kelib chiqadi:

$$\rho(x,y)=\sqrt{\sum_{k=1}^{m}(y_k-x_k)^2}=\sqrt{\sum_{k=1}^{m}(x_k-y_k)^2}=\rho(y,x).$$ ►

3) (1) to‘plamning ixtiyoriy

$$x=(x_1,x_2,...,x_m),\ y=(y_1,y_2,...,y_m),\ z=(z_1,z_2,...,z_m)$$

nuqtalari uchun

$$\rho(x,z)\le\rho(x,y)+\rho(y,z)$$

tengsizlik o‘rinli bo‘ladi.

◄Ma’lumki, ixtiyoriy $a_1,a_2,....,a_m$ va $b_1,b_2,....,b_m$ haqiqiy sonlar uchun

$$\sqrt{\sum_{k=1}^{m}(a_k+b_k)^2}\le\sqrt{\sum_{k=1}^{m}a_k^2}+\sqrt{\sum_{k=1}^{m}b_k^2} \tag{3}$$

bo‘ladi (qaralsin, [1], 12-bob, 1-§; odatda bu tengsizlikni Koshi-Bunyakovskiy tengsizligi deyiladi). (3) tengsizlikda

$$a_k=y_k-x_k\ ,\ b_k=z_k-y_k \qquad (k=1,2,...,m)$$

deb topamiz:

$$\sqrt{\sum_{k=1}^{m}(z_k-x_k)^2}\le\sqrt{\sum_{k=1}^{m}(y_k-x_k)^2}+\sqrt{\sum_{k=1}^{m}(z_k-y_k)^2}\ .$$

Bu esa

$$\rho(x,z)\le\rho(x,y)+\rho(y,z)$$

bo‘linishi bildiriladi.►

Shunday qilib, (1) to‘plamda (to‘plam elementlari orasida) masofa tushunchasining kiritilishini hamda masofa uchta xossaga ega bo‘lishini ko'rdik. Odatda, (1) to'plam $R^m$ fazo deyiladi. Demak,

$$R^m = \{(x_1, x_2, ..., x_m): x_1 \in R\ ,\ x_2 \in R, ..., x_m \in R\}.$$

Endi $R^m$ fazodagi ba'zi bir to'plamlarni keltiramiz.

Aytaylik, biror $a = (a_1, a_2, ..., a_m) \in R^m$ nuqta va $r > 0$ son berilgan bo'lsin.

Ushbu

$$B_r(a) = \left\{(x_1, x_2, ..., x_m) \in R^m : \sqrt{(x_1 - a_1)^2 + ... + (x_m - a_m)^2} < r\right\}$$

qisqacha,

$$B_r(a) = \{x \in R^m : \rho(x, a) < r\}$$

to'plam markazi $a$ nuqta, radiusi $r$ bo'lgan shar ($m$ o'lchovli shar) deyiladi.

Quyidagi $\overline{B_r}(a) = \{x \in R^m : \rho(x, a) \le r\}$ to'plam $R^m$ fazoda yopiq shar,

$$B_r^0(a) = \{x \in R^m : \rho(x, a) = r\}$$

to'plam esa, $R^m$ fazoda sfera ($m$ o'lchovli sfera) deyiladi.

Ravshanki, $\overline{B_r}(a) = B_r(a) \cup B_r^0(a)$ bo'ladi.

Ushbu

$$\Pi(a_1, ..., a_m; b_1, b_2, ..., b_m) = \{(x_1, x_2, ..., x_m) \in R^m : a_1 < x_1 < b_1, a_2 < x_2 < b_2, ..., a_m < x_m < b_m\}$$

to'plam $R^m$ fazoda parallelepiped deyiladi, bunda $a_1, a_2, ..., a_m; b_1, b_2, ..., b_m$ haqiqiy sonlar.

**$2^0$. $R^m$ fazoda nuqtaning atrofi**.

Biror $x^0 = (x_1^0, x_2^0, ..., x_m^0) \in R^m$ nuqta hamda $\varepsilon > 0$ son berilgan bo'lsin.

**1-ta'rif**. Markazi $x^0$ nuqtada radiusi $\varepsilon$ bo'lgan $R^m$ fazodagi shar,

$x^0 \in R^m$ nuqtaning sferik atrofi deyiladi va $U_\varepsilon(x^0)$ kabi belgilanadi:

$$U_\varepsilon(x^0) = \{x \in R^m : \rho(x, x^0) < \varepsilon\}.$$

**2-ta'rif**. Ushbu

$$\Pi(\delta_1, \delta_2 ..., \delta_m) = \{(x_1, x_2, ..., x_m) \in R^m :$$

$$: x_1^0 - \delta_1 < x_1 < x_1^0 + \delta_1, x_2^0 - \delta_2 < x_2 < x_2^0 + \delta_2, ..., x_m^0 - \delta_m < x_m < x_m^0 + \delta_m\}$$

parallelepiped $x^0$ nuqtaning parallelepipedial atrofi deyiladi va $\overline{U}_{\delta_1, \delta_2 ... \delta_m}(x^0)$ kabi belgilanadi, bunda $\delta_1 > 0$, $\delta_2 > 0, ..., \delta_m > 0$.

$R^m$ fazodagi nuqtaning bu atroflari orasidagi munosabatni quyidagi lemma ifodalaydi.

**Lemma.** $x^0 \in R^m$ nuqtaning har qanday $U_\varepsilon(x^0)$ sferik atrofi olinganda ham har doim $x^0$ nuqtaning shunday $\overline{U}_{\delta_1, \delta_2 ... \delta_m}(x^0)$ parallelepipedial atrofi topiladiki,

$$\overline{U}_{\delta_1, \delta_2 ... \delta_m}(x^0) \subset U_\varepsilon(x^0)$$

bo'ladi.

Shuningdek, $x^0$ nuqtaning har qanday $U_{\delta_1, \delta_2 ... \delta_m}(x^0)$ parallelepepidial atrofi olinganda ham har doim $x^0$ nuqtaning shunday $U_\varepsilon(x^0)$ sferik atrofi topiladiki,

$$\overline{U}_\varepsilon(x^0) \subset U_{\delta_1 \delta_2 ... \delta_m}(x^0)$$

bo'ladi.

◄ $x^0 \in R^m$ nuqtaning sferik atrofi

$$U_\varepsilon(x^0) = \{x \in R^m : \rho(x, x^0) < \varepsilon\}$$

berilgan bo'lsin. Demak, $\varepsilon > 0$ son berilgan. Unga ko'ra $\delta < \frac{\varepsilon}{\sqrt{m}}$ tengsizlikni qanoatlantiruvchi $\delta$ sonni olib, $x^0$ nuqtaning ushbu

$$\overline{U}_\delta\left(x^0\right) = \overline{U}_{\delta\delta\ldots\delta}\left(x^0\right) = \{\left(x_1, x_2, \ldots, x_m\right) \in R^m :$$

$$: x_1^0 - \delta < x_1 < x_1^0 + \delta,\ x_2^0 - \delta < x_2 < x_2^0 + \delta, \ldots, x_m^0 - \delta < x_m < x_m^0 + \delta\}$$

paralleipipedial atrofini tuzamiz. Natijadi $x^0$ nuqtaning

$$U_\varepsilon\left(x^0\right) \text{ va } \overline{U}_\delta\left(x^0\right)$$

atroflariga ega bo'lamiz.

Aytaylik, $\forall x \in \overline{U}_\delta\left(x^0\right)$ bo'lsin. Unda

$$\left|x_k - x_k^0\right| < \delta \qquad \left(k = 1, 2, \ldots, m\right)$$

bo'lib,

$$\sqrt{\sum_{k=1}^{m}\left(x_k - x_k^\circ\right)^2} < \sqrt{\sum_{k=1}^{m}\delta^2} = \delta \cdot \sqrt{m}$$

bo'ladi. Yuqoridagi $\delta < \frac{\varepsilon}{\sqrt{m}}$ tengsizlikni e'tiborga olib topamiz:

$$\sqrt{\sum_{k=1}^{m}\left(x_k - x_k^0\right)^2} < \varepsilon .$$

Demak, $\rho\left(x, x_0\right) < \varepsilon$ bo'lib, $x \in U_\varepsilon\left(x^0\right)$ bo'ladi. Bundan

$$\overline{U}_\delta\left(x^0\right) \subset U_\varepsilon\left(x^0\right)$$

bo'lishi kelib chiqadi.

$x^0 \in R^m$ nuqtaning parallelepipedial atrofi

$$\overline{U}_{\delta_1\delta_2\ldots\delta_m}\left(x^0\right) = \{\left(x_1, x_2, \ldots, x_m\right) \in R^m :$$

$$: x_1^0 - \delta_1 < x_1 < x_1^0 + \delta_1,\ x_2^0 - \delta < x_2 < x_2^0 + \delta_2, \ldots, x_m^0 - \delta_m < x_m < x_m^0 + \delta_m\}$$

berilgan bo'lsin. Berilgan $\delta_1, \delta_2, \ldots, \delta_m$ musbat sonlar yordamida

$$\varepsilon = \min\left\{\delta_1, \delta_2, \ldots, \delta_m\right\}$$

sonini topib, $x^0$ nuqtaning ushbu

$$U_\varepsilon(x^0)=\{x\in R^m : \rho(x,x^0)<\varepsilon\}$$

sferik atrofni tuzamiz. Natijada $x^0$ nuqtaning

$$U_\varepsilon(x^0) \text{ va } \overline{U}_{\delta_1\delta_2\ldots\delta_m}(x^0)$$

atroflariga ega bo‘lamiz.

Aytaylik, $\forall x\in U_\varepsilon(x^0)$ bo‘lsin. U holda

$$\rho(x,x^0)=\sqrt{\sum_{k=1}^{m}(x_k-x_k^0)^2}<\varepsilon\le\delta_k \qquad (k=1,2,\ldots,m)$$

bo‘lib,

$$|x_k-x_k^0|<\delta_k \qquad (k=1,2,\ldots,m)$$

bo‘ladi. Bundan esa $x\in\overline{U}_{\delta_1\delta_2\ldots\delta_m}(x^0)$ bo‘lishi kelib chiqadi. Demak,

$$U_\varepsilon(x^0)\subset\overline{U}_{\delta_1\delta_2\ldots\delta_m}(x^0).$$ ►

Bu lemma $R^m$ fazo nuqtasining bir atrofidan ikkinchi atrofiga o‘tishi imkonini beradi.

**$3^0$. $R^m$ fazoda ochiq va yopiq to‘plamlar**.

Aytaylik, $R^m$ fazoda biror $G$ to‘plam $(G\subset R^m)$ berilgan bo‘lib, $x^0\in G$ bo‘lsin.

Agar $x^0$ nuqta $G$ to‘plamga tegishli bo‘lgan $U_\varepsilon(x^0)$ atrofga ega bo‘lsa, $(U_\varepsilon(x^0)\subset G)$ $x^0$ nuqta $G$ to‘plamning ichki nuqtasi deyiladi.

**3-ta’rif** $G$ to‘plamning har bir nuqtasi uning ichki nuqtasi bo‘lsa, u ochiq to‘plam deyiladi.

**1-misol**. $R^m$ fazodagi ushbu

$$B_r(a)=\{x\in R^m : \rho(x,a)<r\}$$

sharning ochiq to‘plam ekanligi ko‘rsatilsin.

◄ $\forall x^0 \in B_r(a)$ nuqtani olamiz. Unda

$$r - \rho(x^0, a)$$

miqdor musbat bo‘ladi. Uni $\delta$ deylik: $\delta = r - \rho(x^0, a)$. (22-chizma)

Endi $x^0$ nuqtaning ushbu

$$U_\delta(x^0) = \{x \in R^m : \rho(x, x^0) < \delta\}$$

atrofini qaraymiz.

Bunda $U_\delta(x^0) \subset B_r(a)$ bo‘ladi. Haqiqatdan ham,

$$\forall x \in U_\delta(x^0) \Rightarrow \rho(x, x^0) < \delta$$

bo‘lib, masofaning 3)-xossasiga ko‘ra

$$\rho(x,a) \le \rho(x, x^0) + \rho(x^0, a) < \delta + \rho(x^0, a) = r$$

bo‘ladi. Demak,

$$\forall x \in U_\delta(x^0) \Rightarrow x \in B_r(x^0)$$

Bundan $U_\delta(x^0) \subset B_r(x^0)$ bo‘lishi kelib chiqadi.

Demak, $B_r(a)$ to‘plamning har bir nuqtasi uning ichki nuqtasi bo‘ladi. Binobarin, $B_r(a)$ ochiq to‘plam.►

Aytaylik, $F \subset R^m$ to‘plam hamda $x^0 \in R^m$ nuqta berilgan bo‘lsin. Agar $x^0$ nuqtaning ixtiyoriy $U_\varepsilon(x^0)$ atrofida $(\forall \varepsilon > 0)$ $F$ to‘plamning $x^0$ dan farqli kamida bitta nuqtasi bo‘lsa, $x^0$ nuqta $F$ to‘plamning limit nuqtasi deyiladi.

Masalan, ushbu

$$B_r(a) = \{x \in R^m : \rho(x,a) < r\}$$

to‘plamning har bir nuqtasi uning limit nuqtasi bo‘ladi. Ayni paytda,

$$B_r^0 = \{x \in R^m : \rho(x,a) = r\}$$

to‘plamning barcha nuqtalari ham shu $B_r(a)$ to‘plamning limit nuqtasi bo‘ladi. Biroq, bu limit nuqtalar $B_r(a)$ to‘plamga tegishli bo‘lmaydi.

**4-ta'rif**. Agar $F \subset R^m$ to‘plamning barcha limit nuqtalari shu to‘plamga tegishli bo‘lsa, $F$ yopiq to‘plam deyiladi.

Masalan,

$$\overline{B}_r(a) = \{x \in R^m : \rho(x,a) \leq r\}$$

to‘plam ( $R^m$ fazodagi yopiq shar) yopiq to‘plam bo‘ladi

Biror $M \subset R^m$ to‘plam hamda $x^0 \in R^m$ nuqtani qaraylik.

Agar $x^0$ nuqtaning ixtiyoriy $U_\varepsilon(x^0)$ atrofida ham $M$ to‘plamning, ham $R^m \setminus M$ to‘plamning nuqtalari bo‘lsa, $x^0$ nuqta $M$ to‘plamning chegaraviy nuqtasi deyiladi. $M$ to‘plamning barcha chegaraviy nuqtalari uning chegarasini tashkil etadi. $M$ to‘plamning chegarasi $\partial(M)$ kabi belgilanadi.

Masalan,

$$B_r^0(a) = \{x \in R^m : \rho(x,a) = r\}$$

to‘plam

$$B_r(a) = \{x \in R^m : \rho(x,a) < r\}$$

to‘plamning chegarasi bo‘ladi:

$$\partial(B_r(a)) = B_r^0(a).$$

Agar $F \subset R^m$ to‘plamning chegarasi $\partial(F)$ shu to‘plamga tegishli bo‘lsa, $F$ yopiq to‘plam bo‘ladi.

Masalan,

$$\overline{B}_r(a) = \{x \in R^m : \rho(x,a) \leq r\}$$

yopiq to‘plam bo‘ladi, chunki

$$\partial(\overline{B}_r(a)) = B_r^0(a) \subset \overline{B}_r(a).$$

## Mashqlar

1. Agar $G_1 \subset R^m$ , $G_2 \subset R^m$ ochiq to‘plamlar bo‘lsa,

$$G_1 \cup R^m \text{ , } G_2 \cap R^m$$

to‘plamlarning ochiq to‘plam bo‘lishi ko‘rsatilsin.

2. Agar $F_1 \subset R^m$ , $F_2 \subset R^m$ yopiq to‘plamlar bo‘lsa,

$$F_1 \cup R^m \text{ , } F_2 \cap R^m$$

to‘plamlarning yopiq to‘plam bo‘lishi ko‘rsatilsin.

## Adabiyotlar


1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’ruzalar, II q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 1*. Hindustan Book Agency, India, 2014.


# Glossariy

**$R$ ning dekart ko‘paytmalaridan tuzilgan to‘plam** - $\underbrace{R\times R\times\cdots\times R}_{m\ ma}=\{(x_1,x_2,...,x_m): x_1\in R,\ x_2\in R,...,x_m\in R\}$.

**Tartiblangan $m$ lik** - $(x_1,x_2,...,x_m)$.

$x$ **va** $y$ **nuqtalar orasidagi masofa** - $\rho(x,y)=\sqrt{\sum_{k=1}^{m}(y_k-x_k)^2}$ .

$R^m$ **fazo** - $R^m=\{(x_1,x_2,...,x_m): x_1\in R,\ x_2\in R,...,x_m\in R\}$.

$m$ **o‘lchovli shar** - $B_r(a)=\{x\in R^m : \rho(x,a)<r\}$.

$m$ **o‘lchovli sfera** - $B_r^0(a)=\{x\in R^m : \rho(x,a)=r\}$.

$R^m$ **fazoda parallelepiped** -
$\Pi(a_1,\ldots,a_m;b_1,b_2,\ldots,b_m)=\{(x_1,x_2,\ldots,x_m)\in R^m : a_1<x_1<b_1, a_2<x_2<b_2,\ldots,a_m<x_m<b_m\}$
$x^0\in R^m$ **nuqtaning sferik atrofi** - Markazi $x^0$ nuqtada radiusi $\varepsilon$ bo‘lgan $R^m$ fazodagi shar.

$x^0$ **nuqta** $G$ **to‘plamning ichki nuqtasi** - Agar $x^0$ nuqta $G$ to‘plamga tegishli bo‘lgan $U_\varepsilon(x^0)$ atrofga ega bo‘lsa.

**Ochiq to‘plam** - To‘plamning har bir nuqtasi uning ichki nuqtasi bo‘lsa.

$x^0$ **nuqta** $F$ **to‘plamning limit nuqtasi** - Agar $x^0$ nuqtaning ixtiyoriy $U_\varepsilon(x^0)$ atrofida $(\forall\varepsilon>0)$ $F$ to‘plamning $x^0$ dan farqli kamida bitta nuqtasi bo‘lsa.

$F$ **yopiq to‘plam** - Agar $F\subset R^m$ to‘plamning barcha limit nuqtalari shu to‘plamga tegishli bo‘lsa.

$x^0$ **nuqta** $M$ **to‘plamning chegaraviy nuqtasi** - Agar $x^0$ nuqtaning ixtiyoriy $U_\varepsilon(x^0)$ atrofida ham $M$ to‘plamning, ham $R^m\setminus M$ to‘plamning nuqtalari bo‘lsa.

## Keys banki

**45-keys.** Masala o`rtaga tashlanadi: Agar $G_1\subset\mathbb{R}^m$, $G_2\subset\mathbb{R}^m$ ochiq to‘plamlar bo‘lsa,

$$G_1\cup\mathbb{R}^m,\ G_2\cap\mathbb{R}^m$$

to‘plamlarning ochiq to‘plam bo‘lishi ko‘rsatilsin.
**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 23-amaliy mashg‘ulot

## Na’muna uchun misollar yechimi

**1-misol.** $R^m$ fazodagi ushbu

$$B_r(a)=\{x\in R^m : \rho(x,a)<r\}$$

sharning ochiq to‘plam ekanligi ko‘rsatilsin.

◄ $\forall x^0 \in B_r(a)$ nuqtani olamiz. Unda

$$r-\rho(x^0,a)$$

miqdor musbat bo‘ladi. Uni $\delta$ deylik: $\delta = r-\rho(x^0,a)$. (22-chizma)

Endi $x^0$ nuqtaning ushbu

$$U_\delta(x^0)=\{x\in R^m : \rho(x,x^0)<\delta\}$$

atrofini qaraymiz.

Bunda $U_\delta(x^0)\subset B_r(a)$ bo‘ladi. Haqiqatdan ham,

$$\forall x\in U_\delta(x^0)\Rightarrow \rho(x,x^0)<\delta$$

bo‘lib, masofaning 3)-xossasiga ko‘ra

$$\rho(x,a)\le \rho(x,x^0)+\rho(x^0,a)<\delta+\rho(x^0,a)=r$$

bo‘ladi. Demak,

$$\forall x\in U_\delta(x^0)\Rightarrow x\in B_r(x^0)$$

Bundan $U_\delta(x^0)\subset B_r(x^0)$ bo‘lishi kelib chiqadi.

Demak, $B_r(a)$ to‘plamning har bir nuqtasi uning ichki nuqtasi bo‘ladi. Binobarin, $B_r(a)$ ochiq to‘plam.►

## Misollar

1. Agar $G_1\subset R^m$, $G_2\subset R^m$ ochiq to‘plamlar bo‘lsa,

$$G_1\cup R^m \text{ , } G_2\cap R^m$$

to‘plamlarning ochiq to‘plam bo‘lishi ko‘rsatilsin.

2. Agar $F_1 \subset R^m$ , $F_2 \subset R^m$ yopiq to‘plamlar bo‘lsa,

$$F_1 \cup R^m \text{ , } F_2 \cap R^m$$

to‘plamlarning yopiq to‘plam bo‘lishi ko‘rsatilsin.

3. Aytaylik $(X,\rho)$ metrik fazo berilgan bo`lsin, $x_0 \in X$ va $r > 0$. $B_r(a) = \{x \in R^m : \rho(x,a) < r\}$ ochiq shar va $C := \{x \in X : \rho(x,x_0) \le r\}$ yopiq shar berilgan.

a) $\overline{B} \subseteq C$ isbotlang.

b) $\overline{B}$ $C$ ga teng bo`lmagan holat uchun $r > 0$ radiusli $(X,\rho)$ metrik fazoga misollar keltiring.

# Test

1. Agar $x^0$ nuqta $G$ to‘plamga tegishli bo‘lgan $U_\varepsilon(x^0)$ atrofga ega bo‘lsa, $(U_\varepsilon(x^0) \subset G)$ $x^0$ nuqta $G$ to‘plamning ………. nuqtasi deyiladi.
   A) ichki B) chegaraviy C) limit D) urinish
2. Agar $x^0$ nuqtaning ixtiyoriy $U_\varepsilon(x^0)$ atrofida $(\forall\varepsilon > 0)$ $F$ to‘plamning $x^0$ dan farqli kamida bitta nuqtasi bo‘lsa, $x^0$ nuqta $F$ to‘plamning ………….. nuqtasi deyiladi.
   A) limit B) chegaraviy C) ichki D) urinish
3. Agar $x^0$ nuqtaning ixtiyoriy $U_\varepsilon(x^0)$ atrofida ham $M$ to‘plamning, ham $R^m \setminus M$ to‘plamning nuqtalari bo‘lsa, $x^0$ nuqta $M$ to‘plamning ………… nuqtasi deyiladi.
   A) chegaraviy B) limit C) ichki D) urinish
4. $G$ to‘plamning har bir nuqtasi uning ichki nuqtasi bo‘lsa, u ………. deyiladi.
   A) ochiq to’plam B) yopiq to’plam C) chegara D) ochiq shar
5. Agar $F \subset R^m$ to‘plamning barcha limit nuqtalari shu to‘plamga tegishli bo‘lsa, $F$ ………… deyiladi.
   A) yopiq to’plam B) ochiq to’plam C) chegara D) ochiq shar
6. $x$ va $y$ nuqtalar orasidagi masofa to’g`ri keltirilgan qatorni ko`rsating.
   A) $\sqrt{\sum_{k=1}^{m}(y_k - x_k)^2}$ B) $\sqrt{\sum_{k=1}^{m}(y_k + x_k)^2}$ C) $\sum_{k=1}^{m}(y_k - x_k)^3$ D) $\sum_{k=1}^{m}(y_k - x_k)^2$
7. $x^0 \in R^m$ nuqtaning sferik atrofini toping.
   A) $U_\varepsilon(x^0) = \{x \in R^m : \rho(x, x^0) < \varepsilon\}$ B) $U_\varepsilon(x^0) = \{x \in R^m : \rho(x, x^0) \le \varepsilon\}$
   C) $U_\varepsilon(x^0) = \{x \in R^m : \rho(x, x^0) > \varepsilon\}$ D) $U_\varepsilon(x^0) = \{x \in R^m : \rho(x, x^0) \ge \varepsilon\}$

8. Agar $G_1 \subset R^m$ , $G_2 \subset R^m$ ochiq to‘plamlar bo‘lsa, u holda quyidagi to`plamlar ichida ochiq to`plamlarni ko`rsating.

A) $G_1 \cup R^m$ , $G_2 \cap R^m$ B) $G_1 \setminus R^m$ , $G_2 \setminus R^m$ C) $G_1 \square R^m$ , $G_2 \square R^m$

D) $G_1 \square R^m$ , $G_2 \setminus R^m$

9. Agar $F_1 \subset R^m$ , $F_2 \subset R^m$ yopiq to‘plamlar bo‘lsa, u holda quyidagi to`plamlar ichida yopiq to`plamlarni ko`rsating.

A) $F_1 \cup R^m$ , $F_2 \cap R^m$ B) $F_1 \setminus R^m$ , $F_2 \setminus R^m$ C) $F_1 \square R^m$ , $F_2 \square R^m$

D) $F_1 \square R^m$ , $F_2 \setminus R^m$

10. $M$ to‘plamning chegarasi to`g`ri belgilangan qatorni qatorni ko`rsating.

A) $\partial(M)$ B) $\overline{M}$ C) $M'$ D) $\mathrm{int}(M)$

# Mavzu. $R^m$ fazoda ketma-ketlik va uning limiti.

## 24-ma'ruza

**Reja**

$1^0$. $R^m$ fazoda ketma-ketlik va uning limiti tushunchalari.

$2^0$. Ketma-ketlik limitining mavjudligi.

$3^0$. Ichma-ich joylashgan yopiq sharlar prinsipi.

$4^0$. Qismiy ketma-ketliklar. Bolsano-Veyershtrass teoremasi.

### $1^0$. $R^m$ fazoda ketma-ketlik va uning limiti tushunchalari.

Aytaylik, biror qoidaga ko'ra har bir natural son $n$ ga $R^m$ fazoning bitta

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}) \qquad (n = 1, 2, \dots)$$

nuqtasi mos qo'yilgan bo'lsin. Bu moslik natijasida $R^m$ fazo nuqtalaridan tashkil topgan ushbu

$$(x_1^{(1)}, x_2^{(1)}, \dots, x_m^{(1)}),\ (x_1^{(2)}, x_2^{(2)}, \dots, x_m^{(2)}),\ \dots,\ (x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)}),\ \dots$$

qisqacha

$$x^{(1)},\ x^{(2)}, \dots,\ x^{(n)}, \dots$$

to'plam hosil bo'ladi. Uni $R^{(m)}$ fazoda ketma-ketlik deyilib, $\{x^{(n)}\}$ kabi belgilanadi. Demak, $\{x^{(n)}\}$ ketma-ketlikning hadlari $R^m$ fazo nuqtalaridan iborat bo'lib, bu nuqtalarning koordinatalari $m$ ta

$$\{x_1^{(n)}\}, \quad \{x_2^{(n)}\}, \ \dots \ \{x_m^{(n)}\}, \qquad (n = 1, 2, ..)$$

sonlar ketma-ketliklarini yuzaga keltiradi.

Faraz qilaylik, $R^m$ fazoda $\{x^{(n)}\}$:

$$x^{(1)}, x^{(2)}, ....., x^{(n)}, .... \qquad (1)$$

ketma-ketlik hamda

$$a=(a_{1,}, a_2, ...a_m) \in R^m$$

nuqta berilgan bo‘lsin.

**1-ta’rif**. Agar $\forall \varepsilon > 0$ olinganda ham, shunday $n_0 \in N$ son topilsaki, barcha $n > n_0$ uchun

$$\rho(x^{(n)}, a) < \varepsilon$$

ya’ni

$$\forall \varepsilon > 0, \qquad \exists n_0 \in N, \qquad \forall n > n_0 : \quad \rho(x^{(n)}, a) < \varepsilon$$

bo‘lsa, $a$ nuqta $\{x^{(n)}\}$ ketma-ketlikning limiti deyiladi va

$$\lim_{n\to\infty} x^{(n)} = a \;\; \text{yoki} \;\; n \to \infty \;\text{da}\; x^{(n)} \to a$$

kabi belgilanadi.

$\forall n > n_0$ da

$$\rho(x^{(n)}, a) < \varepsilon$$

tengsizlikning bajarilishi, (1) ketma-ketlikning $n_0$ dan katta nomerli hadlari $a$ nuqtaning $U_\varepsilon(a)$ atrofiga tegishli bo‘lishini bildiradi. Bu hol (1) ketma-ketlikning limitini quyidagicha ta’riflash imkonini beradi.

**2-ta’rif.** Agar $a \in R^m$ nuqtaning ixtiyoriy $U_\varepsilon(a)$ atrofi olingandan ham, $\{x^{(n)}\}$ ketma-ketlikning biror hadidan keyingi barcha hadlari shu atrofga tegishli bo‘lsa, $a$ nuqta $\{x^{(n)}\}$ ketma-ketlikning limiti deyiladi.

**1-misol.** $R^m$ fazoda ushbu

$$\{x^{(n)}\} = \left\{\frac{1}{n}, \frac{1}{n}, ... \frac{1}{n}\right\}$$

ketma-ketlikning limiti $a = (0, 0, \dots, 0)$ bo‘lishi ko‘rsatilsin.

◀ $\forall\varepsilon>0$ sonini olib, unga ko‘ra $n_0=\left[\frac{\sqrt{m}}{\varepsilon}\right]+1$ ni topamiz.

Unda $\forall n>n_0$ uchun

$$\rho(x^{(n)},a)=\rho((\frac{1}{n},\frac{1}{n},....,\frac{1}{n}),(0,0,...,0))=\frac{\sqrt{m}}{n}<\frac{\sqrt{m}}{n_0}=\frac{\sqrt{m}}{\left[\frac{\sqrt{m}}{\varepsilon}\right]+1}<\varepsilon$$

bo‘ladi. Demak,

$$\lim_{n\to\infty}x^{(n)}=a\ .$$ ▶

**2⁰. Ketma-ketlik limitining mavjudligi.** Faraz qilaylik, $R^m$ fazoda $\{x^{(n)}\}$ ketma-ketlik va $a\in R^m$ nuqta berilgan bo‘lsin.

**1-teorema.** Agar $R^m$ fazoda

$$x^{(n)}=(x_1^{(n)},x_2^{(n)},...,x_m^{(n)})\qquad (n=1,2,...)$$

ketma-ketlik

$$a=(a_1,a_2,....,a_m)$$

limitga ega bo‘lsa;

$$\lim_{n\to\infty}x^{(n)}=a\ ,$$

u holda

$$\lim_{n\to\infty}x_1^{(n)}=a_1,$$
$$\lim_{n\to\infty}x_2^{(n)}=a_2,$$
$$\dots\dots\dots\dots\dots$$
$$\lim_{n\to\infty}x_m^{(n)}=a_m,$$

bo‘ladi.

◀ Aytaylik

$$\lim_{n\to\infty} x^{(n)} = a$$

bo‘lsin. Limit ta’rifiga binoan $\forall n > n_0 \in N$ uchun

$$x^{(n)} \in U_\varepsilon(a) = \{x \in R^m : \rho(x,a) < \varepsilon\} \qquad (\forall \varepsilon > 0)$$

bo‘ladi. Ravshanki,

$$U_\varepsilon(a) \subset \tilde{U}_\varepsilon(a)$$

bunda,

$$\tilde{U}_\varepsilon(a) = x = \{(x_1, x_2, ..., x_m) \in R^m :$$
$$a_1 - \varepsilon < x_1 < a_1 + \varepsilon, a_2 - \varepsilon < x_2 < a_2 + \varepsilon, ..., a_m - \varepsilon < x_m < a_m + \varepsilon\}.$$

Keyingi munosabatlardan, $\forall n > n_0$ uchun

$$a_1 - \varepsilon < x_1^{(n)} < a_1 + \varepsilon \ ,$$
$$a_2 - \varepsilon < x_2^{(n)} < a_2 + \varepsilon \ ,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_m - \varepsilon < x_m^{(n)} < a_m + \varepsilon$$

ya’ni

$$|x_1^{(n)} - a_1| < \varepsilon,$$
$$|x_2^{(n)} - a_2| < \varepsilon,$$
$$\dots\dots\dots\dots\dots\dots$$
$$|x_m^{(n)} - a_m| < \varepsilon$$

bo‘lishini topamiz. Bundan esa

$$\lim_{n\to\infty} x_1^{(n)} = a_1,$$
$$\lim_{n\to\infty} x_2^{(n)} = a_2,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\lim_{n\to\infty} x_m^{(n)} = a_m$$

bo‘lishi kelib chiqadi.►

**2-teorema.** Agar $R^m$ fazodagi

$$\{x^n\}=\{(x_1^{(n)},\ x_2^{(n)},\ ....,\ x_m^{(n)})\} \qquad (n=1,2...)$$

ketma-ketlik va $a=(a_1,a_2,....,a_m)$ nuqta uchun

$$\lim_{n\to\infty} x_1^{(n)}=a_1,$$

$$\lim_{n\to\infty} x_2^{(n)}=a_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lim_{n\to\infty} x_m^{(n)}=a_m$$

bo‘lsa, u holda $\{x^{(n)}\}$ ketma-ketlik limitiga ega bo‘lib,

$$\lim_{n\to\infty} x^{(n)}=a,$$

bo‘ladi.

◄ Teoremaning sharti hamda limit ta’rifidan foydalanib topamiz:

$$\lim_{n\to\infty} x_1^{(n)}=a_1 \Rightarrow \forall\varepsilon>0,\ \exists n_0^{(1)}\in N,\ \forall n>n_0^{(1)}:\left|x_1^{(n)}-a_1\right|<\frac{\varepsilon}{\sqrt{m}} \text{ bo‘ladi.}$$

$$\lim_{n\to\infty} x_2^{(n)}=a_2 \Rightarrow \forall\varepsilon>0,\ \exists n_0^{(2)}\in N,\ \forall n>n_0^{(2)}:\left|x_2^{(n)}-a_2\right|<\frac{\varepsilon}{\sqrt{m}} \text{ bo‘ladi.}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lim_{n\to\infty} x_m^{(n)}=a_m \Rightarrow \forall\varepsilon>0,\ \exists n_0^{(m)}\in N,\ \forall n>n_0^{(m)}:\left|x_m^{(n)}-a_m\right|<\frac{\varepsilon}{\sqrt{m}} \text{ bo‘ladi.}$$

Agar

$$n_0=\max\{n_0^{(1)},\ n_0^{(2)},\ ....,\ n_0^{(m)}\}$$

deyilsa, unda $\forall n>n_0$ da bir yo‘la

$$\left|x_k^{(n)}-a_k\right|<\frac{\varepsilon}{\sqrt{m}} \qquad (k=1,2,..,m)$$

tengsizliklar bajariladi. U holda

$$\sqrt{\sum_{k=1}^{m}(x_k^{(n)}-a_k)^2}<\sqrt{\sum_{k=1}^{m}(\frac{\varepsilon}{\sqrt{m}})^2}=\varepsilon$$

ya'ni,

$$\rho(x^{(n)},a)<\varepsilon$$

bo‘ladi. Demak,

$$\lim_{n\to\infty} x^{(n)}=a\,.\blacktriangleright$$

Bu teoremalardan quyidagi tasdiq kelib chiqadi.

$R^m$ fazoda $\{x^{(n)}\}=\{(\ x_1^{(n)},\ \ x_2^{(n)},\ ....,x_m^{(n)}\ \ )\}$ ketma-ketlik $a=(a_1,a_2,...,a_m)$ limitga,

$$\lim_{n\to\infty} x^{(n)}=a$$

ega bo‘lishi uchun bir yo‘la

$$\lim_{n\to\infty} x_1^{(n)}=a_1,$$

$$\lim_{n\to\infty} x_2^{(n)}=a_2,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\lim_{n\to\infty} x_m^{(n)}=a_m,$$

bo‘lishi zarur va etarli.

Bu muhim tasdiq bo‘lib, u $R^m$ fazodagi ketma-ketliklar limitlarini o‘rganishni sonlar ketma-ketliklar limitlarini o‘rganishga olib keladi. Sonlar ketma-ketliklarning limiti esa 6-8-ma'ruzalarda batafsil bayon etilgan.

Agar (1) ketma-ketlik limitga ega bo‘lsa, u yaqinlashuvchi ketma-ketlik deyiladi.

Yuqoridagi keltirilgan tasdiqdan foydalanib isbotlanadigan muhim teoremani keltiramiz. Avvalo $R^m$ fazoda ketma-ketlikning fundamentalligini ta'riflaymiz.

**3-ta'rif**. $R^m$ fazoda $\{x^{(n)}\}$ ketma-ketlik berilgan bo‘lsin. Agar $\forall\varepsilon>0$ olinganda ham, shunday $n_0\in N$ topilsaki, $\forall n>n_0$, $\forall P>n_0$ lar uchun

$$\rho(x^{(n)},\, x^{(P)}) < \varepsilon$$

tengsizlik bajarilsa, $\{x^{(n)}\}$ fundamental ketma-ketlik deyiladi.

**3-teorema (Koshi teoremasi).** $\{x^{(n)}\}$ ketma-ketlikning yaqinlashuvchi bo‘lishi uchun uning fundamental bo‘lishi zarur va etarli.

Bu teorema 9-ma’ruzada keltirilgan 3-teorema kabi isbotlanadi.

**3⁰. Ichma-ich joylashgan yopiq sharlar prinsipi.**

$R^m$ fazoda markazlari

$$a^{(n)} = (a_1^{(n)}, a_2^{(n)}, \ldots, a_m^{(n)}) \qquad (n = 1,2,\ldots)$$

nuqtalarda, radiuslari $r_n > 0 \quad (n = 1,2,..)$ bo‘lgan ushbu

$$B_1 = \bar{B}_{r_1}(a^{(1)}) = \{x \in R^m : \rho(x, a^{(1)}) \le r_1\}$$

$$B_2 = \bar{B}_{r_2}(a^{(2)}) = \{x \in R^m : \rho(x, a^{(2)}) \le r_2\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$B_n = \bar{B}_{r_n}(a^{(n)}) = \{x \in R^m : \rho(x, a^{(n)}) \le r_n\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

yopiq sharlar ketma-ketligini qaraylik. Agar bu yopiq sharlar ketma-ketligining hadlari uchun quyidagi

$$B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots$$

munosabat o‘rinli bo‘lsa, $\{B_n\}$ ichma-ich joylashgan yopiq sharlar ketma-ketligi deyiladi.

Aytaylik, $\{B_n\}$ $R^m$ fazoda ichma-ich joylashgan yopiq sharlar ketma-ketligi bo‘lsin.

**4-teorema.** Agar $n \to \infty$ da shar radiuslari $r_n$ nolga intilsa, ya’ni

$$\lim_{n\to\infty} r_n = 0$$

bo‘lsa, u holda barcha yopiq sharlarga tegishli bo‘lgan $a$ nuqta $(a \in R^m)$ mavjud va u yagona bo‘ladi.

◄ Shar markazlaridan tuzilgan

$$\{a^{(n)}\} \quad (a^{(n)} \in R^m, \quad n = 1,2,..)$$

ketma-ketlikni qaraylik. Uning fundamental ketma-ketlik bo‘lishini ko‘rsatamiz. Shartga ko‘ra $\lim_{n\to\infty} r_n = 0$ . Unda

$$\forall \varepsilon > 0, \quad \exists n_0 \in N, \quad n > n_0 : r_n < \varepsilon$$

bo‘ladi. Ayni paytda, yopiq sharlar ichma-ich joylashganligidan ixtiyoriy

$$P > n > n_0$$

uchun

$$\overline{B}_{r_\rho}(a^{(P)} \supset \overline{B}_{r_n}(a^{(n)}))$$

bo‘lib,

$$\rho(a^{(n)}, a^{(P)}) \le r_\rho < \varepsilon$$

bo‘ladi.

Demak, $\{a^{(n)}\}$ fundamentalketma-ketlik. Unda Koshi teoremasiga ko‘ra u yaqinlashuvchi bo‘ladi:

$$\lim_{n\to\infty} a^{(n)} = a. \qquad (a \in R^m)$$

Bu $a$ nuqta $\overline{B}_{r_n}(a^{(n)})$ to‘plamning limit nuqtasi va $\overline{B}_{r_n}(a^{(n)})$ yopiq bo‘lganligi uchun $a \in \overline{B}_{r_n}(a^{(n)})$ $(n = 1,2..)$ bo‘ladi. Demak, $a$ barcha sharlarga tegishli bo‘lgan nuqta. Faraz qilaylik, $a$ nuqtadan farqli barcha sharlarga tegishli bo‘lgan $b$ nuqta $(b \in R^m)$ mavjud bo‘lsin: $b \in \overline{B}_{r_n}(a^{(n)})$ $b \ne a$. Masofaning 3-xossasidan foydalanib topamiz:

$$\rho(a,b) \le \rho(a, a^{(n)}) + \rho(a^{(n)}, b) \le 2r_n.$$

Agar $n \to \infty$ da $r_n \to 0$ bo‘lishini e’tiborga olsak, keyingi munosabatdan $\rho(a,b) = 0$, ya’ni $a = b$ bo‘lishi kelib chiqadi.►

Odatda, bu teorema ichma-ich joylashgan yopiq sharlar prinsipi deyiladi.

## $4^0$. Qismiy ketma-ketliklar. Bolsano-Veyershtrass teoremasi.

$R^m$ fazoda $\{x^{(n)}\}$:

$$x^{(1)},\ x^{(2)},\ldots,\ x^{(n)},\ldots$$

ketma-ketlik berilgan bo‘lsin. Ushbu ketma-ketlik

$$x^{(n_1)},\ x^{(n_2)},\ \ldots,\ x^{(n_k)},\ldots,$$

bunda,

$$n_1 < n_2 < \ldots < n_k < \ldots;\ n_k \in N,\quad k = 1,2,\ldots,$$

berilgan $\{x^{(n)}\}$ ketma-ketlikning qismiy ketma-ketligi deyiladi. U $\{x^{(n_k)}\}$ kabi belgilanadi.

Ravshanki, bitta ketma-ketlikning turlicha qismiy ketma-ketliklari bo‘ladi.

Agar $\{x^{(n)}\}$ ketma-ketlik yaqinlashuvchi bo‘lib,

$$\lim_{n\to\infty} x^{(n)} = a$$

bo‘lsa, bu ketma-ketlikning har qanday qismiy ketma-ketligi $\{x^{(n_k)}\}$ ham yaqinlashuvchi bo‘lib,

$$\lim_{k\to\infty} x^{(n_k)} = a$$

bo‘ladi.

Bu tasdiqning isboti ketma-ketlik limiti ta’rifidan bevosita kelib chiqadi.

Aytaylik, $R^m$ fazoda biror $M$ to‘plam berilgan bo‘lsin: $M \subset R^m$. Agar $R^m$ fazoda markazi $(0,0,..,0) \in R^m$, radiusi $r > 0$ bo‘lgan shar

$$U^0 = \{(x_1, x_2, \ldots, x_m) \in R^m : \rho((x_1, x_2, .., x_m,)\ (0,0,\ldots 0)) < r\}$$

topilsaki:

$$M \subset U^0$$

bo‘lsa, $M$ chegaralangan to‘plam deyiladi.

Endi Bolsano-Veyershtrass teoremasini isbotsiz keltiramiz.

**5-teorema (Bolsano-Veyershtrass teoremasi).** $R^m$ fazoda har qanday chegaralangan ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin.

**Xususiy hollar.** $m=1$ bo‘lganda $R^m = R$ bo‘lib, undagi ketma-ketlik sonlar ketma-ketligi bo‘ladi. Ma’lumki, sonlar ketma-ketligi va uning limiti 6-8-ma’ruzalarda batafsil o‘rganilgan.

$m=2$ bo‘lganda $R^m = R^2$ bo‘lib, undagi ketma-ketlik tekislik nuqtalaridan iborat

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), \dots \qquad (x_n \in R, \quad y_n \in R, \quad n=1,2,..)$$

ketma-ketlik bo‘ladi. Bu ketma-ketlikning limiti $\{x_n\}$ va $\{y_n\}$ sonlar ketma-ketliklarining limitlari orqali o‘rganiladi.

Masalan, ushbu

$$\{(-1^n), \ (-1)^n\}:$$

$$(-1,-1), \quad (1,1), \quad (-1,-1), \dots, ((-1)^n, (-1)^n), \dots$$

ketma-ketlik limitga ega bo‘lmaydi, chunki

$$x_n = (-1)^n, \quad y_n = (-1)^n \qquad (n=1,2,..)$$

ketma-ketliklar limitga ega emas.

