O`ZBEKISTON RESPUBLIKASI
OLIY TA`LIM, FAN VA INNOVATSIYALARVAZIRLIGI

NAMANGAN DAVLAT UNIVERSITETI

MATEMATIK ANALIZ KAFEDRASI

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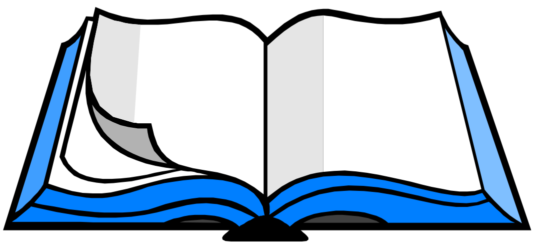
ELEMENTAR FUNKSIYALAR YORDAMIDA BAJARILADIGAN KONFORM AKSLANTIRISHLAR

(Uslubiy qo'llanma)



Namangan-2023

Ushbu uslubiy qo`llanma Namangan Davlat universiteti o`quv-uslubiy Kengashining 20___ yil ___ _______dagi yig'ilishi (№___ sonli bayonnoma) da ko`rib chiqilgan va nashrga tavsiya qilingan.

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Ushbu uslubiy qo`llanma Kompleks o`zgaruvchili funksiyalar nazariyasi fani bo`yicha fizika, matematika yo`nalishlari talabalari uchun mo`ljallangan. Uslubiy qo`llanmada chiziqli, kasr-chiziqli, darajali, ko`rsatkichli, Jukovskiy funksiyasi hamda trigonometrik funksiyalar yordamida bajariladigan conform akslantirishlar tushunchalari bayon etilgan. Har bir mavzuga doir na'munaviy misol va masalalar yechib ko`rsatilgan va mustaqil ishlash ucun mashqlar berilgan.

MUNDARIJA

1. CHIZIQLI FUNKSIYALAR VA ULAR YORDAMIDA BAJARILADIGAN AKSLANTIRISHLAR

Ta'rif-1.1. Ushbu

$$w = az + b \tag{1.1}$$

ko'rinishdagi funksiya *chiziqli akslantirish* deyiladi. Bunda a, b lar o'zgarmas kompleks sonlar va $a \neq 0$.

Bu funksiya $\overline{C_z}$ to'plamda aniqlangan, unga teskari funksiyalar ham chiziqli funksiya bo'lib, u quyidagi

$$z = \frac{1}{a} w - \frac{b}{a} \tag{1.2}$$

ko'rinishga ega.

(1.1) va (1.2) akslantirishlardan $\overline{C_z}$ va $\overline{C_w}$ tekislik nuqtalari o'zaro bir qiymatli moslikda ekanligi kelib chiqadi. Bunda $z = \infty$ da $w = \infty$ bo'ladi va aksincha.

Ravshanki, $w' = (az + b)' = a \neq 0.$

Demak, $w = az + b$ akslantirish $\overline{C_z}$ tekislikni $\overline{C_w}$ tekislikka konform akslantiradi.

Ixtiyoriy $z \in \overline{C_z}$ nuqtani olaylik. Bu z (1.1) akslantirish yordamida w nuqtaga $\left(w \in \overline{C}_w\right)$ o'tadi.

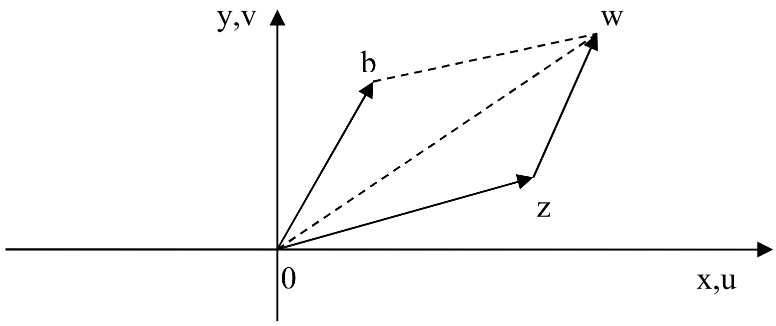
$w = az + b$ akslantirishni o'rganamiz, bunda $a = \alpha_1 + i\alpha_2$, $b = \beta_1 + i\beta_2$. (z), (w) tekisliklarni bir joydan olamiz, ya'ni Ox, Ou o'qlar umumiy boshga ega ustma-ust tushgan haqiqiy, Oy, Ov lar esa mavhum o'qlar.

Quyidagi hollarni ko'rib chiqaylik:

I hol. Aytaylik

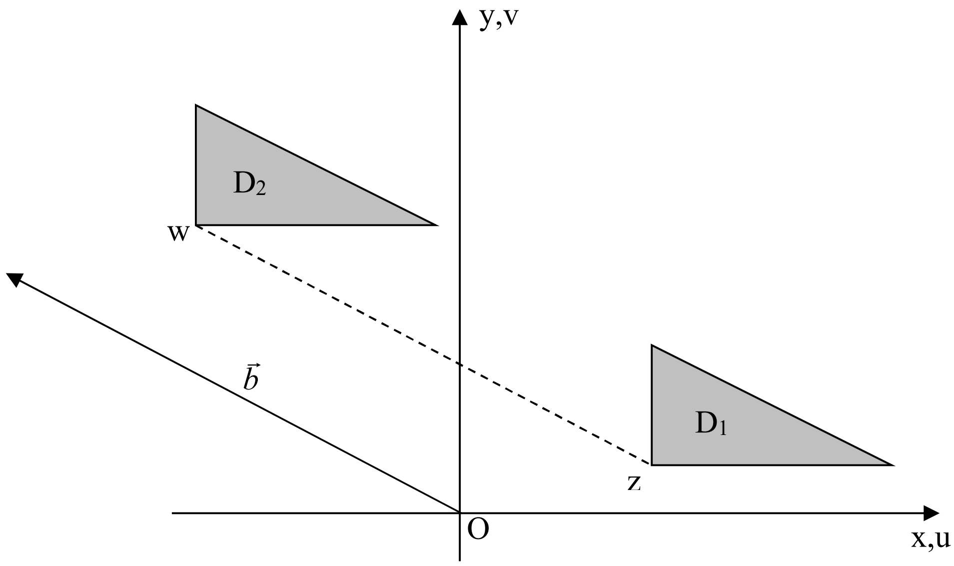
$$w = z + b \tag{1.3}$$

bo'lsin, ya'ni $a = 1$. Buning o'ng tomonidagi yig'indining geometrik ma'nosi, bilamizki, z va b vektorlarni qo'shish demakdir. Uning uchun z nuqtani qo'zg'almas b vektorga parallel qilib $|b|$ masofaga siljitish kifoya. Shu bilan w nuqtani topgan bo'lamiz. Bu jarayon 1.1-chizmada tasvirlangan.



1.1-chizma.

Agar D_1 soha sifatida biror uchburchakni olsak, unga mos D_2 ni topish uchun D_1 ning har bir nuqtasini *b* vektor bo'ylab, yuqorida aytilgandek siljitamiz. Natijada D_1 uchburchak kattaligi o'zgarmay, faqat o'z joyidan siljigan bo'ladi. Bunday akslantirishni parallel ko'chirish deyiladi (1.2-chizmaga qarang).



1.2-chizma.

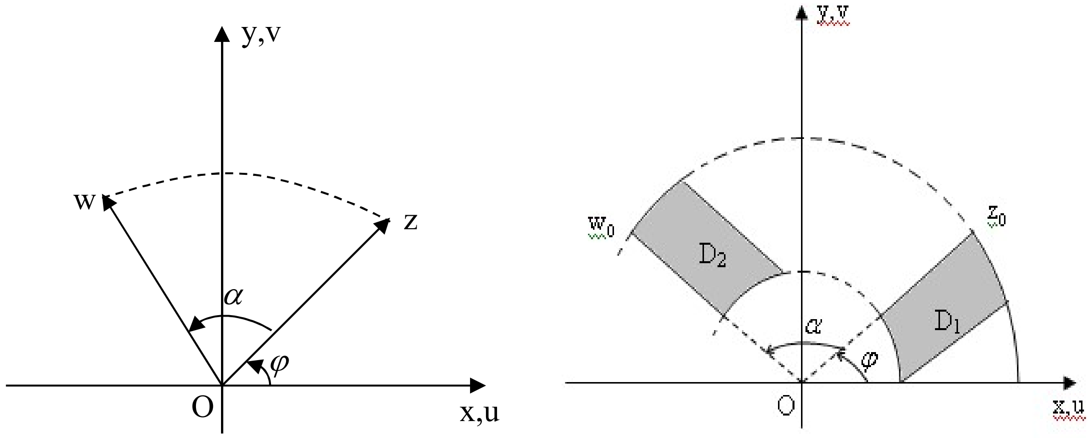
II hol. Aytaylik, $w = e^{i\alpha} z \ (\alpha \in R)$ bo'lsin. Avvalo $e^{i\alpha} = \cos\alpha + i\sin\alpha$ ekanini e'tiborga olib, so'ng