## Mashqlar

1. Agar $x^0 \in R^m$ nuqta $M \subset R^m$ to‘plamning limit nuqtasi bo‘lsa, $M$ to‘plam elementlaridan tashkil topgan va $x^0$ nuqtaga yaqinlashadigan

$$\{x^{(n)}\} \qquad (x^{(n)} \in M, \quad x^{(n)} \neq x_0, \quad n=1,2,\dots)$$

ketma-ketliklarning mavjudligi ko‘rsatilsin.

2. Agar

$$\lim_{n\to\infty} x^{(n)} = a \qquad (x^{(n)} \in R^m, \ a \in R^m, \quad n=1,2,\dots)$$

bo‘lsa, $\{x^{(n)}\}$ ketma-ketlikning chegaralanganligi ko‘rsatilsin.

## Adabiyotlar


1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, II q.* T. "Voris-nashriyot", 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.


# Glossariy

$R^{(m)}$ **fazoda ketma-ketlik** - $(x_1^{(1)}, x_2^{(1)}, ..., x_m^{(1)})$, $(x_1^{(2)}, x_2^{(2)}, ..., x_m^{(2)})$, ... , $(x_1^{(n)}, x_2^{(n)}, ..., x_m^{(n)})$, ... .

$\{x^{(n)}\}$ **ketma-ketlikning limiti** - Agar $\forall\varepsilon > 0$ olinganda ham, shunday $n_0 \in N$ son topilsaki, barcha $n > n_0$ uchun

$$\rho(x^{(n)}, a) < \varepsilon$$

ya'ni

$$\forall\varepsilon > 0, \quad \exists n_0 \in N, \quad \forall n > n_0 : \quad \rho(x^{(n)}, a) < \varepsilon$$

bo'lsa, $a$ nuqta $\{x^{(n)}\}$ ketma-ketlikning limiti deyiladi.

$a$ **nuqta** $\{x^{(n)}\}$ **ketma-ketlikning limiti** - Agar $a \in R^m$ nuqtaning ixtiyoriy $U_\varepsilon(a)$ atrofi olingandan ham, $\{x^{(n)}\}$ ketma-ketlikning biror hadidan keyingi barcha hadlari shu atrofga tegishli bo'lsa, $a$ nuqta $\{x^{(n)}\}$ ketma-ketlikning limiti deyiladi.

**Yaqinlashuvchi ketma-ketlik** - Agar $\{x^{(n)}\}$ketma-ketlik limitga ega bo'lsa.

$\{x^{(n)}\}$ **fundamental ketma-ketlik** - $R^m$ fazoda $\{x^{(n)}\}$ ketma-ketlik berilgan bo'lsin. Agar $\forall\varepsilon > 0$ olinganda ham, shunday $n_0 \in N$ topilsaki, $\forall n > n_0$, $\forall P > n_0$ lar uchun

$$\rho(x^{(n)}, x^{(P)}) < \varepsilon$$

tengsizlik bajarilsa, $\{x^{(n)}\}$ fundamental ketma-ketlik deyiladi.

**Qismiy ketma-ketlik -** Ushbu ketma-ketlik

$$x^{(n_1)},\ x^{(n_2)},\ \ldots,\ x^{(n_k)},\ldots,$$

bunda,

$$n_1 < n_2 < \ldots < n_k < \ldots;\ n_k \in N,\quad k = 1,2,\ldots,\quad .$$

## Keys banki

**46-keys.** Masala o`rtaga tashlanadi: Agar

$$\lim_{n\to\infty} x^{(n)} = a,\quad \left(x^{(n)} \in \mathbb{R}^m,\ \alpha \in \mathbb{R}^m,\ n = 1,2,\ldots\right)$$

bo‘lsa, $\{x^{(n)}\}$ ketma-ketlikning chegaralanganligi ko‘rsatilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 24-amaliy mashg'ulot

## Na'muna uchun misollar yechimi

**1-misol.** $R^m$ fazoda ushbu

$$\left\{x^{(n)}\right\}=\left\{\frac{1}{n},\frac{1}{n},\ \ldots\frac{1}{n}\right\}$$

ketma-ketlikning limiti $a=(0,0,\ldots,0)$ bo'lishi ko'rsatilsin.

◄ $\forall\varepsilon>0$ sonini olib, unga ko'ra $n_0=\left[\frac{\sqrt{m}}{\varepsilon}\right]+1$ni topamiz.

Unda $\forall n>n_0$ uchun

$$\rho(x^{(n)},a)=\rho\left(\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right),(0,0,\ldots,0)\right)=\frac{\sqrt{m}}{n}<\frac{\sqrt{m}}{n_0}=\frac{\sqrt{m}}{\left[\frac{\sqrt{m}}{\varepsilon}\right]+1}<\varepsilon$$

bo'ladi. Demak,

$$\lim_{n\to\infty}x^{(n)}=a\ .$$ ►

## Misollar

1. Agar $x^0\in R^m$ nuqta $M\subset R^m$ to'plamning limit nuqtasi bo'lsa, $M$ to'plam elementlaridan tashkil topgan va $x^0$ nuqtaga yaqinlashadigan

$$\left\{x^{(n)}\right\}\qquad\left(x^{(n)}\in M,\quad x^{(n)}\neq x_0,\qquad n=1,2,\ldots\right)$$

ketma-ketliklarning mavjudligi ko'rsatilsin.

2. Agar

$$\lim_{n\to\infty}x^{(n)}=a\qquad\left(x^{(n)}\in R^m,\ a\in R^m,\qquad n=1,2,\ldots\right)$$

bo'lsa, $\left\{x^{(n)}\right\}$ ketma-ketlikning chegaralanganligi ko'rsatilsin.

3. $x^{(n)}=\left(\frac{13-n^2}{1+2n^2},\frac{2n-1}{2-3n}\right)$ ketma-ketlik limitini toping.

4. $x^{(n)}=\left(\frac{2-3n^2}{1+2n^2},\frac{2n-1}{2+3n}\right)$ ketma-ketlik limitini toping.

5. $x^{(n)}=\left(\frac{3n}{1+2n},\frac{2n-1}{2+3n}\right)$ ketma-ketlik limitini toping.

## Test

1. $x^{(n)}=\left(\frac{13-n^2}{1+2n^2},\frac{2n-1}{2-3n}\right)$ ketma-ketlik limitini toping.

A. *$\left(-\frac{1}{2},-\frac{2}{3}\right)$ B. $\left(\frac{1}{2},-\frac{2}{3}\right)$ C. $\left(-\frac{1}{2},\frac{2}{3}\right)$ D. $\left(\frac{1}{2},\frac{2}{3}\right)$

2. $x^{(n)}=\left(\frac{2-3n^2}{1+2n^2},\frac{2n-1}{2+3n}\right)$ ketma-ketlik limitini toping.

A. $\left(\frac{1}{2},-\frac{2}{3}\right)$ B. *$\left(-\frac{3}{2},\frac{2}{3}\right)$ C. $\left(-\frac{2}{3},\frac{2}{3}\right)$ D. $\left(\frac{3}{2},-\frac{2}{3}\right)$

3. $x^{(n)}=\left(\frac{3n}{1+2n},\frac{2n-1}{2+3n}\right)$ ketma-ketlik limitini toping.

A. $\left(\frac{1}{2},-\frac{2}{3}\right)$ B. $\left(-\frac{3}{2},\frac{2}{3}\right)$ C. $\left(-\frac{2}{3},\frac{2}{3}\right)$ D. *$\left(\frac{3}{2},\frac{2}{3}\right)$

4. $\lim\limits_{\substack{x\to 0\\ y\to a}}\frac{\sin xy}{x}$ limitni hisoblang.

A. 1 B. *$a$ C. $\frac{a}{2}$ D. 0

5. $\lim\limits_{\substack{x\to\infty\\ y\to a}}\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$ limitni hisoblang.

A. $\frac{e}{2}$ B. $a$ C. $\frac{a}{2}$ D. $*e$

6. $x^{(n)}=\left(\frac{1}{n^2},\frac{5}{n}\right)$ ketma-ketlik limitini toping.

A. $*(0,0)$ B. $(0,1)$ C. $(1,0)$ D. $(1,1)$

7. $x^{(n)}=\left(\frac{4n^2+1}{n^2},\frac{2}{n}\cos n\pi\right)$ ketma-ketlik limitini toping.

A. $(1,1)$ B. $(0,4)$ C. $*(4,0)$ D. $(4,1)$

8. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\left(x^2+y^2\right)^{x^2y^2}$ limitni hisoblang.

A. *1 B. 0 C. -1 D. 2

9. $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}}\left(\frac{xy}{x^2+y^2}\right)^{x^2}$ limitni hisoblang.

A. 1 B. -1 C. *0 D. 2

10. $x^{(n)}=\left(\frac{2}{n^2},\frac{5n-1}{n}\right)$ ketma-ketlik limitini toping.

A. $*(0,5)$ B. $(5,5)$ C. $(1,5)$ D. $(0,0)$

# Mavzu. Ko‘p o‘zgaruvchili funksiya va uning limiti

## 25-ma’ruza

**Reja**

$1^0$. Ko‘p o‘zgaruvchili funksiya tushunchasi.

$2^0$. Ko‘p o‘zgaruvchili funksiya limiti (karrali limiti) ta’riflari.

$3^0$. Funksiya limitining mavjudligi.

$4^0$. Takroriy limitlar.

### $1^0$. Ko‘p o‘zgaruvchili funksiya tushunchasi.

Faraz qilaylik, $R^m$ fazoda $E$ to‘plam berilgan bo‘lsin: $E \subset R^m$.

**1-ta’rif.** Agar $E$ to‘plamdagi har bir $x = (x_1, x_2, ... x_m)$ nuqtaga biror $f$ qoidaga ko‘ra bitta haqiqiy $u$ son mos qo‘yilgan bo‘lsa, $E$ to‘plamda ko‘p o‘zgaruvchili ($m$ ta o‘zgaruvchili) funksiya berilgan (aniqlangan) deyiladi. Uni

$$f : x = (x_1, x_2, ..., x_m) \to u \text{ yoki } u = f(x) = f(x_1, x_2, ..., x_m)$$

$$(x = (x_1, x_2, ..., x_m) \in R^m \,,\; u \in R)$$

kabi belgilanadi. Bunda $E$ funksiyaning berilish (aniqlanish) to‘plami, $x_1, x_2, ..., x_m$ lar (erkli o‘zgaruchilar) funksiya argumentlari, $u$ esa $x_1, x_2, ..., x_m$ larning funksiyasi deyiladi.

Masalan, $f$ - har bir

$$x = (x_1, x_2, ... x_m) \in M,$$
$$M = \{x \in R^m : \rho(x, 0) \le 1\}$$

nuqtaga ushbu

$$(x_1, x_2, ..., x_m) \to \sqrt{1 - x_1^2 - x_2^2 - ... - x_m^2}$$

qoida bilan bitta haqiqiy $u$ sonini mos qo‘ysin. Bu holda $M \subset R^m$ to‘plamda aniqlangan

$$u = \sqrt{1 - x_1^2 - x_2^2 - ... - x_m^2}$$

funksiya hosil bo‘ladi.

Aytaylik, $u = f(x_1, x_2, ... x_m)$ funksiya (ko‘p hollarda bu funksiyani $u = f(x)$ kabi yozamiz) $E \subset R^m$ to‘plamda berilgan bo‘lsin. $x^0 = (x_1^0, x_2^0, ..., x_m^0) \in E$ nuqtaga mos keluvchi $u_0$ son $u = f(x)$ funksiyaning $x^0$ nuqtadagi xususiy qiymati deyiladi: $u_0 = f(x^0)$.

Berilgan funksiyaning barcha xususiy qiymatlaridan iborat ushbu

$$\{u = f(x) : x \in E\} \tag{1}$$

sonlar to‘plam $u = f(x)$ funksiya qiymatlari to‘plami deyiladi. Agar (1) to‘plam chegaralangan bo‘lsa, $u = f(x) = f(x_1, x_2, ... x_m)$ funksiya $E$ to‘plamda chegaralangan deyiladi.

$R^{m+1}$ fazodagi ushbu

$$\{(x, f(x)) : x = (x_1, x_2, ... x_m) \in R^m, f(x) \in R\}$$

to‘plam ko‘p o‘zgaruvchili $u = f(x_1, x_2, ... x_m)$ funksiyaning grafigi deyiladi.

Faraz qilaylik, yuqorida qaralayotgan $f(x_1, x_2, ... x_m)$ funksiyada

$$x_1 = \varphi_1(t) = \varphi_1(t_1, t_2, ..., t_k),$$
$$x_2 = \varphi_2(t) = \varphi_2(t_1, t_2, ..., t_k),$$
$$..................................$$
$$x_m = \varphi_m(t) = \varphi_m(t_1, t_2, ..., t_k),$$

bo‘lsin, bunda $\varphi_i(t)$ funksiya $(i = 1, 2, ..., m)$ $T \subset R^k$ to‘plamda aniqlangan bo‘lib, $t = (t_1, t_2, ..., t_k) \in T$ bo‘lganda unga mos $x = (x_1, x_2, ... x_m) \in E$ bo‘lsin.

Natijada

$$f(x(t)) = f(\varphi_1(t_1,...,t_k),\varphi_2(t_1,...,t_k),....,\varphi_m(t_1,...,t_k)) = F(t_1,t_2,...,t_k)$$

funksiya hosil bo‘ladi. Uni murakkab funksiya deyiladi.

**$2^0$. Ko‘p o‘zgaruvchili funksiya limiti (karrali limiti) ta’riflari.**

Faraz qilaylik, $f(x)$ funksiya $(x \in R^m)$ $E \subset R^m$ to‘plamda berilgan, $x^0 \in R^m$ nuqta $E$ ning limit nuqtasi bo‘lsin. U holda $R^m$ fazoda shunday $\{x^{(n)}\}$:

$$x^{(1)}, x^{(2)},...., x^{(n)},....$$

ketma-ketlik topiladiki:

$$1)\ \forall n \in N \text{ da } x^{(n)} \in E, \qquad x^{(n)} \neq x^0,$$

$$2)\ n \to \infty \quad \text{da } x^{(n)} \to x^0$$

bo‘ladi (bunday ketma-ketliklar istalgancha bo‘ladi).

**2-ta’rif (Geyne).** Agar

$$1)\ \forall n \in N \text{ da } x^{(n)} \in E, \qquad x^{(n)} \neq x^0;$$

$$2)\ n \to \infty \quad \text{da } x^{(n)} \to x^0$$

shartlarni qanoatlantiruvchi ixtiyoriy $\{x^{(n)}\}$ ketma-ketlik uchun

$$n \to \infty \quad \text{da } f(x^{(n)}) \to A$$

bo‘lsa, $A$ $f(x) = f(x_1, x_2,...., x_m)$ funksiyaning $x^0 = (x_1^0, x_2^0,..., x_m^0)$ nuqtadagi limiti (karrali limiti) deyiladi. Uni $\lim\limits_{x \to x^0} f(x) = A$ yoki

$$\lim_{\substack{x_1 \to x_1^0 \\ x_2 \to x_2^0 \\ \cdots\cdots\cdots \\ x_m \to x_m^0}} f(x_1, x_2, ..., x_m) = A$$

kabi belgilanadi.

**Eslatma**. Agar

$$\{x^{(n)}\} \quad (x^{(n)} \in E, \quad x^{(n)} \neq x^0, n = 1,2,....),$$
$$\{y^{(n)}\} \quad (y^{(n)} \in E, \quad y^{(n)} \neq x^0, n = 1,2,....)$$

ketma-ketliklar uchun $n \to \infty$ da

$$x^{(n)} \to x^0 \quad , y^{(n)} \to x^0$$

bo‘lib,

$$f(x^{(n)}) \to A, \quad f(y^{(n)}) \to B, \ A \neq B$$

bo‘lsa, $f(x)$ funksiya $x^0$ nuqtada limitga ega bo‘lmaydi.

**3-ta’rif (Koshi).** Agar $\forall \varepsilon > 0$ son olinganda ham shunday $\delta = \delta(\varepsilon) > 0$ topilsaki, $0 < \rho(x, x^0) < \varepsilon$ tengsizlikni qanoatlantiruvchi $\forall x \in E \ \left(E \subset R^m\right)$ da

$$|f(x) - A| < \varepsilon$$

tengsizlik bajarilsa, $A$ son $f(x)$ funksiyaning $x^0$ nuqtadagi limiti (karrali limiti) deyiladi.

Bu ta’rifni qisqacha qilib quyidagicha ham aytsa bo‘ladi.

Agar

$$\forall \varepsilon > 0, \ \exists \delta > 0, \forall x \in E \cap \left(U_\delta(x^0) \setminus \{x^0\}\right): \quad |f(x) - A| < \varepsilon$$

bo‘lsa, $A$ soni $f(x)$ funksiyaning $x^0$ nuqtadagi limiti deyiladi.

**$3^0$. Funksiya limitining mavjudligi.**

Faraz qilaylik, $f(x) = f(x_1, x_2, ...x_m)$ funksiya $E \subset R^m$ to‘plamda berilgan bo‘lib, $x^0 = (x_1^0, x_2^0, ..., x_m^0) \in R^m$ nuqta $E$ to‘plamning limit nuqtasi bo‘lsin.

**1-teorema (Koshi).** $f(x)$ funksiya $x^0$ nuqtada limitga ega bo‘lishi uchun $\forall \varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilib,

$$\forall x' \in E \cap \left(U_\delta(x^0) \setminus \{x^0\}\right), \quad \forall x'' \in E \cap \left(U_\delta(x^0) \setminus \{x^0\}\right)$$

nuqtalarda

$$|f(x'') - f(x')| < \varepsilon$$

tengsizlikning bajarilishi zarur va etarli.

◀ **Zarurligi**. Aytaylik, $f(x)$ funksiya $x^0$ nuqtada $A$ limitga ega bo‘lsin:

$$\lim_{x\to x^0} f(x)=A.$$

Limit ta’rifiga ko‘ra,

$$\forall\varepsilon>0,\ \ \exists\delta>0,\ \forall x\in E\cap\left(U_\delta(x^0)\setminus\{x^0\}\right):$$

$$|f(x)-A|<\frac{\varepsilon}{2}$$

bo‘ladi, Jumladan

$$x'\in E\cap(U_\delta(x^0)\setminus\{x^0\}),\quad x''\in E\cap(U_\delta(x^0)\setminus\{x^0\}):$$

nuqtalar uchun

$$|f(x')-A|<\frac{\varepsilon}{2}\ ,\quad |f(x'')-A|<\frac{\varepsilon}{2}$$

bo‘ladi.

Keyingi tengsizliklardan

$$|f(x'')-f(x')|\le|f(x')-A|+|f(x'')-A|<\varepsilon$$

bo‘lishi kelib chiqadi.

**Yetarliligi.** Aytaylik, $\forall\varepsilon>0$ son olinganda ham shunday $\delta>0$ son topiladiki,

$$\forall x'\in E\cap(U_\delta(x^0)\setminus\{x^0\})\ ,\quad \forall x''\in E\cap(U_\delta(x^0)\setminus\{x^0\})$$

nuqtalar uchun

$$|f(x'')-f(x')|<\varepsilon$$

tengsizlik bajariladi.

$x^0$ nuqtaga intiluvchi ikkita $\{x^{(n)}\}$, $\{y^{(n)}\}$ ketma-ketlik-larni olamiz: $n\to\infty$ da

$$x^{(n)}\to x^0\qquad (x^{(n)}\in E,\quad x^{(n)}\ne x^0,\ n=1,2,....),$$

$$y^{(n)}\to x^0\qquad (y^{(n)}\in E,\quad y^{(n)}\ne x^0,\ n=1,2,....).$$

Bu ketma-ketliklar hadlaridan foydalanib, ushbu

$$x^{(1)},\ y^{(1)},\ x^{(2)},\ y^{(2)},....,x^{(n)},\ y^{(n)},....$$

ketma-ketlikni hosil qilamiz. Uni $\{z^{(n)}\}$ ketma-ketlik deylik. Ravshanki, bu ketma-ketlikning ham limiti $x^0$ bo‘ladi: $n\to\infty$ da

$$z^{(n)}\to x^0 \qquad (z^{(n)}\in E,\quad z^{(n)}\neq x^0,\ n=1,2,....),$$

Limit ta’rifiga binoan yuqoridagi $\delta>0$ songa ko‘ra shunday $n_0\in N$ topiladiki, $\forall n>n_0,\quad \forall m>n_0$ da

$$\begin{aligned} z^{(n)} &\in E\cap\left(U_\delta(x^0)\setminus\{x^0\}\right),\\ z^{(m)} &\in E\cap\left(U_\delta(x^0)\setminus\{x^0\}\right) \end{aligned}$$

bo‘ladi.

Teoremaning shartidan

$$f(z^{(n)})-f(z^{(m)})|<\varepsilon$$

tengsizlikning bajarilishi kelib chiqadi. Demak, $\{f(z^{(n)})\}$ sonlar ketma-ketligi fundamental ketma-ketlik bo‘ladi. Binobarin, u yaqinlashuvchi :

$$n\to\infty \quad \text{da}\ \ f(z^{(n)})\to A.$$

Unda $n\to\infty$ da

$$f(x^{(n)})\to A,\quad f(y^{(n)})\to A$$

bo‘lib, funksiya limitining Geyne ta’rifiga ko‘ra

$$\lim_{x\to x^0} f(x)=A$$

bo‘ladi.►

**4$^0$. Takroriy limitlar.**

Faraz qilaylik, $f(x)=f(x_1,x_2,...x_m)$ funksiya $E\subset R^m$ to‘plamda berilgan bo‘lib, $x^0=(x_1^0,\ x_2^0,...,x_m^0)\in R^m$ shu $E$ to‘plamning limit nuqtasi bo‘lsin.

$m$ ta $x_1,x_2,...x_m$ o‘zgaruvchilarga bog‘liq bo‘lgan $f(x_1,...x_m)$ funksiyada $x_2,x_3,...x_m$ o‘zgaruvchilar tayinlansa, ravshanki, u bitta $x_1$ o‘zgaruvchining funksiyasiga aylanadi. Aytaylik, bu funksiya $x_1 \to x_1^0$ da limiga ega bo‘lsin :

$$\lim_{x_1 \to x_1^0} f(x_1, x_2, ..., x_m) = \varphi_1(x_2, x_3, ..., x_m).$$

Endi $\varphi_1(x_2, x_3, ..., x_m)$ funksiyada $x_3, x_4, ..., x_m$ o‘zgaruvchilari tayinlanib, so‘ng $x_2 \to x_2^0$ limitga o‘tilsa

$$\lim_{x_2 \to x_2^0} \varphi_1(x_2, x_3, ..., x_m) = \varphi_2(x_3, x_4, ..., x_m)$$

bo‘lib, berilgan funksiyaning

$$\lim_{x_2 \to x_2^0} \lim_{x_1 \to x_1^0} f(x_1, x_2, ..., x_m)$$

limiti hosil bo‘ladi.

Xuddi shunga o‘xshash $f(x_1, x_2, ..., x_m)$ funksiyaning

$$x_{i_1}, x_{i_2}, \ldots, x_{i_k}$$

o‘zgaruvchilari mos ravishda $x_{i_1}^0, x_{i_2}^0, ..., x_{i_k}^0$ larga intilgandagi limiti

$$\lim_{x_{i_k} \to x_{i_k}^0} \ldots \lim_{x_{i_1} \to x_{i_1}^0} f(x_1, x_2, ..., x_m)$$

ni ham qarash mumkin.

Odatda, bu limitlar $f(x_1, x_2, ..., x_m)$ funksiyaning takroriy limitlari deyiladi. $f(x_1, x_2, ..., x_m)$ funksiya argumentlari $x_1, x_2, ... x_m$ lar mos ravishda $x_1^0, x_2^0, ..., x_m^0$ sonlarga turli tartibda intilganda funksiyaning turli takroriy limitlari hosil bo‘ladi.

Ko‘p o‘zgaruvchili funksiyaning limiti (karrali limiti) hamda uning takroriy limitlari turlicha munosabatda bo‘ladi. Ular haqida, xususiy holda keyingi banda bayon etamiz.

**Xususiy hollar**. $m = 1$ bo‘lganda bitta o‘zgaruvchiga bog‘liq bo‘lgan $R$ fazodagi biror to‘plamda aniqlangan $u = f(x)$ funksiyaga ega bo‘lamiz. Bu funksiya va uning limiti 11- va 12-ma’ruzalarda o‘rganilgan va to‘liq ma’lumotlar keltirilgan.

$m=2$ bo‘lganda $R^2$ fazodagi (tekislikdagi) biror to‘plamda aniqlangan ikki o‘zgaruvchiga bog‘liq bo‘lgan $u=f(x,y)$ funksiyaga ega bo‘lamiz.

Masalan,

$$u=\frac{\ln(x^2+y^2-1)}{\sqrt{n-x^2-y^2}}\,.$$

Bu funksiyaning aniqlanish to‘plami tekislikning ushbu

$$\begin{cases} x^2+y^2-1>0 \\ 4-x^2-y^2>0 \end{cases}$$

sistemani qanoatlantiradigan nuqtalar to‘plami 24-chizmada tasvirlangan xalqani ifodalaydi:

Ikki o‘zgaruvchiga bog‘liq bo‘lgan $u=f(x,y)$ funksiyaning grafigi umuman $R^3$ fazoda (biz yashab turgan fazoda) sirtni ifodalaydi.

Masalan, ushbu

$$u=x^2+y^2$$

funksiyaning grafigi $R^3$ fazoda 25–chizmada tasvirlangan aylanma paraboloid bo‘ladi:

Aytaylik, $f(x,y)$ funksiya $E\subset R^2$ to‘plamda berilgan bo‘lib, $(x_0,y_0)\in R^2$
u
nuqta $E$ ning limit nuqtasi bo‘lsin. Bu ikki o‘zgaruvchili funksiya limiti ta’riflari quyidagicha bo‘ladi:

Agar

1) $\forall_n\in N$ da $(x_n,y_n)\in E,\ (x_n,y_n)\neq(x_0,y_0)$

2) $n\to\infty$ da $(x_n,y_n)\to(x_0,y_0)$

shartni qanoatlantiruvchi ixtiyoriy $\{(x_n,y_n)\}$ nuqtalar ketma-ketligi uchun

x $\quad n\to\infty$ da $f(x_n,y_n)\to A$

bo‘lsa, $A$ funksiyaning $(x_0, y_0)$ nuqtadagi limiti (karrali limiti) deyiladi va

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = A \text{ yoki } \lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = A$$

kabi belgilanadi.

Agar $\forall\varepsilon > 0$ olinganda ham shunday $\delta > 0$ topilsaki, $0 < \rho\left((x,y),(x_0,y_0)\right) < \delta$ tengsizlikni qanoatlantiruvchi $\forall(x,y)\in E$ da

$$|f(x,y) - A| < \varepsilon$$

tengsizlik bajarilsa, $A$ son $f(x,y)$ funksiyaning $(x_0, y_0)$ nuqtadagi limiti (karrali limiti) deyiladi.

Berilgan funksiyaning ikkita takroriy limitlari

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)\ ,\ \lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$$

bo‘lishi mumkin.

**1-misol.** Ushbu

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, \text{агар } x^2+y^2 \neq 0 \text{ бўлса} \\ 0 \quad\ \ , \text{агар } x^2+y^2 = 0 \text{ бўлса} \end{cases}$$

funksiyaning $(x,y)\to(0,0)$ dagi limiti 0 bo‘lishi ko‘rsatilsin.

◄ Koshi ta’rifidan foydalanib topamiz:

$\forall\varepsilon > 0$ son uchun $\delta = 2\varepsilon$ deyilsa,

$$0 < \rho\left((x,y),(0,0)\right) < \delta$$

tengsizlikni qanoatlantiruvchi $\forall(x,y)\in R^2$ da

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{1}{2}\sqrt{x^2+y^2} =$$

$$= \frac{1}{2}\rho\left((x,y),(0,0)\right) < \frac{1}{2}\delta = \varepsilon$$

bo‘ladi. Demak,

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=0.►$$

**2-misol.** Ushbu

$$f(x,y)=\frac{x^2y^2}{x^2y^2+(x-y)^2}$$

funksiyaning $(0,0)$ nuqtada limiti mavjud emasligi ko‘rsatilsin.

◄ Ravshanki, bu funksiya

$$R^2\setminus\{(0,0)\}$$

to‘plamda aniqlangan va $(0,0)$ nuqta shu to‘plamning limit nuqtasi.

$(0,0)$ nuqtaga intiluvchi $\left\{\left(\frac{1}{n},\frac{1}{n}\right)\right\}, \left\{\left(\frac{1}{n},-\frac{1}{n}\right)\right\}$

ketma-ketliklarni olaylik:

$$\left(\frac{1}{n},\frac{1}{n}\right)\to(0,0)\ ,\ \left(\frac{1}{n},-\frac{1}{n}\right)\to(0,0).$$

$\left(\frac{1}{n},\frac{1}{n}\right)$ hamda $\left(\frac{1}{n},-\frac{1}{n}\right)$ nuqtalarda $(n=1,2,3,....)$ berilgan funksiyaning qiymatlari

$$f\left(\frac{1}{n},\frac{1}{n}\right)=1\ ,\ f\left(\frac{1}{n},-\frac{1}{n}\right)=\frac{1}{4n^2+1}\qquad (n=1,2,....)$$

bo‘lib,

$$f\left(\frac{1}{n},\frac{1}{n}\right)\to 1\ ,\ f\left(\frac{1}{n},-\frac{1}{n}\right)\to 0$$

bo‘ladi. Funksiya limitining Geyne ta’rifidan foydalanib, berilgan funksiyaning $(x,y)\to(0,0)$ da limitga ega emasligini topamiz.►

**3-misol.** Ushbu

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & \text{агар } x^2+y^2\neq 0 \text{ бўлса,}\\ 0 & ,\text{агар } x^2+y^2=0 \text{ бўлса}\end{cases}$$

funksiyaning $(0,0)$ da takroriy limitlari topilsin.

◀Berilgan funksiyaning takroriy limitlarini topamiz:

$$\lim_{x\to 0} f(x,y)=\lim_{x\to 0}\frac{xy}{\sqrt{x^2+y^2}}=0,\ \lim_{y\to 0}\lim_{x\to 0} f(x,y)=0,$$

$$\lim_{y\to 0} f(x,y)=\lim_{y\to 0}\frac{xy}{\sqrt{x^2+y^2}}=0,\ \lim_{x\to 0}\lim_{y\to 0} f(x,y)=0.$$

Demak, berilgan funksiyaning $(0,0)$ nuqtadagi takroriy limitlari bir-biriga teng bo‘lib, ular 0 ga teng.▶

**4-misol.** Ushbu

$$f(x,y)=\begin{cases}\dfrac{2x-y}{\sqrt{x+3y}}, & \text{агар } x+3y\neq 0 \text{ бўлса,}\\ 0 & ,\text{агар } x^2+3y=0 \text{ бўлса}\end{cases}$$

funksiyaning $(0,0)$ nuqtadagi takroriy limitlari topilsin.

◀Berilgan funksiyaning takroriy limitlari quyidagicha bo‘ladi:

$$\lim_{x\to 0}\frac{2x-y}{x+3y}=-\frac{1}{3},\ \lim_{y\to 0}\lim_{x\to 0}\frac{2x-y}{x+3y}=-\frac{1}{3};$$

$$\lim_{y\to 0}\frac{2x-y}{x+3y}=2,\ \lim_{x\to 0}\lim_{y\to 0}\frac{2x-y}{x+3y}=2.$$

Ayni paytda, berilgan funksiya $(x,y)\to(0,0)$ da limitga (karrali limitga) ega bo‘lmaydi, chunki

$$\left(\frac{1}{n},\frac{1}{n}\right)\to(0,0)\ ,\ \left(\frac{5}{n},\frac{4}{n}\right)\to(0,0)$$

ketma-ketliklar uchun

$$f\left(\frac{1}{n},\frac{1}{n}\right)=\frac{1}{4}\to\frac{1}{4},$$

$$f\left(\frac{5}{n},\frac{4}{n}\right)=\frac{6}{17}\to\frac{6}{17}$$

bo‘lib, ular bir-biriga teng emas.►

**5-misol.** Ushbu

$$f(x,y)=\begin{cases} x+y\sin\frac{1}{x}, \text{агар } x\neq 0 \text{ бўлса},\\ 0 \qquad ,\text{агар } x=0 \text{ бўлса}\end{cases}$$

funksiyaning $(0,0)$ nuqtadagi takroriy limitlari topilsin.

◄Bu funksiya uchun

$$\lim_{y\to 0} f(x,y)=x,\ \lim_{x\to 0}\lim_{y\to 0} f(x,y)=0,$$

bo‘lib,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

esa mavjud bo‘lmaydi.

Ayni paytda, $(x,y)\to(0,0)$ da berilgan funksiyaning limiti (karrali limiti) mavjud bo‘ladi, chunki

$$|f(x,y)-0|=\left|x+y\sin\frac{1}{x}\right|\leq|x|+|y| \qquad (x\neq 0)$$

bo‘lib,

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=0$$

bo‘ladi.►

Faraz qilaylik, $f(x,y)$ funksiya $R^2$ fazodagi

$$E=\{(x,y)\in R^2 : |x-x_0|<a, |y-y_0|<b\}$$

to‘plamda berilgan bo‘lsin.

**2-teorema.** Agar

1) $(x,y)\to(x_0,y_0)$ da $f(x,y)$ funksiyaning limiti (karrali limiti) mavjud va

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A,$$

2) har bir tayinlangan $x$ da

$$\lim_{y\to y_0} f(x,y)=\varphi(x) \qquad (2)$$

mavjud bo‘lsa, u holda

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)$$

takroriy limit mavjud va

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=A$$

bo‘ladi.

◀Aytaylik,

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A$$

bo‘lsin. Limit ta’rifiga binoan, $\forall\varepsilon>0$ olinganda ham shunday $\delta>0$ topiladiki, ushbu

$$\{(x,y)\in R^2:\ |x-x_0|<\delta,\ |y-y_0|<\delta\}\subset E$$

to‘plamning barcha $(x,y)$ nuqtalari uchun

$$|f(x,y)-A|<\varepsilon$$

tengsizlik bajariladi. Keyingi tengsizlikdan, $y\to y_0$ da limitga o‘tib topamiz:

$$|\varphi(x)-A|\le\varepsilon.$$

Demak,

$$\lim_{x\to x_0}\varphi(x)=A. \qquad (3)$$

(2) va (3) munosabatlardan

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=A$$

bo‘lishi kelib chiqadi.►

Xuddi shunga o‘xshash quyidagi teorema isbotlanadi.

**3-teorema.** Agar

1) $(x,y)\to(x_0,y_0)$ da $f(x,y)$ funksiyaning limiti (karrali limiti) mavjud va

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A,$$

2) har bir tayinlangan $y$ da

$$\lim_{x\to x_0} f(x,y)=\phi(y)$$

mavjud bo‘lsa, u holda

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)$$

takroriy limit mavjud va

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=A$$

bo‘ladi.

**Natija**. Agar $f(x,y)$ funksiya uchun bir vaqtda yuqoridagi 2,3-teoremalarning shartlari bajarilsa, u holda

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)=\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$$

bo‘ladi.

**Mashqlar**

1. Funksiya limitining Geyne va Koshi ta’riflarining ekvivalentligi ko‘rsatilsin.

2. Limitga ega bo‘lgan funksiyalarning xossalari keltirilsin.

3. Ushbu $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} f(x,y)=A$ limit ta’riflansin.

4. Ushbu $\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}} (x^2,y^2)e^{-(x+y)}$ limit hisoblansin.

**Adabiyotlar**


1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, II q.* T. "Voris-nashriyot", 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.


# Glossariy

**Ko'p o'zgaruvchili ($m$ ta o'zgaruvchili) funksiya** - Agar $E$ to'plamdagi har bir $x=(x_1,x_2,...x_m)$ nuqtaga biror $f$ qoidaga ko'ra bitta haqiqiy $u$ son mos qo'yilgan bo'lsa, $E$ to'plamda ko'p o'zgaruvchili ($m$ ta o'zgaruvchili) funksiya berilgan (aniqlangan) deyiladi.

$u=f(x)$ **funksiyaning** $x^0$ **nuqtadagi xususiy qiymati** - $x^0=(x_1^0, x_2^0,...,x_m^0)\in E$ nuqtaga mos keluvchi $u_0$ son.

$u=f(x)$ **funksiya qiymatlari to'plami** - Berilgan funksiyaning barcha xususiy qiymatlaridan iborat ushbu

$$\{u=f(x): x\in E\}$$

sonlar to'plami.

$u=f(x)=f(x_1,x_2,...x_m)$ **funksiya** $E$ **to'plamda chegaralangan** - Agar funksiya qiymatlari to'plami chegaralangan bo'lsa.

**Ko'p o'zgaruvchili** $u=f(x_1,x_2,...x_m)$ **funksiyaning grafigi** - $R^{m+1}$ fazodagi ushbu

$$\{(x, f(x)) : x = (x_1, x_2, \ldots x_m) \in R^m, f(x) \in R\}$$

to‘plam.

***f(x)* funksiyaning** $x^0 = (x_1^0, x_2^0, \ldots, x_m^0)$ **nuqtadagi limiti (karrali limiti) -** Agar

1) $\forall n \in N$ da $x^{(n)} \in E$, $x^{(n)} \neq x^0$;

2) $n \to \infty$ da $x^{(n)} \to x^0$

shartlarni qanoatlantiruvchi ixtiyoriy $\{x^{(n)}\}$ ketma-ketlik uchun

$$n \to \infty \quad \text{da} \quad f(x^{(n)}) \to A$$

bo‘lsa, $A$ $f(x) = f(x_1, x_2, \ldots, x_m)$ funksiyaning $x^0 = (x_1^0, x_2^0, \ldots, x_m^0)$ nuqtadagi limiti (karrali limiti) deyiladi.

# Keys banki

**47-keys.** Masala o`rtaga tashlanadi: Funksiya limitining Geyne va Koshi ta’riflarining ekvivalentligi ko‘rsatilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 25-amaliy mashg‘ulot

## Na’muna uchun misollar yechimi

**1-misol.** Ushbu

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, \text{агар } x^2+y^2\neq 0 \text{ бўлса}\\ 0 \quad , \text{агар } x^2+y^2=0 \text{ бўлса}\end{cases}$$

funksiyaning $(x,y)\to(0,0)$ dagi limiti 0 bo‘lishi ko‘rsatilsin.

◄ Koshi ta’rifidan foydalanib topamiz:

$\forall\varepsilon>0$ son uchun $\delta=2\varepsilon$ deyilsa,

$$0<\rho\left((x,y),(0,0)\right)<\delta$$

tengsizlikni qanoatlantiruvchi $\forall(x,y)\in R^2$ da

$$|f(x,y)-0|=\left|\frac{xy}{\sqrt{x^2+y^2}}\right|\leq\frac{1}{2}\sqrt{x^2+y^2}=$$

$$=\frac{1}{2}\rho\left((x,y),(0,0)\right)<\frac{1}{2}\delta=\varepsilon$$

bo‘ladi. Demak,

$$\lim_{\substack{x\to0\\ y\to0}} f(x,y)=0. ►$$

**2-misol.** Ushbu

$$f(x,y)=\frac{x^2y^2}{x^2y^2+(x-y)^2}$$

funksiyaning $(0,0)$ nuqtada limiti mavjud emasligi ko‘rsatilsin.

◄ Ravshanki, bu funksiya

$$R^2 \setminus \{(0,0)\}$$

to‘plamda aniqlangan va $(0,0)$ nuqta shu to‘plamning limit nuqtasi.

$(0,0)$ nuqtaga intiluvchi $\left\{\left(\frac{1}{n},\frac{1}{n}\right)\right\}, \left\{\left(\frac{1}{n},-\frac{1}{n}\right)\right\}$

ketma-ketliklarni olaylik:

$$\left(\frac{1}{n},\frac{1}{n}\right) \to (0,0) \ , \left(\frac{1}{n},-\frac{1}{n}\right) \to (0,0).$$

$\left(\frac{1}{n},\frac{1}{n}\right)$ hamda $\left(\frac{1}{n},-\frac{1}{n}\right)$ nuqtalarda $(n=1,2,3,....)$ berilgan funksiyaning qiymatlari

$$f\left(\frac{1}{n},\frac{1}{n}\right)=1 \ , \ f\left(\frac{1}{n},-\frac{1}{n}\right)=\frac{1}{4n^2+1} \qquad (n=1,2,....)$$

bo‘lib,

$$f\left(\frac{1}{n},\frac{1}{n}\right) \to 1 \ , \ f\left(\frac{1}{n},-\frac{1}{n}\right) \to 0$$

bo‘ladi. Funksiya limitining Geyne ta’rifidan foydalanib, berilgan funksiyaning $(x,y) \to (0,0)$ da limitga ega emasligini topamiz.►

**3-misol.** Ushbu

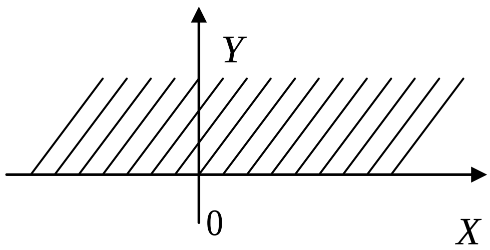
$$z = x + \sqrt{y}$$

Funksiyaning aniqlanish sohasi topilsin.

Bu funksiyaning aniqlanish sohasi tekislikning shunday $(x, y)$ nuqtalaridan iborat to‘plam bo‘lishi kerakki, bu to‘plam nuqtalari uchun $x+\sqrt{y}$ ifoda ma’noga, ya’ni haqiqiy son qiymatga ega bo‘lsin.

Ravshanki, buning uchun $y \geq 0$ bo‘lishi lozim.

Demak, berilgan funksiyaning aniqlanish sohasi $XOY$ tekisligining yuqori yarmidan iborat. (1-chizma).




1-chizma.

## Misollar

1. Quyidagi funksiyalarning aniqlanish soxalari topilsin.

1) $z=x^2+2y$  2) $z=x+\sqrt{y}$  3) $z=\dfrac{4}{x^2+y^2}$  4) $z=\sqrt{xy}$

5) $z=\sqrt{x-y}$  6) $z=\sqrt{1-\left(x^2+y^2\right)}$  7) $z=\ln\left(x+y\right)$  8) $z=\sqrt{x-\sqrt{y}}$

9) $z=\ln\left(y^2-4x+8\right)$  10) $z=\arcsin\dfrac{y}{x}$

2. Quyidagi limitlar topilsin:

1) $\lim\limits_{\substack{x\to1\\ y\to2}}\left(x^2-y^2\right)$  2) $\lim\limits_{\substack{x\to0\\ y\to1}}\dfrac{1-xy}{x^2+y^2}$  3) $\lim\limits_{\substack{x\to0\\ y\to0}}\dfrac{1}{x^2+y^2}$  4) $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\dfrac{1}{x^2+y^2}$

5) $\lim\limits_{\substack{x\to0\\ y\to0}}\dfrac{x}{x+y}$  6) $\lim\limits_{\substack{x\to0\\ y\to0}}\dfrac{\sin xy}{x}$  7) $\lim\limits_{\substack{x\to\infty\\ y\to a}}\left(1+\dfrac{y}{x}\right)^x$  8) $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\dfrac{x+y+2x^2+2y^2}{x^2+y^2}$

9) $\lim\limits_{\substack{x\to0\\ y\to0}}\dfrac{x^2y}{x^2+y^2}$  10) $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\left(x^2+y^2\right)\sin\dfrac{1}{x^2+y^2}$

# Test

1. Limit hisoblans $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y}{x^2+y^2}$ .

A) 0 B) 1 C) $\infty$ D) aniqlab bo`lmaydi

2. $u = x^2 + 2y^2 + 5z^2 + 2xy + 4yz - 20z$ funksiyaning qiymatlar sohasini toping.

A) $[-100; +\infty)$ B) $[-10; 10]$ C) $(-10; 100)$ D) $(-\infty; +\infty)$

3. $u = \ln(1 - x - y - z)$ funksiyaning aniqlanish sohasini toping.