$$z = |z|(\cos\varphi + i\sin\varphi)$$

tenglikdan foydalanib topamiz:

$$w = e^{i\alpha} z = (\cos\alpha + isin\alpha)\cdot|z|\cdot(\cos\varphi + isin\varphi) = |z|\cdot\left[\cos(\varphi+\alpha) + i\sin(\varphi+\alpha)\right].$$

Demak, $|w| = |z|$, $\arg w = \varphi + \alpha = \arg z + \alpha$ bo'ladi. Bu holda z ga ko'ra uning aksi w vektor z vektorni α burchakka burish bilan topilar ekan. Bu jarayon 1.3-chizmada tasvirlangan:



1.3-chizma

III hol. Aytaylik, $w = mz$, $m > 0$, ya'ni $a = m, b = 0$ bo'lsin. $w = mz$ funksiya D$_1$ to'plamni D$_2$ ga akslantirsin. D$_1$ soha s ifatida uchburchak olsak, uning obrazi D$_2$ ni topish uchun quyidagicha mulohaza qilamiz.

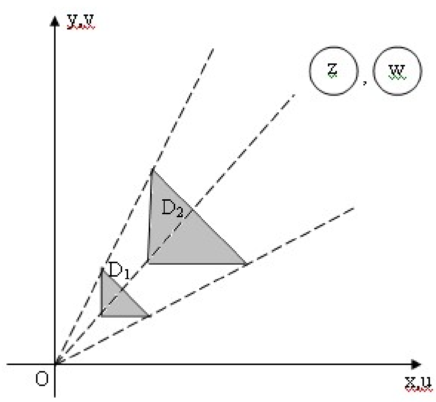
$$z = x + iy = r(\cos\varphi + i\sin\varphi),\ w = u + iv = \rho(\cos\theta + i\sin\theta)$$

bo'lsin. U holda

$$mz = w = \rho(\cos\theta + i\sin\theta) = mr(\cos\varphi + i\sin\varphi).$$

Bundan $\rho = mr$, $\theta = \varphi$.

Endi z va w sonlarning har biriga bittadan vektor mos keladi deb qaraymiz. U holda z ning w obrazini topish uchun z ga tegishli vektorning r uzunligini m marta cho'zish kerak. Lekin vektor burilmaydi, chunki $\theta = \varphi$. Agar $0 < m < 1$ bo'lsa, D$_2$ kichrayadi (1.4-chizmada $m > 1$ olindi).



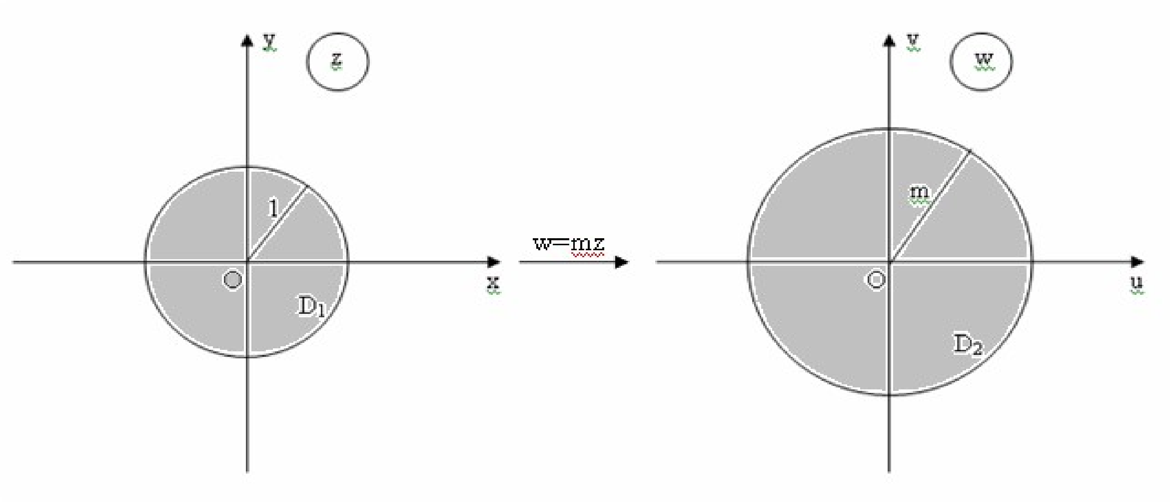
Endi D$_1$ soha $|z| \leq 1$ doira bo'lsin.

U holda $|w| = |mz| = m|z| \le m$ ga ega bo'lamiz.Bundan D_2 soha $|w| \le m$

ko'rinishdagi 1.4-chizma

m radiusli doira bo'ladi.

Xususan, agar $|z| = 1$ bo'lsa, $|w| = m$ bo'ladi. 1.5-chizmada $m > 1$. Bunday akslantirish o'xshashlik akslantirishi deyiladi.



1.5-chizma.

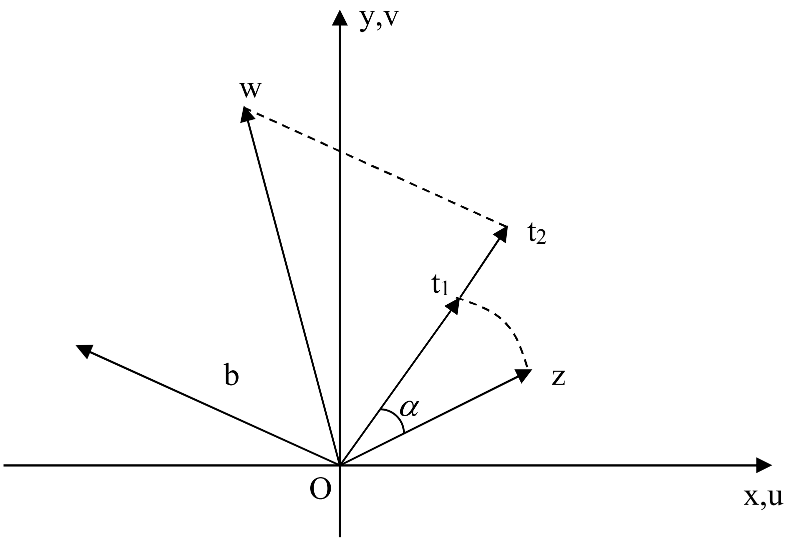
Umumiy hol. Faraz qilaylik, $w = az + b$ akslantirishda $a = m \cdot e^{i\alpha} \ne 0$ bo'lsin. U holda bu akslantirishni uch holga ajratish mumkin:

$$w = t_2 + b,\ t_2 = mt_1,\ t_1 = e^{i\alpha} \cdot z.$$

Bunda t_1 akslantirish yordamida z vektorni α burchakka burib t_1 vektorga ega bo'lamiz, t_2 yordamida t_1 vektorni m marta uzaytirib, t_2 nuqtaga ega bo'lamiz. So'ngra w akslantirish yordamida t_2 ni b vektorga parallel siljitib w nuqtani hosil qilamiz (1.6-chizma):

$$w = t_2 + b = mt_1 + b = m \cdot e^{i\alpha} \cdot z + b = az + b.$$

Yuqorida keltirilgan hollarda ko'rinadiki, $w = az + b$ chiziqli funksiya yordamida akslantirish $\overline{C_z}$ tekislikdagi sohani «parallel ko'chirish», «burchakka burish» hamda «cho'zish yoki siqishni» amalga oshirish ekan.