A) $[12; +\infty)$ B) $(-\infty; 12)$ C) $(-\infty; +\infty)$ D) $[-12; 12)$

4. Hisoblang $\lim\limits_{y\to\infty}(\lim\limits_{x\to\infty}\frac{x^2+y^2}{x^4+y^2})$ .

*A)* 0 B) 1 C) 2 D) *e*

5. Hisoblang. $\lim\limits_{x\to\infty}(\lim\limits_{y\to\infty}\frac{x^2+y^2}{x^4+y^2})$ .

A) 1 B) 0 C) 2 D) *e*

6. Hisoblang $\lim\limits_{\substack{x\to 0\\ y\to a}} \frac{\sin xy}{x}$ .

A) $a$ B) 0 C) 1 D) -1

7. Hisoblang $\lim\limits_{\substack{x\to +\infty\\ y\to +\infty}} \frac{x+y}{e^{x+y}}$ .

A) 0 B) 1 C) 2 D) *e*

8. $u = \sqrt{x^2 + y^2 - 1} + \sqrt{4 - x^2 - y^2}$ funksiyaning aniqlanish sohasini toping.

A) $1 \le x^2 + y^2 \le 4$ B) $1 \le x^2 + y^2$ C) $x^2 + y^2 \le 4$ D)

$x^2 + y^2 \le 5$

9. $u = x^2 + \sqrt{y} + 1$ funksiyaning aniqlanish sohasini toping.

A) $-\infty < x < +\infty,\ 0 \le y < +\infty$ B) $-\infty < x < +\infty$ C) $0 \le y < +\infty$

D) $-\infty < x < 0,\ y \le 0$

10. $u = \sqrt{1 - x^2 - y^2}$ funksiyaning aniqlanish sohasini toping.

A) $x^2 + y^2 \le 1$ B) $x^2 + y^2 \ge 1$ C) $x^2 + y^2 \le 4$ D) $x^2 + y^2 \le 16$

# Mavzu. Ko‘p o‘zgaruvchili funksiyaning uzluksizligi. Tekis uzluksizlik. Kantor teoremasi.

## 26-ma’ruza

**Reja**

$1^0$. Ko‘p o‘zgaruvchili funksiya uzluksizligi tushunchasi.

$2^0$. Uzluksiz funksiyalarning sodda xossalari.

$3^0$. To‘plamda uzluksiz bo‘lgan funksiyalarning xossalari.

$4^0$. Funksiyaning tekis uzluksizligi. Kantor teoremasi.

**$1^0$. Ko‘p o‘zgaruvchili funksiya uzluksizligi tushunchasi**.

Faraz qilaylik, $f(x)=f(x_1,x_2,...,x_m)$ funksiya $R^m$ fazodagi $E$ to‘plamda berilgan bo‘lib, $x^0\in E$ nuqta $E$ to‘plamning limit nuqtasi bo‘lsin.

**1-ta’rif.** Agar

$$\lim_{x\to x_0} f(x)=f\left(x^0\right) \tag{1}$$

bo‘lsa, $f(x)$ funksiya $x^0$ nuqtada uzluksiz deyiladi.

**2-ta’rif (Geyne).** Agar

1) $\forall n\in N$ da $x^{(n)}\in E$ ;

2) $n\to\infty$ da $x^{(n)}\to x^0$

shartlarni qanoatlantiruvchi ixtiyoriy $\left\{x^{(n)}\right\}$ ketma-ketlik uchun

$$n\to\infty \text{ da } f\left(x^{(n)}\right)\to f\left(x^0\right)$$

bo‘lsa, $f(x)$ funksiya $x^0$ nuqtada uzluksiz deyiladi.

**3-ta'rif (Koshi).** Agar

$$\forall \varepsilon>0, \exists \delta>0, \quad \forall x \in E \bigcap U_{\delta}\left(x^{0}\right), \left| f(x)-f\left(x^{0}\right)\right|<\varepsilon$$

bo'lsa, $f(x)$ funksiya $x^{0}$ nuqtada uzluksiz deyiladi.

Umuman, $u=f(x)$ funksiyaning $x^{0} \in E$ nuqtadagi uzluksizligi quyidagini anglatadi:

$$\forall \varepsilon>0, \exists \delta>0, \quad \forall x \in E \bigcap U_{\delta}\left(x^{0}\right), \ f(x) \in U_{\varepsilon}\left(f\left(x^{0}\right)\right).$$

Odatda, ushbu

$$\Delta u=f(x)-f\left(x^{0}\right) \qquad \left(x=\left(x_{1}, x_{2}, \ldots, x_{m}\right), \ x^{0}=\left(x_{1}^{0}, \ldots, x_{k}^{0}\right)\right)$$

ayirma, $u=f(x)$ funksiyaning $x^{0}$ nuqtadagi orttirmasi (to'liq orttirmasi) deyiladi.

Agar

$$\Delta x_{1}=x_{1}-x_{1}^{0}, \ \Delta x_{2}=x_{2}-x_{2}^{0}, \ \ldots, \ \Delta x_{m}=x_{m}-x_{m}^{0}$$

deyilsa, unda

$$\Delta u=f\left(x_{1}^{0}+\Delta x_{1}, \ x_{2}^{0}+\Delta x_{2}, \ \ldots, \ x_{m}^{0}+\Delta x_{m}\right)-f\left(x_{1}^{0} \ x_{2}^{0} \ \ldots, x_{m}^{0}\right)$$

bo'ladi. Yuqoridagi (1) munosabatdan foydalanib quyidagi tasdiqni ayta olamiz:

$f(x)$ funksiyaning $x^{0}$ nuqtada uzluksiz bo'lishi uchun

$$\lim_{x \to x_{0}} \Delta u=0, \text{ ya'ni } \lim_{\substack{\Delta x_{1} \to 0 \\ \Delta x_{2} \to 0 \\ \cdots\cdots\cdots \\ \Delta x_{m} \to 0}} \Delta u=0$$

bo'lishi zarur va yetarli.

Yuqoridagi ta'riflar ekvivalent ta'riflar bo'ladi.

Agar (1) munosabat bajarilmasa $f(x)$ funksiya $x^{0}$ nuqtada uzilishga ega deyiladi.

**4-ta'rif.** Agar $f(x)$ funksiya $E$ to'plamning har bir nuqtasida uzluksiz bo'lsa, funksiya shu $E$ to'plamda uzluksiz deyiladi.

Ko'p o'zgaruvchili funksiyalarda funksiyaning nuqtadagi to'liq orttirmasi tushunchasi bilan bir qatorda uning xususiy orttirmalari tushunchalari ham kiritiladi.

Ushbu

$$\Delta_{x_1} u = f(x_1^0 + \Delta x_1, x_2^0, x_3^0, \ldots, x_m^0) - f(x_1^0, x_2^0, \ldots, x_m^0),$$
$$\Delta_{x_2} u = f(x_1^0, x_2^0 + \Delta x_2, x_3^0, \ldots, x_m^0) - f(x_1^0, x_2^0, \ldots, x_m^0),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Delta_{x_m} u = f(x_1^0, x_2^0, \ldots, x_{m-1}^0, x_m^0 + \Delta x_m) - f(x_1^0, x_2^0, \ldots, x_m^0),$$

ayirmalar mos ravishda $f(x)$ funksiyaning $x^0$ nuqtadagi $x_1, x_2, \ldots, x_m$ o'zgaruvchilar bo'yicha xususiy orttirmalari deyiladi.

Ravshanki,

$$\lim_{x \to x^0} \Delta u = 0 \quad \Rightarrow \begin{cases} \lim\limits_{\Delta x_1 \to 0} \Delta_{x_1} u = 0\ , \\ \lim\limits_{\Delta x_2 \to 0} \Delta_{x_2} u = 0\ , \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \lim\limits_{\Delta x_m \to 0} \Delta_{x_m} u = 0 \end{cases}$$

bo'ladi. Biroq, $\Delta x_k \to 0$ da $\Delta_{x_k} u \to 0$ $(k = 1, 2, \ldots, m)$ bo'lishidan

$$\lim_{x \to x_0} \Delta u = 0$$

bo'lishi har doim kelib chiqavermaydi (bunga misol keyingi punktda keltiriladi).

### $2^0$. Uzluksiz funksiyalarning sodda xossalari.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $E \subset R^m$ to'plamda berilgan bo'lib, $x^0 \in E$ nuqtada uzluksiz bo'lsin. U holda

$$c \cdot f(x), \quad f(x) \pm g(x), \quad f(x) \cdot g(x), \quad \frac{f(x)}{g(x)} \quad \left(g(x^0) \neq 0\right)$$

funksiyalar ham $x^0$ nuqtada uzluksiz bo'ladi, bunda $c = const$.

Bu tasdiqning isboti 15-ma'ruzadagi mos tasdiqning isboti kabidir.

Aytaylik,

$$\begin{aligned} x_1 &= \varphi_1(t_1,t_2,....,t_k) = \varphi_1(t), \\ x_2 &= \varphi_2(t_1,t_2,....,t_k) = \varphi_2(t), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m &= \varphi_m(t_1,t_2,....,t_k) = \varphi_m(t) \end{aligned} \qquad (2)$$

funksiyalarning har biri $M \subset R^k$ to‘plamda aniqlangan bo‘lsin. Bu (2) munosabat natijasida $M$ to‘plamning har bir $t=(t_1,t_2,....,t_k)$ nuqtasiga mos keluvchi $R^m$ fazoning $x=(x_1,x_2,....,x_m)$ nuqtasi hosil bo‘ladi. Bunday nuqtalar to‘plamini $E$ deylik. Ravshanki, $E \subset R^m$ bo‘ladi.

Faraz qilaylik, $E$ to‘plamda

$$u = f(x) = f(x_1,x_2,....,x_m)$$

funksiya aniqlangan bo‘lsin. Natijada

$$t \in M \to x \in \mathrm{E} \to u \in R,$$

ya’ni

$$(t_1,t_2,....,t_k) \in M \to (x_1,x_2,....,x_m) \in \mathrm{E} \to u \in R$$

bo‘lib,

$$u = f(x(t)) = f\left(\varphi_1(t_1,...,t_k),\varphi_2(t_1,...,t_k),...,\varphi_m(t_1,...,t_k)\right) = \Phi(t) = \Phi(t_1,t_2,...,t_k)$$

funksiya hosil bo‘ladi. Uni murakkab funksiya deyilar edi.

**1-teorema.** Agar

$$x_1 = \varphi_1(t),\ \ x_2 = \varphi_2(t),....,x_m = \varphi_m(t) \qquad (t=(t_1,....,t_k))$$

funksiyalar $t^0 = (t_1^0,...,t_k^0) \in M \subset R^k$ nuqtada uzluksiz, $u = f(x)$ funksiya $x^0 = (x_1^0,.x_2^0,...,x_m^0) \in E \subset R^m$ nuqtada $(x_1^0 = \varphi_1(t^0),\ x_2^0 = \varphi_2(t^0),...,x_m^0 = \varphi_m(t^0))$ uzluksiz bo‘lsa, u holda $f(x(t)) = f(\varphi_1,\varphi_2,....,\varphi_m)$ murakkab funksiya $t^0$ nuqtada uzluksiz bo‘ladi.

◄ Shartga ko‘ra $f(x)$ funksiya $x^0$ nuqtada uzluksiz. Unda ta’rifga binoan

$$\forall\varepsilon>0,\ \exists\delta>0,\ \forall x\in E\cap U_\delta(x^0):\quad \left|f(x)-f(x^0)\right|<\varepsilon \qquad (3)$$

bo‘ladi. Ravshanki.

$$\forall x\in E\cap U_\delta(x^0)\Rightarrow\left|x_i-x_i^0\right|<\delta \qquad (i=1,2,.....,m) \qquad (4)$$

bo‘ladi.

Shartga ko‘ra $\varphi_i(t)$ $(i=1,2,....,m)$ funksiyalar $t^0$ nuqtada uzluksiz. Ta’rifga ko‘ra $\delta>0$ uchun shunday $\delta_1>0$ topiladiki,

$$\forall t\in M\cap U_{\delta_i}(t^0):\left|\varphi_i(t)-\varphi_i(t^0)\right|<\delta$$

bo‘ladi, $i=1,2,...,m$

Endi $\min\{\delta_1,\delta_2,....,\delta_m\}=\delta$ deb olamiz. U holda $\forall t\in M\cap U_\delta(t^0)$ va barcha $i=1,2,...,m$ lar uchun

$$\left|\varphi_i(t)-\varphi_i(t^0)\right|<\delta\text{, ya’ni }\left|x_i-x_i^\circ\right|<\delta$$

bo‘ladi. (3) va (4) munosabatlardan $\forall t\in M\cap U_\delta(t^0)$ uchun

$$\left|f(x(t))-f(x(t_\circ)\right|<\varepsilon$$

bo‘lishi kelib chiqadi. Demak, murakkab $f(x(t))$ funksiya $t^\circ$ nuqtada uzluksiz.►

**$3^0$. To‘plamda uzluksiz bo‘lgan funksiyalarning xossalari.**

Endi to‘plamda uzluksiz bo‘lgan funksiyalarning xossalarini keltiramiz.

**2- teorema**. Agar $f(x)$ funksiya chegaralangan yopiq $E\subset R^m$ to‘plamda uzluksiz bo‘lsa, funksiya $E$ da chegaralangan bo‘ladi.

◄Aytaylik, funksiya shu $E$ to‘plamda chegaralanmagan bo‘lsin. Unda

$$\forall n\in N,\ \ \exists x^{(n)}\in E\ :\left|f(x^{(n)})\right|\geq n \qquad (n=1,2,.....)$$

bo‘ladi. Ravshpnki, $\{x^{(n)}\}$ ketma-ketlik chegaralangan. Bolsano-Veyershtrass teoremasiga ko‘ra yaqinlashuvchi

$$\{x^{(n_k)}\} \qquad \left(x^{(n_k)}\in E,\ \ k=1,2,.....\right)$$

qismiy ketma-ketlik mavjud:

$$k \to \infty \text{ da } x^{(n_k)} \to x^0 \text{ va } x^0 \in E.$$

Ayni paytda, $f(x)$ funksiyaning $E$ da uzluksizligidan

$$k \to \infty \text{ da } f(x^{(n_k)}) \to f(x^0) \ .$$

bo‘lishi kelib chiqadi. Bu esa

$$k \to \infty \text{ da } \left| f(x^{(n_k)}) \right| \ge n_k \to +\infty$$

deyilishiga zid. Ziddiyat $f(x)$ funksiyaning $E$ da chegaralanmagan deyilishidan kelib chiqdi. Demak, $f(x)$ funksiya $E$ da chegaralangan.►

**3-teorema.** Agar $f(x)$ funksiya chegaralgan yopiq $E \subset R^m$ to‘plamda uzluksiz bo‘lsa, funksiya shu to‘plamda o‘zining aniq yuqori hamda aniq quyi chegaralariga erishadi, ya’ni

$$\exists x^{(*)} \in E, \quad \sup_{x \in E} \{f(x)\} = f(x^{(*)}),$$
$$\exists x^{(**)} \in E, \quad \inf_{x \in E} \{f(x)\} = f(x^{(**)})$$

bo‘ladi.

◄ Yuqoridagi teoremaga ko‘ra $f(x)$ funksiya $E$ to‘plamda chegaralangan bo‘ladi. Unda bu funksiya aniq chegaralarga ega:

$$\sup_{x \in E} \{f(x)\} = a, \qquad \inf_{x \in E} \{f(x)\} = b\,.$$

Aniq yuqori chegara ta’rifiga ko‘ra

$$\forall n \in N, \ \ \exists x^{(n)} \in E : a - \frac{1}{n} < f(x^{(n)}) \le a \qquad (n = 1, 2, \ldots)$$

bo‘ladi. Ravshanki, $\{x^{(n)}\}$ chegaralangan ketma-ketlik bo‘lib, undan $\{x^{(n_k)}\}$ qismiy ketma-ketlik ajratish mumkinki,

$$k \to \infty \text{ da } x^{(n_k)} \to x^{(*)} \text{ va } x^{(*)} \in E \tag{5}$$

bo‘ladi. Berilgan funksiyaning uzluksizligidan foydalanib topmiz:

$$k \to \infty \text{ da } f(x^{(n_k)}) \to f(x^{(*)}).$$

Ayni paytda,

$$\forall n \in N \text{ da } a - \frac{1}{n_k} < f(x^{(n_k)}) \le a$$

bo‘lib, undan $k \to \infty$ da

$$f(x^{(n_k)}) \to a \qquad (6)$$

bo‘lishi kelib chiqadi.

(5) va (6) munosabatlaradan

$$f(x^{(*)}) = a = \sup\{f(x)\} \quad (x^* \in E)$$

bo‘lishini topamiz.

Xuddi shunga o‘xshash

$$f(x^{(**)}) = b = \inf\{f(x)\} \quad (x^{(**)} \in E)$$

bo‘lishi isbotlanadi.►

**4-teorema**. Faraz qilaylik. $f(x) = f(x_1, x_2, ...., x_m)$ funksiya bog‘lamli $E \subset R^m$ to‘plamda berilgan bo‘lsin.

Agar

1) $f(x)$ funksiya $E$ da uzluksiz,

2) $a = (a_1, a_2, ...., a_m) \in E$ , $b = (b_1, b_2, ...., b_m) \in E$

nuqtalarda turli ishorali qiymatlarga ega

$$(f(a) > 0, \ f(b) < 0 \ \text{ yoki } \ f(a) < 0, \ f(b) > 0)$$

bo‘lsa, u holda shunday $c = (c_1, c_2, ...., c_m) \in E$ nuqta topiladiki

$$f(c) = 0$$

bo‘ladi.

◄ Aytaylik, $f(x)$ funksiya bog‘lamli $E \subset R^m$ to‘plamda uzluksiz bo‘lib,

$$f(a) < 0, \quad f(b) > 0$$

bo‘lsin.

$E$ bog‘lamli to‘plam. Binobarin, $a$ va $b$ nuqtalarni birlashtiruvchi va shu to‘plamga tegishli siniq chiziq topiladi. Agar bu siniq chiziq uchlarini ifodalovchi nuqtalarning birida $f(x)$ funksiya nolga aylansa teorema isbotlanadi.

Agar siniq chiziq uchlarida $f(x)$ funksiya nolga aylanmasa, u holda siniq chiziqning shunday kesmasi topiladiki, uning bir uchi $a' = (a_1', a_2', ...., a_m')$ da $f(a') < 0$, ikkinchi uchi $b' = (b_1', b_2', ...., b_m')$ da $f(b') > 0$ bo‘ladi.

Endi $f(x) = f(x_1, x_2, ...., x_m)$ ni shu kesma

$$k = \{(x_1, x_2, \dots, x_m) \in R^m : x_1 = a_1' + t(b_1' - a_1'),$$
$$x_2 = a_2' + t(b_2' - a_2'), \dots, x_m = a_m' + t(b_m' - a_m')\}$$

$(0 \le t \le 1)$ da qaraymiz. Unda

$$f(a_1' + t(b_1' - a_1'),\ a_2' + t(b_2' - a_2'), \dots a_m' + t(b_m' - a_m')) = F(t)$$

bo‘lib, bitta $t$ o‘zgaruvchiga bog‘liq funksiya hosil bo‘ladi. Bu funksiya $[0, 1]$ segmentda uzluksiz va

$$F(0) = f(a') < 0 \quad , \ F(1) = f(b') > 0$$

bo‘ladi. Unda 16-ma’ruzada keltirilgan teoremaga ko‘ra, shunday $t_0 \in (0,1)$ nuqta topiladiki,

$$F(t_0) = 0$$

ya’ni

$$f(a_1' + t_\circ(b_1' - a_1'),\ a_2' + t_\circ(b_2' - a_2'), \dots a_m' + t_\circ(b_m' - a_m')) = 0$$

bo‘ladi. Agar

$$c_1 = a_1' + t_\circ(b_1' - a_1'),\ c_2 = a_2' + t_\circ(b_2' - a_2'), \dots, c_m = .a_m' + t_\circ(b_m' - a_m'))$$

deyilsa, unda $c = (c_1, c_2, ...., c_m) \in E$ nuqtada

$$f(c) = 0$$

bo‘ladi.►

**5-teorema**. Faraz qilaylik, $f(x)$ funksiya bog‘lamli $E\subset R^m$ to‘plamda berilgan bo‘lsin. Agar

1) $f(x)$ funksiya $E$ da uzluksiz,

2) $a\in E, b\in E$ nuqtalarda $f(a)=A$, $f(b)=B$

qiymatlarga ega va $A\neq B$ bo‘lsa, u holda A bilan V orasida har qanday $C$ son olinsa ham, shunday $c\in E$ nuqta topiladiki,

$$f(c)=C$$

bo‘ladi.

◀Bu teorema yuqoridagi 4–teorema kabi isbotlanadi.▶

**$4^0$ Funksiyaning tekis uzluksizligi. Kantor teoremasi**.

Aytaylik, $f(x)$ funksiya $E\subset R^m$ to‘plamda berilgan bo‘lsin.

**5-ta'rif.** Agar $\forall\varepsilon>0$ son olinganda ham shunday $\delta=\delta(\varepsilon)>0$ son topilsaki,

$$\rho(x',x'')<\delta$$

tengsizlikni qanoatlantiruvchi ixtiyoriy $x'\in E, x''\in E$ uchun

$$|f(x'')-f(x')|<\varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya $E$ to‘plamda tekis uzluksiz deyiladi.

Agar $f(x)$ funksiya $E$ to‘plamda tekis uzluksiz bo‘lsa, u shu to‘plamda uzluksiz bo‘ladi.

◀ Haqiqatdan ham, yuqoridagi ta'rifda $x''$ nuqta sifatida $x^0\in E$ olinsa, funksiyaning $x^0$ nuqtada uzluksizligi, binobarin $E$ to‘plamda uzluksizligi kelib chiqadi.▶

$f(x)$ funksiyaning $E\subset R^m$ to‘plamda tekis uzluksiz emasligi quyidagicha:

$$\exists\varepsilon_0>0,\ \forall\delta>0, \exists x'\in E, \exists x''\in E, \rho(x',x'')<\delta :\ |f(x'')-f(x')|\geq\varepsilon_0$$

bo‘ladi.

**6-teorema. (Kantor).** Agar $f(x)$ funksiya chegaralangan yopiq $E \subset R^m$ to‘plamda uzluksiz bo‘lsa, funksiya shu to‘plamda tekis uzluksiz bo‘ladi.

◀Faraz qilaylik, $f(x)$ funksiya chegaralangan yopiq $E \subset R^m$ to‘plamda uzluksiz bo‘lib, u shu to‘plamda tekis uzluksiz bo‘lmasin. Unda biror $\varepsilon_0 > 0$ son va $\forall n \in N$ uchun $E$ to‘plamda

$$\rho(x^{(n)}, y^{(n)}) < \frac{1}{n} \qquad (n = 1,2,3,\ldots)$$

tengsizlikni qanoatlantiruvchi shunday

$$x^{(n)} \in E, \qquad y^{(n)} \in E$$

nuqtalar topiladiki,

$$\left| f(x^{(n)}) - f(y^{(n)}) \right| \geq \varepsilon_0$$

bo‘ladi. Ravshanki,

$$\{x^{(n)}\} \qquad (x^{(n)} \in E,\ n = 1,2,3,\ldots)$$

ketma-ketlik chegaralangan. Undan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin:

$$k \to \infty \text{ da } x^{(n_k)} \to x^0 \text{ va } x^0 \in E.$$

Masofa xossasidan foydalanib topamiz:

$$\rho(y^{(n_k)}, x^{\circ}) \leq \rho(y^{(n_k)}, x^{(n_k)}) + \rho(x^{(n_k)}, x^0) < \frac{1}{n_k} + \rho(x^{(n_k)}, x^0).$$

Keyingi munosabatdan, $k \to \infty$ da limitga o‘tish bilan

$$y^{(n_k)} \to x^0$$

bo‘lishini topamiz. $f(x)$ funksiya $E$ to‘plamda, jumladan $x^0 \in E$ nuqtada uzluksiz. Unda $k \to \infty$ da

$$f(x^{(n_k)}) \to f(x^0), \quad f(y^{(n_k)}) \to f(x^0)$$

bo‘lib, undan

$$f(x^{(n_k)}) - f(y^{(n_k)}) \to 0$$

bo‘lishi kelib chiqadi. Bu esa

$$\left| f(x^{(n_k)}) - f(y^{(n_k)}) \right| \geq \varepsilon_0$$

deb qilingan farazga ziddir. Demak, $f(x)$ funksiya $E$ to‘plamda tekis uzluksiz.►

Aytaylik, $R^m$ fazoda biror $E$ to‘plam berilgan bo‘lsin: $E \subset R^m$. Ushbu

$$\alpha = \sup_{x' \in E, x'' \in E} \rho(x', x'')$$

miqdor $E$ to‘plamning diametri deyiladi.

**6-ta’rif.** $f(x)$ funksiya $E \subset R^m$ to‘plamda aniqlangan bo‘lsin. Unda

$$\omega(f, E) = \sup_{x' \in E, x'' \in E} \left\{ \left| f(x') - f(x'') \right| \right\}$$

son $f(x)$ funksiyaning $E$ to‘plamidagi tebranishi deyiladi.

**Natija.** $f(x)$ funksiya chegaralangan yopiq $E \subset R^m$ to‘plamda uzluksiz bo‘lsa, u holda $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ son topiladiki, $E$ to‘plamning diametri $\delta$ dan kichik bo‘lgan $E_k$ to‘plamlarga shunday ajratish mumkinki,

$$\bigcup_k E_k = E, \; E_k \cap E_i = \phi \quad (k \neq i),$$

har bir $E_k$ da

$$\omega(f; E_k) \leq \varepsilon$$

bo‘ladi.

◄Natijaning shartidan $f(x)$ funksiyaning $E$ to‘plamda tekis uzluksizligi kelib chiqadi. Unda ta’rifga binoan $\forall \varepsilon > 0$ uchun shunday $\delta > 0$ topiladiki, $\rho(x', x'') < \delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy $x' \in E, x'' \in E$ nuqtalarda

$$\left| f(x'') - f(x') \right| < \varepsilon$$

bo‘ladi.

Ravshanki, $\forall x' \in E_k, \forall x'' \in E_k$ nuqtalar uchun

$$\rho(x', x'') < \delta$$

tengsizlik bajariladi. Demak,

$$\left| f(x'') - f(x') \right| < \varepsilon .$$

Keyingi tengsizlikdan

$$\sup_{x' \in E_k, x'' \in E_k} \left\{ \left| f(x') - f(x'') \right| \right\} \le \varepsilon ,$$

ya'ni

$$\omega(f; E_k) \le \varepsilon$$

bo'lishi kelib chiqadi.►

**Xususiy hollar**. $m=1$ bo'lganda $R^m = R$ va bundagi to'plamda berilgan funksiyaning uzluksizligi bir o'zgaruvchili $u = f(x)$ funksiyaning $(x \in R, u \le R)$ uzluksizligi bo'lib, uning xossalari 15-17-ma'ruzalarda o'rganilgan. $m=2$ bo'lganda $R^m = R^2$ va undan $M$ to'plamda berilgan funksiyaning uzluksizligi, ikki o'zgaruvchili $u = f(x, y)$ funksiyaning $\left((x, y) \in E \subset R^2, u \in R\right)$ uzluksizligi bo'lib, uning $(x_0, y_0) \in E$ nuqtadagi uzluksizligi quyidagicha bo'ladi: Agar

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_\circ, y_\circ)$$

bo'lsa,

yoki

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x, y) \in \mathrm{E} \cap U_\delta((x_0, y_0)):\ \left| f(x, y) - f(x_0, y_0) \right| < \varepsilon$$

bo'lsa, yoki

$$\forall n \in N \text{ da } (x_n, y_n) \in E \text{ bo'lgan va } n \to \infty \text{ da}$$

$(x_n, y_n) \to (x_0, y_0)$ bo'ladigan ixtiyoriy $\{(x_n, y_n)\}$ ketma-ketlik uchun

$f(x_n, y_n) \to f(x_0, y_0)$ bo‘lsa yoki

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[ f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \right] = 0$$

bo‘lsa, $f(x,y)$ funksiya $(x_0, y_0)$ nuqtada uzluksiz bo‘ladi.

**1-misol.** Ushbu

$$f(x,y) = x + y$$

funksiyaning $R^2$ da uzluksiz bo‘lishi ko‘rsatilsin.

◀ $\forall \varepsilon > 0$ sonini olamiz. Unga ko‘ra $\delta > 0$ soni $\delta = \frac{\varepsilon}{2}$ deyilsa, u holda

$$\rho((x,y),(x_0,y_0)) = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

tengsizlikni qanoatlantiruvchi $\forall (x,y) \in R^2$ nuqtalarda

$$|f(x,y) = f(x_0,y_0)| = |x + y - (x_0 + y_0)| \le |x - x_0| + |y - y_0| \le 2\sqrt{(x-x_0)^2} < 2\delta = \varepsilon$$

bo‘ladi. Bu esa ta’rifga ko‘ra berilgan funksiyaning $\forall (x_0, y_0) \in R^2$ nuqtada uzluksiz bo‘lishini bildiradi.▶

Aytaylik, $f(x,y)$ funksiya $E \subset R^2$ to‘plamda berilgan bo‘lib, $(x_0, y_0) \in E$ bo‘lsin. Ma’lumki, bu funksiyaning to‘liq orttirmasi

$$\Delta f(x_0,y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0,y_0),$$

xususiy orttirmalari

$$\Delta_x f(x_0,y_0) = f(x_0 + \Delta x, y_0) - f(x_0,y_0),$$

$$\Delta_y f(x_0,y_0) = f(x_0, y_0 + \Delta y) - f(x_0,y_0),$$

bo‘ladi $\left( (x_0 + \Delta x, y_0 + \Delta y) \in E,\ (x_0 + \Delta x, y_0) \in E,\ (x_0, y_0 + \Delta y) \in E \right)$.

Agar

$$\lim_{\Delta x \to 0} \Delta_x f(x_0,y_0) = 0, \quad \left( \lim_{\Delta y \to 0} \Delta_y f(x_0,y_0) = 0 \right)$$

bo‘lsa, $f(x,y)$ funksiya $(x_0, y_0)$ nuqtada $x$ o‘zgaruvchi bo‘yicha ( $y$

o'zgaruvchi bo'yicha) uzluksiz deyiladi. Ravshanki, $f(x,y)$ funksiya $(x_0,y_0)$ nuqtada uzluksiz bo'lsa, funksiya shu nuqtada har bir o'zgaruvchisi bo'yicha uzluksiz bo'ladi.

Biroq, $f(x,y)$ funksiyaning $(x_0,y_0)$ nuqtada har bir o'zgaruvchisi bo'yicha uzluksiz bo'lishidan uning shu nuqtada uzluksiz bo'lishi har doim kelib chiqavermaydi

**2-misol.** Ushbu

$$f(x,y)=\begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{агар } x^2+y^2\neq 0 \text{ бўлса,} \\ 0, & \text{агар } x^2+y^2=0 \text{ бўлса} \end{cases}$$

funksiya $(0,0)$ nuqtada uzluksizlikka tekshirilsin.

◄ Berilgan funksiyaning $(0,0)$ nuqtadagi xususiy orttirmalari

$$\Delta_x f(0,0)=f(0+\Delta x,0)-f(0,0)=0\ ,$$

$$\Delta_y f(0,0)=f(0,0+\Delta y,)-f(0,0)=0$$

bo'lib

$$\lim_{\Delta x\to 0}\Delta_x f(0,0)=0\ ,\quad \lim_{\Delta y\to 0}\Delta_y f(0,0)=0$$

bo'ladi. Demak, $f(x,y)$ funksiya $(0,0)$ nuqtada har bir o'zgaruvchisi bo'yicha uzluksiz.

Qaralayotgan funksiyaning $(0,0)$ nuqtadagi to'liq orttirmasi

$$\Delta f(0,0)=f(0+\Delta x\,,0+\Delta y)-f(0,0)=\frac{2\Delta x\cdot\Delta y}{\Delta x^2+\Delta y^2}$$

bo'ladi. Ushbu

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\Delta f(0,0)$$

limit mavjud bo'lmaydi, chunki

$$\Delta x=\frac{1}{n}\to 0, \quad \Delta y=\frac{2}{n}\to 0 \quad \text{da} \quad \Delta f(0,0)=\frac{2\cdot\frac{1}{n}\cdot\frac{2}{n}}{\frac{1}{n^2}+\frac{4}{n^2}}=\frac{4}{5}\to\frac{4}{5}$$

$$\Delta x=\frac{1}{n}\to 0, \quad \Delta y=\frac{1}{n}\to 0 \quad \text{da} \quad \Delta f(0,0)=\frac{2\cdot\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}}=1\to 1 .$$

Demak, berilgan funksiya $(0,0)$ nuqtada uzluksiz bo‘lmaydi.►

**3-misol.** Ushbu

$$f(x,y)=\frac{1}{\sin^2\pi x+\sin^2\pi y}$$

funksiyaning uzilish nuqtalari topilsin.

◄Bu funksiya $R^2$ to‘plamning

$$\sin\pi x=0 ,$$
$$\sin\pi y=0$$

sistemasini qanoatlantiruvchi $(x,y)$ nuqtalarida uzilishga ega bo‘ladi.Ravshanki, sistemaning echimi

$$\{(x,y)\in R^2;\ x=n\in Z,\ y=m\in Z\}$$

to‘plam nuqtalaridan iborat. Demak, berilgan funksiyaning uzilish nuqtalari cheksiz ko‘p bo‘lib, ular

$$\{(n m)\in R^2 ;\ n\in Z ,\ m\in Z\}$$

to‘plamni tashkil etadi.►

## Mashqlar

1. Agar biror $E\subset R^m$ to‘plam va $x\in R^m$ uchun

$$\rho(x,E)=\inf_{y\in E}\rho(x,y)$$

bo‘lsa, ushbu

$$f(x)=\rho(x,E)$$

funksiyaning $R^m$ da uzluksiz bo‘lishi isbotlansin.

2. Ushbu

$$f(x,y)=\begin{cases}\dfrac{2x^2y}{x^4+y^2}, & \text{агар } x^2+y^2>0 \text{ бўлса,}\\ 0, & \text{агар } x=y=0 \text{ бўлса}\end{cases}$$

funksiya uzluksizlikka tekshirilsin.

**Adabiyotlar**


1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’ruzalar, II q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.


# Glossariy

$f(x)$ **funksiya** $x^0$ **nuqtada uzluksiz** - $\lim_{x\to x_0} f(x)=f(x^0)$ yoki

$\forall\varepsilon>0,\ \exists\delta>0,\ \ \forall x\in E\cap U_\delta(x^0),\ f(x)\in U_\varepsilon(f(x^0))$.

$u=f(x)$ **funksiyaning** $x^0$ **nuqtadagi orttirmasi** -

$\Delta u=f(x)-f(x^0)$ $\quad (x=(x_1,x_2,\ldots,x_m),\ x^0=(x_1^0,\ldots,x_k^0))$.

**Funksiya** $E$ **to‘plamda uzluksiz** - Agar $f(x)$ funksiya $E$ to‘plamning har bir nuqtasida uzluksiz bo‘lsa.

**Xusuisy orttirmalar** - 

$$\Delta_{x_1} u = f(x_1^0 + \Delta x_1, x_2^0, x_3^0, ..., x_m^0) - f(x_1^0, x_2^0, ..., x_m^0),$$
$$\Delta_{x_2} u = f(x_1^0, x_2^0 + \Delta x_2, x_3^0, ..., x_m^0) - f(x_1^0, x_2^0, ..., x_m^0),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{x_m} u = f(x_1^0, x_2^0, ..., x_{m-1}^0, x_m^0 + \Delta x_m) - f(x_1^0, x_2^0, ..., x_m^0),$$

**$f(x)$ funksiya $E$ to'plamda tekis uzluksiz** - Agar $\forall\varepsilon > 0$ son olinganda ham shunday $\delta = \delta(\varepsilon) > 0$ son topilsaki, $\rho(x', x'') < \delta$ tengsizlikni qanoatlantiruvchi ixtiyoriy $x' \in E, x'' \in E$ uchun

$$|f(x'') - f(x')| < \varepsilon$$

tengsizlik bajarilsa, $f(x)$ funksiya $E$ to'plamda tekis uzluksiz deyiladi.

**$f(x)$ funksiyaning $E$ to'plamidagi tebranishi** - $f(x)$ funksiya $E \subset R^m$ to'plamda aniqlangan bo'lsin. Unda

$$\omega(f, E) = \sup_{x' \in E, x'' \in E} \{|f(x') - f(x'')|\}$$

son $f(x)$ funksiyaning $E$ to'plamidagi tebranishi deyiladi.

# Keys banki

**48-keys.** Masala o`rtaga tashlanadi: Ushbu

$$f(x,y) = \begin{cases} \dfrac{2x^2 y}{x^4 + y^2}, & \text{agar } x^2 + y^2 > 0 \text{ bo`lsa} \\ 0, & \text{agar } x = y = 0 \text{ bo`lsa} \end{cases}$$

funksiya uzluksizlikka tekshirilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 26-amaliy mashg‘ulot

## Na’muna uchun misollar yechimi

**1-misol.** Ushbu

$$f(x,y)=x+y$$

funksiyaning $R^2$ da uzluksiz bo‘lishi ko‘rsatilsin.

◀ $\forall\varepsilon>0$ sonini olamiz. Unga ko‘ra $\delta>0$ soni $\delta=\dfrac{\varepsilon}{2}$ deyilsa, u holda

$$\rho((x,y),(x_0,y_0))=\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$$

tengsizlikni qanoatlantiruvchi $\forall(x,y)\in R^2$ nuqtalarda

$$|\,f(x,y)-f(x_0,y_0)|=|x+y-(x_0+y_0)|\le|x-x_0|+|y-y_0|\le 2\sqrt{(x-x_0)^2}<2\delta=\varepsilon$$

bo‘ladi. Bu esa ta’rifga ko‘ra berilgan funksiyaning $\forall(x_0,y_0)\in R^2$ nuqtada uzluksiz bo‘lishini bildiradi.►

**2-misol.** Ushbu

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & \text{агар } x^2+y^2\neq 0 \text{ бўлса,}\\ 0, & \text{агар } x^2+y^2=0 \text{ бўлса}\end{cases}$$

funksiya $(0,0)$ nuqtada uzluksizlikka tekshirilsin.

◀ Berilgan funksiyaning $(0,0)$ nuqtadagi xususiy orttirmalari

$$\Delta_x f(0,0)=f(0+\Delta x,0)-f(0,0)=0\ ,$$

$$\Delta_y f(0,0)=f(0,0+\Delta y,)-f(0,0)=0$$

bo‘lib

$$\lim_{\Delta x\to 0}\Delta_x f(0,0)=0\ ,\quad \lim_{\Delta y\to 0}\Delta_y f(0,0)=0$$

bo‘ladi. Demak, $f(x,y)$ funksiya $(0,0)$ nuqtada har bir o‘zgaruvchisi bo‘yicha uzluksiz.

Qaralayotgan funksiyaning $(0,0)$ nuqtadagi to‘liq orttirmasi

$$\Delta f(0,0)=f(0+\Delta x,0+\Delta y)-f(0,0)=\frac{2\Delta x\cdot\Delta y}{\Delta x^2+\Delta y^2}$$

bo‘ladi. Ushbu

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\Delta f(0,0)$$

limit mavjud bo‘lmaydi, chunki

$$\begin{matrix}\Delta x=\frac{1}{n}\to 0\\ \Delta y=\frac{2}{n}\to 0\end{matrix}\quad \text{da}\quad \Delta f(0,0)=\frac{2\cdot\frac{1}{n}\cdot\frac{2}{n}}{\frac{1}{n^2}+\frac{4}{n^2}}=\frac{4}{5}\to\frac{4}{5}$$

$$\begin{matrix}\Delta x=\frac{1}{n}\to 0\\ \Delta y=\frac{1}{n}\to 0\end{matrix}\quad \text{da}\quad \Delta f(0,0)=\frac{2\cdot\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}}=1\to 1\,.$$

Demak, berilgan funksiya $(0,0)$ nuqtada uzluksiz bo‘lmaydi.►

**3-misol.** Ushbu

$$f(x,y)=\frac{1}{\sin^2\pi x+\sin^2\pi y}$$

funksiyaning uzilish nuqtalari topilsin.

◄Bu funksiya $R^2$ to‘plamning

$$\sin\pi x=0\,,$$
$$\sin\pi y=0$$

sistemasini qanoatlantiruvchi $(x,y)$ nuqtalarida uzilishga ega bo‘ladi.Ravshanki, sistemaning echimi

$$\{(x,y)\in R^2;\ x=n\in Z,\ y=m\in Z\}$$

to‘plam nuqtalaridan iborat. Demak, berilgan funksiyaning uzilish nuqtalari cheksiz ko‘p bo‘lib, ular

$$\{(n m)\in R^2 \;;\; n\in Z \;,\; m\in Z\}$$

to‘plamni tashkil etadi.►

**Misollar**

Quyidagi funksiyalar uzluksizlikka tekshirilsin.

1) $f(x,y)=\dfrac{x-y}{1+x^2+y^2}$ 2) $f(x,y)=\dfrac{x-y}{x+y}$ 3) $f(x,y)=\dfrac{2x-3}{x^2+y^2-4}$

4) $f(x,y)=\dfrac{1}{2y+x+1}$ 5) $f(x,y)=\dfrac{2}{x^2+y^2}$ 6) $f(x,y)=\dfrac{x-y^2}{x+y^2}$

7) $f(x,y)=\dfrac{x-y}{x^3-y^3}$ 8) $f(x,y)=\ln(9-x^2-y^2)$ 9) $f(x,y)=\dfrac{x^2+y^2}{x^2-y^2}$

10) $f(x,y)=\sin\dfrac{1}{x+y}$ 11) $f(x,y)=\dfrac{x}{|y|}$ 12) $f(x,y)=\cos\dfrac{1}{x^2+y^2-9}$

13) $f(x,y)=\dfrac{1}{xy}$ 14) $f(x,y)=\dfrac{x+y}{\sqrt{x^2+y^2-4}}$ 15) $f(x,y)=\ln\sqrt{x^2+y^2}$

# Test

1. Ko`rsatilgan funksiyalardan qaysi biri $(0,0)$ nuqtada limitga ega emas?

A) $u=\dfrac{x-y}{x+y}$ B) $u=x^2+y^2+1$ C) $u=x+y$ D) $u=x-y$

2. $u=\dfrac{1}{x^2+y^2}$ funksiyaning uzilish nuqtalarini toping.

A) $(0,0)$ B) $(-1,1)$ C) $(1,-1)$ D) $(0,1)$

3. $u=\dfrac{1}{x-y}$ funksiyaning uzilish nuqtalarini toping.

A) $y=x$ B) $(0,2)$ C) $(0,0)$ D) $y=-x$

4. Agar $f(x)$ funksiya chegaralangan yopiq $E\subset R^m$ to‘plamda uzluksiz bo‘lsa, funksiya $E$ da ………………… bo‘ladi.

A) chegaralangan B) monoton C) quyidan chegaralgan D) yuqoridan chegaralangan

5. $\lim\limits_{\substack{x\to0\\ y\to99}}\left(xy+x^2\right)-?$

A) 0 B) 5 C) 2 D) 1

6. $\lim\limits_{\substack{x\to0\\ y\to111}}\left(4\sin xy-7xy^2\right)-?$

A) 1 B) 0 C) 2 D) 3

7. $\lim\limits_{\substack{x\to1\\ y\to1}}\left(3xye^{x-y}-2y^2\right)-?$

A) 1 B) 0 C) -1 d) 2

8. $\lim\limits_{\substack{x\to0\\ y\to1}}\left(\dfrac{x}{y}\sin y+9x\right)-?$

A) 0 B) 1 C) 2 D) -1

9. $\lim\limits_{\substack{x\to 0\\ y\to 1}}\left(e^{\sin xy}+y^2\right)-?$

A) 1 B) 2 C) 2 D) $e$

10. $\lim\limits_{\substack{x\to 4\\ y\to 1}}\left(xy-y^2+10\right)-?$

A) 13 B) 1 C) 0 D) 2

# Mavzu. Ko‘p o‘zgaruvchili funksiyaning xususiy hosilalari. Funksiyaning differensiallanuvchiligi

## 27-28-ma’ruza

**Reja**

$1^0$. Funksiyaning xususiy hosilalari tushunchasi.

$2^0$. Ko‘p o‘zgaruvchili funksiyaning differensiallanuvchiligi. Zaruriy shart.

$3^0$. Funksiya differensiallanuvchiligining yetarli sharti.

$4^0$. Murakkab funksiyaning differensiallanuvchiligi.