1.6-chizma

Faraz qilaylik, $w = f(z)$ funksiya biror E sohada $(E \subset \overline{C})$ berilgan bo’lsin.

Ta’rif-1.2. Agar $a \in E$ nuqtada $f(a) = a$ tenglik bajarilsa, $z = a$ nuqta $w = f(z)$ akslantirishning *qo’zg’almas nuqtasi* deyiladi.

Yuqorida keltirilgan

$$w = az + b$$

chiziqli akslantirish:

1) $a = 1$ bo’lganda $z = \infty$ qo’zg’almas nuqtaga ega.

2) $a \neq 1$ ikkita $z_1 = \infty,\ z_2 = \dfrac{b}{1-a}$ qo’zg’almas nuqtalarga ega bo’ladi.

Faraz qilaylik (1.1) akslantirish uchun $z = z_0$ nuqta qo’zg’olmas nuqta bo’lsin ya’ni $z_0 = az_0 + b$ bo’lsin. U holda quyidagi ko’rinishni hosil qilishimiz mumkin:

$$w - z_0 = a(z - z_0)\,. \qquad (1.4)$$

Ta’rif-1.3. $w - z_0 = a(z - z_0)$ ko’rinish chiziqli akslantirishni *kanonik ko’rinishi* deyiladi.

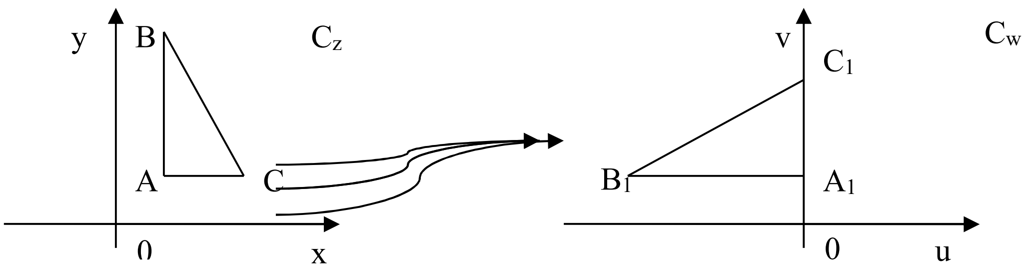
Na'munaviy misol va masalalar yechimi

1.1-misol. Uchlari $A = 1 + i, \quad B = 1 + 3i, \quad C = 2 + i$ nuqtalarda bo'lgan ABC burchakni $w = iz + 1$ chiziqli funksiya yordamida akslantiring.

Yechilishi. Ravshanki, bu $w = iz + 1$ chiziqli funksiya C_z tekislikdagi ABC uchburchakni C_w tekislikdagi $A_1B_1C_1$ uchburchakka akslantiradi. Uning A_1, B_1, C_1 uchlari mos ravishda A,B,C nuqtalarning aksi bo'ladi:

$A_1 = w(A) = i(1 + i) + 1 = i$, $B_1 = w(B) = i(1 + 3i) + 1 = i - 1$, $C_1 = w(C) = i(2 + i) + 1 = 2i$.

Demak, $w = iz + 1$ funksiya uchlari $1 + i$; $1 + 3i$; $2 + i$ nuqtalarida bo'lgan ABC uchburchakka akslantiradi.



1.7-chizma.

1.2-misol. C_z tekislikdagi

$$D = \{z \in C_z |z - z_0 :| < r\}$$

doirani C_w tekislikdagi $\{z \in C_w |w| < 1\}$ birlik doiraga akslantiruvchi chiziqli funksiyani toping.

Yechilishi. Ushbu

$$w_1 = z - z_0$$

chiziqli funksiyani qaraylik. Ravshanki, bu funksiya C_z tekislikdagi

$$D = \{z \in C_z, |z - z_0| < r\}$$

doirani C_{w_1} tekislikdagi

$$\{z \in C_w |w_1| < r\}$$

doiraga akslantiradi.

Quyidagi $w=\frac{1}{r}w_1$ chiziqli funksiya esa,

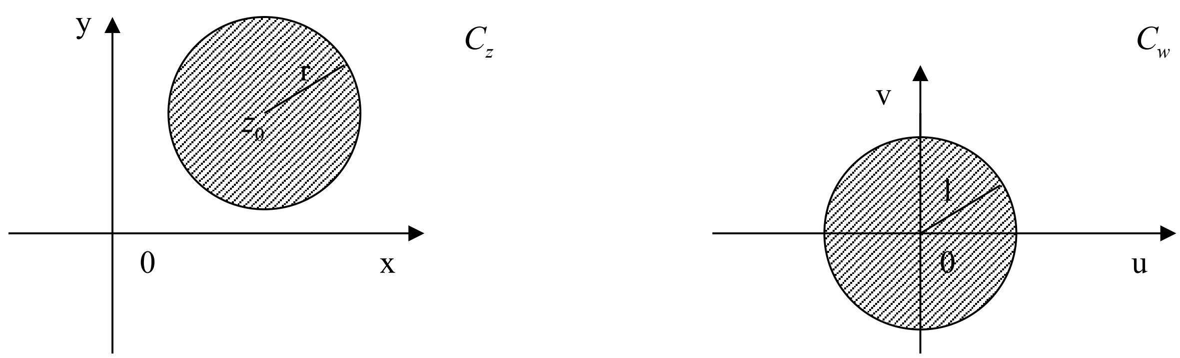
$$\{z\in C_w |w_1|<r\}$$

doirani

$$\{z\in C_w : |w|<1\}$$

birlik doiraga akslantiradi.

Shunday qilib berilgan D sohani C_w tekislikdagi birldik doiraga akslantiruvchi chiziqli akslantirish $w=\frac{1}{r}(z-z_0)$ ko'rinishga ega bo'ladi.



1.8-chizma

1.3-misol. C_z tekislikdagi $z_0=1+i$ nuqtani ko'zg'almas qoldirib, $z_1=2+i$ nuqtani esa $w_1=4-3i$ nuqtaga o'tkazadigan chiziqli akslantirishni toping.

Yechilishi. Topilishi lozim bo'lgan chiziqli akslantirishni quyidagi

$$w=az+b \tag{1.4}$$

ko'rinishda izlaymiz. $z_0=1+i$ nuqta qo'zg'almas bo'lganli uchun

$$az_0+b=z_0 \tag{1.5}$$

bo'ladi. (1.4) va (1.5) munosabatlardan

$$w-z_0=a(z-z_0)$$

bo'lishi kelib chiqadi.

z_1 nuqta akslantirish natijasida w_1 nuqtaga o'tishidan foydalanib,

$$w_1 - z_0 = a(z_1 - z_0),$$

ya'ni

$$4 - 3i - (1+i) = a\left[2 + i - (1+i)\right]$$

bo'lishini topamiz. Bu tenglikdan $a = 3 - 4i$ bo'lishi kelib chiqadi.

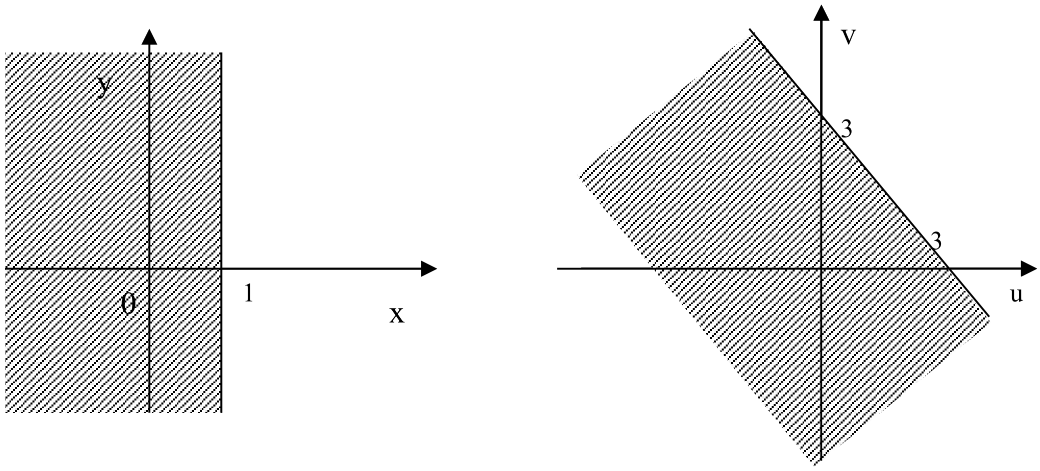
Shunday qilib, izlanayotgan chiziqli akslantirish

$$w = z_0 + a(z - z_0) = 1 + i + (3-4i)\cdot\left[z - (1+i)\right] = (3-4i)z - 6 + 2i$$

bo'ladi.