Murakkab funksiyaning hosilasi.

### $1^0$. Funksiyaning xususiy hosilalari tushunchasi

Faraz qilaylik, $f(x)=f(x_1,x_2,...,x_m)$ funksiya $E\subset R^m$ to‘plamda berigan bo‘lib,

$$x^0=(x_1^0,x_2^0,...,x_m^0)\in E,\quad (x_1^0+\Delta x_1,x_2^0,...,x_m^0)\in E \qquad (\Delta x_1 \gtrless 0)$$

bo‘lsin. Bu funksiyaning $x^0$ nuqtadagi $x_1$ o‘zgaruvchi bo‘yicha xususiy ortirmasi

$$\Delta_{x_1} f(x^0)=f(x_1^0+\Delta x_1,x_2^0,...,x_m^0)-f(x_1^0,x_2^0,...,x_m^0)$$

$\Delta x_1$ ga bog‘liq bo‘ladi.

**1-ta’rif.** Ushbu

$$\lim_{\Delta x_1\to 0}\frac{\Delta_{x_1} f\left(x^0\right)}{\Delta x_1}$$

limit mavjud bo‘lsa, bu limit $f(x)=f(x_1,x_2,...,x_m)$ funksiyaning $x^0=(x_1^0,x_2^0,...,x_m^0)$ nuqtadagi $x_1$ o‘zgaruvchisi bo‘yicha xususiy hosilasi deyiladi. Uni

$$\frac{\partial f\left(x^0\right)}{\partial x_1} \text{ yoki } f'_{x_1}\left(x^0\right)$$

kabi belgilanadi:

$$\frac{\partial f\left(x^0\right)}{\partial x_1}=f'_{x_1}=\lim_{\Delta x_1\to 0}\frac{\Delta_{x_1} f\left(x^0\right)}{\Delta x_1}=$$

$$=\lim_{\Delta x_1\to 0}\frac{f\left(x_1^0+\Delta x_1, x_2^0,..., x_m^0\right)-f\left(x_1^0, x_2^0,..., x_m^0\right)}{\Delta x_1}.$$

Berilgan funksiyaning bu xususiy hosilasini quyidagicha

$$f'_{x_1}\left(x^0\right)=\lim_{x_1\to x_1^0}\frac{f\left(x_1, x_2^0,..., x_m^0\right)-f\left(x_1^0, x_2^0,..., x_m^0\right)}{x_1-x_1^0}$$

ta’riflasa ham bo‘ladi.

Xuddi shunga o‘xshash $f(x_1,x_2,...,x_m)$ funksiyaning boshqa $x_2,x_3,...,x_m$ o‘zgaruvchilari bo‘yicha xususiy hosilalari ta’riflanadi:

$$\frac{\partial f\left(x^0\right)}{\partial x_2}=\lim_{\Delta x_2\to 0}\frac{\Delta_{x_2} f\left(x^0\right)}{\Delta x_2}=$$

$$=\lim_{\Delta x_2\to 0}\frac{f\left(x_1^0, x_2^0+\Delta x_2, x_3^0,..., x_m^0\right)-f\left(x_1^0, x_2^0,..., x_m^0\right)}{\Delta x_2}, \ ...$$

$$\frac{\partial f\left(x^0\right)}{\partial x_m}=\lim_{\Delta x_m\to 0}\frac{\Delta_{x_m} f\left(x^0\right)}{\Delta x_m}=$$

$$=\lim_{\Delta x_m\to 0}\frac{f\left(x_1^0, x_2^0,..., x_{m-1}^0, x_m^0+\Delta x_m\right)-f\left(x_1^0, x_2^0,..., x_m^0\right)}{\Delta x_m}. \quad \left(\Delta x_k \gtrless 0,\ k=2,3,\dots,m\right)$$

Yuqorida keltirilgan ta’riflardan ko‘p o‘zgaruvchili funksiyaning xususiy hosilalari bir o‘zgaruvchili funksiyaning hosilasi kabi ekanligi ko‘rinadi.

Demak, ko‘p o‘zgaruvchili funksiyaning xususiy hosilalarini topishda ma’lum jadval va qoidalardan foydalanish mumkin. Jumladan, agar

$$f(x)=f(x_1,x_2,...,x_m) \quad , \quad g(x)=g(x_1,x_2,...,x_m)$$

funksiyalar $E\subset R^m$ to‘plamda berilgan bo‘lib, $x\in E$ nuqtada xususiy hosilalarga ega bo‘lsa, u holda:

1) $\forall c\in R$: $\quad \dfrac{\partial(c f(x))}{\partial x_k}=c\dfrac{\partial f(x)}{\partial x_k}$;

2) $\dfrac{\partial(f(x)+g(x))}{\partial x_k}=\dfrac{\partial f(x)}{\partial x_k}+\dfrac{\partial g(x)}{\partial x_k}$;

3) $\dfrac{\partial(f(x)\cdot g(x))}{\partial x_k}=\dfrac{\partial f(x)}{\partial x_k}g(x)+f(x)\dfrac{\partial g(x)}{\partial x_k}$;

4) $\dfrac{\partial\left(\dfrac{f(x)}{g(x)}\right)}{\partial x_k}=g^{-2}(x)\left(\dfrac{\partial f(x)}{\partial x_k}g(x)-f(x)\dfrac{\partial g(x)}{\partial x_k}\right)$

$$(g(x)\neq 0), \quad k=1,2,...,m$$

bo‘ladi.

**2⁰. Ko‘p o‘zgaruvchili funksiyaning differensiallanuvchiligi. Zaruriy shart** Aytaylik, $f(x)=f(x_1,x_2,...,x_m)$ funksiya $E\subset R^m$ to‘plamda berilgan bo‘lib,

$$x^0=(x_1^0,x_2^0,...,x_m^0)\in E,\ (x_1^0+\Delta x_1,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m)\in E$$

bo‘lsin. Ma’lumki, berilgan funksiyaning $x^0$ nuqtadagi to‘la orttirmasi

$$\Delta f(x^0)=f(x^0+\Delta x_1,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m)-f(x_1^0,x_2^0,...,x_m^0)$$

bo‘lib, u $\Delta x_1,\Delta x_2,...,\Delta x_m$ larga bog‘liq bo‘ladi.

**2-ta’rif.** Agar $\Delta x_1,\Delta x_2,...,\Delta x_m$ orttirmalarga bog‘liq bo‘lmagan shunday $A_1,A_2,....,A_m$ sonlari topilib, funksiyaning $x^0$ nuqtadagi to‘liq orttirmasi ushbu

$$\Delta f(x^0) = A_1\Delta x_1 + A_2\Delta x_2 + \dots + A_m\Delta x_m + \\ + \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + \dots + \alpha_m\Delta x_m \quad (1)$$

ko'rinishda ifodalansa, $f(x)$ funksiya $x^0$ nuqtada differensiallanuvchi deyiladi, bunda $\alpha_1, \alpha_2, \dots, \alpha_m$ lar $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ larga bog'liq va $\Delta x_1 \to 0,\ \Delta x_2 \to 0, \dots,\ \Delta x_m \to 0$ da cheksiz kichik miqdorlar.

Agar $(x_1^0, x_2^0, \dots, x_m^0)$ hamda $(x_1^0 + \Delta x_1, x_2^0 + \Delta x_2, \dots, x_m^0 + \Delta x_m)$ nuqtalar orasidagi masofa

$$\rho = \sqrt{\Delta x_1^2 + \Delta x_2^2 + \dots + \Delta x_m^2}$$

uchun, $\Delta x_1 \to 0$, $\Delta x_2 \to 0, \dots, \Delta x_m \to 0$ da

$$\alpha_1\Delta x_1 + \alpha_2\Delta x_2 + \dots + \alpha_m\Delta x_m = 0(\rho)$$

bo'lishini e'tiborga olsak, (1) munosabat ushbu

$$\Delta f(x^0) = A_1\Delta x_1 + A_2\Delta x_2 + \dots + A_m\Delta x_m + 0(\rho) \quad (2)$$

ko'rinishga keladi.

Odatda, (1) va (2) munosabatlar $f(x)$ funksiyaning $x^0$ nuqtada differensiallanuvchi sharti deyiladi.

**1-misol.** Ushbu

$$f(x) = f(x_1, x_2, \dots, x_m) = x_1^2 + x_2^2 + \dots + x_m^2$$

funksiyaning $\forall (x_1^0, x_2^0, \dots, x_m^0) \in R^m$ nuqtada differensiallanuvchi bo'lishi ko'rsatilsin.

◄Berilgan funksiyaning $x^0 = (x_1^0, x_2^0, \dots, x_m^0)$ nuqtadagi to'liq orttirmasini topamiz:

$$\Delta f(x^0) = (x_1^0 + \Delta x_1)^2 + (x_2^0 + \Delta x_2)^2 + \dots + (x_m^0 + \Delta x_m)^2 - \\ - \left(x_1^{0^2} + x_2^{0^2} + \dots + x_m^{0^2}\right) = 2x_1^0\Delta x_1 + 2x_2^0\Delta x_2 + \dots + \\ + 2x_m^0\Delta x_m + \Delta x_1^2 + \Delta x_2^2 + \dots + \Delta x_m^2.$$

Agar

$$A_1=2x_1^0,\ A_2=2x_2^0,\dots,\ A_m=2x_m^0,$$
$$\alpha_1=\Delta x_1, \alpha_2=\Delta x_2,\dots,\ \alpha_m=\Delta x_m$$

deyilsa, unda

$$\Delta f\left(x^0\right)=A_1\Delta x_1+A_2\Delta x_2+\dots+A_m\Delta x_m+\alpha_1\Delta x_1+\alpha_2\Delta x_2+\dots+\alpha_m\Delta x_m$$

bo‘ladi. Demak, berilgan funksiya $\forall x^0\in R^m$ nuqtada differensiallanuvchi.►

Agar $f(x)$ funksiya $E\subset R^m$ to‘plamning har bir nuqtasida differensiallanuvchi bo‘lsa, funksiya $E$ to‘plamda differensiallanuvchi deyiladi.

**1-teorema**. Agar $f(x)$ funksiya $x^0\in E\subset R^m$ nuqtada differensiallanuvchi bo‘lsa, u holda funksiya shu nuqtada uzluksiz bo‘ladi.

◄SHartga ko‘ra $f(x)$ funksiya $x^0$ nuqtada differensiallanuvchi. Demak, funksiyaning shu nuqtadagi to‘liq orttirmasi

$$\Delta f\left(x^0\right)=A_1\Delta x_1+A_2\Delta x_2+\dots+A_m\Delta x_m+\alpha_1\Delta x_1+\alpha_2\Delta x_2+\dots+\alpha_m\Delta x_m$$

bo‘ladi. Bu tenglikdan

$$\lim_{\substack{\Delta x_1\to 0\\ \Delta x_2\to 0\\ \dots\dots\dots\\ \Delta x_m\to 0}}\Delta f\left(x^0\right)=0$$

bo‘lishini topamiz. Demak, $f(x)$ funksiya $x^0$ nuqtada uzluksiz.►

**2-teorema**. Agar $f(x)$ funksiya $x^0$ nuqtada differensiallanuvchi bo‘lsa, u holda funksiya shu nuqtada barcha xususiy hosilalarga ega va

$$f'_{x_1}\left(x^0\right)=A_1,\ \ f'_{x_2}\left(x^0\right)=A_2,\dots,\ \ f'_{x_m}\left(x^0\right)=A_m$$

bo‘ladi.

◄Shartga ko‘ra $f(x)$ funksiya $x^0$ nuqtada differensiallanuvchi. Binobarin, (1) shart bajariladi. Unda

$$\Delta x_1\neq 0\ \ ,\ \Delta x_2=\Delta x_3=\dots=\Delta x_m=0$$

deb olinsa, quyidagi

$$\Delta_{x_1} f\left(x^0\right)=A_1\,\Delta x_1+\alpha_1\,\Delta x_1$$

tenglik hosil bo‘ladi. Bu tenglikdan topamiz:

$$\lim_{\Delta x_1\to 0}\frac{\Delta_{x_1} f\left(x^0\right)}{\Delta x_1}=\lim_{\Delta x_1\to 0}\left(A_1+\alpha_1\right)=A_1 .$$

Demak,

$$f'_{x_1}\left(x^0\right)=A_1 .$$

Xuddi shunga o‘xshash $f(x)$ funksiyaning $x^0$ nuqtada $f'_{x_2}\left(x^0\right), f'_{x_3}\left(x^0\right),\dots, f'_{x_m}\left(x^0\right)$ xususiy hosilalarining mavjudligi hamda

$$f'_{x_2}\left(x^0\right)=A_2\,,\ f'_{x_3}\left(x^0\right)=A_3\,,\dots,\ f'_{x_m}\left(x^0\right)=A_m$$

bo‘lishi ko‘rsatiladi.►

Bu teoremadan $x^0$ nuqtada differensiallanuvchi $f(x)$ funksiyaning orttirmasi uchun

$$\Delta f\left(x^0\right)=f'_{x_1}\left(x^0\right)\cdot\Delta x_1+f'_{x_2}\left(x^0\right)\Delta x_2+\ldots+f'_{x_m}\left(x^0\right)\Delta x_m+0(\rho)$$

bo‘lishi kelib chiqadi.

**Eslatma.** $f(x)$ funksiyaning biror $x^0$ nuqtada barcha xususiy hosilalari

$$f'_{x_1}\left(x^0\right), f'_{x_2}\left(x^0\right), f'_{x_3}\left(x^0\right),\ldots, f'_{x_m}\left(x^0\right)$$

ning mavjud bo‘lishidan, uning shu nuqtada differensiallanuvchi bo‘lishi har doim kelib chiqavermaydi. (bunga misol keyingi punktda keltiriladi).

Yuqorida keltirilgan teorema va eslatmadan $f(x)$ funksiyaning $x^0$ nuqtada barcha xususiy hosilalarga ega bo‘lish funksiyaning shu nuqtada differensiallanuvchi bo‘lishining zaruriy sharti ekanligi kelib chiqadi.

### $3^0$. Funksiya differensiallanuvchiligining yetarli sharti

Faraz qilaylik, $f(x)$ funksiya $E\subset R^m$ to‘plamda berilgan bo‘lib, $U_\delta\left(x^0\right)\subset E$ bo‘lsin ($\delta>0$).

**3-teorema**. Agar $f(x)$ funksiya $U_\delta(x^0)$da barcha xususiy hosilalarga ega bo‘lib, bu xususiy hosilalar $x^0$ nuqtada uzluksiz bo‘lsa, $f(x)$ funksiya $x^0$ nuqtada differensiallanuvchi bo‘ladi.

◀Ushbu

$$\left(x_1^0+\Delta x_1,\, x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)\in U_\delta\left(x^0\right).$$

nuqtani olib, berilgan funksiyaning $x^0=\left(x_1^0,x_2^0,\dots,x_m^0\right)$ nuqtadagi to‘liq orttirmasini qaraymiz:

$$\Delta f\left(x^0\right)=f\left(x_1^0+\Delta x_1,x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,\dots,x_m^0\right).$$

Bu orttirmani quyidagicha yozib olamiz:

$$\begin{aligned}\Delta f\left(x^0\right)=&\left[f\left(x_1^0+\Delta x_1,x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)\right]+\\ +&\left[f\left(x_1^0,x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,x_3^0+\Delta x_3,\dots,x_m^0+\Delta x_m\right)\right]+\\ +&\dots+\left[f\left(x_1^0,x_2^0,\dots,x_{m-1}^0,\,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,\dots,x_m^0\right)\right].\end{aligned} \qquad (3)$$

Lagranj teoremasidan foydalanib topamiz:

$$\begin{aligned}&f\left(x_1^0+\Delta x_1,x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)=\\ &\qquad=f'_{x_1}\left(x_1^0+\theta_1\cdot\Delta x_1,x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)\cdot\Delta x_1,\\ &f\left(x_1^0,x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,x_3^0+\Delta x_3,\dots,x_m^0+\Delta x_m\right)=\\ &\qquad=f'_{x_2}\left(x_1^0,x_2^0+\theta_2\cdot\Delta x_2,x_3^0+\Delta x_3,\dots,x_m^0+\Delta x_m\right)\cdot\Delta x_2;\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &f\left(x_1^0,x_2^0,\dots,x_{m-1}^0,x_m^0+\Delta x_m\right)-f\left(x_1^0,x_2^0,\dots,x_m^0\right)=\\ &\qquad=f'_{x_m}\left(x_1^0,x_2^0,\dots,x'_{m-1},x_m^0+\theta_m\,\Delta x_m\right)\cdot\Delta x_m.\\ &\left(0<\theta_k<1\ ,\ k=1,2,\dots,m\right)\end{aligned} \qquad (4)$$

Shartga ko‘ra $f'_{x_1},f'_{x_2},\dots,f'_{x_m}$ xususiy hosilalar $x^0=\left(x_1^0,x_2^0,\dots,x_m^0\right)$ nuqtada uzluksiz. Unda

$$f'_{x_1}\left(x_1^0+\theta_1\Delta x_1, x_2^0+\Delta x_2,\dots,x_m^0+\Delta x_m\right)=f'_{x_1}\left(x^0\right)+\alpha_1,$$
$$f'_{x_2}\left(x_1^0, x_2^0+\theta_2\Delta x_2, x_3^0+\Delta x_3,\dots,x_m^0+\Delta x_m\right)=f'_{x_2}\left(x^0\right)+\alpha_2, \qquad (5)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f'_{x_m}\left(x_1^0, x_2^0,\dots,x_{m-1}^0, x_m^0+\theta_m\Delta x_m\right)=f'_{x_m}\left(x^0\right)+\alpha_m,$$

bo‘ladi. Bunda

$$\Delta x_1\to 0,\ \Delta x_2\to 0,\dots,\Delta x_m\to 0 \quad \partial a \quad \alpha_1\to 0,\ \alpha_2\to 0,\dots,\alpha_m\to 0.$$

Yuqoridagi (3), (4) va (5) munosabatlardan

$$\Delta f\left(x^0\right)=f'_{x_1}\left(x^0\right)\Delta x_1+f'_{x_2}\left(x^0\right)\Delta x_2+\dots+f'_{x_m}\left(x^0\right)\Delta x_m+$$
$$+\alpha_1\Delta x_1+\alpha_2\Delta x_2+\dots+\alpha_m\Delta x_m$$

bo‘lishi kelib chiqadi. Demak, $f(x)$ funksiya $x^0$ nuqtada differensiallanuvchi.►

Bu teorema $f(x)$ funksiyaning $x^0$ nuqtada differensiallanuvchi bo‘lishining etarli shartini ifodalaydi.

## $4^0$. Murakkab funksiyaning differensiallanuvchiligi. Murakkab funksiyaning hosilasi

Aytaylik, ushbu

$$x_1=\varphi_1(t)=\varphi_1(t_1,t_2,\dots,t_k),$$
$$x_2=\varphi_2(t)=\varphi_2(t_1,t_2,\dots,t_k),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x_m=\varphi_m(t)=\varphi_m(t_1,t_2,\dots,t_k)$$

funksiyalarning har bir $M\subset R^k$ to‘plamda,

$$u=f(x_1,x_2,\dots,x_m)$$

funksiya esa

$$E=\left\{(x_1,x_2,\dots,x_m)\in R^m;\ x_1=\varphi_1(t),\ x_2=\varphi_2(t),\dots,x_m=\varphi_m(t)\right\}$$

to‘plamda berilgan bo‘lib, ular yordamida

$$f\left(\varphi_1(t),\varphi_2(t),...,\varphi_m(t)\right)=F\left(t_1,t_2,...,t_k\right)$$

murakkab funksiya hosil qilingan bo‘lsin.

**4-teorema.** Agar $x_i=\varphi_i\left(t_1,t_2,...,t_k\right)$ funksiyalarning har biri $(i=1,2,...,m)$, $\left(t_1^0,t_2^0,...,t_k^0\right)\in M$ nuqtada differensiallanuvchi bo‘lib, $f\left(x_1,x_2,...,x_m\right)$ funksiya mos $\left(x_1^0,x_2^0,...,x_m^0\right)$ nuqtada

$$\left(x_1^0=\varphi_1\left(t_1^0,t_2^0,...,t_k^0\right),\ x_2^0=\varphi_2\left(t_1^0,...,t_k^0\right),...,\ x_m^0=\varphi_m\left(t_1^0,...,t_k^0\right)\right)$$

differensiallanuvchi bo‘lsa, u holda murakkab

$$f\left(\varphi_1\left(t_1,...,t_k\right)\varphi_2\left(t_1,...,t_k\right),...,\varphi_m\left(t_1,...,t_k\right)\right)$$

funksiya $\left(t_1^0,t_2^0,...,t_k^0\right)$ nuqtada differensiallanuvchi bo‘ladi.

◄ $\left(t_1^0,t_2^0,...,t_k^0\right)\in M$ nuqtaning koordinatalariga mos ravishda $\Delta t_1,\Delta t_2,...,\Delta t_k$ orttirmalar beraylikki

$$\left(t_1^0+\Delta t_1,\ t_2^0+\Delta t_2,...,t_k^0+\Delta t_k\right)\in M$$

bo‘lsin. Unda har bir $x_i=\varphi_i\left(t_1,t_2,...,t_k\right)$ funksiya $(i=1,2,...,m)$ ham $\Delta x_i$ $(i=1,2,...,m)$ orttirmalarga va nihoyat $f(x)$ funksiya $\Delta f$ orttirmaga ega bo‘ladi.

Shartga ko‘ra $x_i=\varphi_i\left(t_1,t_2,...,t_k\right)$ funksiyalarning har biri $\left(t_1^0,t_2^0,...,t_k^0\right)$ nuqtada differensiallanuvchi. Demak,

$$\begin{aligned}
&\Delta x_1=\frac{\partial x_1}{\partial t_1}\Delta t_1+\frac{\partial x_1}{\partial t_2}\Delta t_2+...+\frac{\partial x_1}{\partial t_k}\Delta t_k+0(\rho)\ ,\\
&\Delta x_2=\frac{\partial x_2}{\partial t_1}\Delta t_1+\frac{\partial x_2}{\partial t_2}\Delta t_2+...+\frac{\partial x_2}{\partial t_k}\Delta t_k+0(\rho)\ ,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\Delta x_m=\frac{\partial x_m}{\partial t_1}\Delta t_1+\frac{\partial x_m}{\partial t_2}\Delta t_2+...+\frac{\partial x_m}{\partial t_k}\Delta t_k+0(\rho)
\end{aligned}\tag{6}$$

bo‘ladi, bunda

$$\frac{\partial x_i}{\partial t_j} \qquad (i=1,2,\dots,m; \quad j=1,2,\dots,k)$$

xususiy hosilalarning $(t_1^0, t_2^0, \dots, t_k^0)$ nuqtadagi qiymatlari olingan va

$$\rho = \sqrt{\Delta t_1^2 + \Delta t_2^2 + \dots + \Delta t_k^2}\,.$$

Shartga ko‘ra $f(x_1, x_2, \dots, x_m)$ funksiya $(x_1^0, x_2^0, \dots, x_m^0)$ nuqtada differensiallanuvchi. Demak,

$$\Delta f = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \dots + \frac{\partial f}{\partial x_m}\Delta x_m + \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + \dots + \alpha_m\Delta x_m \qquad (7)$$

bo‘ladi, bunda $\frac{\partial f}{\partial x_i}$, $(i=1,2,\dots,m)$ xususiy hosilalarning $(x_1^0, x_2^0, \dots, x_m^0)$ nuqtadagi qiymatlari olingan va

$$\Delta x_1 \to 0,\ \Delta x_2 \to 0, \dots, \Delta x_m \to 0 \quad \partial a \quad \alpha_1 \to 0,\ \alpha_2 \to 0, \dots, \alpha_m \to 0\,.$$

(6), (7) munosabatlardan topamiz:

$$\Delta f = \frac{\partial f}{\partial x_1}\left[\frac{\partial x_1}{\partial t_1}\Delta t_1 + \frac{\partial x_1}{\partial t_2}\Delta t_2 + \dots + \frac{\partial x_1}{\partial t_k}\Delta t_k + 0(\rho)\right] +$$

$$+ \frac{\partial f}{\partial x_2}\left[\frac{\partial x_2}{\partial t_1}\Delta t_1 + \frac{\partial x_2}{\partial t_2}\Delta t_2 + \dots + \frac{\partial x_2}{\partial t_k}\Delta t_k + 0(\rho)\right] +$$

$$+ \dots + \frac{\partial f}{\partial x_m}\left[\frac{\partial x_m}{\partial t_1}\Delta t_1 + \frac{\partial x_m}{\partial t_2}\Delta t_2 + \dots + \frac{\partial x_m}{\partial t_k}\Delta t_k + 0(\rho)\right] +$$

$$+ \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + \dots + \alpha_m\Delta x_m = \left(\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1} +\right.$$

$$\left. + \dots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_1}\right)\Delta t_1 + \left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \dots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_2}\right)\Delta t_2 +$$

$$+ \dots + \left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \dots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_k}\right)\Delta t_k +$$

$$+ \left(\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \dots + \frac{\partial f}{\partial x_m}\right)\cdot 0(\rho) + \alpha_1\Delta x_1 + \alpha_2\Delta x_2 + \dots + \alpha_m\Delta x_m\,.$$

Bu tenglikdan

$$\left(\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\ldots+\frac{\partial f}{\partial x_m}\right)\cdot 0(\rho)=0(\rho),$$

$\Delta t_1 \to 0$ , $\Delta t_2 \to 0$ , ....,$\Delta t_k \to 0$ , ya'ni $\rho \to 0$ da

$\Delta x_1 \to 0$ , $\Delta x_2 \to 0$ , ....,$\Delta x_m \to 0$ , va $\alpha_1 \to 0$ , $\alpha_2 \to 0$ , ...,$\alpha_m \to 0$

bo‘lgani sababli

$$\alpha_1 \Delta x_1+\alpha_2 \Delta x_2+\ldots+\alpha_m \Delta x_m=0(\rho)$$

bo‘lishi hamda quyidagi

$$A_j=\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_j}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_j}+\ldots.+\frac{\partial f}{\partial x_m}\,\frac{\partial x_m}{\partial t_j} \tag{8}$$

$(j=1,2,\ldots,k)$belgilashlar natijasida

$$\Delta f=A_1\Delta t_1+A_2\Delta t_2+\ldots.+A_m\Delta t_m+0(\rho) \tag{9}$$

bo‘ladi. Demak, murakkab funksiya $t^0$ nuqtada differensiallanuvchi.►

Aytaylik, $f(x(t))$ murakkab funksiya yuqoridagi teoremaning shartlarini qanoatlantirsin. U holda

$$\Delta f(t)=\frac{\partial f}{\partial t_1}\Delta t_1+\frac{\partial f}{\partial t_2}\Delta t_2+\ldots+\frac{\partial f}{\partial t_k}\Delta t_k+0(\rho)$$

bo‘ladi. Bu hamda (8), (9) munosabatlardan foydalanib murakkab funksiyaning xususiy hosilalari quyidagicha

$$\frac{\partial f}{\partial t_1}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1}+\ldots+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_1},$$

$$\frac{\partial f}{\partial t_2}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2}+\ldots+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_2},$$

..........................................................................

$$\frac{\partial f}{\partial t_k}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k}+\ldots+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_k}$$

bo‘lishini topamiz.

**Xususiy hollar**. $m=1$ bo‘lganda bir o‘zgaruvchili $u=f(x)$ $(x\in R\,,\, u\in R)$ funksiya hosilasi tushunchasiga kelamiz. $m=2$ bo‘lsin.

Bu holda ikki o'zgaruvchili $u=f(x,y)$ $\left((x,y)\in E\subset R^2,\ u\in R\right)$ funksiyaning xususiy hosilalari

$$\frac{\partial f}{\partial x}=\lim_{\Delta x\to 0}\frac{\Delta_x f}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x},$$

$$\frac{\partial f}{\partial y}=\lim_{\Delta y\to 0}\frac{\Delta_y f}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

hamda quyidagi

$$\Delta f=f(x+\Delta x,y+\Delta y)-f(x,y)=$$
$$=A\Delta x+B\Delta y+\alpha_1\Delta x+\alpha_2\Delta y$$

differensiallanuvchilik shartiga ega bo'lamiz.

**2-misol**. Ushbu

$$f(x,y)=\ln tg\,\frac{x}{y}$$

funksiyaning xususiy hosilalar topilsin.

◀Berilgan funksiyaning xususiy hosilalari quyidagicha bo'ladi:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\ln tg\,\frac{x}{y}\right)=\frac{1}{tg\,\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\frac{1}{y}=\frac{2}{y\sin\frac{2x}{y}};$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\ln tg\,\frac{x}{y}\right)=\frac{1}{tg\,\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\left(-\frac{x}{y^2}\right)=\frac{-2}{y^2\sin\frac{2x}{y}}\ ▶$$

**3-misol**. Ushbu

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & \text{агар}\quad (x,y)\neq(0,0)\ \text{бўлса},\\ 0\quad, & \text{агар}\quad (x,y)=(0,0)\ \text{бўлса}\end{cases}$$

funksiyaning xususiy hosilalari topilsin.

◀ Aytaylik, $(x,y)\neq(0,0)$ bo'lsin. U holda

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\frac{2xy}{x^2+y^2}\right)=\frac{2y\left(x^2+y^2\right)-2xy\cdot 2x}{\left(x^2+y^2\right)^2}=\frac{2y\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2};$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\frac{2xy}{x^2+y^2}\right)=\frac{2x\left(x^2+y^2\right)-2xy\cdot 2y}{\left(x^2+y^2\right)^2}=\frac{2x\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

bo‘ladi.

Aytaylik, $(x,y)=(0,0)$ bo‘lsin. Bu holda ta’rifdan foydalanib topamiz:

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\Delta x\cdot 0}{\Delta x^3}=0,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{2\Delta y\cdot 0}{\Delta y^3}=0.$$ ►

**4-misol**. Ushbu

$$f(x,y)=\sqrt{x^2+y^2}$$

funksiyaning xususiy hosilalari topilsin.

◄Aytaylik, $(x,y)\neq(0,0)$ bo‘lsin. Bu holda

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\sqrt{x^2+y^2}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}},$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\sqrt{x^2+y^2}=\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}}$$

bo‘ladi.

Aytaylik, $(x,y)=(0,0)$ bo‘lsin. Ta’rifga ko‘ra

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x},$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{|\Delta y|}{\Delta y}$$

bo‘lib, bu limitlar mavjud bo‘lmaganligi sababli berilgan funksiya $(0,0)$ nuqtada xususiy hosilalarga ega bo‘lmaydi.►

**5-misol**. Ushbu

$$f(x,y)=\begin{cases} \dfrac{x^3y}{x^6+y^2}, \text{ агар } & (x,y)\neq(0,0) \text{ бўлса,} \\ 0 \quad, \text{ агар } & (x,y)=(0,0) \text{ бўлса} \end{cases}$$

funksiyaning $(0,0)$ nuqtadagi xususiy hosilalari topilsin.

◀Ta’rifga ko‘ra

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0 \ ,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0\,,0+\Delta y)-f(0,0)}{\Delta y}=0$$

bo‘ladi. Biroq berilgan funksiya $(0,0)$ nuqtada uzluksiz bo‘lmaydi, chunki

$$\left(\frac{1}{n},\frac{1}{n^3}\right)\to(0,0) \text{ da } f\left(\frac{1}{n},\frac{1}{n^3}\right)=\frac{1}{2}\to\frac{1}{2}\neq f(0,0).$$ ▶

**6-misol**. Ushbu

$$f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, \text{ агар } & (x,y)\neq(0,0) \text{ бўлса,} \\ 0 \quad, \text{ агар } & (x,y)=(0,0) \text{ бўлса} \end{cases}$$

funksiyaning $(0,0)$ nuqtada xususiy hosilalarning mavjudligi ammo shu nuqtada differensiallanuvchi emasligi ko‘rsatilsin.

◀Ravshanki,

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,\Delta y,)-f(0,0)}{\Delta y}=0.$$

Demak, berilgan funksiyaning $(0,0)$ nuqtada xususiy hosilalari mavjud va ular 0 ga teng.

Bu funksiya $(0,0)$ nuqtada differensiallanuvchi bo‘lmaydi. Shuni isbotlaymiz. Teskarisini faraz qilaylik, qaralayotgan funksiya $(0,0)$ nuqtada differensiallanuvchi bo‘lsin:

$$\Delta f(0,0)=\frac{\partial f(0,0)}{\partial x}\Delta x+\frac{\partial f(0,0)}{\partial y}\Delta y+\alpha_1\Delta x+\alpha_2\Delta y=$$

$$=\alpha_1\Delta x+\alpha_2\Delta y \ . \qquad (\Delta x\to 0,\ \Delta y\to 0 \ \partial a \ \alpha_1\to 0\ ,\ \alpha_2\to 0).$$

Ayni paytda,

$$\Delta f(0,0)=f(0+\Delta x,\ 0+\Delta y)-f(0,0)=f(\Delta x,\Delta y)=\frac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}$$

bo‘ladi. Demak,

$$\frac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\alpha_1\Delta x+\alpha_2\Delta y.$$

Bu tenglikdan, $\Delta x=\Delta y>0$ bo‘lganda

$$\alpha_1+\alpha_2=\frac{1}{\sqrt{2}}$$

bo‘lishi kelib chiqadi. Bu esa $\Delta x=\Delta y>0$ da $\alpha_1\to 0,\alpha_2\to 0$ bo‘lishigi zid. Demak, berilgan funksiya $(0,0)$ nuqtada differensiallanuvchi emas.►

**7-misol**. Agar $f(x,y)$ funksiya $R^2$ differensiallanuvchi bo‘lib, $x=r\cos\varphi$, $y=r\sin\varphi$ bo‘lsa, $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \varphi}$ lar topilsin.

◄Ravshanki,

$$f(x,y)=f(r\cos\varphi, r\sin\varphi).$$

Murakkab funksiyaning xususiy hosilalarini topish qoidasiga ko‘ra

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}=\cos\varphi\frac{\partial f}{\partial x}+\sin\varphi\frac{\partial f}{\partial y}=\frac{1}{\sqrt{x^2+y^2}}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial f}{\partial \varphi}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \varphi}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \varphi}=-r\sin\varphi\frac{\partial f}{\partial x}+r\cos\varphi\frac{\partial f}{\partial y}=-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}.$$ ►

## O`rta qiymat haqida teorema. Yo`nalish bo`yicha hosila

**$1^0$. O`rta qiymat haqida teorema**. Faraz qilaylik, $f(x)=f(x_1,x_2,...,x_m)$

funksiya $E \subset R^m$ to`plamda berilgan bo`lsin. Bu $E$ to`plamda shunday

$$a = (a_1, a_2, ..., a_m), \qquad b = (b_1, b_2, ..., b_m)$$

nuqtalarni qaraymizki, bu nuqtalarni birlashtiruvchi to`g`ri chiziq kesmasi $E$ to`plamga tegishli bo`lsin.

Ravshanki, bu kesma ushbu

$$K = \{(x_1, x_2, ..., x_m) \in R^m : \ x_1 = a_1 + t(b_1 - a_1),$$
$$x_2 = a_2 + t(b_2 - a_2), ..., x_m = a_m + t(b_m - a_m) \ , \quad (0 \le t \le 1)\}$$

nuqtalar to`plami bilan ifodalanadi: $K \subset E$.

**1-teorema.** Agar $f(x)$ funksiya $K$ kesmaning $a$ va $b$ nuqtalarida uzluksiz bo`lib, kesmaning qolgan barcha nuqtalarida differentsiallanuvchi bo`lsa, u holda $K$ kesmada shunday $c = (c_1, c_2, ..., c_m)$ nuqta topiladiki,

$$f(b) - f(a) = f'_{x_1}(c)(b_1 - a_1) + f'_{x_2}(c)(b_2 - a_2) + .... + f'_{x_m}(c)(b_m - a_m) \quad (1)$$

bo`ladi.

◀ $f(x)$ funksiya $K \subset E$ kesmada quyidagi

$$f(x) = f(x_1, x_2, ...., x_m) =$$
$$= f(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2), ...., a_m + t(b_m - a_m)) \quad (0 \le t \le 1)$$

ko`rinishda bo`ladi. Bu $t$ o`zgaruvchining funksiyasini $F(t)$ bilan belgilaylik:

$$F(t) = f(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2), ...., a_m + t(b_m - a_m)).$$

Ravshanki, $F(t)$ funksiya $[0,1]$ segmentda uzluksiz bo`lib, $(0,1)$ da

$$F'(t) = f'_{x_1} \cdot (b_1 - a_1) + f'_{x_2} \cdot (b_2 - a_2) + .... + f'_{x_m} \cdot (b_m - a_m)$$

hosilaga ega bo`ladi. Bunda $f'_{x_1}, f'_{x_2}, ...., f'_{x_m}$ xususiy hosilalarning

$$(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2), ...., a_m + t(b_m - a_m))$$

nuqtadagi qiymatlari olingan.

Lagranj teoremasidan foydalanib topamiz:

$$F(1) - F(0) = F^1(t_0). \qquad (0 < t_0 < 1) \qquad (2)$$

Agar

$$F(0)=f(a),\qquad F(1)=f(b) \tag{3}$$

hamda

$$\begin{aligned} F'(t_\circ)&=f'_{x_1}(a_1+t_\circ(b_1-a_1),\ a_2+t_\circ(b_2-a_2),\dots,a_m+t_\circ(b_m-a_m))\cdot\\ &\cdot(b_1-a_1)+f'_{x_2}(a_1+t_\circ(b_1-a_1),a_2+t_\circ(b_2-a_2),\dots,a_m+t_\circ(b_m-a_m))\cdot\\ &\cdot(b_2-a_2)+\dots+f'_{x_m}(a_1+t_\circ(b_2-a_2),\dots,a_m+t_\circ(b_m-a_m))\cdot(b_m-a_m)\end{aligned} \tag{4}$$

bo`lishini e`tiborga olsak, so`ng ushbu

$$\begin{aligned} &a_1+t_\circ(b_1-a_1)=c_1,\\ &a_2+t_\circ(b_2-a_2)=c_2,\\ &\dots\dots\dots\dots\dots\dots\\ &a_m+t_\circ(b_m-a_m)=c_m\end{aligned}$$

belgilashlarini bajarsak, unda

$$c=(c_1,c_2,\dots,c_m)\in K$$

bo`lib, (2), (3) va (4) munosabatlardan

$$f(b)-f(a)=f'_{x_1}(c)\cdot(b_1-a_1)+f'_{x_2}(c)\cdot(b_2-a_2)+\dots+f'_{x_m}(c)\cdot(b_m-a_m)$$

bo`lishi kelib chiqadi.►

Odatda, (1) formula Lagranjning chekli orttirmalar formulasi deyiladi.

**2⁰. Xususiy hollar. Yo`nalish bo`yicha hosila.**

$m=1$ bo`lganda yuqoridagi teoremada keltirilgan formula

$$f(b)-f(a)=f'(c)\cdot(b-a)$$

$(a\in R, b\in R,\ a<c<b)$ ko`rinishga keladi. Bu Lagranj teorema-sini ifolovchi formula bo`lib, 21-ma`ruzada o`rganilgan.

$m=2$ bo`lganda (1) formula

$$f(b)-f(a)=f'_x(c)\cdot(b_1-a_1)+f'_y(c)(b_2-a_2)$$
$$\left(b=(b_1,b_2)\in R^2\quad a=(a_1,a_2)\in R^2\quad c=(c_1,c_2)\in R^2\right)$$

ko\`rinishida bo\`ladi.

Ma\`lumki, $u=f(x)$ $(x\in R\,,u\in R)$ funksiyaning hosilasi $f'(x)$ shu funksiyaning o\`zgarishini (o\`zgarish tezligini) ifodalar edi. $u=f(x,y)$ $\left((x,y)\in R^2\ ,\ u\in R\right)$ ikki o\`zgaruv-chili funksiyaning $f'_x(x,y)$ , $f'_y(x,y)$ xususiy hosilalari funksiyaning mos ravishda $OX$ hamda $OY$ o\`qlar bo\`yicha o\`zgarish tezligini bildiradi. Boshqacha aytganda $f(x,y)$ funksiyaning xususiy hosilalari koordinata o\`qlari yo\`nalishi bo\`yicha hosilalar bo\`ladi.

Endi $f(x,y)$ funksiyaning tekislikdagi ixtiyoriy tayin yo\`nalishi bo\`yicha hosilasi tushunchasini keltiramiz.

Faraz qilaylik, $f(x,y)$ funksiya $E\subset R^2$ to\`plamda berilgan bo\`lsin. Bu funksiyani Dekart koordinatalar sistemasida tasvirlangan $A_0=(x_0,y_0)$ nuqtaning $U_\delta(A_0)\subset E$ $(\delta>0)$ atrofida qaraymiz. Ushbu $A=(x,y)\subset U_\delta(A_0)$ nuqtani olib, $A_0$ va $A$ nuqtalari orqali to\`g\`ri chiziq o\`tkazamiz. Undagi ikki yo\`nalishdan birini musbat yo\`nalish (26-chizmada ko\`rsatilgandek), ikkinchisini esa manfiy yo\`nalish deb qabul qilamiz. Bu yo\`nalgan to\`g\`ri chiziqni $l$ bilan belgilaymiz. $A_0$ va $A$ nuqtalar orasidagi masofa

$$\rho=\rho(A_0,A)=\sqrt{(x-x_0)^2+(y-y_0)^2}$$

bo\`lib, bu masofa $\overrightarrow{A_0A}$ vektorning yo\`nalishi $l$ ning yo\`nalishi bilan bir hil bo\`lsa, musbat ishora bilan aks holda manfiy ishora bilan olinadi.

Agar $l$ ning musbat yo\`nalishi bilan $OX$ va $OY$ koordinata o\`qlarining musbat yo\`nalishilari orasidagi burchakni mos ravishda $\alpha$ va $\beta$ deyilsa, (26-chizma) unda

$$\frac{x-x_0}{\rho}=\cos\alpha,\ \frac{y-y_0}{\rho}=\cos\beta$$

bo\`lishi topiladi.

**1-ta\`rif.** Agar

$$\lim_{A\to A_0}\frac{f(A)-f(A_0)}{\rho}$$

limit mavjud bo\`lsa, bu limit $f(x,y)$ funksiyaning $A_0=(x_0,y_0)$ nuqtadagi $l$

yo`nalish bo`yicha hosila deyiladi. Uni

$$\frac{\partial f(A_0)}{\partial l} \text{ yoki } \frac{\partial f(x_0, y_0)}{\partial l}$$

kabi belgilanadi. Demak,

$$\frac{\partial f(A_0)}{\partial l} = \lim_{A \to A_0} \frac{f(A) - f(A_0)}{\rho}.$$

**1-misol.** Ushbu

$$f(x,y) = \sqrt[3]{x^2 y}$$

funksiyaning $(0,0)$ nuqtada barcha yo`nalishlar bo`icha hosilalarining mavjudligi ko`rsatilgan.

◀ Aytaylik, $\alpha \neq \pm\frac{\pi}{2}$ bo`lsin. Bu holda

$$y = tg\,\alpha \cdot x$$

bo`lib,

$$f(x,y) = \sqrt[3]{x^3 tg\,\alpha} = x\sqrt[3]{tg\,\alpha},$$

$$\rho((x,y),(0,0)) = \sqrt{x^2 + x^2 tg^2\alpha} = |x|\sqrt{1 + tg^2\alpha}$$

bo`ladi. Unda berilgan funktsining $(0,0)$ nuqtadagi $\alpha \neq \pm\frac{\pi}{2}$ bo`lgan ixtiyoriy yo`nalish bo`yicha hosilasi, ta`rifga binoan

$$\frac{\partial f(0,0)}{\partial l} = \lim_{\rho \to 0} \frac{f(x,y) - f(0,0)}{\rho} = \lim_{\rho \to 0} \frac{\sqrt[3]{tg\,\alpha} \cdot x}{\sqrt{1 + tg^2\alpha} \cdot |x|}$$

bo`ladi.

Agar $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ bo`lsa, unda $x > 0$, $|x| = x$ bo`lib,

$$\frac{\partial f(0,0)}{\partial l} = \frac{\sqrt[3]{tg\,\alpha}}{\sqrt{1 + tg^2\,\alpha}}$$

bo`ladi.

Agar $\frac{\pi}{2}<\alpha<\frac{3\pi}{2}$ bo`lsa, unda $x<0$ , $|x|=-x$ bo`lib,

$$\frac{\partial f(0,0)}{\partial l}=-\frac{\sqrt[3]{tg\alpha}}{\sqrt{1+tg^2\alpha}}$$

bo`ladi.

Aytaylik, $\alpha=\pm\frac{\pi}{2}$ bo`lsin. Bu holda $x=0$ , $f(0,y)=0$ bo`lib, bu yo`nalishlar bo`yicha hosila

$$\frac{\partial f(0,0)}{\partial l}=0$$

bo`ladi.►

**1-teorema.** Agar $f(x,y)$ funksiya $A_0=(x_0,y_0)$ nuqtada differentsiallanuvchi bo`lsa, u holda funksiya shu nuqtada har qanday yo`nalish bo`yicha hosilaga ega va

$$\frac{\partial f(A_0)}{\partial l}=\frac{\partial f(x_0,y_0)}{\partial l}=\frac{\partial f(x_0,y_0)}{\partial x}\cos\alpha+\frac{\partial f(x_0,y_0)}{\partial y}\cos\beta \qquad (5)$$

bo`ladi.

◄ Aytaylik, $f(x,y)$ funksiya $A_0=(x_0,y_0)$ nuqtada dif-ferentsiallanuvchi bo`lsin. U holda

$$f(A)-f(A_0)=f(x,y)-f(x_0,y_0)$$

orttirma uchun

$$f(A)-f(A_0)=\frac{\partial f(A_0)}{\partial x}\cdot(x-x_0)+\frac{\partial f(A_0)}{\partial y}\cdot(y-y_0)+0(\rho)$$

bo`ladi, bunda

$$\rho=\sqrt{(x-x_0)^2+(y-y_0)^2}\,.$$

Keyingi tenglikning har ikki tomonini $\rho$ ga bo`lamiz:

$$\frac{f(A)-f(A_0)}{\rho}=\frac{\partial f(A_0)}{\partial x}\cdot\frac{x-x_0}{\rho}+\frac{\partial f(A_0)}{\partial y}\cdot\frac{y-y_0}{\rho}+\frac{0(\rho)}{\rho}.$$

Ma`lumki,

$$\frac{x-x_0}{\rho}=\cos\alpha,\ \frac{y-y_0}{\rho}=\cos\beta.$$

SHuni e`tiborga olib, $\rho\to 0$ da limitga o`tib topamiz:

$$\lim_{\rho\to 0}\frac{f(A)-f(A_0)}{\rho}=\frac{\partial f(A_0)}{\partial x}\cos\alpha+\frac{\partial f(A_0)}{\partial y}\cos\beta.$$

Demak,

$$\frac{\partial f(A_0)}{\partial l}=\frac{\partial f(x_0,y_0)}{\partial x}\cos\alpha+\frac{\partial f(x_0,y_0)}{\partial y}\cos\beta.$$ ►

**2-misol.** Ushbu

$$f(x,y)=x^2+y^2$$

funksiyaning $(1,1)$ nuqtada $\vec{r}=\vec{i}+2\vec{j}$ vektor yo`nalish bo`yicha hosilasi topilsin.

◄ Ravshanki, bu holda

$$\cos\alpha=\frac{1}{\sqrt{5}},\ \cos\beta=\frac{2}{\sqrt{5}},$$

$$\frac{\partial f(x,y)}{\partial x}=\frac{\partial(x^2+y^2)}{\partial x}=2x,\ \frac{\partial f(1,1)}{\partial x}=2\ ,$$

$$\frac{\partial f(x,y)}{\partial y}=\frac{\partial(x^2+y^2)}{\partial y}=2y,\ \frac{\partial f(1,1)}{\partial y}=2\ ,$$

bo`ladi. (5) formuladan foydalanib topamiz:

$$\frac{\partial f(1,1)}{\partial l}=2\cdot\frac{1}{\sqrt{5}}+2\cdot\frac{2}{\sqrt{5}}=\frac{6}{\sqrt{5}}.$$ ►

Faraz qilaylik, $f(x,y)$ funksiya ochiq $E\subset R^2$ to`plamda differentsiallanuvchi bo`lsin. Binobarin, funksiya $E$ to`plamning har bir $(x,y)\in E$ nuqtasida

$$\frac{\partial f(x,y)}{\partial x}, \quad \frac{\partial f(x,y)}{\partial y}$$

xususiy hosilalarga ega bo`ladi. Koordinatalari shu xususiy hosilalardan iborat bo`lgan vektorni tuzamiz:

$$\frac{\partial f(x,y)}{\partial x}\cdot\vec{i}+\frac{\partial f(x,y)}{\partial y}\cdot\vec{j} \qquad (6)$$

bunda, $\vec{i}$ va $\vec{j}$ koordinata o`qlari bo`yicha yo`nalgan birlik vektorlar. (6) vektor $f(x,y)$ funksiyaning gradienti deyiladi va $grad\, f$ kabi belgilanadi:

$$grad\, f=\frac{\partial f(x,y)}{\partial x}\cdot\vec{i}+\frac{\partial f(x,y)}{\partial y}\cdot\vec{j}\,.$$

Demak, $grad\, f$ $E$ to`plamning har bir $(x,y)$ nuqtasiga bitta vektorni mos qo`yuvchi qoida, boshqacha aytganda ikki o`zgaruvchili vektor funksiya bo`ladi.

$f(x,y)$ funksiyaning $\vec{e}=(\cos\alpha\ \cos\beta)$ vektor yo`nalishi bo`yicha $\frac{\partial f(x,y)}{\partial l}$ hosilasini uning gradienti orqali ifodalash mumkin. Haqiqatan ham, $grad\, f$ va $\vec{e}$ vektorlarning skalyar ko`paytmasi

$$\vec{e}\, grad\, f=\cos\alpha\frac{\partial f(x,y)}{\partial x}+\cos\beta\frac{\partial f(x,y)}{\partial y} \qquad (7)$$

bo`lib, u (5) formulaga ko`ra $\frac{\partial f(x,y)}{\partial l}$ ga teng bo`ladi:

$$\vec{e}\, grad\, f=\frac{\partial f(x,y)}{\partial x}.$$

Ayni paytda, $\vec{e}$ va $grad\, f$ vektorlarning skalyar ko`paytmasi shu vektor uzunliklari ko`paytmasini ular orasidagi burchak kosinusiga ko`paytirilganiga teng bo`ladi:

$$\vec{e}\, grad\, f=|grad\, f|\cdot\left|\vec{e}\right|\cdot\cos\left(\vec{e}\,\hat{,}\, grad\, f\right) \qquad (8)$$

Ravshanki, $\left|\vec{e}\right|=1$.

(7) va (8) munosabatlardan

$$\frac{\partial f(x,y)}{\partial l}=|grad\, f(x,y)|\cdot\cos\left(\vec{e}, grad\, f(x,y)\right)$$

bo`lishi kelib chiqadi.

Keyingi tenglikdan ko`rinadiki, $\vec{e}$ hamda $grad\, f(x,y)$ vektorlar parallel bo`lganda $\dfrac{\partial f(x,y)}{\partial l}$ ning qiymati eng katta va u

$$|grad\, f(x,y)|=\sqrt{f_x'^2(x,y)+f_y'^2(x,y)}$$

ga teng bo`ladi.

SHunday qilib, $f(x,y)$ funksiyaning gradienti $grad\, f$ funksiyaning $(x,y)$ nuqtadagi eng tez o`sadigan tomonga yo`nalgan bo`lib, uning uzunligi shu yo`nalish bo`yicha o`sish tezligiga teng ekan.

**3-misol.** Ushbu

$$f(x,y)=x^2+2y^2$$

funksiyaning $(1,1)$ nuqtada eng tez o`sadigan yo`nalishi aniqlansin va shu yo`nalish bo`yicha o`sish tezligi topilsin.

◄Ravshanki,

$$\frac{\partial f(x,y)}{\partial x}=\frac{\partial(x^2+2y^2)}{\partial x}=2x,\qquad \frac{\partial f(1,1)}{\partial x}=2;$$

$$\frac{\partial f(x,y)}{\partial y}=\frac{\partial(x^2+2y^2)}{\partial y}=4y,\qquad \frac{\partial f(1,1)}{\partial y}=4;$$

bo`lib,

$$grad\, f(1,1)=2\vec{i}+4\vec{j},$$

$$|grad\, f(1,1)|=\sqrt{2^2+4^2}=2\sqrt{5}$$

bo`ladi.►

## Mashqlar

1. Ushbu

$$f(x,y)=(\frac{y}{x})^x, \qquad f(x,y)=\ln\sin\frac{x+1}{\sqrt{y}}$$

funksiyalarning xususiy hosilalari topilsin.

2. Agar

$$f(x,y)=x\sin y+y\sin x, \qquad x=\frac{u}{v},\ y=u\cdot v$$

bo‘lsa, $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ lar topilsin.

3. Aytaylik, $f(x)$ va $g(x)$ funksiyalar $U_\delta(x^0)\subset R^m$ da aniqlangan bo‘lib,

1) $f(x)$ funksiya $x^0$ nuqtada differensiyallanuvchi va $f(x^0)=0$,

2) $g(x)$ funksiya $x^0$ nuqtada uzluksiz bo‘lsa, $f(x)\cdot g(x)$ funksiyaning $x^0$ nuqtada differensiallanuvchi bo‘lishi ko‘rsatilsin.

4. Ushbu

$$f(x,y)=\sqrt{|xy|}$$

funksiyaning $(0,0)$ nuqtada differensiallanuvchi emasligi isbotlansin.

## Adabiyotlar

1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma’ruzalar, II q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.

# Glossariy

$f(x)=f(x_1,x_2,...,x_m)$ **funksiyaning** $x^0=(x_1^0,x_2^0,...,x_m^0)$ **nuqtadagi** $x_1$ **o'zgaruvchisi bo'yicha xususiy hosilasi** - Ushbu

$$\lim_{\Delta x_1\to 0}\frac{\Delta_{x_1} f(x^0)}{\Delta x_1}$$

limit mavjud bo'lsa, bu limit $f(x)=f(x_1,x_2,...,x_m)$ funksiyaning $x^0=(x_1^0,x_2^0,...,x_m^0)$ nuqtadagi $x_1$ o'zgaruvchisi bo'yicha xususiy hosilasi deyiladi.

$u=f(x)$ **funksiyaning** $x^0$ **nuqtadagi orttirmasi** -

$\Delta u=f(x)-f(x^0)$ $\quad (x=(x_1,x_2,....,x_m)\ ,\ x^0=(x_1^0,...,x_k^0))$.

**Funksiyaning** $x^0$ **nuqtadagi to'la orttirmasi** -

$\Delta f(x^0)=f(x^0+\Delta x_1,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m)-f(x_1^0,x_2^0,...,x_m^0)$.

$f(x)$ **funksiya** $x^0$ **nuqtada differensiallanuvchi** - Agar $\Delta x_1,\Delta x_2,...,\Delta x_m$ orttirmalarga bog'liq bo'lmagan shunday $A_1,A_2,....,A_m$ sonlari topilib, funksiyaning $x^0$ nuqtadagi to'liq orttirmasi ushbu

$$\Delta f(x^0)=A_1\Delta x_1+A_2\Delta x_2+....+A_m\Delta x_m+$$
$$+\alpha_1\Delta x_1+\alpha_2\Delta x_2+...+\alpha_m\Delta x_m$$

ko'rinishda ifodalansa, $f(x)$ funksiya $x^0$ nuqtada differensiallanuvchi deyiladi.

$(x_1^0,x_2^0,...,x_m^0)$ **hamda** $(x_1^0+\Delta x_1,x_2^0+\Delta x_2,...,x_m^0+\Delta x_m)$ **nuqtalar orasidagi masofa** - $\rho=\sqrt{\Delta x_1^2+\Delta x_2^2+...+\Delta x_m^2}$ .

# Keys banki

**49-keys.** Masala o`rtaga tashlanadi: Ushbu

$$f(x,y)=\sqrt{|xy|}$$

funksiyaning $(0,0)$ nuqtada differensiallanuvchi emasligi isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 27-28-amaliy mashg'ulot

## Na'muna uchun misollar yechimi

**1-misol**. Ushbu

$$z = x^2 x^2 y + y^3$$

funksiyaning xususiy hosilalari topilsin.

Berilgan funksiyada $y$ ni o'zgarmas deb qarab, $x$ o'zgaruvchi bo'yicha hosilasini topamiz. Bunda ma'lum bo'lgan qoida va formulalardan foydalanamiz:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3 + x^2 y + y^3) = 3x^2 + 2xy + 0 = 3x^2 + 2xy$$

Huddi shunga o'xshash $x$ ni o'zgarmas hisoblab topamiz:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3 + x^2 y + y^3) = 0 + x^2 + 3y^2 = x^2 + 3y^2$$

**2-misol**. Ushbu

$$f(x, y) = \sqrt{x^2 - y^2}$$

funksiyaning $(5; -3)$ nuqtadagi xususiy hosilalari topilsin.

Avvalo berilgan funksiyaning xususiy hosilalarini topamiz:

$$f'_x(x, y) = \frac{\partial}{\partial x}(\sqrt{x^2 - y^2}) = \frac{1}{2\sqrt{x^2 - y^2}} \cdot 2x = \frac{x}{\sqrt{x^2 - y^2}},$$

$$f'_y(x, y) = \frac{\partial}{\partial x}(\sqrt{x^2 - y^2}) = \frac{1}{2\sqrt{x^2 - y^2}} \cdot (-2x) = \frac{-y}{\sqrt{x^2 - y^2}}$$

Endi bu xususiy hosilalarning ko'rsatilgan $(5; -3)$ nuqtadagi qiymatlarini topamiz:

$$f'_x(5, -3) = \frac{5}{\sqrt{5^2 - (-3)^2}} = \frac{5}{\sqrt{25 - 9}} = \frac{5}{4},$$

$$f'_y(5,-3)=\frac{-(-3)}{\sqrt{5^2-(-3)^2}}=\frac{3}{\sqrt{25-9}}=\frac{3}{4}$$

Demak, $f'_x(5;-3)=\frac{5}{4}$, $f'_y(5,-3)=\frac{3}{4}$ ►

**3-misol.** Ushbu

$$f(x)=f(x_1,x_2,...,x_m)=x_1^2+x_2^2+...+x_m^2$$

funksiyaning $\forall(x_1^0,x_2^0,...,x_m^0)\in R^m$ nuqtada differensiallanuvchi bo‘lishi ko‘rsatilsin.

◄Berilgan funksiyaning $x^0=(x_1^0,x_2^0,...,x_m^0)$ nuqtadagi to‘liq orttirmasini topamiz:

$$\Delta f(x^0)=(x_1^0+\Delta x_1)^2+(x_2^0+\Delta x_2)^2+...+(x_m^0+\Delta x_m)^2-$$
$$-(x_1^{0^2}+x_2^{0^2}+...+x_m^{0^2})=2x_1^0\Delta x_1+2x_2^0\Delta x_2+...+$$
$$+2x_m^0\Delta x_m+\Delta x_1^2+\Delta x_2^2+...+\Delta x_m^2.$$

Agar

$$A_1=2x_1^0,\ A_2=2x_2^0,...,\ A_m=2x_m^0,$$
$$\alpha_1=\Delta x_1,\alpha_2=\Delta x_2,...,\ \alpha_m=\Delta x_m$$

deyilsa, unda

$$\Delta f(x^0)=A_1\Delta x_1+A_2\Delta x_2+....+A_m\Delta x_m+\alpha_1\Delta x_1+\alpha_2\Delta x_2+...+\alpha_m\Delta x_m$$

bo‘ladi. Demak, berilgan funksiya $\forall x^0\in R^m$ nuqtada differensiallanuvchi.►

**4-misol**. Ushbu

$$f(x,y)=\ln tg\,\frac{x}{y}$$

funksiyaning xususiy hosilalar topilsin.

◄Berilgan funksiyaning xususiy hosilalari quyidagicha bo‘ladi:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\ln tg\frac{x}{y}\right)=\frac{1}{tg\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\frac{1}{y}=\frac{2}{y\sin\frac{2x}{y}};$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\ln tg\frac{x}{y}\right)=\frac{1}{tg\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\left(-\frac{x}{y^2}\right)=\frac{-2}{y^2\sin\frac{2x}{y}}$$ ►

## Misollar

**Quyidagi funksiyalarning xususiy hosilalari topilsin.**

1. $a)\, z=x^3+y^2-2xy,\qquad b)\, z=5x^2+8xy^2+y^3$

2. $a)\, z=x^3+y^2-3axy,\qquad b)\, z=x^2-2xy+y^2$

3. $a)\, z=\frac{xy}{x+y},\qquad b)\, z=\frac{x-y}{x+y}$

4. $a)\, z=\frac{y}{x},\qquad b)\, z=\sqrt{\frac{x}{y}}$

5. $a)\, z=\sqrt{x^2-y^2},\qquad b)\, z=\sqrt{x+3y}$

6. $a)\, z=\frac{1}{\sqrt{x}-\sqrt{y}},\qquad b)\, z=\frac{x}{\sqrt{x^2+y^2}}$

7. Agar $f(x,y)=\frac{1-xy}{1+xy}$ bo\`lsa, $f'_x(0,1),\ f'_y(0,1)$ topilsin.

8. Agar $f(x,y)=\sqrt{xy+\frac{x}{y}}$ bo\`lsa, $f'_x(2,1),\ f'_y(2,1)$ topilsin.

9. Agar $f(x,y)=\ln\frac{x+y}{x-y}$ bo\`lsa, $f'_x(2e,e),\ f'_y(2e,e)$ topilsin.

10. Agar $f(x,y)=y\sin x+\cos(x-y)$ bo\`lsa, $f'_x(\pi,0),\ f'_y(\pi,0)$ topilsin.

Quyidagi murakkab funksiyalarning xosilalari topilsin.

1. Agar $z=e^{x^2+2y^2}$ bo'lib, $x=\sin t,\quad y=\cos t$ bo'lsa, $\frac{dz}{dt}$ topilsin.

2. Agar $z=\frac{x}{y}$ bo'lib, $x=e^t,\quad y=\ln t$ bo'lsa, $\frac{dz}{dt}$ topilsin.

3. Agar $z=\frac{x^2}{y}$ bo'lib, $x=u-2v,\quad y=v+2v$ bo'lsa, $\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}$ lar topilsin.

4. Agar $z=\ln\sin\frac{x}{\sqrt{y}}$ bo'lib, $x=3t^2,\quad y=\sqrt{t^2+1}$ bo'lsa, $\frac{\partial z}{\partial t}$ topilsin.

5. Agar $z=x^2-y^2$ bo'lib, $x=u\cos v,\quad y=u\sin v$ bo'lsa, $\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}$ lar topilsin.

6. Agar $z=u^v$ bo'lib, $u=\sin x,\quad v=\cos x$ bo'lsa, $\frac{\partial z}{\partial x}$ topilsin.

7. Agar $z=\ln\sqrt{\frac{u}{v}}$ bo'lib, $u=ax+by,\quad v=ax-by$ bo'lsa, $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$ topilsin.

8. Agar $z=e^{\frac{y^2}{x}}$ bo'lib, $x=x(u,v),\quad y=y(u,v)$ bo'lsa, $\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}$ topilsin.

# Test

1. $u = x^2 + y + 1$ funksiyaning birinchi tartibli xususiy hosilalarini toping.

A) $\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 1$ B) $\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2x + 1$

C) $\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial u}{\partial y} = 1$ D) $\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 0$

2. $u = xy + \frac{1}{x}$ funksiyaning birinchi tartibli xususiy hosilalarini toping.

A) $\frac{\partial u}{\partial x} = y - \frac{1}{x^2}, \quad \frac{\partial u}{\partial y} = x$ B) $\frac{\partial u}{\partial x} = x + \frac{1}{x^2}, \quad \frac{\partial u}{\partial y} = x$

C) $\frac{\partial u}{\partial x} = x - \frac{1}{x^2}, \quad \frac{\partial u}{\partial y} = y + 1$ D) $\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x + 1$

3. $u = x^2 + y + 1$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping.

A) $\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x \partial y} = 0$ B) $\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = 1, \quad \frac{\partial^2 u}{\partial x \partial y} = 1$

C) $\frac{\partial^2 u}{\partial x^2} = 2x, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x \partial y} = 3$ D) $\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x \partial y} = 0$

4. $u = x^y$ funksiyaning birinchi tartibli xususiy hosilalarini toping.

A) $\frac{\partial u}{\partial x} = y\,x^{y-1}, \quad \frac{\partial u}{\partial y} = x^y \ln x$ B) $\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x^y$

C) $\frac{\partial u}{\partial x} = x^y, \quad \frac{\partial u}{\partial y} = \ln x$ D) $\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x$

5. $u = xy + \frac{1}{x}$ funksiyaning ikkinchi tartibli xususiy hosilalarini toping.

A) $\frac{\partial^2 u}{\partial x^2} = \frac{2}{x^3}, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x \partial y} = 1$ B) $\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = 1, \quad \frac{\partial^2 u}{\partial x \partial y} = 1$

C) $\frac{\partial^2 u}{\partial x^2} = \frac{2}{x^3}, \quad \frac{\partial^2 u}{\partial y^2} = 1, \quad \frac{\partial^2 u}{\partial x \partial y} = 3$ D) $\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x \partial y} = 0$

6. $u(x,y)=x+y\ u_x(x,y)-?$

A) 1 B) $x$ C) $y$ D) 0

7. $u(x,y)=2x+y\ u_y(x,y)-?$

A) 1 B) $x$ C) $y$ D) 0

8. $u(x,y)=x^2-y^2\ u_x(x,y)-?$

A) $2x$ B) $4y$ C) $5x+y$ D) $x+2y$

9. $u(x,y)=xy+x^2,\ u_x(x,y)-?$

A) $2x+y$ B) $x^5+y+3$ C) $y+5x+9$ D) $2x+2y+5$

10. $u(x,y)=xy+x^2, u_y(x,y)-?$

A) $x$ B) $x^5+y+3$ C) $y+5x+9$ D) $2x+2y+5$

# Mavzu. Ko‘p o‘zgaruvchili funksiyaning differensiali

## 29-ma’ruza

**Reja**

$1^0$. Funksiya differensiali tushunchasi.

$2^0$. Murakkab funksiyaning differensiali. Differensial shaklning invariantligi.

$3^0$. Sodda qoidalar.

$4^0$. Xususiy hollar. Funksiya differensialining geometrik ma’nosi.

**$1^0$. Funksiya differensiali tushunchasi.**

Faraz qilaylik, $f(x)=f(x_1,x_2,\ldots,x_m)$ funksiya $E\subset R^m$ da berilgan bo‘lib, $x^0=(x_1^0,x_2^0,\ldots,x_m^0)\in E$ nuqtada differensiallanuvchi bo‘lsin. Unda ta’rifga ko‘ra funksiyaning $x^0$ nuqtadagi to‘liq orttirmasi

$$\Delta f(x^0)=\frac{\partial f(x^0)}{\partial x_1}\Delta x_1+\frac{\partial f(x^0)}{\partial x_2}\Delta x_2+\cdots+\frac{\partial f(x^0)}{\partial x_m}\Delta x_m+o(\rho) \qquad (1)$$

bo‘ladi. Bu munosabatda

$$\rho=\sqrt{\Delta x_1^2+\Delta x_2^2+\cdots+\Delta x_m^2}$$

bo‘lib, $\Delta x_1\to 0$, $\Delta x_2\to 0$, $\ldots$, $\Delta x_m\to 0$ da $\rho\to 0$.

**1-ta’rif.** $f(x)$ funksiyaning $\Delta f(x^0)$ orttirmasidagi

$$\frac{\partial f(x^0)}{\partial x_1}\Delta x_1+\frac{\partial f(x^0)}{\partial x_2}\Delta x_2+\cdots+\frac{\partial f(x^0)}{\partial x_m}\Delta x_m$$

ifoda $f(x)$ funksiyaning $x^0$ nuqtadagi differensiali (to‘liq differensiali) deyiladi va

$$df\left(x^0\right) \text{ yoki } df\left(x_1^0, x_2^0, \dots, x_m^0\right)$$

kabi belgilanadi:

$$df\left(x^0\right)=\frac{\partial f\left(x^0\right)}{\partial x_1}\Delta x_1+\frac{\partial f\left(x^0\right)}{\partial x_2}\Delta x_2+\dots+\frac{\partial f\left(x^0\right)}{\partial x_m}\Delta x_m .$$

Demak, $f(x)$ funksiyaning $x^0$ nuqtadagi differensiali $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ larga bog‘liq va ularning chiziqli funksiyasi bo‘ladi.

Agar

$$\Delta x_1 = dx_1 , \Delta x_2 = dx_2 , \dots , \Delta x_m = dx_m$$

deyilsa, $f(x)$ funksiyaning $x^0$ nuqtadagi differensiali ushbu

$$df\left(x^0\right)=\frac{\partial f\left(x^0\right)}{\partial x_1}dx_1+\frac{\partial f\left(x^0\right)}{\partial x_2}dx_2+\dots+\frac{\partial f\left(x^0\right)}{\partial x_m}dx_m \qquad (2)$$

ko‘rinishga keladi. Demak,

$$\Delta f\left(x^0\right)=df\left(x^0\right)+0(\rho).$$

Keyingi tenglikdan $\rho \to 0$ da

$$\Delta f\left(x^0\right)\approx df\left(x^0\right)$$

bo‘lishi kelib chiqadi. Bu taqribiy formulaning mohiyati shundaki, funksiyaning orrtirmasi $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ larning, umuman aytganda murakkab funksiyasi bo‘lgan holda funksiyaning differensiali $\Delta x_1, \Delta x_2, \dots, \Delta x_m$ larning chiziqli funksiyai bo‘lishidadir.

**$2^0$. Murakkab funksiyaning differensiali. Differensial shaklning invariantligi.**

Aytaylik,

$$\begin{aligned} x_1 &= \varphi_1(t)=\varphi_1(t_1,t_2,\dots,t_k), \\ x_2 &= \varphi_2(t)=\varphi_1(t_1,t_2,\dots,t_k), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_m &= \varphi_m(t)=\varphi_1(t_1,t_2,\dots,t_k) \end{aligned}$$

funksiyalarning har biri $M \subset R^k$ to‘plamda berilgan bo‘lib,

$$E = \{(x_1, x_2, \ldots, x_m) \in R^m : x_1 = \varphi_1(t) = \varphi_1(t_1, t_2, \ldots, t_k),$$

$$x_2 = \varphi_2(t) = \varphi_1(t_1, t_2, \ldots, t_k), \ldots, x_m = \varphi_m(t) = \varphi_1(t_1, t_2, \ldots, t_k)\}$$

to‘plamda esa $f(x_1, x_2, ..., x_m)$ funksiya aniqlangan bo‘lsin. Bular yordamida

$$f(x(t)) = f(x_1(t), x_2(t), ..., x_m(t)) = F(t_1, t_2, ..., t_k)$$

murakkab funksiya hosil qilingan bo‘lsin.

Ma’lumki, $x_i = \varphi_i(t_1, \ldots, t_k)$ funksiyalar $(i = 1, 2, \ldots m)$ $t^0 = (t_1^0, \ldots, t_k^0)$ nuqtada differensiallanuvchi bo‘lib, $f(x) = f(x_1, x_2, \ldots, x_m)$ funksiya mos $x^0 = (x_1^0, x_2^0, \ldots, x_m^0)$ nuqtada $(x_1^0 = \varphi_1(t^0), x_2^0 = \varphi_2(t^0), \ldots, x_m^0 = \varphi_m(t^0))$ differensiallanuvchi bo‘lsa, murakkab funksiya $t^0 = (t_1^0, \ldots, t_k^0)$ nuqtada differensialanuvchi bo‘ladi.

Modomiki, $f(x(t))$ funksiya $t_1, t_2, \ldots, t_k$ o‘zgaruvchilarga bog‘liq ekan, unda

$$df = \frac{\partial f}{\partial t_1} dt_1 + \frac{\partial f}{\partial t_2} dt_2 + \cdots + \frac{\partial f}{\partial t_m} dt_m \qquad (3)$$

bo‘ladi.

Murakkab funksiyaning xususiy hosilalarini hisoblash formulalaridan foydalanib topamiz:

$$\frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_1} + \ldots + \frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_1},$$

$$\frac{\partial f}{\partial t_2} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \ldots + \frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_2},$$

..........................................................

$$\frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_k} + \ldots + \frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_k}$$

Bu xususiy hosilalarni (3) ifodadagi $\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, ...., \frac{\partial f}{\partial t_k}$ larning o‘rniga qo‘yamiz.

Natijada

$$df=\left[\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_1}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_1}+\cdots+\frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_1}\right]dt_1+$$

$$+\left[\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_2}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_2}+\cdots+\frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_2}\right]dt_2+$$

$$+\cdots+\left[\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_k}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_k}+\cdots+\frac{\partial f}{\partial x_m}\cdot\frac{\partial x_m}{\partial t_k}\right]dt_k=$$

$$=\frac{\partial f}{\partial x_1}\left[\frac{\partial x_1}{\partial t_1}dt_1+\frac{\partial x_1}{\partial t_2}dt_2+\cdots+\frac{\partial x_1}{\partial t_k}dt_k\right]+$$

$$+\frac{\partial f}{\partial x_2}\left[\frac{\partial x_2}{\partial t_1}dt_1+\frac{\partial x_2}{\partial t_2}dt_2+\cdots+\frac{\partial x_2}{\partial t_k}dt_k\right]+\cdots+\frac{\partial f}{\partial x_m}\left[\frac{\partial x_m}{\partial t_1}dt_1+\frac{\partial x_m}{\partial t_2}dt_2+\cdots+\frac{\partial x_m}{\partial t_k}dt_k\right]$$

bo‘ladi.

Ravshanki,

$$\frac{\partial x_1}{\partial t_1}dt_1+\frac{\partial x_1}{\partial t_2}dt_2+\ldots+\frac{\partial x_1}{\partial t_k}dt_k=dx_1,$$

$$\frac{\partial x_2}{\partial t_1}dt_1+\frac{\partial x_2}{\partial t_2}dt_2+\ldots+\frac{\partial x_2}{\partial t_k}dt_k=dx_2,$$

..........................................................

$$\frac{\partial x_m}{\partial t_1}dt_1+\frac{\partial x_m}{\partial t_2}dt_2+\ldots+\frac{\partial x_m}{\partial t_k}dt_k=dx_k.$$

Demak, murakkab funksiyaning differensiali

$$df=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+\ldots+\frac{\partial f}{\partial x_m}dx_m \qquad (4)$$

bo‘ladi.

Biz yuqorida $f(x)$ hamda $f(x(t))$ murakkab funksiyaning differensallari uchun (2) va (4) ifodalarni topdik. Bu ifodalarni solishtirib ularning formasi (shakli, ko‘rinishi) bir xil, ya’ni (2) va (4) formulalarda funksiyaning differensiali xususiy hosilalarni mos differensiallarga ko‘paytmalardan tuzilgan yig‘indiga teng ekanligini payqaymiz. Bu xossa differensial shaklning **invariantligi** deyiladi.

**Eslatma.** $f(x)$ funksiya differensialining (2) ifodasidagi $dx_1, dx_2, ..., dx_m$ lar mos ravishda $\Delta x_1, \Delta x_2, ..., \Delta x_m$ lar bo‘lsa, $f(x(t))$ funksiya differensialidagi $dx_1, dx_2, ..., dx_m$ lar $t_1, t_2, ..., t_k$ o‘zgaruvchilarning funksiyalari bo‘ladi. Demak, (2) va (4) formulalarning ko‘rinishlarigina bir xil bo‘ladi.

**3⁰. Sodda qoidalar**. Aytaylik,

$$u = u(x_1, x_2, ..., x_m), \quad v = v(x_1, x_2, ..., x_m)$$

funksiyalari $E \subset R^m$ to‘plamda berilgan bo‘lib, $x^\circ = (x_1^\circ, x_2^\circ, ..., x_m^\circ) \in E$ nuqtada differensiallanuvchi bo‘lsin. U holda:

$$1)\ d(u+v) = du + dv,$$

$$2)\ d(u \cdot v) = vdu + udv,$$

$$3)\ d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2} \qquad (v \neq 0)$$

bo‘ladi.

Bu munosabatlardan birini, masalan, 3) ning isbotini keltiramiz.

◀Aytaylik,

$$F = \frac{u}{v}$$

bo‘lsin. Bu holda $F$ funksiya $u$ va $v$ larga va $u$ va $v$ lar o‘z navbatida $x_1, x_2, ..., x_m$ o‘zgaruvchilarga bog‘liq bo‘lib, murakkab funksiyaga ega bo‘lamiz. Differensial shaklning invariantli xossasiga ko‘ra

$$dF = \frac{\partial F}{\partial u} du + \frac{\partial F}{\partial v} dv$$

bo‘ladi. Ravshanki,

$$\frac{\partial F}{\partial u} = \frac{1}{v}, \quad \frac{\partial F}{\partial v} = -\frac{u}{v^2}$$

Demak,

$$dF = \frac{1}{v} du - \frac{u}{v^2} dv = \frac{vdu - ydv}{v^2},$$

ya'ni

$$d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$$

bo'ladi.►

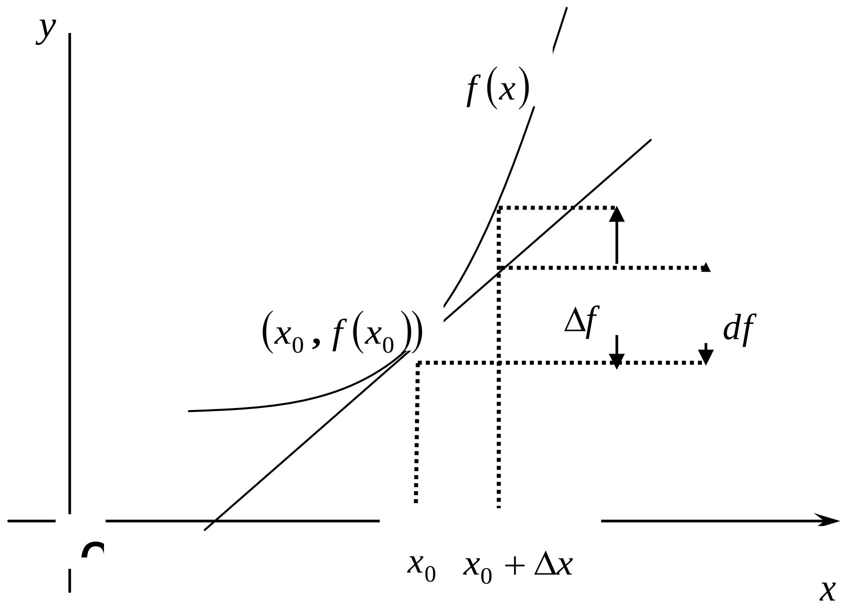
### $4^0$. Xususiy hollar. Funksiya differensialining geometrik ma'nosi.

Aytaylik, $m=1$ bo'lsin. Bu holda $u=f(x)$ $(x\in R, u\in R)$ funksiya va uning differensiali

$$du=df=f'(x)dx$$

ga ega bo'lamiz.

Ma'lumki, $u=f(x)$ funksiyaning differensiali shu funksiya tasvirlangan egri chiziqqa $(x_\circ, f(x))$ nuqtada o'tkazilgan urinmaning ordinatasining orttirmasini ifodalaydi (27-chizma):




27-chizma

$m=2$ bo'lsin. Bu holda ikki o'zgaruvchili $u=f(x,y)$ $\left((x,y)\in R^2, u\in R\right)$ funksiyaga ega bo'lib, uning $(x_\circ, y_\circ)$ nuqtadagi differensial

$$du = df(x_\circ, y_\circ) = \frac{\partial f(x_\circ, y_\circ)}{\partial x}dx + \frac{\partial f(x_\circ, y_\circ)}{\partial y}dy \qquad (5$$

bo‘ladi, bunda $dx = \Delta x$ , $dy = \Delta y$ .

$\Delta x$ va $\Delta y$ lar yetarlicha kichik bo‘lganda

$$\Delta f(x_\circ, y_\circ) \approx df(x_\circ, y_\circ)$$

ya’ni

$$f(x_\circ + \Delta x, y_\circ + \Delta y) \approx f(x_\circ, y_\circ) + \frac{\partial f(x_\circ, y_\circ)}{\partial x_1}\Delta x + \frac{\partial f(x_\circ, y_\circ)}{\partial y}\Delta y$$

taqribiy formula hosil bo‘ladi.

**1-misol.** Ushbu

$$u = x^y$$

funksiyaning differensiali topilsin.

◀Ravshanki,

$$\frac{\partial u}{\partial x} = yx^{y-1} \quad , \quad \frac{\partial u}{\partial y} = x^y \ln x.$$

Unda (5) formulaga ko‘ra

$$du = yx^{y-1}dx + x^y \ln x dy$$

bo‘ladi.▶

**2-misol.** Tomonlari $x = 6м$ va $н = 8м$ bo‘lgan to‘g‘ri to‘rtburchak berilgan. Agar bu to‘g‘ri to‘rtburchakning $x$ tomonini 5 sm. ga oshirilsa, u tomonini 10 sm. ga kamaytirilsa, to‘rtburchakning dioganali qanchaga o‘zgaradi?

◀ Agar berilgan to‘g‘ri to‘rtburchakning dioganalini $u$ desak, unda

$$u = \sqrt{x^2 + y^2}$$

bo‘ladi. Endi

$$\Delta u(x_\circ, y_\circ) \approx \frac{x_\circ}{\sqrt{x_\circ^2 + y_\circ^2}}\Delta x + \frac{y_\circ}{\sqrt{x_\circ^2 + y_\circ^2}}\Delta y$$

bo‘lishini e’tiborga olib, topamiz:

$$\Delta u(x_0, y_0) \approx \frac{x_0}{\sqrt{x_0^{\ 2}+y_0^{\ 2}}}\cdot\Delta x+\frac{y_0}{\sqrt{x_0^{\ 2}+y_0^{\ 2}}}\cdot\Delta y=\frac{x_0\cdot\Delta x+y_0\cdot\Delta y}{\sqrt{x_0^{\ 2}+y_0^{\ 2}}}$$

Bu munosabatda

$$x_\circ = 6\,м\ ,\ \Delta x = 0,05\,м,\ y_\circ = 8\,м\ ,\ \Delta y = -0,10\,м$$

deyilsa, unda

$$\Delta u \approx \frac{6\cdot 0,06+8\cdot(-0,10)}{\sqrt{36+64}}\,м = -0,05\,м$$

bo‘lishi kelib chiqadi.

Demak, to‘g‘ri to‘rtburchakning diogonali taxminan 5 sm. ga kamayar ekan.►

Endi $f(x,y)$ funksiya differensialining geometrik ma’nosini keltiramiz.

Aytaylik,

$$z = f(x,y)$$

funksiya ochiq $E\subset R^2$ to‘plamda differensiallnuvchi bo‘lsin. Bu funksiya grafigi $R^3$ fazoda biror $\Gamma(f)$ sirti ifodalasin. $\Gamma(f)=\{(x,y,z)\in R^3:(x,y)\in E, z=f(x,y)\}$ sirtda $(x_\circ,y_\circ,z_\circ)\in\Gamma(f)$ $(z_\circ=f(x_\circ,y_\circ))$ nuqtani va shu nuqtadan o‘tuvchi, qaralayotgan sirtga tegishli bo‘lgan silliq

$$\Gamma=\{x=x(t),y=y(t),z=z(t):\alpha\le t\le\beta\}$$

egri chiziqni olamiz. Modomiki, egri chiziq sirtda yotar ekan, unda

$$z(t)=f(x(t),y(t))$$

$$\left(\left(x(t_\circ),y(t_\circ),z(t_\circ)\right)=(x_\circ,y_\circ,z_\circ),t_\circ\in(\alpha,\beta)\right)$$

bo‘ladi. Ravshanki,

$$z(t)=f(x(t),y(t))$$

murakkab funksiya bo‘lib, uning $t_\circ$ nuqtadagi differensiali, differensial shaklning invariantligi xossasiga binoan, ushbu

$$df(x_\circ,y_\circ)=dz=\frac{\partial f(x_\circ,y_\circ)}{\partial x}dx+\frac{\partial f(x_\circ,y_\circ)}{\partial y}dy \qquad (6)$$

ko‘rinishga ega.

Koordinatalari $dx, dy, dz$ bo‘lgan vektor $\Gamma$ egri chiziqqa $(x_\circ, y_\circ, z_\circ)$ nuqtada o‘tkazilgan urinma vektor bo‘ladi.

Endi koordinatalari

$$-\frac{\partial f(x_\circ, y_\circ)}{\partial x},\ -\frac{\partial f(x_\circ, y_\circ)}{\partial y},\ 1$$

bo‘lgan $\vec{n}$ vektorni qaraylik. Yuqoridagi $(6)$ munosabat $\vec{n}$ vektor urinma vektorga $(x_\circ, y_\circ, z_\circ)$ nuqtada ortogonal bo‘lishini bildiradi. Shuning uchun $\vec{n}$ vektor egri chiziqqa $(x_\circ, y_\circ, z_\circ)$ nuqtada ortogonal deyiladi.

Ma’lumki, $\Gamma$ egri chiziq $(x_\circ, y_\circ, z_\circ)$ nuqtadan o‘tuvchi va $\Gamma(f)$ sirtda yotuvchi ixtiyoriy egri chiziq edi. Binobarin, $\vec{n}$ vektor shu $(x_\circ, y_\circ, z_\circ)$ nuqtadan o‘tuvchi va $\Gamma(f)$ sirtda yotuvchi ixtiyoriy egri chiziqqa ortogonal bo‘ladi. Shuning uchun $\vec{n}$ vektor $\Gamma(f)$ sirtning nuqtasidagi normal vetori deyiladi.

Sirtning $(x_\circ, y_\circ, z_\circ)$ nuqtasida o‘tuvchi va sirtning normal vektoriga ortogonal bo‘lgan tekislik, $\Gamma(f)$ sirtga $(x_\circ, y_\circ, z_\circ)$ nuqtada o‘tkazilgan urinma tekislik deyiladi. Uning tenglamasi

$$Z - z_\circ = \frac{\partial f(x_\circ, y_\circ)}{\partial x}(X - x_\circ) + \frac{\partial f(x_\circ, y_\circ)}{\partial y}(Y - y_\circ)$$

bo‘ladi, bunda $(X, Y, Z)$ urinma tekislikdagi o‘zgaruvchi nuqta. Bu tenglikdan foydalanib,

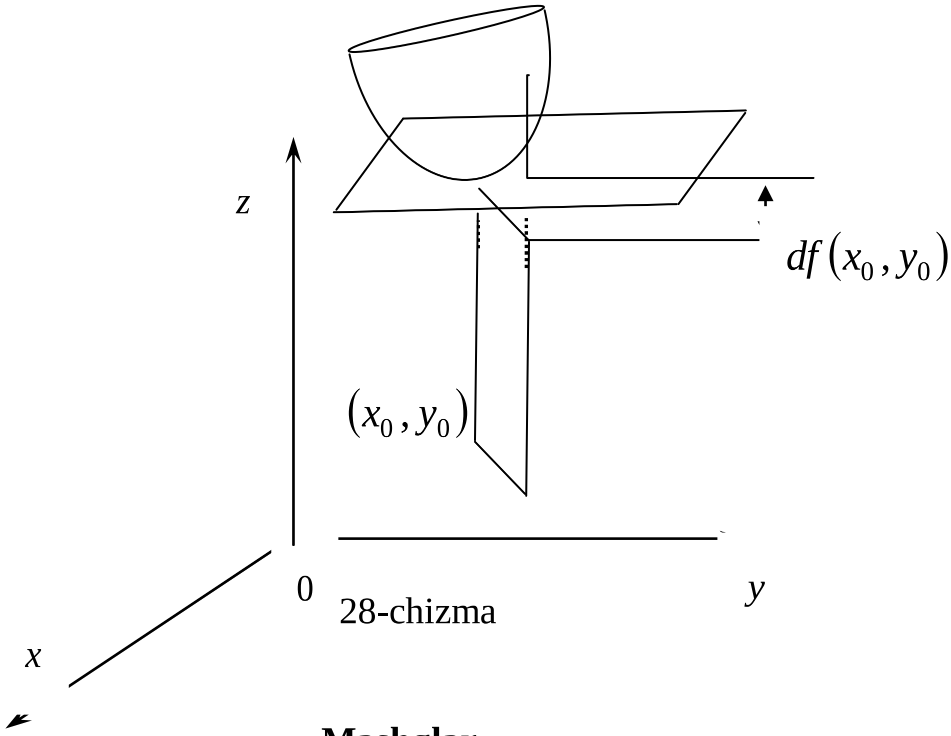
$$Z - z_\circ = \frac{\partial f(x_\circ, y_\circ)}{\partial x}(x - x_\circ) + \frac{\partial f(x_\circ, y_\circ)}{\partial y}(y - y_\circ)$$

bo‘lishini olamiz. Keltirilgan tenglik va (6) munosabatdan

$$df(x_\circ, y_\circ) = Z - z_\circ$$

bo‘lishi kelib chiqadi.

Shunday qilib, $z = f(x, y)$ funksiyaning $(x_\circ, y_\circ)$ nuqtadagi differensiali $df(x_\circ, y_\circ)$ bu funksiya grafigiga $(x_\circ, y_\circ f(x_\circ, y_\circ))$ nuqtasida urinma tekislik applikatasiyaning orttirmasini ifodalar ekan (28-chizma)




28-chizma

**Mashqlar**

1. Ushbu $f\left(x^2+y^2, arctg\,\dfrac{y}{x}\right)$ $\left(x^2+y^2>0\right)$ funksiyaning differensiali topilsin.

2. Ushbu $\alpha=(1,02)^{3,01}$ miqdorning taqribiy qiymati topilsin.

**Adabiyotlar**

1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, II q.* T. “Voris-nashriyot”, 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.

# Glossariy

$f(x)$ **funksiyaning** $x^0$ **nuqtadagi differensiali (to'liq differensiali) -** $f(x)$ funksiyaning $\Delta f\left(x^0\right)$ orttirmasidagi

$$\frac{\partial f\left(x^0\right)}{\partial x_1}\Delta x_1+\frac{\partial f\left(x^0\right)}{\partial x_2}\Delta x_2+\cdots+\frac{\partial f\left(x^0\right)}{\partial x_m}\Delta x_m$$

ifoda.

$\Gamma(f)$ **sirtning nuqtasidagi normal vetori -** $\vec{n}$ vektor $(x_\circ, y_\circ, z_\circ)$ nuqtadan o'tuvchi va $\Gamma(f)$ sirtda yotuvchi ixtiyoriy egri chiziqqa ortogonal bo'lsa.

**Urinma vektor -** Koordinatalari $dx, dy, dz$ bo'lgan vektor $\Gamma$ egri chiziqqa $(x_\circ, y_\circ, z_\circ)$ nuqtada o'tkazilgan urinma vektor bo'ladi.

**Funksiya grafigiga** $(x_\circ, y_\circ f(x_\circ, y_\circ))$ **nuqtasida urinma tekislik applikatasiyaning orttirmasi -** $z=f(x,y)$ funksiyaning $(x_\circ, y_\circ)$ nuqtadagi differensiali $df(x_\circ, y_\circ)$. **Urinma tekislik -** Sirtning $(x_\circ, y_\circ, z_\circ)$ nuqtasida o'tuvchi va sirtning normal vektoriga ortogonal bo'lgan tekislik, $\Gamma(f)$ sirtga $(x_\circ, y_\circ, z_\circ)$ nuqtada o'tkazilgan urinma tekislik deyiladi.

## Keys banki

**50-keys.** Masala o`rtaga tashlanadi: Ushbu $\alpha=(1,02)^{3,01}$ miqdorning taqribiy qiymati topilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 29-amaliy mashg'ulot

## Na'muna uchun misollar yechimi

**1-misol.** Ushbu

$$u = x^y$$

funksiyaning differensiali topilsin.

◀Ravshanki,

$$\frac{\partial u}{\partial x} = yx^{y-1} \quad , \quad \frac{\partial u}{\partial y} = x^y \ln x.$$

Unda (5) formulaga ko'ra

$$du = yx^{y-1}dx + x^y \ln x dy$$

bo'ladi.▶

**2-misol.** Tomonlari $x = 6м$ va $н = 8м$ bo'lgan to'g'ri to'rtburchak berilgan. Agar bu to'g'ri to'rtburchakning $x$ tomonini 5 sm. ga oshirilsa, u tomonini 10 sm. ga kamaytirilsa, to'rtburchakning dioganali qanchaga o'zgaradi?

◀ Agar berilgan to'g'ri to'rtburchakning dioganalini $u$ desak, unda

$$u = \sqrt{x^2 + y^2}$$

bo'ladi. Endi

$$\Delta u(x_\circ, y_\circ) \approx \frac{x_\circ}{\sqrt{x_\circ^2 + y_\circ^2}} \Delta x + \frac{y_\circ}{\sqrt{x_\circ^2 + y_\circ^2}} \Delta y$$

bo'lishini e'tiborga olib, topamiz:

$$\Delta u(x_0, y_0) \approx \frac{x_0}{\sqrt{x_0{}^2 + y_0{}^2}} \cdot \Delta x + \frac{y_0}{\sqrt{x_0{}^2 + y_0{}^2}} \cdot \Delta y = \frac{x_0 \cdot \Delta x + y_0 \cdot \Delta y}{\sqrt{x_0{}^2 + y_0{}^2}}$$

Bu munosabatda

$$x_\circ = 6м \ , \ \Delta x = 0,05м, \ y_\circ = 8м \ , \ \Delta y = -0,10м$$

deyilsa, unda

$$\Delta u \approx \frac{6\cdot 0,06+8\cdot(-0,10)}{\sqrt{36+64}}\,\text{м} = -0,05\text{м}$$

bo‘lishi kelib chiqadi.

Demak, to‘g‘ri to‘rtburchakning diogonali taxminan 5 sm. ga kamayar ekan.►

**3-misol**. Ushbu

$$\alpha = 1,07^{3,97}$$

Miqdor taqribiy xisoblansin.

◄Ravshanki, $\alpha = 1,07^{3,97}$ son ushbu

$$f(x,y) = x^y$$

Funksiyaning $x = 1,07,\ y = 3,97$ dagi xususiy qiymatidan iborat. Bu funksiya uchun $f(1,4) = 1$ bo‘lishini e’tiborga olib

$$x_0 = 1,\quad y_0 = 4$$

deb olamiz. Unda

$$\Delta x = x - x_0 = 1,07 - 1 = 0,07,$$

$$\Delta y = y - y_0 = 3,97 - 4 = -0,03$$

bo‘ladi. Ushbu

$$f(x_0+\Delta x,\ y_0+\Delta y) \approx f(x_{0,},y_0) + f'_x(x_0,y_0)\cdot\Delta x + f'_y(x_0,y_0)\cdot\Delta y$$

formuladan foydalanib topamiz:

$$f(x_0+\Delta x,\ y_0+\Delta y) = f(1+0,7,\ 4-0,03) = f(1,7,\ 3,97) = 1,7^{3,97}$$

$$f(x_0,y_0) = f(1,4) = 1^4 = 1$$

$$f'_x(x_0,y_0) = \frac{\partial}{\partial x}(x^y)\Big|_{x_0,y_0} = y\cdot x^{y-1}\Big|_{\substack{x=1\\ y=4}} = 4\cdot 1 = 4$$

$$f'_y(x_0,y_0) = \frac{\partial}{\partial y}(x^y)\Big|_{x_0,y_0} = x^y\ln x\Big|_{\substack{x=1\\ y=4}} = 1^4\cdot\ln 1 = 0$$

$$1,7^{3,97} \approx 1 + 4\cdot 0,07 + 0\cdot(-0,03) = 1 + 0,28 = 1,28$$

Demak,

$$\alpha = 1,7^{3,97} \approx 1,28.$$ ►

## Misollar

**Quyidagi funksiyalarning differensiallari topilsin.**

1. $z = x^2 y - xy^2 + 3$  2. $z = \left(x^2 + y^2\right)^3$  3. $z = \sin^2 x + \cos^2 y$

4. $z = arctg\left(xy\right)$  5. $z = e^{12x+5y}$  6. $z = \left(\sin x\right)^{\cos y}$

7. $z = \ln\left(1 + \frac{x}{y}\right)$  8. $z = \frac{x}{y} e^{xy}$  9. $x = arctg\sqrt{xy}$

10. Agar $f\left(x,y\right) = \frac{x}{y^2}$ bo'lsa, $df\left(1,1\right)$ topilsin.

11. Ushbu $f\left(x,y\right) = e^{xy}$ funksiya to'liq differensialini $x = 1,\quad y = 2,\quad dx = -0,1,\quad dy = 0,1$ bo'lgandagi qiymati topilsin.

12. Ushbu $f\left(x,y\right) = \frac{x}{x-y}$ funksiya to'liq differensialini $x = 2,\quad y = 1,\quad dx = -\frac{1}{3},\quad dy = \frac{1}{2}$ bo'lgandagi qiymati topilsin.

Quyidagi miqdorlar taqribiy hisoblansin.

13. 1) $1,08^{3,96}$,  2) $1,94e^{0,12}$

14. 1) $\sin 1,59 \cdot tg 3,09$,  2) $2,68e^{\sin 0,05}$

# Test

1. Ko'rsatilgan funksiyalardan qaysi birlari har bir nuqtada differensiyallanuvchi?

A) $u = x + y^2 + 1$ B) $u = \dfrac{1}{x-y}$ C) $u = \dfrac{1}{x+y}$ D) $u = \dfrac{1}{2+3x}$

2. $u = x^3 + y^2 + z + 1$ funksiyaning birinchi tartibli $du$ differensialini toping.

A) $du = 3x^2dx + 2ydy + dz$ B) $du = dx + 3dz$

C) $du = dy + 2dz$ D) $du = xdx + 2ydy$

3. $u = \sqrt{x^2 + y^2}$ funksiya uchun $(1,0)$ nuqtada $du$ differensialini hisoblang.

A) $dx$ B) $-2dx + 3dy$ C) $-7dy$ D) $dx - dy$

4. $u = x^2y^3$ funksiyaning ikkinchi tartibli $d^2u$ differensialini toping.

A) $d^2u = 2y^3\,dx^2 + 6xy^2dx\,dy + 6x^2y\,dy^2$ B) $d^2u = 2y^3\,dx^2$

C) $d^2u = 2y^3\,dx^2 + y^2dy^2$ D) $d^2u = y\,dx^2 + 3xdy^2$

5. $u = x^3 + y^3$ funksiyaning uchinchi tartibli $d^3u$ differensialini toping.

A) $d^3u = 6\,dx^3 + 6\,dy^3$ B) $d^3u = y^3\,dx^3$ C) $d^3u = 2y^3\,dx^3 + y^2dy^3$

D) $d^3u = 3x^2dxdy^2$

6. Quyidagi funksiyalardan qaysi biri $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ tenglamani qanoatlantiradi.

A) $u = x^2 - y^2$ B) $u = x^3 - y^2$ C) $u = x^2 - y^3$ D) $u = x^2 + y^2$

7. Noto'g'ri javobni ko'rsating.

A) Agar funksiya biror nuqtada uzluksiz bo'lsa, u holda bu nuqtada funksiya differensiallanuvchi bo'ladi

B) Agar funksiya biror nuqtada differensiallanuvchi bo'lsa, u holda uning bu nuqtada har bir o'zgaruvchi bo'yicha xususiy hosilalari mavjud bo'ladi

C) Agar funksiya biror nuqtada differensiallanuvchi bo'lsa, u holda bu nuqtada funksiya uzluksiz bo'ladi

D) Agar funksiya biror nuqtada uzluksiz xususiy hosilalarga ega bo'lsa, u holda bu nuqtada funksiya differensiallanuvchi bo'ladi

8. $u(x,y)=\sin 2y+\sin 2x$, $u_y(x,y)-?$

A) $2\cos 2y$ B) $x^5+y+3$ C) $y+5\sin x+9$ D) $2x+2y+5$

9. $u(x,y)=x^2-y^2$ $u_{xx}(x,y)-?$

A) 2 B) 0 C) -1 D) 1

10. $u(x,y)=x^2+y^2$ $u_{xy}(x,y)-?$

A) 0 B) 1 C) -1 D) 2

# Mavzu. Ko‘p o‘zgaruvchili funksiyaning yuqori tartibli hosila va differensiallari. Teylor formulasi.

## 30-ma’ruza

**Reja**

$1^0$. Yuqori tartibli xususiy hosilalar.

$2^0$. Yuqori tartibli differensialar.

$3^0$. Murakkab funksiyaning yuqori tartibli differensiallari.

### $1^0$. Yuqori tartibli xususiy hosilalar.

Faraz qilaylik, $f(x)=f(x_1,x_2,....,x_m)$ funksiya ochiq $E\subset R^m$ to‘plamning har bir $x=(x_1,x_2,....,x_m)\in E$ nuqtasida

$$\frac{\partial f(x)}{\partial x_i}=f'_{x_i} \qquad (i=1,2,....,m)$$

xususiy hosilalarga ega bo‘lsin. Bu xususiy hosilalar $x_1,x_2,.....,x_m$ o‘zgaruvchilarning funksiyasi bo‘lib, ular ham xususiy hosilalarga ega bo‘lishi mumkin:

$$\frac{\partial}{\partial x_k}\left(\frac{\partial f(x)}{\partial x_i}\right)=\left(f'_{x_i}(x)\right)'_{x_k} \qquad (i,k=1,2,....,m).$$

Bu xususiy hosilalar berilgan $f(x)$ funksiyaning ikkinchi tartibli xususiy hosilalari deyiladi va

$$\frac{\partial^2 f(x)}{\partial x_k\,\partial x_i} \qquad \ddot{e}\kappa u \qquad f''_{x_i x_k}(x) \qquad (i,k=1,2,....,m)$$

kabi belgilanadi:

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_i} = f''_{x_i x_k}(x) = \frac{\partial}{\partial x_k}\left(\frac{\partial f(x)}{\partial x_i}\right).$$

Agar $i \neq k$ bo‘lsa,

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_i}$$

ikkinchi tartibli xususiy hosila aralash hosila deyiladi.

Agar $i = k$ bo‘lsa, ikkinchi tartibli xususiy hosilalar

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_i} = f''_{x_i x_k}(x)$$

quyidagicha

$$\frac{\partial^2 f(x)}{\partial x_i^2} = f''_{x_i^2}(x)$$

yoziladi.

$f(x)$ funksiyaning uchinchi, to‘rtinchi va h.k. tartibdagi xususiy hosilalari xuddi yuqoridagiga o‘xshash ta’riflanadi. Umuman, $f(x) = f(x_1, x_2, ..., x_m)$ funksiyaning $x_i, x_{i_2}, ....., x_{i_{n-1}}, x_{i_n}$ o‘zgaruvchilari bo‘yicha $n$-tartibli xususiy hosilasi berilgan funksiyaning $(n-1)$ – tartibli xususiy hosilasi

$$\frac{\partial^{n-1} f(x)}{\partial x_{i_{n-1}} \partial x_{i_{n-2}} ... \partial x_{i_1}} \qquad (i_1 + i_2 + .... + i_{n-1} = n-1)$$

ning $x_{i_n}$ o‘zgaruvchi bo‘yicha xususiy hosilasi sifatida ta’riflanadi:

$$\frac{\partial^n f(x)}{\partial x_{i_n} \partial x_{i_{n-1}} ... \partial x_{i_2} \partial x_{i_1}} = \frac{\partial}{\partial x_{i_n}}\left(\frac{\partial^{n-1} f(x)}{\partial x_{i_{n-1}} ... \partial x_{i_2} \partial x_{i_1}}\right).$$

Bu holda ham $i_1, i_2, ...., i_n$ lar bir-biriga teng bo‘lmaganda

$$\frac{\partial^n f}{\partial x_{i_n} ... \partial x_{i_2} \partial x_{i_1}}$$

aralash hosila deyiladi.

Agar $i_1 = i_2 = \ldots = i_n = k$ bo'lsa, $n$– tartibli xususiy hosilalar quyidagicha

$$\frac{\partial^n f(x)}{\partial x_k^n}$$

yoziladi. Ushbu

$$\frac{\partial^2 f}{\partial x_k \partial x_i} \quad , \quad \frac{\partial^2 f}{\partial x_i \partial x_k} \qquad (i \neq k)$$

aralash hosilalar funksiyaning turli o'zgaruvchilari bo'yicha differensiallash tartibi bilan farq qiladi:

$$\frac{\partial^2 f}{\partial x_k \partial x_i}$$

da $f(x_1, x_2, \ldots, x_m)$ funksiyaning avval $x_i$ o'zgaruvchisi bo'yicha, so'ng $x_k$ o'zgaruvchisi bo'yicha xususiy hosilasi hisoblangan bo'lsa,

$$\frac{\partial^2 f}{\partial x_i \partial x_k}$$

da esa avval $x_k$ o'zgaruvchisi bo'yicha, so'ng $x_i$ o'zgaruvchisi bo'yicha xususiy hosilasi hisoblangan. Ular bir-biriga teng ham bo'lishi mumkin, teng bo'lmasdan qolishi ham mumkin (misollar keyingi punktda keltiriladi).

Aralash hosilalarning tengligini quyidagi teorema ifodalaydi.

**1-teorema**. Faraz qilaylik, $f(x_1, x_2, \ldots, x_m)$ funksiya $x^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in E \subset R^m$ nuqtada $n$ marta differensiallanuvchi bo'lsin. U holda $x^0$ nuqtada $f(x_1, x_2, \ldots, x_m)$ funksiyaning ixtiyoriy $n$-tartibli aralash hosilalarning qiymati $x_1, x_2, \ldots, x_m$ o'zgaruvchilar bo'yicha qanday tartibda differensiallanishiga bog'liq bo'lmaydi.

◄Bu teoremaning isboti, keyingi punktda ikki o'zgaruvchili funksiya uchun keltiriladigan teorema isboti kabi bo'ladi.►

**$2^0$. Yuqori tartibli differensialar**.

Faraz qilaylik, $f(x)=f(x_1,x_2,....,x_m)$ funksiya ochiq $E\subset R^m$ to‘plamda berilgan, $x\in E$ nuqtada ikki marta differensiallanuvchi bo‘lsin.

**1-ta’rif**. $f(x)$ funksiya differensiali $d\,f(x)$ ning differensiali berilgan funksiyaning $x$ nuqtadagi ikkinchi tartibli differensiali deyiladi va $d^2 f(x)$ kabi belgilanadi:

$$d^2 f(x)=d(d\,f(x)).$$

Endi funksiya ikkinchi tartibli differensialini uning xususiy hosilalari orqali ifodalanishini ko‘rsatamiz.

Ravshanki,

$$d\,f(x)=\frac{\partial f(x)}{\partial x_1}d\,x_1+\frac{\partial f(x)}{\partial x_2}d\,x_2+\ldots+\frac{\partial f(x)}{\partial x_m}d\,x_m$$

bo‘lib, u $x=(x_1,x_2,...,x_m)$ ga hamda $dx_1,dx_2,...,dx_m$ larga bog‘liq bo‘ladi. Bu tenglikda $d\,x_1,d\,x_2,....d\,x_m$ larni tayinlangan deb hisoblab, $d\,f(x)$ ni $x_1,x_2,...,x_m$ o‘zgaruvchilarning funksiyasi siftaida qarab, uning differensialini topamiz:

$$d(d\,f)=d\left(\frac{\partial f}{\partial x_1}d\,x_1+\frac{\partial f}{\partial x_2}d\,x_2+\ldots+\frac{\partial f}{\partial x_m}d\,x_m\right)=$$

$$=d\,x_1\left[\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)\cdot d\,x_1+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right)\cdot d\,x_2+\ldots+\frac{\partial}{\partial x_m}\left(\frac{\partial f}{\partial x_1}\right)\cdot d\,x_m\right]+$$

$$+d\,x_2\left[\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)\cdot d\,x_1+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_2}\right)d\,x_2+\ldots.+\frac{\partial}{\partial x_m}\left(\frac{\partial f}{\partial x_2}\right)d\,x_m\right]+$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+d\,x_m\left[\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_m}\right)d\,x_1+\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_m}\right)d\,x_2+\ldots+\frac{\partial}{\partial x_m}\left(\frac{\partial f}{\partial x_m}\right)d\,x_m\right]=$$

$$=\frac{\partial^2 f}{\partial x_1^2}dx_1^2+\frac{\partial^2 f}{\partial x_2^2}dx_2^2+\ldots+\frac{\partial^2 f}{\partial x_m^2}dx_m^2+2\left[\frac{\partial^2 f}{\partial x_1\partial x_2}dx_1dx_2+\right.$$

$$+\frac{\partial^2 f}{\partial x_1\partial x_3}dx_1dx_3+\ldots+\frac{\partial^2 f}{\partial x_1\partial x_m}dx_1dx_m+\frac{\partial^2 f}{\partial x_2 d x_3}dx_2dx_3+\frac{\partial^2 f}{\partial x_2\partial x_4}dx_2dx_4+$$

$$\left.+\ldots.+\frac{\partial^2 f}{\partial x_2\partial x_m}dx_2dx_m+\ldots.+\frac{\partial^2 f}{\partial x_{m-1}\partial x_m}dx_{m-1}dx_m\right]=$$

$$=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+\ldots+\frac{\partial}{\partial x_m}dx_m\right)^2 f.$$

Bunda simvollik ravishda yozilishidan foydalaniladi. U quyidagicha tushuniladi: qavs ichidagi yig‘indi kvadratga ko‘tarilib, so‘ng $f$ ga "ko‘paytiriladi". Keyin daraja ko‘rsatkichlari xususiy hosilalar tartibi deb hisoblanadi. Demak,

$$d^2 f(x)=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+\ldots+\frac{\partial}{\partial x_m}dx_m\right)^2 f. \qquad (1)$$

$f(x)$ funksiyaning $x$ nuqtadagi uchinchi, to‘rtinchi va h.k. tartibdagi differensiallari ham yuqoridagidek ta’riflanadi.

Umuman, $f(x)$ funksiyaning $x$ nuqtadagi $(n-1)$-tartibli differensiali $d^{n-1}f(x)$ ning differensiali $f(x)$ ning $n$-tartibli differensiali deyiladi va $d^n f(x)$ kabi belgilanadi:

$$d^n f(x)=d\left(d^{n-1}f(x)\right).$$

Agar, $f(x)$ funksiya $x$ nuqtada $n$ marta differensialanuvchi bo‘lsa, u holda

$$d^n f(x)=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+\ldots+\frac{\partial}{\partial x_m}\right)^n f \qquad (2)$$

bo‘ladi.

### $3^0$. Murakkab funksiyaning yuqori tartibli differensiallari.

Biz yuqorida funksiyaning yuqori tartibli differensallarini bayon etdik. Unda funksiya argumenti $x_1, x_2, \ldots, x_m$ lar erkli o‘zgaruvchilar edi.

Aytaylik, $f(x)=f(x_1,x_2,...,x_m)$ funksiyada $x_1,x_2,...,x_m$ o‘zgaruvchilarning har biri $t_1,t_2,...,t_k$ o‘zgaruvchilarning funksiyalari bo‘lsin $(x_i=\varphi_i(t_1,t_2,...,t_k))$.

Qaralayotgan $f(x)$ va $x_i=\varphi_i(t)$ $(i=1,2,....,m)$ funksiyalar $n$ marta differensiallanuvchilik shartlarini bajargan deb, murakkab $f(x(t))$ funksiyaning yuqori tartibli differensiallarini hisoblaymiz.

Ma’lumki, differensial shaklning invariantligi xossasiga binoan, murakkab funksiyaning differensiali

$$d\,f=\frac{\partial f}{\partial x_1}d\,x_1+\frac{\partial f}{\partial x_2}d\,x_2+....+\frac{\partial f}{\partial x_m}d\,x_m$$

bo‘ladi. Differensiallash qoidalaridan foydalanib funksiyaning ikkinchi tartibli differensialini topamiz:

$$d^2f=d(d\,f)=d\left(\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+...+\frac{\partial f}{\partial x_m}dx_m\right)=$$

$$=d\left(\frac{\partial f}{\partial x_1}dx_1\right)+d\left(\frac{\partial f}{\partial x_2}dx_2\right)+....+d\left(\frac{\partial f}{\partial x_m}dx_m\right)=$$

$$=d\left(\frac{\partial f}{\partial x_1}\right)\cdot dx_1+\frac{\partial f}{\partial x_1}d(dx_1)+d\left(\frac{\partial f}{\partial x_2}\right)\cdot dx_2+\frac{\partial f}{\partial x_2}\cdot d(dx_2)+$$

$$+....+d\left(\frac{\partial f}{\partial x_m}\right)\cdot dx_m+\frac{\partial f}{\partial x_m}d(dx_m)=$$

$$=d\left(\frac{\partial f}{\partial x_1}\right)\cdot dx_1+d\left(\frac{\partial f}{\partial x_2}\right)dx_2+.....+d\left(\frac{\partial f}{\partial x_m}\right)dx_m+$$

$$+\frac{\partial f}{\partial x_1}d^2x_1+\frac{\partial f}{\partial x_2}d^2x_2+....+\frac{\partial f}{\partial x_m}d^2x_m=$$

$$=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+...+\frac{\partial}{\partial x_m}dx_m\right)^2 f+$$

$$+\frac{\partial f}{\partial x_1}d^2x_1+\frac{\partial f}{\partial x_2}d^2x_2+....+\frac{\partial f}{\partial x_m}d^2x_m.$$

Demak,

$$d^2f=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+....+\frac{\partial}{\partial x_m}dx_m\right)^2 f+$$
$$+\frac{\partial f}{\partial x_1}d^2x_1+\frac{\partial f}{\partial x_2}d^2x_2+....+\frac{\partial f}{\partial x_m}d^2x_m \ . \qquad (3)$$

Shu yo'l bilan berilgan murakkab funksiyaning keyingi tartibdagi differensiallari topiladi.

**1-eslatma**. (1) va (3) formulalarni solishtirib, ikkinchi tartibli differensialarda differensial shaklning invariantligi xossasi o'rinli emasligii ko'ramiz.

**2-eslatma**. Agar $f(x_1,x_2,...,x_m)$ funksiya argumentlari $x_1,x_2,...,x_m$ larning har biri $t_1,t_2,...,t_k$ o'zgaruvchilarning chiziqli funksiyasi bo'lsa, u holda $f(x)$ funksiyaning ikkinchi tartibli (umuman, $n$-tartibli) differensiali differensial shaklning invariantlik xossasiga ega bo'ladi.

◄ Aytaylik,

$$x_i=b_i+a_{i_1}t_1+a_{i_2}t_2+...+a_{i_k}t_k \qquad (i=1,2,...,k)$$

bo'lsin. U holda, ravshanki,

$$d^2t_1=d^2t_2=.....=d^2t_k=0$$

bo'lib,

$$d^2x_i=a_{i_1}d^2t_1+a_{i_2}d^2t_2+...+a_{i_k}d^2t_k=0$$

bo'ladi. (3) formuladan foydalanib topamiz:

$$d^2f=\left(\frac{\partial}{\partial x_1}dx_1+\frac{\partial}{\partial x_2}dx_2+...+\frac{\partial}{\partial x_m}\right)^2 f.$$

Bu esa $d^2f$ ning (1) formula ko'rinishiga ega ekanligini bildiradi.►

## Mashqlar

1. Ushbu

$$u = f(x, y)\ ,\ \ x = t^2 + s^2\ \ ,\ y = t \cdot s$$

funksiyaning ikkinchi tartibli differensiali topilsin.

2. Ushbu

$$u = y \cdot \varphi\left(x^2 - y\right)$$

funksiya quyidagi

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} = \frac{u}{y^2}$$

tenglikni qanoatlantirishi isbotlansin.

## Adabiyotlar


1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, II q.* T. "Voris-nashriyot", 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.

# Glossariy

$f(x)$ **funksiyaning ikkinchi tartibli xususiy hosilalari** –

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_i} \quad \text{ёки} \quad f''_{x_i x_k}(x) \qquad (i,k=1,2,....,m)$$

**Aralash hosila** - $\frac{\partial^2 f(x)}{\partial x_k \partial x_i} = f''_{x_i x_k}(x) = \frac{\partial}{\partial x_k}\left(\frac{\partial f(x)}{\partial x_i}\right)$, agar $i \neq k$ bo‘lsa,

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_i}$$

$f(x)$ **funksiyaning** $x$ **nuqtadagi ikkinchi tartibli differensiali** - $f(x)$ funksiya differensiali $d\, f(x)$ ning differensiali berilgan funksiyaning $x$ nuqtadagi ikkinchi tartibli differensiali deyiladi.

$f(x)$ **funksiyaning** $x$ **nuqtadagi** $(n-1)$**-tartibli differensiali** - $d^{n-1} f(x)$ ning differensiali $f(x)$ ning $n$-tartibli differensiali deyiladi, $d^n f(x) = d\left(d^{n-1} f(x)\right)$.

## Keys banki

**51-keys.** Masala o`rtaga tashlanadi: Ushbu

$$u = f(x, y)\ ,\ x = t^2 + s^2\ ,\ y = t \cdot s$$

funksiyaning ikkinchi tartibli differensiali topilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma’lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 30-amaliy mashg‘ulot

## Na’muna uchun misollar yechimi

1-misol. $z=z(x,y)$ *bo`lsa* $\frac{\partial z}{\partial x}$ *va* $\frac{\partial z}{\partial y}$ *lar topilsin:*

$$F(y-zx, x-zy, z-xy)=0$$

◀ $\xi=y-zx,\quad \eta=x-zy,\quad \zeta=z-xy$ deb belgilab, berilgan tenglamani differensiallash yordamida topamiz:

$$\begin{cases} F'_\xi\cdot\left(-z-x\frac{\partial z}{\partial x}\right)+F'_\eta\cdot\left(1-y\frac{\partial z}{\partial x}\right)+F'_\zeta\cdot\left(\frac{\partial z}{\partial x}-y\right)=0, \\ F'_\xi\cdot\left(1-x\frac{\partial z}{\partial y}\right)+F'_\eta\cdot\left(-z-y\frac{\partial z}{\partial y}\right)+F'_\zeta\cdot\left(\frac{\partial z}{\partial y}-x\right)=0 \end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases} \frac{\partial z}{\partial x}\cdot\left(-xF_\xi-yF'_\eta+F'_\zeta\right)=z\cdot F'_\xi-F'_\eta+y\cdot F'_\zeta \\ \frac{\partial z}{\partial y}\cdot\left(-xF'_\xi-yF'_\eta+F'_\zeta\right)=-F'_\xi+z\cdot F'_\eta+x\cdot F'_\zeta. \end{cases}\Rightarrow\begin{cases} \frac{\partial z}{\partial x}=\frac{-z\cdot F'_\xi+F'_\eta-y\cdot F'_\zeta}{x\cdot F'_\xi+y\cdot F'_\eta-F'_\zeta}, \\ \frac{\partial z}{\partial y}=\frac{F'_\xi-z\cdot F'_\eta-x\cdot F'_\zeta}{x\cdot F'_\xi+y\cdot F'_\eta-F'_\zeta} \end{cases}$$

▶

$z=z(x,y)$ **bo`lsa,** $\frac{\partial z}{\partial x}$ **va** $\frac{\partial z}{\partial y}$ **lar topilsin.**

**1.** $F(y-zx, x-zy, z-xy)=0$.

**2.** $F(xyz, x+y)=0$.

## Ko`rsatilgan tartibdagi xususiy hosilalar va differensiallar hisoblansin.

**1.** $u=\frac{x+y}{x-y};\ \frac{\partial^{m+n}u}{\partial x^m\partial y^n}$.

**2.** $u=x^m y^n;\ \frac{\partial^{m+n}u}{\partial x^m\partial y^n}$.

**3.** $u=e^{2x}\sin y+e^x\cos\frac{y}{2};\ \frac{\partial^{m+n}u}{\partial x^m\partial y^n}$.

**4.** $u=e^{xyz};\ \frac{\partial^3 u}{\partial x\,\partial y\partial z}$.

**5.** $u=\sin x\cdot\cos 2y;\ \frac{\partial^{10}u}{\partial x^4\partial y^6}$.

**6.** $u=x^4\cos y+y^4\sin x;\ \frac{\partial^8 u}{\partial x^4\partial y^4}$.

**7.** $u=(x^2+y)^{10}tgx;\ \dfrac{\partial^{10}u}{\partial x\ \partial y^9}$.

**8.** $u=\sin xy;\ \dfrac{\partial^3 u}{\partial x^2\partial y}$ **va** $\dfrac{\partial^3 u}{\partial x\ \partial y^2}$.

**9.** $u=\sqrt{x^2+y^2}\cdot e^{xy};\ d^2u$.

**10.** $u=\left(\dfrac{x}{y}\right)^z;\ d^2u$.

**11.** $u=x^{yz};\ d^2u$.

**12.** $u=f(x+y,x^2+y^2);\ d^2u$.

**13.** $u=f(xy)\cdot g(xz);\ d^2u$.

**14.** $u=f(\sin x+\cos y);\ d^2u$.

**15.** $u=f(x+y,z^2);\ d^2u$.

**16.** $u=f(xy,x^2+y^2);\ d^2u$.

**17.** $u=f(2x-3y+4z);\ d^nu$.

**18.** $u=f(2x,3y,2z);\ d^nu$.

**Funksiya differensialini ko`rsatilgan nuqtalarda toping.**

**1.** $u=\dfrac{yz}{x},\ M(x,y,z)$ **va** $M_0(1,2,3)$.

**2.** $u=\cos(xy+xz),\ M(x,y,z)$ **va** $M_0\left(1,\dfrac{\pi}{6},\dfrac{\pi}{6}\right)$.

**3.** $u=x^y,\ M(x,y,)$ **va** $M_0(2,3)$.

**4.** $u=x\ln(xy),\ M(x,y,)$ **va** $M_0(-1,-1)$.

**Agar *f*-ixtiyoriy differensiallanuvchi funksiya bo`lsa,** $u(x,y)$ **funksiya mos tenglamani qanoatlantirishini tekshiring.**

**1.** $u=f(x^2+y^2)$**;** $y\dfrac{\partial u}{\partial x}-x\dfrac{\partial u}{\partial y}=0$.

**2.** $u=x^n\cdot f\left(\dfrac{y}{x}\right)$**;** $x\dfrac{\partial u}{\partial y}-2y\dfrac{\partial u}{\partial y}=nu$.

**3.** $u=yf(x^2-y^2)$ $y^2\dfrac{\partial u}{\partial x}+xy\dfrac{\partial u}{\partial y}=xu$.

**4.** $u=\dfrac{y^2}{3x}+f(xy),\quad x^2\dfrac{\partial u}{\partial x}-xy\dfrac{\partial u}{\partial y}+y^2=0$.

# Test

1. $u = x^2 y^4$ funksiyaning ikkinchi tartibli $d^2u$ differensialini toping.

A) $d^2u = 2y^3\,dx^2 + 8xy^3 dx\,dy + 12x^2y^2\,dy^2$ B) $d^2u = 2y^3\,dx^2$
C) $d^2u = 2y^3\,dx^2 + y^2dy^2$ D) $d^2u = y\,dx^2 + 3xdy^2$

2. $u = x^4 + y^3$ funksiyaning uchinchi tartibli $d^3u$ differensialini toping.

A) $d^3u = 24xdx^3 + 6\,dy^3$ B) $d^3u = y^3\,dx^3$ C) $d^3u = 2y^3\,dx^3 + y^2dy^3$

D) $d^3u = 3x^2dxdy^2$

6. $u = x^3 + y^2 + z + 1$ funksiyaning ikkinchi tartibli $d^2u$ differensialini toping.

A) $du = 6xdx^2 + 2dy^2$ B) $du = dx + 3dz$ C) $du = dy + 2dz$ D) $du = xdx + 2ydy$

7. $u = x^3 + 2y^3 + 3z^2 + 5$ funksiyaning ikkinchi tartibli $d^2u$ differensialini toping.

A) $du = 6xdx^2 + 12ydy^2 + 6dz^2$ B) $du = 12dx + 4dy + 3dz$ C) $du = 4dy + 2dz$

D) $du = x^2dx + 2ydy + dz$

8. $u(x,y) = \sin 2y + \sin 2x$, $u_{yy}(x,y) - ?$

A) $-4\sin 2y$ B) $x^5 + y + 3$ C) $y + 5\sin x + 9$ D) $2x + 2y + 5$

9. $u(x,y) = x^3 - 4y^2$ $u_{xx}(x,y) - ?$

A) $6x$ B) $3x$ C) 0 D) 1

10. $u(x,y) = 2x^2 + y^3$ $u_{xy}(x,y) - ?$

A) 0 B) 1 C) -1 D) 2

# Mavzu. Teylor formulasi

## 31-ma’ruza

**Reja**

$1^0$. Ko‘p o‘zgaruvchili funksiyaning Teylor formulasi.

$2^0$. Xususiy hollar. Aralash hosilaning tengligi haqida teorema.

### $1^0$. Ko‘p o‘zgaruvchili funksiyaning Teylor formulasi.

Aytaylik, $f(x)=f(x_1,x_2,...,x_m)$ funksiya ochiq $E\subset R^m$ to‘plamda berilgan bo‘lib, $U_\delta(x^0)\subset E$ bo‘lsin, bunda $x^0=(x_1^0,x_2^0,...,x_m^0)$ va $\delta>0$. Ravshanki,

$$\forall x=(x_1,x_2,...,x_m)\in U_\delta(x^0),\quad x^0=(x_1^0,x_2^0,...,x_m^0)$$

nuqtalarni birlashtiruvchi to‘g‘ri chiziq kesmasi

$$A=\{x_1^0+t(x_1-x_1^0),\ x_2^0+t(x_2-x_2^0),...,x_m^0+t(x_m-x_m^0);\ 0\le t\le 1\}$$

shu $U_\delta(x^0)$ ga tegishli bo‘ladi.

Faraz qilaylik, $f(x_1,x_2,....,x_m)$ funksiya $U_\delta(x^0)$ to‘plamda $(n+1)$ marta differensialllanuvchi bo‘lsin. Bu funksiyani $A$ to‘plamda qarasak, $[0,1]$ segmentda aniqlangan ushbu

$$F(t)=f\left(x_1^0+t(x_1-x_1^0),x_2^0+t(x_2-x_2^0),...,x_m^0+t(x_m-x_m^0)\right)$$

funksiyaga ega bo‘lamiz. $F(t)$ funksiya $[0,1]$ da hosilaga ega bo‘lib,

$$F'(t)=\frac{\partial f}{\partial x_1}\cdot\left(x_1-x_1^0\right)+\frac{\partial f}{\partial x_2}\cdot\left(x_2-x_2^0\right)+\ldots+\frac{\partial f}{\partial x_m}\cdot\left(x_m-x_m^0\right)=$$

$$=\left(\frac{\partial}{\partial x_1}\cdot\left(x_1-x_1^0\right)+\frac{\partial}{\partial x_2}\cdot\left(x_2-x_2^0\right)+\ldots+\frac{\partial}{\partial x_m}\left(x_m-x_m^0\right)\right)f$$

bo‘ladi, bunda $f(x)$ funksiyaning barcha xususiy hosilalari

$$\left(x_1^0+t\left(x_1-x_1^0\right),\ x_2^0+t\left(x_2-x_2^0\right),\ldots,x_m^0+t\left(x_m-x_m^0\right)\right) \qquad (4)$$

nuqtada hisoblangan.

Umuman, hosil qilingan $F(t)$ funksiya $k$-tartibli $(k=1,2,\ldots,n+1)$ hosilalarga ega va u

$$F^{(k)}(t)=\left(\frac{\partial}{\partial x_1}\cdot\left(x_1-x_1^0\right)+\frac{\partial}{\partial x_2}\cdot\left(x_2-x_2^{\circ}\right)+\ldots+\frac{\partial}{\partial x_m}\left(x_m-x_m^0\right)\right)^k f$$

ga teng, bundagi barcha xususiy hosilalar (4) nuqtada hisoblangan. Bu munosabatning to‘g‘riligi matematik induksiya usuli yordamida isbotlanadi.

Shunday qilib, $F(t)$ funksiya $F'(t)$, $F''(t),\ldots,F^{(n+1)}(t)$ hosilalarga ega bo‘ladi. Teylor formulasiga ko‘ra (qaralsin, 24-ma’ruza) $t_0$ nuqtada $(0\le t_0\le 1)$

$$F(t)=F(t_0)+F'(t_0)(t-t_0)+\frac{1}{2!}F''(t_0)(t-t_0)^2+\ldots+$$
$$+\frac{1}{n!}F^{(n)}(t_0)\cdot(t-t_0)^n+\frac{1}{(n+1)!}F^{(n+1)}(c)\cdot(t-t_0)^{n+1} \qquad (5)$$

bo‘ladi, bunda $c=t_0+\theta(t-t_0)$, $\quad 0<\theta<1$. Bu tenglikda $t_0=0,\ t=1$ deyilsa, unda

$$F(1)=F(0)+\frac{1}{1!}F'(0)+\frac{1}{2!}F''(0)+\ldots+\frac{1}{n!}F^{(n)}(0)+\frac{1}{(n+1)!}F^{(n+1)}(\theta)$$

bo‘lishi kelib chiqadi.

Ayni paytda,

$$F(0)=f\left(x_1^0,x_2^0,\ldots,x_m^0\right),$$
$$F(1)=f\left(x_1,x_2,\ldots,x_m\right), \qquad (6)$$

$$F^{(k)}(0)=\left(\frac{\partial}{\partial x_1}\cdot\left(x_1-x_1^0\right)+\frac{\partial}{\partial x_2}\cdot\left(x_2-x_2^0\right)+\ldots+\frac{\partial}{\partial x_m}\cdot\left(x_m-x_m^0\right)\right)f$$

(bunda $f$ funksiyaning barcha xususiy hosilalari $\left(x_1^0, x_2^0, \ldots, x_m^0\right)$ nuqtada hisoblangan) bo‘lishini e’tiborga olsak, u holda (5) va (6) tengliklardan ushbu

$$f\left(x_1, x_2, \ldots, x_m\right)=f\left(x_1^0, x_2^0, \ldots, x_m^0\right)+$$

$$+\sum_{k=1}^{n}\frac{1}{k!}\left(\frac{\partial}{\partial x_1}\cdot\left(x_1-x_1^0\right)+\frac{\partial}{\partial x_2}\cdot\left(x_2-x_2^0\right)+\ldots+\frac{\partial}{\partial x_m}\left(x_m-x_m^0\right)\right)^k f\left(x_1^0, x_2^0, \ldots, x_m^0\right)+$$

$$+\frac{1}{(n+1)!}\left(\frac{\partial}{\partial x_1}\left(x_1-x_1^0\right)+\frac{\partial}{\partial x_2}\left(x_2-x_2^0\right)+\ldots+\frac{\partial}{\partial x_m}\left(x_m-x_m^0\right)\right)^{n+1}\times$$

$$\times f\left(x_1^0+\theta\left(x_1-x_1^0\right), x_2^0+\theta\left(x_2-x_2^0\right), \ldots, x_m^0+\theta\left(x_m-x_m^0\right)\right)$$

$(0<\theta<1)$ tenglikka kelamiz. Bu ko‘p o‘zgaruvchili $f\left(x_1, x_2, \ldots, x_m\right)$ funksiyaning Lagranj ko‘rinishidagi qoldiq hadli Teylor formulasi deyiladi.

### $2^0$. Xususiy hollar. Aralash hosilaning tengligi haqida teorema.

$m=1$ bo‘lsin. bu holda $u=f(x)$ $(x\in R, u\in R)$ funksiyaning yuqori tartibli hosila va differensiallariga kelamiz. Ular 23-ma’ruzada batafsil bayon etilgan.

$m=2$ bo‘lganda $u=f(x,y)$ $\left((x,y)\in R^2, u\in R\right)$ ikki o‘zgaruvchili funksiya bo‘lib. Uning ikkinchi tartibli xususiy hosilalari (ular 4 ta bo‘ladi) quyidagicha bo‘ladi:

$$\frac{\partial^2 f(x,y)}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right),$$

$$\frac{\partial^2 f(x,y)}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right),$$

$$\frac{\partial^2 f(x,y)}{\partial y\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right),$$

$$\frac{\partial^2 f(x,y)}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right).$$

**1-misol**. Ushbu

$$f(x,y)=arctg\frac{x}{y}\qquad (y\neq 0)$$

funksiyaning ikiinchi tartibli xususiy hosilalari topilsin.

◀Ravshaki,

$$\frac{\partial f}{\partial x}=\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}=\frac{y}{x^2+y^2}\quad,\quad \frac{\partial f}{\partial y}=\frac{1}{1+\frac{x^2}{y^2}}\cdot\left(-\frac{x}{y^2}\right)=-\frac{x}{x^2+y^2}$$

bo‘ladi.

Endi ta’rifdan foydalanib berilgan funksiyaning ikkinchi tartibli xususiy hosilalarini topamiz:

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)=-\frac{2xy}{(x^2+y^2)^2}\ ,$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{x^2-y^2}{(x^2+y^2)^2}\ ,$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right)=\frac{x^2-y^2}{(x^2+y^2)^2}\ ,$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right)=\frac{2xy}{(x^2+y^2)^2}\ .$$ ▶

**2-misol**. Ushbu

$$f(x,y)=\begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}\ , & \text{агар}\ \ x^2+y^2>0\ \ \text{бўлса}\ ,\\ 0\ , & \text{агар}\ \ x^2+y^2=0\ \ \text{бўлса}\end{cases}$$

funksiyaning $(0,0)$ nuqtadagi aralash hosilalari topilsin.

Aytaylik, $(x,y)\neq(0,0)$ bo‘lsin. Bu holda

$$\frac{\partial f}{\partial x}=y\left(\frac{x^2-y^2}{x^2+y^2}+\frac{4x^2y^2}{\left(x^2+y^2\right)^2}\right),\quad \frac{\partial f}{\partial y}=x\left(\frac{x^2-y^2}{x^2+y^2}-\frac{4x^2y^2}{\left(x^2+y^2\right)^2}\right),$$

$$\frac{\partial^2 f}{\partial y\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{x^2-y^2}{x^2+y^2}\cdot\left(1+\frac{8x^2y^2}{\left(x^2+y^2\right)^2}\right),$$

$$\frac{\partial^2 f}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{x^2-y^2}{x^2+y^2}\cdot\left(1+\frac{8x^2y^2}{\left(x^2+y^2\right)^2}\right)$$

bo‘ladi.

Aytaylik, $(x,y)=(0,0)$ bo‘lsin. Bu holda funksiyaning hosilalarini ta’rifga ko‘ra hisoblaymiz:

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0,$$

$$\frac{\partial^2 f(0,0)}{\partial y\,\partial x}=\lim_{\Delta y\to 0}\frac{\frac{\partial f(0,\Delta y)}{\Delta x}-\frac{\partial f(0,0)}{\Delta x}}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y^3}{\Delta y^3}=-1,$$

$$\frac{\partial^2 f(0,0)}{\partial x\,\partial y}=\lim_{\Delta x\to 0}\frac{\frac{\partial f(\Delta x,0)}{\Delta y}-\frac{\partial f(0,0)}{\Delta y}}{\Delta x}=\lim_{\Delta x\to 0}\frac{-\Delta y^3}{\Delta x^3}=1 \ ►$$

Yuqorida keltirilgan misollardan ko‘rinadiki, $f(x,y)$ funksiyaning

$$\frac{\partial^2 f}{\partial x\,\partial y},\quad \frac{\partial^2 f}{\partial y\,\partial x}$$

aralash hosilalari bir-biriga teng ham bo‘lishi mumkin, teng bo‘lmasdan qolishi ham mumkin ekan.

**2-teorema.** Faraz qilaylik, $f(x,y)$ funksiya $(x_0,y_0)\in R^2$ nuqtaning

$U_\delta((x_0,y_0))$ atrofida

$$\frac{\partial^2 f(x,y)}{\partial x \partial y}, \frac{\partial^2 f(x,y)}{\partial y \partial x} \qquad ((x,y)\in U_\delta((x_0,y_0)))$$

aralash hosilalarga ega bo‘lib, bu hosilalar $(x_0,y_0)$ nuqtada uzluksiz bo‘lsin. U holda $f(x,y)$ funksiyaning aralash hosilalari $(x_0,y_0)$ nuqtada teng bo‘ladi:

$$\frac{\partial^2 f(x_0,y_0)}{\partial y \partial x}=\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}.$$

◀Aytaylik, $(x_0+\Delta x, y_0+\Delta y)$, $(x_0+\Delta x, y_0)$ , $(x_0, y_0+\Delta y)$ nuqtalar $(x_0,y_0)$ nuqtaning atrofiga tegishli bo‘lsin:

$$(x_0+\Delta x, y_0+\Delta y)\in U_\delta((x_0,y_0)),\ (x_0+\Delta x, y_0)\in U_\delta((x_0,y_0)),$$
$$(x_0, y_0+\Delta y)\in U_\delta((x_0,y_0)).$$

Ushbu

$$\Phi(\Delta x,\Delta y)=f(x_0+\Delta x, y_0+\Delta y)-f(x_0+\Delta x, y_0)-f(x_0, y_0+\Delta y)+f(x_0,y_0),$$
$$\varphi(x)=f(x,y_0+\Delta y)-f(x,y_0)$$

funksiyalarni qaraylik

Ravshanki,

$$\Phi(\Delta x,\Delta y)=\varphi(x_0+\Delta x)-\varphi(x_0)$$

bo‘ladi. Bu tenglikning o‘ng tomoniga Lagranj teoremasini ikki marta qo‘llab topamiz:

$$\varphi(x_0+\Delta x)-\varphi(x_0)=\varphi'(x_0+\theta_1\cdot\Delta x)\cdot\Delta x=$$
$$=\left[\frac{\partial f(x_0+\theta_1\Delta x, y_0+\Delta y)}{\partial x}-\frac{\partial f(x_0+\theta_1\cdot\Delta x, y_0)}{\partial x}\right]\Delta x=$$
$$=\frac{\partial^2 f(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)}{\partial y \partial x}\cdot\Delta x\cdot\Delta y. \qquad (0<\theta_1,\theta_2<1)$$

Shartga ko‘ra aralash hosila $(x_0,y_0)$ nuqtada uzluksiz. Demak, $\Delta x\to 0, \Delta y\to 0$ da

$$\frac{\partial^2 f(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)}{\partial y\,\partial x}\Delta x\,\Delta y = \frac{\partial^2 f(x_0,y_0)}{\partial y\,\partial x}\Delta x\,\Delta y + 0(1)$$

bo‘lib,

$$\Phi(\Delta x,\Delta y) = \frac{\partial^2 f(x_0,y_0)}{\partial y\,\partial x}\Delta x\,\Delta y + 0(1) \qquad (7)$$

bo‘ladi.

Endi $\Phi(\Delta x,\Delta y)$ funksiya bilan birga quyidagi

$$\psi(y) = f(x_0+\Delta x, y) - f(x_0, y)$$

funksiyani qaraymiz. Ravshanki,

$$\Phi(\Delta x,\Delta y) = \psi(y_0+\Delta y) - \psi(y_0)$$

bo‘ladi. Yuqoridagidek, bu tenglikning o‘ng tomoniga Lagranj teoremasini ikki marta qo‘llab, so‘ng aralash hosilaning $(x_0, y_0)$ nuqtada uzluksizligidan foydalanib topamiz:

$$\psi(y_0+\Delta y) - \psi(y_0) = \left[\frac{\partial f(x_0+\Delta x, y_0+\theta_1'\Delta y)}{\partial y} - \frac{\partial f(x_0, y_0+\theta_1'\Delta y)}{\partial y}\right]\Delta y =$$

$$= \frac{\partial^2 f(x_0+\theta_2'\Delta x, y_0+\theta_1'\Delta y)}{\partial x\,\partial y}\Delta x\,\Delta y = \frac{\partial^2 f(x_0,y_0)}{\partial x\,\partial y}\Delta x\,\Delta y + 0(1)$$

$(0<\theta_1',\theta_2'<1\ ,\ \Delta x\to 0\ ,\Delta y\to 0)$. Demak,

$$\Phi(\Delta x,\Delta y) = \frac{\partial^2 f(x_0,y_0)}{\partial x\,\partial y}\Delta x\,\Delta y + 0(1)\ . \qquad (8)$$

(7) va (8) munosabatlardan

$$\frac{\partial^2 f(x_0,y_0)}{\partial x\,\partial y} = \frac{\partial^2 f(x_0,y_0)}{\partial y\,\partial x}$$

bo‘lishi kelib chiqadi. ►

## Mashqlar

1. Ushbu

$$u = f(x, y)\ ,\ \ x = t^2 + s^2\ \ ,\ y = t \cdot s$$

funksiyaning ikkinchi tartibli differensiali topilsin.

2. Ushbu

$$u = y \cdot \varphi\left(x^2 - y\right)$$

funksiya quyidagi

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} = \frac{u}{y^2}$$

tenglikni qanoatlantirishi isbotlansin.

## Adabiyotlar

1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, II q.* T. "Voris-nashriyot", 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.

# Glossariy

$f(x_1, x_2, \dots, x_m)$ **funksiyaning Lagranj ko'rinishidagi qoldiq hadli Teylor formulasi** –

$$f(x_1, x_2, \dots, x_m) = f(x_1^0, x_2^0, \dots, x_m^0) +$$

$$+\sum_{k=1}^{n}\frac{1}{k!}\left(\frac{\partial}{\partial x_1}\cdot(x_1 - x_1^0) + \frac{\partial}{\partial x_2}\cdot(x_2 - x_2^0) + \dots + \frac{\partial}{\partial x_m}(x_m - x_m^0)\right)^k f(x_1^0, x_2^0, \dots, x_m^0) +$$

$$+\frac{1}{(n+1)!}\left(\frac{\partial}{\partial x_1}(x_1 - x_1^0) + \frac{\partial}{\partial x_2}(x_2 - x_2^0) + \dots + \frac{\partial}{\partial x_m}(x_m - x_m^0)\right)^{n+1} \times$$

$$\times f\left(x_1^0 + \theta(x_1 - x_1^0), x_2^0 + \theta(x_2 - x_2^0), \dots, x_m^0 + \theta(x_m - x_m^0)\right).$$

$f(x)$ **funksiyaning** $x$ **nuqtadagi** $(n-1)$**-tartibli differensiali** - $d^{n-1} f(x)$ ning differensiali $f(x)$ ning $n$-tartibli differensiali deyiladi, $d^n f(x) = d\left(d^{n-1} f(x)\right)$.

# Keys banki

**52-keys.** Masala o`rtaga tashlanadi: Ushbu $u = y \cdot \varphi(x^2 - y)$ funksiya quyidagi

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} = \frac{u}{y^2}$$

tenglikni qanoatlantirishi isbotlansin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 31-amaliy mashg'ulot

**Na'muna uchun misollar yechimi**

**1-misol**. Ushbu

$$f(x,y)=arctg\frac{x}{y} \qquad (y\neq 0)$$

funksiyaning ikiinchi tartibli xususiy hosilalari topilsin.

◀Ravshaki,

$$\frac{\partial f}{\partial x}=\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}=\frac{y}{x^2+y^2}\quad ,\quad \frac{\partial f}{\partial y}=\frac{1}{1+\frac{x^2}{y^2}}\cdot\left(-\frac{x}{y^2}\right)=-\frac{x}{x^2+y^2}$$

bo'ladi.

Endi ta'rifdan foydalanib berilgan funksiyaning ikkinchi tartibli xususiy hosilalarini topamiz:

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)=-\frac{2xy}{\left(x^2+y^2\right)^2}\ ,$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\ ,$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right)=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\ ,$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right)=\frac{2xy}{\left(x^2+y^2\right)^2}\ .$$ ▶

**2-misol**. Ushbu

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{агар } x^2+y^2>0 \text{ бўлса}, \\ 0, & \text{агар } x^2+y^2=0 \text{ бўлса} \end{cases}$$

funksiyaning $(0,0)$ nuqtadagi aralash hosilalari topilsin.

Aytaylik, $(x,y)\neq(0,0)$ bo‘lsin. Bu holda

$$\frac{\partial f}{\partial x}=y\left(\frac{x^2-y^2}{x^2+y^2}+\frac{4x^2y^2}{(x^2+y^2)^2}\right), \quad \frac{\partial f}{\partial y}=x\left(\frac{x^2-y^2}{x^2+y^2}-\frac{4x^2y^2}{(x^2+y^2)^2}\right),$$

$$\frac{\partial^2 f}{\partial y\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{x^2-y^2}{x^2+y^2}\cdot\left(1+\frac{8x^2y^2}{(x^2+y^2)^2}\right),$$

$$\frac{\partial^2 f}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{x^2-y^2}{x^2+y^2}\cdot\left(1+\frac{8x^2y^2}{(x^2+y^2)^2}\right)$$

bo‘ladi.

Aytaylik, $(x,y)=(0,0)$ bo‘lsin. Bu holda funksiyaning hosilalarini ta’rifga ko‘ra hisoblaymiz:

$$\frac{\partial f(0,0)}{\partial x}=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0,$$

$$\frac{\partial f(0,0)}{\partial y}=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0,$$

$$\frac{\partial^2 f(0,0)}{\partial y\,\partial x}=\lim_{\Delta y\to 0}\frac{\dfrac{\partial f(0,\Delta y)}{\Delta x}-\dfrac{\partial f(0,0)}{\Delta x}}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y^3}{\Delta y^3}=-1,$$

$$\frac{\partial^2 f(0,0)}{\partial x\,\partial y}=\lim_{\Delta x\to 0}\frac{\dfrac{\partial f(\Delta x,0)}{\Delta y}-\dfrac{\partial f(0,0)}{\Delta y}}{\Delta x}=\lim_{\Delta x\to 0}\frac{-\Delta y^3}{\Delta x^3}=1 \;\blacktriangleright$$

**Misollar**

1. Ushbu

$$u = f(x, y)\ ,\ \ x = t^2 + s^2\ \ ,\ y = t \cdot s$$

funksiyaning ikkinchi tartibli differensiali topilsin.

3. Ushbu

$$u = y \cdot \varphi(x^2 - y)$$

funksiya quyidagi

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} = \frac{u}{y^2}$$

tenglikni qanoatlantirishi isbotlansin.

4. Quyidagi funksiyalarning ikkinchi tartibli differensiallari topilsin.

1. $z = 2x^2 - 3xy - y^2$
2. $z = e^{xy}$
3. $z = \dfrac{x}{y}$
4. $z = \ln\sqrt{x^2 + y^2}$
5. Agar $f(x, y) = x^2 + xy + y^2 - 4\ln x - 10\ln y$ bo'lsa, $d^2 f(1, 2)$ topilsin.
6. $z = y\ln x$
7. $z = \sin x \cos y$

# Test

1. $f(x;y)=-x^2+2xy+3y^2-6x-2y-4$ funksiya Teylor formulasi bo`yicha $(-2;1)$ nuqta atrofida yoyilsin.

A) $f(x;y)=1-(x+2)^2+(x+2)(y-1)+3(y-1)^2$

B) B) $f(x;y)=2(x-1)^2-(x-1)(y+2)-2(y+2)^3$

C) $f(x;y)=1-(x+2)^3+(x+2)(y-1)+6(y-1)^2$

D) $f(x;y)=2(x-1)^2-(x-1)(y+2)-(y+2)^2$

2. $f(x;y)=2x^2-xy-y^2-6x-3y$ funksiya Teylor formulasi bo`yicha $(1;-2)$ nuqta atrofida yoyilsin.

A) $f(x;y)=2(x-1)^2-(x-1)(y+2)-(y+2)^2$

B) B) $f(x;y)=1-(x+2)^2+(x+2)(y-1)+3(y-1)^2$

C) $f(x;y)=1-(x+2)^3+(x+2)(y-1)+6(y-1)^2$

D) $f(x;y)=2(x-1)^2-(x-1)(y+2)-2(y+2)^3$

3. $f(x,y)=e^x\sin y$ funksiya Teylor formulasi bo`yicha (0,0) nuqta atrofida yoyilsin.

A) $y+xy+\frac{1}{2}x^2y-\frac{1}{6}y^3+\dots$ B) $y+xy-\frac{1}{2}x^2y-\frac{1}{6}y^3+\dots$ C) $y+xy-\frac{1}{2}x^2y+\frac{1}{6}y^3+\dots$ D) $y-xy-\frac{1}{2}x^2y+\frac{1}{6}y^3+\dots$

4. $f(x;y)=2x^2-xy-y^2+2x-5y-4$ funksiya Teylor formulasi bo`yicha $(-1;-2)$ nuqta atrofida yoyilsin.

A) $f(x;y)=2(x+1)^2-(x+1)(y+2)-(y+2)^2$

B) $f(x;y)=2(x+1)^2+(x+1)(y+2)-(y+2)^2$

C) $f(x;y)=2(x+1)^2-2(x+1)(y+2)-(y+2)^2$

D) $f(x;y)=(x+1)^2-(x+1)(y+2)-(y+2)^2$

5. $f(x;y)=x^2+xy-2y^2-3x-6y$ funksiya Teylor formulasi bo`yicha $(2;-1)$ nuqta atrofida yoyilsin.

A) $f(x;y)=(x-2)^2+(x-2)(y+1)-2(y+1)^2$

B) $f(x;y)=(x-2)^2-(x-2)(y+1)-2(y+1)^2$

C) $f(x;y)=(x-2)^2+(x-2)(y+1)+2(y+1)^2$

D) $f(x;y)=(x-2)^2-2(x-2)(y+1)-2(y+1)^2$

6. $f(x;y)=x^2-xy+y^2-6x+3y+9$ funksiya Teylor formulasi bo\`yicha $(3;0)$ nuqta atrofida yoyilsin.

A) $f(x;y)=(x-3)^2-(x-3)y+y^2$ B) $f(x;y)=(x-3)^2-2(x-3)y-y^2$

C) $f(x;y)=(x-3)^2+(x-3)y+y^2$ D) $f(x;y)=(x-3)^2-3(x-3)y+y^2$

7. $z=\dfrac{1}{1-x-y+xy}$ funksiya Teylor formulasi bo\`yicha (0,0) nuqta atrofida yoyilsin.

A) $1+(x+y)+...+\dfrac{x^{n+1}-y^{n+1}}{x-y}+...$

B) $1+(x+y)+...+\dfrac{x^{n+1}+y^{n+1}}{x-y}+...$

C) $1+2(x+y)+...+\dfrac{x^{n+1}-y^{n+1}}{x-y}+...$

D) $1+(x+y)+...+\dfrac{x^{n+1}-y^{n+1}}{x+y}+...$

8. $z=\ln(1-x)\ln(1-y)$ funksiya Teylor formulasi bo\`yicha (0,0) nuqta atrofida yoyilsin.

A) $\sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty}\dfrac{x^n y^m}{nm}$ B) $\sum\limits_{n=1}^{\infty}\sum\limits_{m=0}^{\infty}\dfrac{x^n y^m}{nm}$ C) $\sum\limits_{n=0}^{\infty}\sum\limits_{m=1}^{\infty}\dfrac{x^n y^m}{nm}$

D) $\sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{\infty}\dfrac{x^n y^m}{nm}$

9. $z=\ln\dfrac{1-x-y+xy}{1-x-y}$ funksiya Teylor formulasi bo\`yicha (0,0) nuqta atrofida yoyilsin.

A) $\sum\limits_{n=2}^{\infty}\dfrac{(x+y)^n-x^n-y^n}{n}$ B) $\sum\limits_{n=2}^{\infty}\dfrac{(x+y)^n-x^n+y^n}{n}$

C) $\sum\limits_{n=2}^{\infty}\dfrac{(x+y)^n+x^n-y^n}{n}$ D) $\sum\limits_{n=2}^{\infty}\dfrac{(x+y)^n+x^n+y^n}{n}$

10. $z=\sin(x^2+y^2)$ funksiya Teylor formulasi bo\`yicha (0,0) nuqta atrofida yoyilsin.

A) $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(x^2+y^2)^{2n+1}}{(2n+1)!}$ B) $\sum\limits_{n=0}^{\infty}\dfrac{(x^2+y^2)^{2n+1}}{(2n+1)!}$ C) $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(x^2+y^2)^{2n}}{(2n)!}$

D) $\sum\limits_{n=0}^{\infty}\dfrac{(x^2+y^2)^{2n}}{(2n)!}$

# Mavzu. Ko'p o'zgaruvchili funksiyaning ekstremumlari

## 32-ma'ruza

**Reja**

$1^0$. Funksiya ekstremumi tushunchasi. Zaruriy shart.

$2^0$. Funksiya ekstremumga erishishining yetarli sharti.

### $1^0$. Funksiya ekstremumi tushunchasi. Zaruriy shart.

Faraz qilaylik, $f(x)=f(x_1,x_2,...,x_m)$ funksiya $E\subset R^m$ to'plamda berilgan bo'lib, $x^0=(x_1^0,x_2^0,...,x_m^0)\in E$ bo'lsin.

**1-ta'rif.** Agar shunday $\delta>0$ son topilsaki,

$$U_\delta(x^0)\subset E \text{ bo'lib, } \forall x\in U_\delta(x^0) \text{ da } f(x)\le f(x^0)$$

bo'lsa, $f(x)$ funksiya $x^0$ nuqtada lokal maksimumga, $f(x)\ge f(x^0)$ bo'lsa, $f(x)$ funksiya $x^0$ nuqtada lokal minimumga erishadi deyiladi.

**2-ta'rif.** Agar shunday $\delta>0$ son topilsaki, $U_\delta(x^0)\subset E$ bo'lib, $\forall x\in U_\delta(x^0)\setminus\{x^0\}$ da $f(x)<f(x^0)$ bo'lsa, $f(x)$ funksiya $x^0$ nuqtada qat'iy lokal maksimumga, $f(x)>f(x^0)$ bo'lsa, $f(x)$ funksiya $x^0$ nuqtada lokal qat'iy minimumga erishadi deyiladi.

Funksiyaning lokal maksimumi, lokal minimumi umumiy nom bilan lokal ekstremumi deyiladi. Bunda $x^0$ nuqta $f(x)$ funksiyaning lokal ekstremum nuqtasi, $f(x^0)$ga esa funksiyaning lokal ekstremum qiymati deyiladi.

Funksiyaning maksimum (minimum) qiymati quyidagicha belgilanadi:

$$f\left(x^0\right)=\max_{x\in U_\delta\left(x^0\right)} f(x) \qquad \left(f\left(x^0\right)=\min_{x\in U_\delta\left(x^0\right)} f(x)\right).$$

Ma'lumki,

$$\Delta f\left(x^0\right)=f(x)-f\left(x^0\right)$$

ayirma $f(x)$ funksiyaning $x^0$ nuqtadagi to'liq orttirmasi deyilar edi.

$f(x)$ funksiya $x^0$ nuqtada lokal maksimumga erishsa, unda $\forall x\in U_\delta\left(x^0\right)$ da

$$\Delta f\left(x^0\right)\le 0$$

bo'ladi va aksincha.

Shuningdek, $f(x)$ funksiya $x^0$ nuqtada lokal minimumga erishsa, unda $\forall x\in U_\delta\left(x^0\right)$ da

$$\Delta f\left(x^0\right)\ge 0$$

bo'ladi va aksincha.

**1-teorema.** Agar $f(x)=f(x_1,x_2,....,x_m)$ funksiya $x^0=\left(x_1^0,x_2^0,...,x_m^0\right)$ nuqtada lokal estremumga erishsa va shu nuqtada barcha

$$\frac{\partial f}{\partial x_i} \qquad (i=1,2,...,m)$$

xususiy hosilalarga ega bo'lsa, u holda

$$\frac{\partial f\left(x^0\right)}{\partial x_i}=0 \qquad (i=1,2,...,m)$$

bo'ladi.

◄Aytaylik, $f(x)=f(x_1,x_2,....,x_m)$ funksiya $x^0=\left(x_1^0,x_2^0,...,x_m^0\right)$ nuqtada lokal minimumga erishsin. U holda

$$\forall x=(x_1,x_2,....,x_m)\in U_\delta\left(x^0\right) \text{ da } f(x_1,x_2,...,x_m)\ge f\left(x_1^0,x_2^0,...,x_m^0\right)$$

tengsizlik bajariladi. Jumladan

$$f\left(x_1,x_2^0,x_3^0,...,x_m^0\right)\ge f\left(x_1^0,x_2^0,...,x_m^0\right)$$

bo‘ladi. Agar

$$\varphi(x_1)=f\left(x_1,x_2^0,x_3^0,...,x_m^0\right)$$

deyilsa, $\forall x_1\in\left(x_1^0-\delta,x_1^0+\delta\right)$ da

$$\varphi(x_1)\ge\varphi\left(x_1^0\right)$$

bo‘lib, bir o‘zgaruvchili $\varphi(x_1)$ funksiya $x_1^0$ nuqtada lokal minimumga erishadi. Unda 25-ma’ruzada keltirilgan teoremaga ko‘ra

$$\varphi'\left(x_1^0\right)=0,\ \text{ya'ni}\ \frac{\partial f\left(x^0\right)}{\partial x_1}=0$$

bo‘ladi.

Xuddi shunga o‘xshash

$$\frac{\partial f\left(x^0\right)}{\partial x_2}=0,...,\frac{\partial f\left(x^0\right)}{\partial x_m}=0$$

bo‘lishi isbotlanadi.►

**1-eslatma**. Agar $f(x)$ funksiya biror $x^0$ nuqtada lokal ekstremumga erishsa va shu nuqtada differensiallanuvchi bo‘lsa, u holda

$$d\ f\left(x^0\right)=0$$

bo‘ladi.

**2-eslatma**. $f(x)=f(x_1,x_2,...,x_m)$ funksiyaning biror $x^0$ nuqtada barcha xususiy hosilalarga ega va

$$\frac{\partial f\left(x^0\right)}{\partial x_i}=0\qquad(i=1,2,...,m)$$

bo‘lishidan berilgan funksiyaning shu nuqtada lokal estremumga erishishi har doim kelib chiqavermaydi. (misollar keyingi punktda keltiriladi).

Demak, 1-teorema funksiyaning lokal ekstremumga erishishining zaruriy shartini ifodalaydi.

$f(x)$ funksiya xususiy hosilalarini nolga aylantiradigan nuqtalar uning statsionar nuqtalari deyiladi.

**$2^0$. Funksiya ekstremumga erishishining yetarli sharti**.

Aytaylik, $f(x)=f(x_1,x_2,...,x_m)$ funksiya $x^0\in R^m$ nuqtaning biror $U_\delta(x^0)$ atrofida berilgan, shu atrofda barcha ikkinchi tartibli uzluksiz xususiy hosilalarga ega va

$$\frac{\partial f(x^0)}{\partial x_i}=0 \qquad (i=1,2,...,m)$$

bo‘lsin. Bu funksiyaning Teylor formulasi (62-ma’ruzada keltirilgan Teylor formulasida $n=2$ bo‘lgan hol),

$$\frac{\partial f(x^0)}{\partial x_i}=0 \qquad (i=1,2,...,m)$$

shartni hisobga olgan holda, quyidagicha

$$f(x)=f(x^0)+\frac{1}{2}\sum_{i,k=1}^{m}\frac{\partial^2 f}{\partial x_i\,\partial x_k}\Delta x_i\,\Delta x_k \qquad (1)$$

bo‘ladi, bunda ikkinchi tartibli xususiy hosilalar

$$(x_1^0+\theta\cdot\Delta x_1,\ x_2^0+\theta\cdot\Delta x_2,...,\ x_m^0+\theta\cdot\Delta x_m)$$

$(0<\theta<1)$ nuqtada hisoblangan va

$$\Delta x_1=x_1-x_1^0,\ \Delta x_2=x_2-x_2^0,...,\Delta x_m=x_m-x_m^0.$$

Berilgan $f(x)$ funksiya ikkinchi tartibli xususiy hosilalarning statsionar nuqta $x^0$ dagi qiymatlarini

$$a_{ik}=\frac{\partial^2 f(x^0)}{\partial x_i\,\partial x_k} \qquad (i,k=1,2,...,m)$$

bilan belgilaymiz. Barcha ikkinchi tartibli xususiy hosilalar

$$\frac{\partial^2 f}{\partial x_i\partial x_k}$$

larning $x^0=\left(x_1^0,x_2^0,\ldots,x_m^0\right)$ nuqtada uzluksizligidan

$$a_{ik}=a_{ki}$$

hamda

$$\frac{\partial^2 f\left(x_1^0+\theta\Delta x_1,\, x_2^0+\theta\cdot\Delta x_2,\ldots,x_m^0+\theta\cdot\Delta x_m\right)}{\partial x_i\,\partial x_k}=\frac{\partial^2 f\left(x^0\right)}{\partial x_i\,\partial x_k}+\alpha_{ik}=a_{ik}+\alpha_{ik}$$

bo‘lishi kelib chiqadi, bunda

$$\Delta x_i\to 0 \qquad \left(i=1,2,\ldots,m\right)\ \partial a\quad \alpha_{ik}\to 0.$$

Natijada (1) tenglik ushbu

$$\Delta f\left(x^0\right)=f\left(x\right)-f\left(x^0\right)=\frac{1}{2}\left[\sum_{i,k=1}^{m}a_{ik}\,\Delta x_i\,\Delta x_k+\sum_{i,k}^{m}\alpha_{ik}\,\Delta x_i\,\Delta x_k\right]$$

ko‘rinishga keladi.

Agar

$$\rho=\sqrt{\Delta x_1^2+\Delta x_2^2+\ldots+\Delta x_m^2}\ ,$$

$$\Delta x_i=\rho\cdot\zeta_i \qquad \left(i=1,2,\ldots,m\right)$$

deyilsa, so‘ng $\Delta x_i\to 0$ $\left(i=1,2,\ldots,m\right)$ da, ya’ni $\rho\to 0$ da

$$\sum_{i,k=1}^{m}\alpha_{ik}\Delta x_k\,\Delta x_i=\rho^2\sum_{i,k}^{m}\alpha_{ik}\zeta_i\,\zeta_k=\rho^2\cdot\alpha\left(\rho\right)$$

(bunda, $\rho\to 0$ da $\alpha\left(\rho\right)\to 0$) bo‘lishini e’tiborga olsak, u holda

$$\Delta f\left(x^0\right)=\frac{\rho^2}{2}\left[\sum_{i,k=1}^{m}a_{i,k}\zeta_i\,\zeta_k+\alpha\left(\rho\right)\right] \qquad (2)$$

bo‘lishini topamiz.

Ma’lumki, $\Delta f\left(x^0\right)=f\left(x\right)-f\left(x^0\right)$ ayirma $U_\delta\left(x^0\right)$da ishora saqlasa, ya’ni $\forall x\in U_\delta\left(x^0\right)$ da

$$\Delta f\left(x^0\right)\ge 0$$

bo‘lsa, $f\left(x\right)$ funksiya $x^0$ nuqtada lokal minimumga,

$$\Delta f\left(x^0\right)\le 0$$

bo‘lsa, $f(x)$ funksiya $x^0$ nuqtada lokal maksimumga erishadi.

Yuqoridagi (2) tenglikdan ko‘rinadiki, $\Delta f(x^0)$ ning ishorasi koeffitsientlari

$$a_{ik}=\frac{\partial^2 f(x^0)}{\partial x_i x_k} \quad (i,k=1,2,...,m)$$

bo‘lgan

$$\sum_{i,k=1}^{m} a_{ik}\zeta_i\zeta_k \tag{3}$$

kvadratik formaga bog‘liq bo‘ladi.

**2-teorema**. Agar (3) kvadratik forma musbat aniqlangan bo‘lsa, $f(x)$ funksiya $x^0$ nuqtada lokal minimumga, manfiy aniqlangan bo‘lsa, lokal maksimumga erishadi.

Agar (3) kvadratik forma noaniq bo‘lsa, $f(x)$ funksiya $x^0$ nuqtada lokal estremumga erishmaydi.

◄ Bu teorema, keyingi punktda, xususiy holda ya’ni ikki o‘zgaruvchili funksiyalar uchun isbotlanadi.►(qaralsin, [1], 13-bob)

**Xususiy hollar**. $m=1$ bo‘lsin. bu holda $u=f(x)$ $(x\in R, u\in R)$ funksiyaning lokal ekstremumlari, ekstremumning zaruriy va yetarli shartlari kabi tushuncha va tasdiqlarga kelamiz. Ular 25-ma’ruzada bayon etilgan.

$m=2$ bo‘lsin. Bu holda $u=f(x,y)\left((x,y)\in R^2, u\in R\right)$ ikki o‘zgaruvchili funksiyaning lokal ekstremum tushunchalari yuzaga kelib, bu hol uchun ularning ta’riflari quyidagicha bo‘ladi.

Aytaylik, $u=f(x,y)$ funksiya $E\subset R^2$ to‘plamda berilgan bo‘lib, $(x_0,y_0)\in E$ bo‘lsin.

Agar shunday $\delta>0$ son topilsaki, $U_\delta((x_0,y_0))\subset E$ bo‘lib, $\forall(x,y)\in U_\delta((x_0,y_0))$ uchun

$$f(x,y)\geq f(x_0,y_0) \quad (f(x,y)\leq f(x_0,y_0))$$

bo‘lsa, $f(x,y)$ funksiya $(x_0,y_0)$ nuqtada lokal minimumga (lokal maksimumga) erishadi deyiladi. $(x_0,y_0)$ nuqta $f(x,y)$ funksiyaning lokal minimum (maksimum) nuqtasi, $f(x_0,y_0)$ miqdor esa funksiyaning minimum (maksimum) qiymati deyiladi.

Agar shunday $\delta>0$ son topilsaki, $U_\delta((x_0,y_0))\subset E$ bo‘lib, $\forall(x,y)\in U_\delta((x_0,y_0))\setminus\{(x_0,y_0)\}$ uchun

$$f(x,y)> f(x_0,y_0) \quad (f(x,y)< f(x_0,y_0))$$

bo‘lsa, $f(x,y)$ funksiya $(x_0,y_0)$ nuqtada qat’iy lokal minimumga (qat’iy lokal maksimumga) erishadi deyiladi.

**1-misol.** Ushbu

$$f(x,y)=\sqrt{1-x^2-y^2}$$

funksiyaning $(0,0)$ nuqtada qat’iy maksimumga erishishi ko‘rsatilsin.

◄ $\delta>0$ $(0<\delta<1)$sonni olib, $(0,0)$ nuqtaning $U_\delta((0,0))$ atrofini hosil qilamiz. Unda $\forall(x,y)\in U_\delta((0,0))\setminus\{(0,0)\}$ uchun

$$f(x,y)=\sqrt{1-x^2-y^2}<f(0,0)=1$$

bo‘ladi. Demak, berilgan funksiya $(0,0)$ nuqtada maksimumga erishadi.►

Agar $f(x,y)$ funksiya $(x_0,y_0)\in E\subset R^2$ nuqtada lokal ekstremumga erishsa va shu nuqtada

$$\frac{\partial f}{\partial x}\ ,\ \frac{\partial f}{\partial y}$$

xususiy hosilalarga ega bo‘lsa, u holda

$$\frac{\partial f(x_0,y_0)}{\partial x}=0\,,\quad \frac{\partial f(x_0,y_0)}{\partial y}=0$$

bo‘ladi.

Biroq, $f(x,y)$ funksiyaning biror $(x^*,y^*)$ nuqtada $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ xususiy hosilalari mavjud bo‘lib, ular shu nuqtada nolga teng bo‘lsa, qaralayotgan funksiya $(x^*,y^*)$ nuqtada ekstremumga erishmasdan qolishi mumkin. Masalan,

$$f(x,y)=xy$$

funksiya

$$\frac{\partial f}{\partial x}=y, \quad \frac{\partial f}{\partial y}=x$$

xususiy hosilalarga ega bo‘lib, ular $(0,0)$ nuqtada nolga teng:

$$\frac{\partial f(0,0)}{\partial x}=0, \quad \frac{\partial f(0,0)}{\partial y}=0$$

bo‘lsa ham, bu funksiya $(0,0)$ nuqtada ekstremumga erishmaydi (funksiya grafigi-giperbolik paraboloidni tasavvur qiling).

Aytaylik, $f(x,y)$ funksiya $(x_0,y_0)\in R^2$ nuqtaning biror $U_\delta((x_0,y_0))$ atrofida $(\delta>0)$ berilgan bo‘lib, quyidagi shartlarni bajarsin:

1) $f(x,y)$ funksiya $U_\delta((x_0,y_0))$ da uzulksiz va uzluksiz $f'_x, f'_y, f''_{x^2}, f''_{xy}, f''_{y^2}$ xususiy hosilalarga ega,

2) $(x_0,y_0)$ statsionar nuqta:

$$f'_x(x_0,y_0)=0, \quad f'_y(x_0,y_0)=0.$$

Bu $f(x)$ funksiya uchun $2^0$ da yuritilgan mulohazalarni qo‘llab

$$\Delta f(x_0,y_0)=f(x,y)-f(x_0,y_0)=$$
$$=\frac{1}{2}\left(a_{11}\Delta x^2+2a_{12}\Delta x\Delta y+a_{22}\Delta y^2+\alpha_{11}\Delta x^2+2\alpha_{12}\Delta x\Delta y+\alpha_{22}\Delta y^2\right) \quad (*)$$

bo‘lishini topamiz, bunda

$$a_{11}=f''_{x^2}(x_0,y_0),\ a_{12}=f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0),\ a_{22}=f''_{y2}(x_0,y_0)$$

bo‘lib,

$$\Delta x \to 0 \text{ , } \Delta y \to 0 \ \partial a \ \alpha_{11} \to 0 \text{ , } \alpha_{12} \to 0 \text{ , } \alpha_{22} \to 0$$

bo‘ladi.

**3-teorema**. Agar

$$a_{11}\Delta x^2 + 2a_{12}\,\Delta x\,\Delta y + a_{22}\Delta y^2 \qquad (4)$$

kvadratik forma musbat aniqlangan, ya’ni

$$a_{11} > 0 \text{ , } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 > 0$$

bo‘lsa, $f(x,y)$ funksiya $(x_0, y_0)$ nuqtada lokal minimumga erishadi, agar (4) kvadratik forma manfiy aniqlangan, ya’ni

$$a_{11} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}^2 > 0$$

bo‘lsa, $f(x,y)$ funkuiya $(x_0, y_0)$ nuqtada lokal maksimumga erishadi.

◀ Ma’lumki, $f(x,y)$ funksiyaning $(x_0, y_0)$ nuqtada ekstremumga erishishi $U_\delta((x_0, y_0))$ da ushbu

$$\Delta f(x_0, y_0) = f(x,y) - f(x_0, y_0)$$

ayirmaning ishora saqlashi bilan bog‘liq:

$$\forall (x,y) \in U_\delta((x_0, y_0)) \text{ da } \Delta f(x_0, y_0) > 0 \text{ bo‘lsa,}$$

$(x_0, y_0)$ nuqtada lokal minimum, $\Delta f(x_0, y_0) < 0$ bo‘lsa. $(x_0, y_0)$ nuqtada lokal maksimum sodir bo‘ladi.

$\Delta f(x_0, y_0)$ ayirmaning ishorasini aniqlash qulay bo‘lishi maqsadida (4) da

$$\Delta x = \rho \cdot \cos\varphi \text{ , } \Delta y = \rho \cdot \sin\varphi$$

almashtirishni bajaramiz, bunda

$$\rho = \sqrt{\Delta x^2 + \Delta y^2} \text{ .}$$

Natijada (*) munosabat ushbu

$$\Delta f(x_0,y_0)=\frac{\rho^2}{2}\left[\left(a_{11}\cos^2\varphi+2a_{12}\cos\varphi\sin\varphi+a_{22}\sin^2\varphi\right)+\right.$$
$$\left.+\left(\alpha_{11}\cos^2\varphi+2\alpha_{12}\cos\varphi\sin\varphi+\alpha_{22}\sin^2\varphi\right)\right] \qquad (5)$$

ko‘rinishga keladi.

Aytaylik,

$$a_{11}>0\ ,\ a_{11}a_{22}-a_{12}^2>0$$

bo‘lsin.

Ravshanki,

$$a_{11}\cos^2\varphi+2a_{12}\cos\varphi\sin\varphi+a_{22}\sin^2\varphi=$$
$$=\frac{1}{a_{11}}\left[(a_{11}\cos\varphi+a_{12}\sin\varphi)^2+\left(a_{11}a_{22}-a_{12}^2\right)\cdot\sin^2\varphi\right].$$

Ayni paytda, bu funksiya, $\varphi$ ning funksiyasi sifatida $[0,2\pi]$ da uzluksiz bo‘lib, o‘zining eng kichik qiymati (uni $m$ bilan belgilaylik) $m$ ga ega bo‘ladi:

$$\left|a_{11}\cos^2\varphi+2a_{12}\cos\varphi\sin\varphi+a_{22}\sin^2\varphi\right|\geq m>0.$$

Ikkinchi tomondan, $\Delta x\to 0$ , $\Delta y\to 0$ ya’ni $\rho\to 0$ da $\alpha_{11}\to 0,\ \alpha_{12}\to 0,\ \alpha_{22}\to 0$ bo‘lganligi sababli, $\rho$ ning etardli kichik qiymatlarida

$$\left|\alpha_{11}\cos^2\varphi+2\alpha_{12}\cos\varphi\sin\varphi+\alpha_{22}\sin^2\varphi\right|\leq|\alpha_{11}|+2|\alpha_{12}|+|\alpha_{22}|<m$$

bo‘laoladi.

Demak, $a_{11}>0$ , $a_{11}a_{22}-a_{12}^2>0$ bo‘lganda (5) tenglikning o‘ng tomonidagi ifoda musbat bo‘ladi. Binobarin,

$$\Delta f(x_0,y_0)>0$$

bo‘lib, $f(x,y)$ funksiya $(x_0,y_0)$ nuqtada lokal minimumga erishadi.

Aytaylik,

$$a_{11}<0\ ,\ a_{11}a_{22}-a_{12}^2>0$$

bo‘lsin. Bu holda (5) tenglikning o‘ng tomonidagi ifoda manfiy bo‘ladi.

Binobarin,

$$\Delta f(x_0, y_0) < 0$$

bo'lib, $f(x, y)$ funksiya $(x_0, y_0)$ nuqtada lokal maksimumga erishadi.►

**3-eslatma.** Agar

$$a_{11} a_{22} - a_{12}^2 < 0$$

bo'lsa, $f(x, y)$ funksiya $(x_0, y_0)$ nuqtada ekstremumga erishmaydi.

**4-eslatma.** Agar

$$a_{11} a_{22} - a_{12}^2 = 0$$

bo'lsa, $f(x, y)$ funksiya $(x_0, y_0)$ nuqtada ekstremumga erishishi ham mumkin, erishmasligi ham mumkin (qaralsin, $[1]$, 13-bob).

**2-misol.** Ushbu

$$f(x, y) = x^2 + xy + y^2 - 2x - 3y$$

funksiya ekstremumga tekshirilsin.

◄Avvalo berilgan funksiyaning statsionar nuqtalarini topamiz:

$$f_x'(x, y) = 2x + y - 2, \qquad 2x + y - 2 = 0, \quad x_0 = \frac{1}{3}$$

$$f_y'(x, y) = x + 2y - 3, \qquad x + 2y - 3 = 0, \quad y_\circ = \frac{4}{3}.$$

Demak, $\left(\frac{1}{3}, \frac{4}{3}\right)$ statsionar nuqta.

Ravshanki,

$$f''_{x^2}(x, y) = 2, \quad f''_{xy}(x, y) = 1, \quad f''_{y^2}(x, y) = 2.$$

Demak,

$$a_{11} = 2, \quad a_{12} = 1, \quad a_{22} = 2$$

$a_{11}=2>0$ va $a_{11}a_{22}-a_{12}^2=3>0$ bo‘lganligi uchun berilgan funksiya $\left(\frac{1}{3},\frac{4}{3}\right)$ nuqtada lokal minimumga erishadi va

$$\min f(x,y)=f\left(\frac{1}{3},\frac{4}{3}\right)=-\frac{7}{3}$$

bo‘ladi.►

**3-misol.** Ushbu

$$f_1(x,y)=x^4+y^4\ ,$$

$$f_2(x,y)=-(x^4+y^4),$$

$$f_3(x,y)=x^3+y^3\ ,$$

funksiyalar ekstremumga tekshirilsin.

◄Berilgan funksiyalar uchun $(0,0)$ statsionar nuqta bo‘ladi. Bu funksiyalar uchun

$$a_{11}a_{22}-a_{12}^2=0$$

bo‘ladi. Ravshanki, $(0,0)$ nuqtada $f_1(x,y)$ funksiya minimumga, $f_2(x,y)$ funksiya esa maksimumga erishadi. $f_3(x,y)$ funksiya $(0,0)$ nuqtada ekstremumga ega bo‘lmaydi.►

**Mashqlar**

1. 3-teoremada keltirilgan $f(x,y)$ funksiya uchun $(x_0,y_0)$ statsionar nuqtada

$$f''_{x^2}(x_0,y_0)\cdot f''_{y^2}(x_0,y_0)-\left[f''_{xy}(x_0,y_0)\right]^2<0$$

bo‘lsa, $f(x,y)$ funksiya $(x_0,y_0)$ nuqtada ekstremumga erishmasligi isbotlansin.

2. Ushbu

$$f(x,y)=(y-x)^2+(y+2)^3$$

funksiya ekstremumga tekshirilsin.

**Adabiyotlar**

1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, II q*. T. "Voris-nashriyot", 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.

# Glossariy

$f(x)$ **funksiya** $x^0$ **nuqtada lokal maksimumga erishadi** – Agar shunday $\delta > 0$ son topilsaki, $U_\delta(x^0) \subset E$ bo'lib, $\forall x \in U_\delta(x^0)$ da $f(x) \le f(x^0)$ bo'lsa.

$f(x)$ **funksiya** $x^0$ **nuqtada lokal minimumga erishadi -** Agar shunday $\delta > 0$ son topilsaki, $U_\delta(x^0) \subset E$ bo'lib, $\forall x \in U_\delta(x^0)$ da $f(x) \ge f(x^0)$ bo'lsa.

$f(x)$ **funksiya** $x^0$ **nuqtada qat'iy lokal maksimumga erishadi -** Agar shunday $\delta > 0$ son topilsaki, $U_\delta(x^0) \subset E$ bo'lib, $\forall x \in U_\delta(x^0) \setminus \{x^0\}$ da $f(x) < f(x^0)$ bo'lsa.

$f(x)$ **funksiya** $x^0$ **nuqtada lokal qat'iy minimumga erishadi** - Agar shunday $\delta > 0$ son topilsaki, $U_\delta(x^0) \subset E$ bo'lib, $\forall x \in U_\delta(x^0) \setminus \{x^0\}$ da $f(x) > f(x^0)$ bo'lsa.

**Lokal ekstremum -** Funksiyaning lokal maksimumi, lokal minimumi umumiy nom bilan lokal ekstremumi deyiladi.

**Funksiyaning maksimum (minimum) qiymati -**

$$f\left(x^0\right)=\max_{x\in U_\delta\left(x^0\right)} f\left(x\right) \qquad \left( f\left(x^0\right)=\min_{x\in U_\delta\left(x^0\right)} f\left(x\right) \right).$$

**Statsionar nuqtalar -** $f(x)$ funksiya xususiy hosilalarini nolga aylantiradigan nuqtalar uning statsionar nuqtalari deyiladi.

## Keys banki

**53-keys.** Masala o`rtaga tashlanadi: Ushbu

$$f(x,y)=(y-x)^2+(y+2)^3$$

funksiya ekstremumga tekshirilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 32-amaliy mashg`ulot

**1-misol.** Ushbu

$$f(x,y)=\sqrt{1-x^2-y^2}$$

funksiyaning $(0,0)$ nuqtada qat'iy maksimumga erishishi ko'rsatilsin.

◀ $\delta>0$ $(0<\delta<1)$ sonni olib, $(0,0)$ nuqtaning $U_\delta((0,0))$ atrofini hosil qilamiz. Unda $\forall(x,y)\in U_\delta((0,0))\setminus\{(0,0)\}$ uchun

$$f(x,y)=\sqrt{1-x^2-y^2}<f(0,0)=1$$

bo'ladi. Demak, berilgan funksiya $(0,0)$ nuqtada maksimumga erishadi.▶

**2-misol.** Ushbu

$$f(x,y)=x^2+xy+y^2-2x-3y$$

funksiya ekstremumga tekshirilsin.

◀Avvalo berilgan funksiyaning statsionar nuqtalarini topamiz:

$$f'_x(x,y)=2x+y-2,\qquad 2x+y-2=0,\quad x_0=\frac{1}{3}$$

$$f'_y(x,y)=x+2y-3,\qquad x+2y-3=0,\quad y_\circ=\frac{4}{3}.$$

Demak, $\left(\frac{1}{3},\frac{4}{3}\right)$ statsionar nuqta.

Ravshanki,

$$f''_{x^2}(x,y)=2,\ f''_{xy}(x,y)=1,\ f''_{y^2}(x,y)=2.$$

Demak,

$$a_{11}=2\ ,\ a_{12}=1\ ,\ a_{22}=2$$

$a_{11}=2>0$ va $a_{11}a_{22}-a_{12}^2=3>0$ bo'lganligi uchun berilgan funksiya $\left(\frac{1}{3},\frac{4}{3}\right)$

nuqtada lokal minimumga erishadi va

$$\min f(x,y)=f\left(\frac{1}{3},\frac{4}{3}\right)=-\frac{7}{3}$$

bo‘ladi.►

**3-misol.** Ushbu

$$f_1(x,y)=x^4+y^4\ ,$$

$$f_2(x,y)=-\left(x^4+y^4\right),$$

$$f_3(x,y)=x^3+y^3\ ,$$

funksiyalar ekstremumga tekshirilsin.

◄Berilgan funksiyalar uchun $(0,0)$ statsionar nuqta bo‘ladi. Bu funksiyalar uchun

$$a_{11}a_{22}-a_{12}^2=0$$

bo‘ladi. Ravshanki, $(0,0)$ nuqtada $f_1(x,y)$ funksiya minimumga, $f_2(x,y)$ funksiya esa maksimumga erishadi. $f_3(x,y)$ funksiya $(0,0)$ nuqtada ekstremumga ega bo‘lmaydi.►

## Mashqlar

1. $f(x,y)$ funksiya uchun $(x_0,y_0)$ statsionar nuqtada

$$f''_{x^2}(x_0,y_0)\cdot f''_{y^2}(x_0,y_0)-\left[f''_{xy}(x_0,y_0)\right]^2<0$$

bo‘lsa, $f(x,y)$ funksiya $(x_0,y_0)$ nuqtada ekstremumga erishmasligi isbotlansin.

2. Ushbu

$$f(x,y)=(y-x)^2+(y+2)^3$$

funksiya ekstremumga tekshirilsin.

Quyidagi ko`p o`zgaruvchili funksiyalarni ekstremumga tekshirilsin:

1. $z = x^2 + (y-1)^2$

2. $z = (x - y + 1)^2$

3. $z = x^2 y^3 (6 - x - y)$

4. $z = x^4 + y^4 - x^2 - 2xy - y^2$

5. $z = xy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ (a>0,b>0)

6. $z = 1 - \sqrt{x^2 + y^2}$

7. $z = e^{x^2 - y}(5 - 2x + y)$

8. $z = \sin x \sin y \sin(x + y)$

9. $z = x^2 + xy + y^2 - 4\ln x - 10\ln y$

10. $z = xy\ln(x^2 + y^2)$

11. $u = (x + y + z)e^{-(x+2y+3z)}$ funksiyaning x>0,y>0 sohada inf va sup topilsin.

# Test

1. $f(x;y)=e^{2x}(x+y^2+2y)$ funksiya kritik nuqtalarini toping.

A) $\left(\frac{1}{2};-1\right)$ B) $\left(\frac{1}{2};1\right)$ C) $\left(-\frac{1}{2};-1\right)$ D) $\left(-\frac{1}{2};1\right)$

2. $f(x;y)=2x^3+xy^2+5x^2+y^2$ funksiya kritik nuqtalarini toping.

A) $(0,0)$, $\left(-\frac{5}{3},0\right)$, (-1,2), $(-1,-2)$ B) $(0,0)$, $\left(\frac{5}{3},0\right)$, (-1,2), $(1,-2)$

C) $(0,0)$, $\left(\frac{5}{3},0\right)$, (-1,2), $(-1,-2)$ D) $(0,0)$, $\left(\frac{5}{3},0\right)$, $(1,2)$, $(1,-2)$

3. $f(x;y)=\sin x+\sin y+\cos(x+y)$ $\left(0\le x\le\frac{\pi}{4}, 0\le y\le\frac{\pi}{4}\right)$ funksiya kritik nuqtalarini toping.

A) $\left(\frac{\pi}{6},\frac{\pi}{4}\right)$ B) $\left(\frac{\pi}{6},\frac{\pi}{6}\right)$ C) $\left(\frac{\pi}{4},\frac{\pi}{6}\right)$ D) $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$

4. $f(x,y)=y\sqrt{1+x}+x\sqrt{1+y}$ funkisya kritik nuqtalarini toping.

A) $\left(-\frac{2}{3},-\frac{2}{3}\right)$ B) $\left(-\frac{2}{3},\frac{2}{3}\right)$ C) $\left(\frac{2}{3},\frac{2}{3}\right)$ D) $\left(\frac{2}{3},-\frac{2}{3}\right)$

5. $u=2x^2+y^2+2z-xy-xz$ funksiyaning statsionar nuqtalarini toping.

A) $(2,1,7)$ B) $(2,0,7)$ C) $(-2,1,7)$ D) $(-2,-1,-7)$

6. $z=4(x-y)-x^2-y^2$ funksiyaning ekstremum nuqtasini toping.

A) $(2,-2)$ B) $(-2,-2)$ C) $(-2,2)$ D) $(2,2)$

7. $z=x^2+xy+y^2+x-y+1$ funksiyaning ekstremum nuqtasini toping.

A) $(0,1)$ B) $(1,1)$ C) $(-1,1)$ D) $(-1,-1)$

8. $u=3\ln x+2\ln y+5\ln z+\ln(22-x-y-z)$ funksiyaning statsionar nuqtalarini toping.

A) $(6,4,10)$ B) $(6,4,-10)$ C) $(-6,-4,-10)$ D) $(-6,-4,10)$

9. $f(x,y)=x^2+xy+y^2-2x-3y$ funksiyani ekstremumga tekshiring.

A) $\min f(x,y)=-\frac{7}{3}$ B) $\min f(x,y)=\frac{7}{3}$ C) $\max f(x,y)=\frac{7}{3}$ D) $\max f(x,y)=-\frac{7}{3}$

10. $z=x^2+xy+y^2+x-y+1$ funksiyani ekstremumga tekshiring.

A) 0 B) 1 C) -1 D) 2

# Mavzu. Oshkormas funksiyalar

## 33-34-ma'ruza

**Reja**

$1^0$. Oshkormas funksiya tushunchasi.

$2^0$. Oshkormas funksiyaning mavjudligi.

$3^0$. Oshkormas funksiyaning hosilalari.

### $1^0$. Oshkormas funksiya tushunchasi.

Ma'lumki, $x \subset R, Y \subset R$ to'plamlar va biror $f$ qoida berilgan holda har bir $x \in X$ songa $f$ qoidaga ko'ra bitta $y \in Y$ son mos qo'yilsa, $X$ to'plamda $y = f(x)$ funksiya aniqlangan deyilar edi.

$x$ va $y$ o'zgaruvichlarni bog'lovchi qoida turlicha jumladan analitik ifodalar yordamida, jadval yordamida, egri chiziq yordamida bo'lishi mumkin.

Endi $x$ va $y$ o'zgaruvchilar tenglama yordamida bog'langan holda funksiya yuzaga kelishini ko'rsatamiz.

Aytaylik, $x$ va $y$ o'zgaruvchilarning $F(x, y)$ funksiyasi

$$E = \{(x, y) \in R^2 : a < x < b,\ c < y < d\}$$

to'plamda berilgan bo'lsin. Ushbu

$$F(x, y) = 0 \qquad (1)$$

teglamani qaraylik. Har bir tayinlangan $x = x_0$ da (1) tenglama $y$ ga isbatan tenglamaga aylanadi. Bu tenglama yagona $y_0$ echimga ega bo'lsin. Demak,

$$F(x_0, y_0) = 0.$$

Bunday xususiyatga ega bo‘lgan $x_0$ nuqtalar bir qancha bo‘lishi mumkin. Ulardan tashkil topgan to‘plamni $X$ deylik. Ravshanki, bunda $X \subset (a,b)$ bo‘ladi.

Endi $X$ to‘plamdan olingan har bir $x$ ga $(x \in X)$ (1) teglamaning yagona echimi $y$ ni mos qo‘yaylik. Natijada $X$ da aniqlangan funksiya hosil bo‘ladi. Uni $\varphi(x)$ deylik. Demak,

$$\varphi : x \to y \text{ va } F(x,\varphi(x)) \equiv 0 .$$

Bu $\varphi(x)$ oshkormas (oshkormas ko‘rinishda berilgan) funksiya deyiladi.

**1-misol.** Ushbu

$$F(x,y) = y\sqrt{x^2 - 1} - 2 = 0 \tag{2}$$

tenglama yordamida oshkormas fuksiya aniqlanishi ko‘rsatilsin.

◄Ravshanki, (2) tenglama har bir $x \in (-\infty,-1) \cup (1,+\infty)$ da yagona

$$y = \varphi(x) = \frac{2}{\sqrt{x^2 - 1}}$$

echimga ega va

$$F(x,\varphi(x)) \equiv 0 .$$

Demak, (2) tenglama $\varphi(x)$ oshkormas funksiyani aniqlaydi.►

**2-misol.** Ushbu

$$F(x,y) = x - y + \frac{1}{2}\sin y = 0$$

tenglama yordamida oshkormas funksiya aniqlanishi ko‘rsatilsin.

◄Berilgan tenglamani quyidagicha yozib olamiz:

$$x = y - \frac{1}{2}\sin y = \alpha(y) \ , \qquad (y \in (-\infty,+\infty)).$$

Bu $\alpha(y)$ funksiya $R$ da uzuluksiz va $\alpha'(y) = 1 - \frac{1}{2}\cos y > 0$ bo‘ladi. Unda $\alpha(y)$ funksiya $(-\infty,+\infty)$ da teskari $y = \alpha^{-1}(x)$ funksiyaga ega va

$$F\left(x, \alpha^{-1}(x)\right)=0$$

bo‘ladi. Demak, bu tenglama ushbu

$$\varphi: x \to \alpha^{-1}(x)$$

oshkormas funksiyani aniqlaydi.►

**3-misol.** Ushbu

$$F(x,y)=x^2+y^2-\ln y=0 \qquad (y>0)$$

tenglama $y$ ni $x$ ning oshkormas funksiyasi sifatida aniqlaydimi?

◄Aniqlamaydi, chunki har bir $x\in(-\infty,+\infty)$ da $y^2-\ln y>0$ bo‘lganligi sababli, echimga ega emas.►

Oshkormas funksiyalarni o‘rganishda quyidagi masalalar muhimdir:

1) $F(x,y)$ funksiya biror $E\subset R^2$ to‘plamda berilgan holda $y=\varphi(x)$ oshkormas funksiya mavjud bo‘ladimi va bu funksiyaning aniqlanish to‘plami qanday bo‘ladi?

2) (1) tenglama bilan aniqlangan oshkormas funksiya $y=\varphi(x)$ qanday xossalarga ega va bu xossalar $F(x,y)$ funksiya xossalari bilan qanday bog‘langan?

## $2^0$. Oshkormas funksiyaning mavjudligi.

**1-teorema.** Faraz qilaylik, $F(x,y)$ funksiya $(x_0,y_0)$ nuqtaning biror atrofi

$$U_{hk}((x_0,y_0))=\{(x,y)\in R^2: x_0-h<x<x_0+h \quad y_0-k<y<y_0+k\}$$

da $(h>0, k>0)$ berilgan bo‘lib, quyidagi shartlarni bajarsin:

1) $F(x,y)$ funksiya $U_{hk}((x_0,y_0))$ da uzluksiz;

2) Har bir tayin $x\in(x_0-h, x_0+h)$ da $y$ o‘zgaruvchining funksiyasi sifatida o‘suvchi;

3) $F(x_0,y_0)=0$.

U holda $(x_0, y_0)$ nuqtaning shunday atrofi

$$U_{\delta\varepsilon}((x_0,y_0))=\{(x,y)\in R^2 : x_0-\delta<x<x_0+\delta,\ y_0-\varepsilon<y<y_0+\varepsilon\}$$

topiladiki, $(0<\delta<h,\ 0<\varepsilon<k)$ ,

a) $\forall x\in(x_0-\delta, x_0+\delta)$ da

$$F(x,y)=0$$

tenglama yagona $y$ $(y\in(y_0-\varepsilon\ ,\ y_0+\varepsilon))$ echimga ega, ya’ni $F(x,y)=0$ tenglama yordamida oshkormas $y=\varphi(x)$ funksiya aniqlanadi,

b) $\varphi(x_0)=y_0$ bo‘ladi

v) $y=\varphi(x)$ funksiya $(x_0-\delta,\ x_0+\delta)$ da uzluksiz bo‘ladi.

◀ $U_{hk}((x_0,y_0))$ atrofga tegishli bo‘lgan

$$(x_0, y_0-\varepsilon)\quad (x_0,y_0+\varepsilon)\qquad\qquad (0<\varepsilon<k)$$

nuqtalarni olib, $[y_0-\varepsilon\ ,\ y_0+\varepsilon]$ segmentda

$$\psi(y)=F(x_0,y)$$

funksiyani qaraymiz. Teoremaning 2)-shartiga ko‘ra $\psi(y)$ o‘suvchi, 3)-shartiga ko‘ra $\psi(y_0)=F(x_0,y_0)=0$ bo‘ladi. Bunda esa

$$\psi(y_0-\varepsilon)=F(x_0,y_0-\varepsilon)<0,$$
$$\psi(y_0+\varepsilon)=F(x_0,y_0+\varepsilon)>0$$

bo‘lishi kelib chiqadi.

Teoremaning 1)-shartiga ko‘ra $F(x,y)$ funksiya $U_{hk}((x_0,y_0))$ da uzluksiz. Unda uzluksiz funksiyaning xossasiga ko‘ra, $x_0$ nuqtaning shunday $(x_0-\delta\ ,\ x_0+\delta)$ atrofi $(0<\delta<h)$ topiladiki, $\forall x\in(x_0-\delta,\ x_0+\delta)$ da

$$\begin{aligned}&F(x,y_0-\varepsilon)<0,\\&F(x,y_0+\varepsilon)>0\end{aligned}\qquad (3)$$

bo‘ladi.

Endi $(x_0,y_0)$ nuqtaning

$$U_{\delta\varepsilon}((x_0,y_0))=\{(x,y)\in R^2 : x_0-\delta<x<x_0+\delta\ ,\ y_0-\varepsilon<y<y_0+\varepsilon\}$$

atrofida

$$F(x,y)=0$$

tenglama $y$ ni $x$ ning oshkormas funksiyasi sifatida aniqlashini ko‘rsatamiz.

Ixtiyoriy $x^* \in (x_0-\delta,\ x_0+\delta)$ nuqtani olib, $[y_0-\varepsilon, y_0+\varepsilon]$ da ushbu

$$g(y)=F(x^*,y)$$

funksiyani qaraymiz. Ravshanki, bu funksiya $[y_0-\varepsilon, y_0+\varepsilon]$ segmentda uzluksiz va ayni paytda (3) munosabatga binoan

$$g(y_0-\varepsilon)=F(x^*,y_0-\varepsilon)<0\,,$$

$$g(y_0+\varepsilon)=F(x^*,y_0+\varepsilon)>0$$

bo‘ladi. Unda Bolsano-Koshining teoremasiga ko‘ra shunday $y^* \in [y_0-\varepsilon, y_0+\varepsilon]$ nuqta topiladiki,

$$g(y^*)=F(x^*,y^*)=0$$

bo‘ladi.

Ayni paytda, $g(y)$ funksiya $[y_0-\varepsilon, y_0+\varepsilon]$ da o‘suvchi (qat’iy o‘suvchi) bo‘lganligi sababli $y$ shu oraliqqa bittadan ortiq nuqtada nolga aylanmaydi.

Shunday qilib, ixtiyoriy $x\in(x_0-\delta\ ,\ x_0+\delta)$ uchun yagona $y\in(y_0-\varepsilon\ ,\ y_0+\varepsilon)$ topiladiki,

$$F(x,y)=0$$

bo‘ladi. Bu esa $U_{\delta\varepsilon}((x_0,y_0))$ da $F(x,y)=0$ tenglama $y$ ni $x$ ning oshkormas funksiyasi sifatida aniqlashni bildiradi:

$$y=\varphi(x)\ :\ \ F(x,\varphi(x))=0.$$

Aytaylik, $x=x_0$ bo‘lsin. Unda teoremaning 3) shartiga ko‘ra

$$F(x_0,y_0)=0$$

bo‘ladi. Binobarin, aniqlangan oshkormas funksiyaning $x_0$ nuqtadagi qiymati

$\varphi(x_0) = y_0$ bo‘ladi.

Modomiki, $\forall x \in (x_0 - \delta, x_0 + \delta)$ uchun $\varphi(x)$ ga ko‘ra unga mos keladigan $y \in (y_0 - \varepsilon\,,\, y_0 + \varepsilon)$ bo‘lar ekan, unda

$$|x - x_0| < \delta \quad \Rightarrow \quad |y - y_0| = |\varphi(x) - \varphi(x_0)| < \varepsilon$$

bo‘ladi. Demak, oshkormas funksiya $x_0$ nuqtada uzluksiz.

Oshkormas funksiyaning $\forall x^* \in (x_0 - \delta, x_0 + \delta)$ nuqtada uzluksiz bo‘lishini ko‘rsatish bu funksiyaning $x_0$ nuqtada uzluksiz bo‘lishini ko‘rsatish kabidir. Demak, mavjudligi ko‘rsatilgan oshkormas funksiya $(x_0 - \delta, x_0 + \delta)$ da uzluksiz bo‘ladi.►

## $3^0$. Oshkormas funksiyaning hosilalari.

Oshkormas funksiyaning hosilasini aniqlaydigan teoremani keltiramiz.

**2-teorema.** Faraz qilaylik, $F(x, y)$ funksiya $(x_0, y_0)$ nuqtaning biror atrofi $U_{hk}((x_0, y_0))$ da $(h > 0\,,\, k > 0)$ berilgan bo‘lib, quyidagi shartlarni bajarsin:

1) $U_{hk}((x_0, y_0))$ da uzluksiz;

2) $U_{hk}((x_0, y_0))$ da uzluksiz $F'_x(x, y)$ , $F'_y(x, y)$ xususiy hosilalarga ega va $F'_y(x_0, y_0) \neq 0$;

3) $F(x_0, y_0) = 0$.

U holda $(x_0, y_0)$ nuqtaning shunday $U_{\delta\varepsilon}((x_0, y_0))$ atrofi $(0 < \delta < h\,,\, 0 < \varepsilon < k)$ topiladiki, $F(x, y) = 0$ tenglama $y$ ni $x$ ning oshkormas $y = \varphi(x)$ funksiyasi sifatida aniqlaydi va bu $y = \varphi(x)$ funksiya $(x_0 - \delta, x_0 + \delta)$ da uzluksiz differensiallanuvchi bo‘lib,

$$\varphi'(x) = -\frac{F'_x(x, \varphi(x))}{F'_y(x, \varphi(x))}$$

bo‘ladi.

◀Teoremaning shartiga ko‘ra $F_y'(x,y)$ funksiya $U_{hk}((x_0,y_0))$ da uzluksiz va $F_y'(x_0,y_0)\neq 0$. Aytaylik, $F_y'(x_0,y_0)>0$ bo‘lsin. Uzluksiz funksiya xossasiga ko‘ra $(x_0,y_0)$ nuqtaning shunday $U_{\delta\varepsilon}((x_0,y_0))$ atrofi $(0<\delta<h\,,\ 0<\varepsilon<k)$ topiladiki, $\forall(x,y)\in U_{\delta\varepsilon}((x_0,y_0))$ da $F_y'(x,y)>0$ bo‘ladi. Bundan esa har bir tayin $x\in(x_0-\delta\,,\ x_0+\delta)$ da $F(x,y)$ funksiya $y$ o‘zgaruvchining funksiyai sifatida o‘suvchi bo‘lishi kelib chiqadi. U holda 1-teoremaga ko‘ra $F(x,y)=0$ tenglama $(x_0-\delta\,,\ x_0+\delta)$ da $y$ ni $x$ ning oshkormas $y=\varphi(x)$ funksiyasi sifatida aniqlaydi va $y=\varphi(x)$ oshkormas funksiya $x\in(x_0-\delta\,,\ x_0+\delta)$ da uzluksiz bo‘lib, $\varphi(x_0)=y_0$ bo‘ladi.

Aytaylik, $x\in(x_0-\delta\,,\ x_0+\delta)$, $x+\Delta x\in(x_0-\delta\,,\ x_0+\delta)$ bo‘lsin. Ravshanki,

$$F(x,y)=0\,,\ F(x+\Delta x\,,\ y+\Delta y)=0$$

bo‘lib,

$$\Delta F(x,y)=F(x+\Delta x\,,\ y+\Delta y)-F(x,y)=0 \tag{4}$$

bo‘ladi.

Teoremaning shartidan $F(x,y)$ funksiyaning $(x,y)$ nuqtada differensialanuvchi bo‘lishi kelib chiqadi. Binobarin,

$$\Delta F(x,y)=F_x'(x,y)\Delta x+F_y'(x,y)\Delta y+\alpha\cdot\Delta x+\beta\cdot\Delta y \tag{5}$$

bo‘lib, $\Delta x\to 0$, $\Delta y\to 0$ da $\alpha\to 0$ , $\beta\to 0$ bo‘ladi.

(4) va (5) munosabatlardan topamiz:

$$\frac{\Delta y}{\Delta x}=-\frac{F_x'(x,y)+\alpha}{F_y'(x,y)+\beta}.$$

Keyingi tenglikda $\Delta x\to 0$ da limitga o‘tsak, unda

$$\varphi'(x) = y' = -\frac{F_x'(x,y)}{F_y'(x,y)}$$

hosil bo‘ladi.

$U_{\delta\varepsilon}((x_0, y_0))$ da $F_x'(x,y)$ , $F_y'(x,y)$ xususiy hosilalar uzluksiz va $F_y'(x,y) \neq 0$ bo‘lishidan oshkormas funksiyaning hosilasi

$$\phi'(x) = -\frac{F_x'(x,y)}{F_y'(x,y)}$$

ning $(x_0 - \delta\ ,\ x_0 + \delta)$ da uzluksiz bo‘lishi kelib chiqadi.►

**4-misol.** Ushbu

$$F(x,y) = e^y + y\sin x - x^3 + 7 = 0$$

tenglama $(2,0)$ nuqtaning atrofida $y$ ni $x$ ning oshkormas funksiyasi sifatida aniqlashi va bu oshkormas funksiyaning hosilasi topilsin.

◄ Ravshanki,

$$F(x,y) = e^y + y\sin x - x^3 + 7$$

funksiya $R^2$ da aniqlangan va uzluksiz. Binobarin, u $(2,0)$ nuqtaning atrofida uzluksiz, $F(x,y)$ funksiyaning xususiy hosilalari quyidagicha bo‘ladi:

$$\frac{\partial F(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(e^y + y\sin x - x^3 + 7\right) = y\cos x - 3x^2\ ,$$

$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(e^y + y\sin x - x^3 + 7\right) = e^y + \sin x\,.$$

Demak, $F(x,y)$ funksiyaning xususiy hosilalari $R^2$ da, jumladan $(2,0)$ nuqtaning atrofida uzluksiz.

So‘ng

$$\frac{\partial F(2,0)}{\partial y} = \left(e^y + \sin x\right)_{x=2,\,y=0} = 1 + \sin 2 \neq 0\,.$$

Va nihoyat,

$$F(2,0)=\left(e^y+y\sin x-x^3+7\right)_{x=2,\,y=0}=0$$

bo‘ladi. Unda 2- teoremaga ko‘ra

$$F(x,y)=e^y+y\sin x-x^3+7=0$$

tenglama $(2,0)$ nuqtaning atrofida $y$ ni $x$ ning oshkormas funksiyasi sifatida aniqlaydi va bu oshkormas $\varphi(x)$ funksiyaning hosilasi

$$\varphi'(x)=-\frac{F'_x(x,y)}{F'_y(x,y)}=-\frac{y\cos x-3x^2}{e^y+\sin x}$$

bo‘ladi.►

**1-eslatma**. Oshkormas funksiyaning hosilasini quyidagicha ham hisoblasa bo‘ladi:

$$F(x,y)=0$$

ni ($y$ o‘zgaruvchi $x$ ning funksiyasi ekanini hisobga olib) differensiallab topamiz:

$$F'_x(x,y)+F'_y(x,y)\cdot y'=0.$$

Keyingi tenglikdan esa

$$y'=-\frac{F'_x(x,y)}{F'_y(x,y)}$$

bo‘lishi kelib chiqadi.

Aytaylik, $F(x,y)$ funksiya $U_{\delta\varepsilon}((x_0,y_0))$ da uzluksiz ikkinchi tartibli

$$F''_{x^2}(x,y)\ ,\quad F''_{xy}(x,y)\ ,\quad F''_{y^2}(x,y)$$

xususiy hosilalarga ega bo‘lsin.

Ma’lumki,

$$y'=-\frac{F'_x(x,y)}{F'_y(x,y)}.$$

Buni differensiallab topamiz:

$$y'' = -\frac{\left(F'_x(x,y)\right)'_x \cdot F'_y(x,y) - \left(F'_y(x,y)\right)'_x \cdot F'_x(x,y)}{\left(F'_y(x,y)\right)^2}.$$

Agar

$$\begin{aligned} \left(F'_x(x,y)\right)'_x &= F''_{x^2}(x,y) + F''_{xy}(x,y)\cdot y', \\ \left(F'_y(x,y)\right)'_x &= F''_{yx}(x,y) + F''_{y^2}(x,y)\cdot y' \end{aligned} \qquad (6)$$

ekanini hisobga olsak. Unda

$$y'' = \frac{\left(F''_{yx}(x,y) + F''_{y^2}(x,y)\cdot y'\right)F'_x(x,y) - \left(F''_{x^2}(x,y) + F''_{xy}(x,y)\cdot y'\right)\cdot F'_y(x,y)}{\left(F'_y(x,y)\right)^2} =$$

$$= \frac{F''_{yx}(x,y)\cdot F'_x(x,y) - F''_{x^2}(x,y)\cdot F'_y(x,y) + \left[F''_{y^2}(x,y)\cdot F'_x(x,y) - F''_{xy}(x,y)\cdot F'_y(x,y)\right]y'}{\left(F'_y(x,y)\right)^2}$$

bo‘ladi. Bu ifodadagi $y'$ ning o‘rniga

$$-\frac{F'_x(x,y)}{F'_y(x,y)}$$

ni qo‘yib, oshkormas funksiyaning ikkinchi tartibli hosilasi uchun quyidagi munosabatga (formulaga) kelamiz:

$$y'' = \frac{2F'_x \cdot F'_y \cdot F''_{xy} - F'^2_y \cdot F''_{x^2} - F'^2_x \cdot F''_{y^2}}{F'^2_y}.$$

**2-eslatma.** Oshkormas funksiyaning yuqori tartibli hosilalarini quyidagicha ham hisoblasa bo‘ladi.

Yuqorida

$$F(x,y) = 0$$

ni dfferensiallab,

$$F'_x(x,y) + F'_y(x,y)\cdot y' = 0$$

bo‘lishini topgan edik. Buni yana bir marta differensiallab topamiz:

$$\left[F_x'(x,y)+F_y'(x,y)\cdot y'\right]_x'=\left(F_x'(x,y)\right)_x'+y'\cdot\left(F_y'(x,y)\right)_x'+F_y'(x,y)\cdot y''=0$$

Agar (6) munsabatlardan foydalansak, keyingi tenglik ushbu

$$F_{x^2}''(x,y)+2F_{xy}''(x,y)\cdot y'+F_{y^2}''(x,y)\cdot y'^2+F_y'(x,y)\cdot y''=0$$

tenglikka keladi. Undan esa

$$y''=-\frac{F_{x2}''(x,y)+2F_{xy}''(x,y)\cdot y'+F_{y^2}''(x,y)\cdot y'^2}{F_y'(x,y)}$$

bo‘lishi kelib chiqadi.

**5-misol.** Ushbu

$$F(x,y)=xe^y+ye^x-2=0$$

tenglama bilan aniqlanadigan oshkormas funksiyaning ikkinchi tartibli hosilasi topilsin.

◀Differensiallab topamiz:

$$\left(F(x,y)\right)_x'=\left(xe^y+ye^x-2\right)_x'=0 \quad ,$$

$$e^y+ye^x+\left(xe^y+e^x\right)\cdot y'=0 \qquad (7)$$

$$y'=-\frac{e^y+ye^x}{e^x+xe^y} \quad . \qquad (8)$$

Endi (7) ni yana bir marta differensiallaymiz:

$$e^y\cdot y'+y'e^x+ye^x+e^y\cdot y'+xe^y y'\cdot y'+xe^y\cdot y''+y''e^x+y'e^x=0.$$

Keyingi tenglikdan

$$y''=-\frac{2e^y y'+2e^x y'+xe^y\cdot y'^2+ye^x}{xe^y+e^x}$$

bo‘lishi kelib chiqadi. Bu tenglikdan $y'$ ning o‘rniga (8) da ifodalangan qiymatini qo‘yib, oshkormas funksiyanin ikkinchi tartibli hosilasi topiladi.▶

### Mashqlar

1. Ushbu

$$y^5 + y - x = 0$$

tenglama bilan aniqlangan $y = \varphi(x)$ oshkormas funksiyaning grafigi yasalsin.

2. Ushbu

$$x^y = y^x \quad (x \neq y)$$

tenglama bilan aniqlanadigan $y = \varphi(x)$ oshkormas funksiyaning $y'$ va $y''$ hosilalari topilsin.

### Adabiyotlar


1. **Xudoyberganov G., Vorisov A. K., Mansurov X. T., Shoimqulov B. A.** *Matematik analizdan ma'ruzalar, II q.* T. "Voris-nashriyot", 2010.
2. **Fixtengols G. M.** *Курс дифференциального и интегрального исчисления, 1 т.* М. «ФИЗМАТЛИТ», 2001.
3. **Tao T.** *Analysis 2*. Hindustan Book Agency, India, 2014.


## Glossariy

$\varphi(x)$ **oshkormas (oshkormas ko'rinishda berilgan) funksiya** –

$\varphi : x \to y$ va $F(x, \varphi(x)) \equiv 0$.

**Oshkormas funksiyaning hosilasini -** $y' = -\dfrac{F'_x(x,y)}{F'_y(x,y)}$

**Oshkormas funksiyaning ikkinchi tartibli hosilalasi -**

$$y'' = -\frac{F''_{x^2}(x,y) + 2F''_{xy}(x,y)\cdot y' + F''_{y^2}(x,y)\cdot y'^2}{F'_y(x,y)}.$$

# Keys banki

**54-keys.** Masala o`rtaga tashlanadi: Ushbu

$$x^y = y^x \quad (x \neq y)$$

tenglama bilan aniqlanadigan $y = \varphi(x)$ oshkormas funksiyaning $y'$ va $y''$ hosilalari topilsin.

**Keysni bajarish bosqichlari va topshiriqlar:**

- keysdagi muammoni hal qilish mumkin bo`lgan asosiy formula, tushuncha va tasdiqlarni keltiring (individual va kichik guruhlarda);
- to`plangan ma'lumotlardan foydalanib, qo`yilgan masalani yeching (individual).

# 33-34-amaliy mashg`ulot

**1-misol.** Ushbu

$$F(x,y)=y\sqrt{x^2-1}-2=0 \qquad (2)$$

tenglama yordamida oshkormas fuksiya aniqlanishi ko‘rsatilsin.

◀Ravshanki, (2) tenglama har bir $x\in(-\infty,-1)\cup(1,+\infty)$ da yagona

$$y=\varphi(x)=\frac{2}{\sqrt{x^2-1}}$$

echimga ega va

$$F(x,\varphi(x))\equiv 0.$$

Demak, (2) tenglama $\varphi(x)$ oshkormas funksiyani aniqlaydi.▶

**2-misol.** Ushbu

$$F(x,y)=x-y+\frac{1}{2}\sin y=0$$

tenglama yordamida oshkormas funksiya aniqlanishi ko‘rsatilsin.

◀Berilgan tenglamani quyidagicha yozib olamiz:

$$x=y-\frac{1}{2}\sin y=\alpha(y)\ , \qquad (y\in(-\infty,+\infty)).$$

Bu $\alpha(y)$ funksiya $R$ da uzuluksiz va $\alpha'(y)=1-\frac{1}{2}\cos y>0$ bo‘ladi. Unda $\alpha(y)$ funksiya $(-\infty,+\infty)$ da teskari $y=\alpha^{-1}(x)$ funksiyaga ega va

$$F\left(x,\alpha^{-1}(x)\right)=0$$

bo‘ladi. Demak, bu tenglama ushbu

$$\varphi : x\to\alpha^{-1}(x)$$

oshkormas funksiyani aniqlaydi.▶

**3-misol.** Ushbu

$$F(x,y)=x^2+y^2-\ln y=0 \qquad (y>0)$$

tenglama $y$ ni $x$ ning oshkormas funksiyasi sifatida aniqlaydimi?

◀Aniqlamaydi, chunki har bir $x\in(-\infty,+\infty)$ da $y^2-\ln y>0$ bo‘lganligi sababli, yechimga ega emas.▶

**4-misol.** Ushbu

$$F(x,y)=e^y+y\sin x-x^3+7=0$$

tenglama $(2,0)$ nuqtaning atrofida $y$ ni $x$ ning oshkormas funksiyasi sifatida aniqlashi va bu oshkormas funksiyaning hosilasi topilsin.

◀ Ravshanki,

$$F(x,y)=e^y+y\sin x-x^3+7$$

funksiya $R^2$ da aniqlangan va uzluksiz. Binobarin, u $(2,0)$ nuqtaning atrofida uzluksiz, $F(x,y)$ funksiyaning xususiy hosilalari quyidagicha bo‘ladi:

$$\frac{\partial F(x,y)}{\partial x}=\frac{\partial}{\partial x}\left(e^y+y\sin x-x^3+7\right)=y\cos x-3x^2\ ,$$

$$\frac{\partial F(x,y)}{\partial y}=\frac{\partial}{\partial y}\left(e^y+y\sin x-x^3+7\right)=e^y+\sin x\,.$$

Demak, $F(x,y)$ funksiyaning xususiy hosilalari $R^2$ da, jumladan $(2,0)$ nuqtaning atrofida uzluksiz.

So‘ng

$$\frac{\partial F(2,0)}{\partial y}=\left(e^y+\sin x\right)_{x=2,\,y=0}=1+\sin 2\neq 0\,.$$

Va nihoyat,

$$F(2,0)=\left(e^y+y\sin x-x^3+7\right)_{x=2,\,y=0}=0$$

bo‘ladi. Unda 2- teoremaga ko‘ra

$$F(x,y)=e^y+y\sin x-x^3+7=0$$

tenglama $(2{,}0)$ nuqtaning atrofida $y$ ni $x$ ning oshkormas funksiyasi sifatida aniqlaydi va bu oshkormas $\varphi(x)$ funksiyaning hosilasi

$$\varphi'(x)=-\frac{F'_x(x,y)}{F'_y(x,y)}=-\frac{y\cos x-3x^2}{e^y+\sin x}$$

bo‘ladi.►

**5-misol.** Ushbu

$$F(x,y)=xe^y+ye^x-2=0$$

tenglama bilan aniqlanadigan oshkormas funksiyaning ikkinchi tartibli hosilasi topilsin.

◄Differensiallab topamiz:

$$(F(x,y))'_x=\left(xe^y+ye^x-2\right)'_x=0 \quad ,$$

$$e^y+ye^x+\left(xe^y+e^x\right)\cdot y'=0 \tag{7}$$

$$y'=-\frac{e^y+ye^x}{e^x+xe^y} \quad . \tag{8}$$

Endi (7) ni yana bir marta differensiallaymiz:

$$e^y\cdot y'+y'e^x+ye^x+e^y\cdot y'+xe^y y'\cdot y'+xe^y\cdot y''+y''e^x+y'e^x=0 .$$

Keyingi tenglikdan

$$y''=-\frac{2e^y y'+2e^x y'+xe^y\cdot y'^2+ye^x}{xe^y+e^x}$$

bo‘lishi kelib chiqadi. Bu tenglikdan $y'$ ning o‘rniga (8) da ifodalangan qiymatini qo‘yib, oshkormas funksiyanin ikkinchi tartibli hosilasi topiladi.►

## Mashqlar

1. Ushbu

$$y^5 + y - x = 0$$

tenglama bilan aniqlangan $y = \varphi(x)$ oshkormas funksiyaning grafigi yasalsin.

2. Ushbu

$$x^y = y^x \quad (x \neq y)$$

tenglama bilan aniqlanadigan $y = \varphi(x)$ oshkormas funksiyaning $y'$ va $y''$ hosilalari topilsin.

Quyiadgi funksiyalarni oshkormas funksiyaga tekshirilsin.

1) $F(x, y) = x^2 + y^3 - \cos(xy)$
2) $F(x, y) = \sin x - xy - x^2$
3) $F(x, y) = e^x + \sin y + x$
4) $F(x, y) = y + x + e^{x+y}$
5) $F(x, y) = \ln y + x^2$

Oshkormas funksiyaning hosilasi topilsin.

1) $F(x, y) = xe^y + y\sin x + y^2$
2) $F(x, y) = \ln(xy) + tgx - x^2$
3) $F(x, y) = y\cos x - e^{y+x} - x + 1$
4) $F(x, y) = x^3 y^5 \ln(x^2 + y)$
5) $F(x, y) = \sin y + \cos x - 4^{y-x}$

# Test

1. $f(x;y)=\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ funksiyaning aniqlanish sohasini toping.

A) $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$ B) $\frac{x^2}{a^2}+\frac{y^2}{b^2}>1$ C) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ D) $\frac{x^2}{a^2}+\frac{y^2}{b^2}\ge 1$

2. $x^2+y^2=64$ funksiya hosilasini toping.

A) $y'=-\frac{x}{y}$ B) $y'=\frac{x}{y}$ C) $y'=-\frac{2x}{y}$ D) $y'=-\frac{x}{2y}$

3. $x^2-2y^2+z^2-4x+2z-5=0$ $\frac{\partial z}{\partial x}-?$

A) $\frac{\partial z}{\partial x}=\frac{2-x}{z+1}$ B) $\frac{\partial z}{\partial x}=\frac{2+x}{z+1}$ C) $\frac{\partial z}{\partial x}=\frac{2-x}{z-1}$ D) $\frac{\partial z}{\partial x}=\frac{1-x}{z+1}$

4. $x^2-2y^2+z^2-4x+2z-5=0$ $\frac{\partial z}{\partial y}-?$

A) $\frac{\partial z}{\partial x}=\frac{2y}{z+1}$ B) $\frac{\partial z}{\partial x}=\frac{y}{z+1}$ C) $\frac{\partial z}{\partial x}=\frac{2-y}{z+1}$ D) $\frac{\partial z}{\partial x}=\frac{2y}{z-1}$

5. $x^3y-y^3x=a^4$ $\frac{dy}{dx}-?$

A) $\frac{3x^2y-y^3}{3xy^2-x^3}$ B) $\frac{3x^2y+y^3}{3xy^2-x^3}$ C) $\frac{3x^2y-y^3}{3xy^2+x^3}$ D) $\frac{3x^2y+y^3}{3xy^2+x^3}$

6. $x^2-y^2=25$ funksiya hosilasini toping.

A) $\frac{x}{y}$ B) $-\frac{x}{y}$ C) $\frac{2x}{y}$ D) $-\frac{2x}{y}$

7. $x^2y^2-x^4-y^4=a^4$ $\frac{dy}{dx}-?$

A) $\frac{x(y^2-2x^2)}{y(2y^2-x^2)}$ B) $\frac{x(y^2+2x^2)}{y(2y^2-x^2)}$ C) $\frac{x(y^2-2x^2)}{y(y^2-x^2)}$ D) $\frac{x(y^2-x^2)}{y(2y^2-x^2)}$

8. $xy-\ln y=a$ $\frac{dy}{dx}-?$

A) $\frac{y^2}{1-xy}$ B) $\frac{y^2}{1+xy}$ C) $\frac{y}{1-xy}$ D) $\frac{y}{1+xy}$

9. $e^z - xyz = 0$ $\frac{\partial z}{\partial y} - ?$

A) $\frac{z}{y(z-1)}$ B) $\frac{z}{y(z+1)}$ C) $\frac{z}{y(y-1)}$ D) $\frac{z}{y(y+1)}$

10. $e^z - xyz = 0$ $\frac{\partial z}{\partial x} - ?$

A) $\frac{z}{x(z-1)}$ B) $\frac{z}{y(x-1)}$ C) $\frac{z}{x(z+1)}$ D) $\frac{z}{y(x+1)}$