1.4-misol. Berilgan $D = \{\operatorname{Re} z < 1\}$ sohaning $w = (1+i)z + 1$ akslantirish yordamida aksini toping.

Yechilishi. Berilgan sohani aksini topish uchun teskari akslantirishdan foydalanamiz. Ya'ni $w = (1+i)z + 1$ga teskari akslantirish quramiz. Teskari akslantirish $z = \dfrac{w-1}{1+i}$ ko'rinishida bo'ladi. Agar biz $w = u + iv$ deb, teskari akslantirishni haqiqiy va mavhum qismlarini ajratsak $z = \dfrac{u-1+v}{2} + i\dfrac{v+1-u}{2}$ bo'ladi. Natijada $\operatorname{Re} z = \dfrac{u-1+v}{2}$ bo'ladi. Dastlabki $D = \{\operatorname{Re} z < 1\}$ sohaga olib kelib qo'ysak $D_1 = \left\{\dfrac{u-1+v}{2} < 1\right\}$ yoki $D = \{u + v < 3\}$ hosil bo'ladi. Bu esa C_w tekisligidagi $D = \{\operatorname{Re} w + \operatorname{Im} w < 3\}$ to'g'ri chiziqni hosil qiladi. (1.9-chizma)



1.9-chizma.

1.5-misol. Yuqori yarim tekislikni o'zini o'ziga akslantiruvchi akslantirishning umumiy ko'rinishini toping.

Yechilishi. Ma'lumki yuqori yarim tekislikni tenglamasi $D = \{\operatorname{Im} z > 0\}$ dir. O'xshashlik akslantirishi yordamida akslantirish $m > 0$ bo'lganda bu tekislikni yana o'zini o'ziga akslantiradi. Ikkinchi tomondan esa yuqori yarim tekislikni $b = (b,0)$ vektor bo'yicha parallel ko'chirsak yana yuqori yarim tekislik hosil bo'ladi. Natijada yuqori yarim tekislikni o'zini o'ziga akslantiruvchi akslantirishning umumiy ko'rinishini: $w = mz + b;\ m, b \in R$ va $m > 0$ bo'ladi.

1.6-misol. Yuqori yarim tekislikni quyi yarim tekislikka akslantiruvchi akslantirishning umumiy ko'rinishini toping.

Yechilishi. Yuqori yarim tekislikni tenglamasi $D = \{\operatorname{Im} z > 0\}$. Bu tenglamani o'xshashlik akslantirishi yordamida akslantirish $m < 0$ bo'lganda bu tekislikni quyi yarim tekislikka akslantiradi. Ikkinchi tomondan esa quyi yarim tekislikni $b = (b,0)$ vektor bo'yicha parallel ko'chirsak yana quyi yarim tekislik hosil bo'ladi. Natijada yuqori yarim tekislikni quyi yarim tekislikka akslantiruvchi akslantirishning umumiy ko'rinishini: $w = mz + b;\ m, b \in R$ va $m < 0$ bo'ladi.

1.7-misol. Yuqori yarim tekislikni o'ng yarim tekislikka akslantiruvchi akslantirishning umumiy ko'rinishini toping.

Yechilishi. Ma'lumki, yuqori yarim tekislikni tenglamasi $D = \{\operatorname{Im} z > 0\}$ dir. $w = e^{ia} z$ akslantirishi yordamida akslantirish a burchakka burishni hosil qilardi. Yuqori yarim tekislikni $-\frac{\pi}{2}$ burchakka bursak o'ng yarim tekislikka akslanadi. Demak, $a = -\frac{\pi}{2}$ bo'lganda

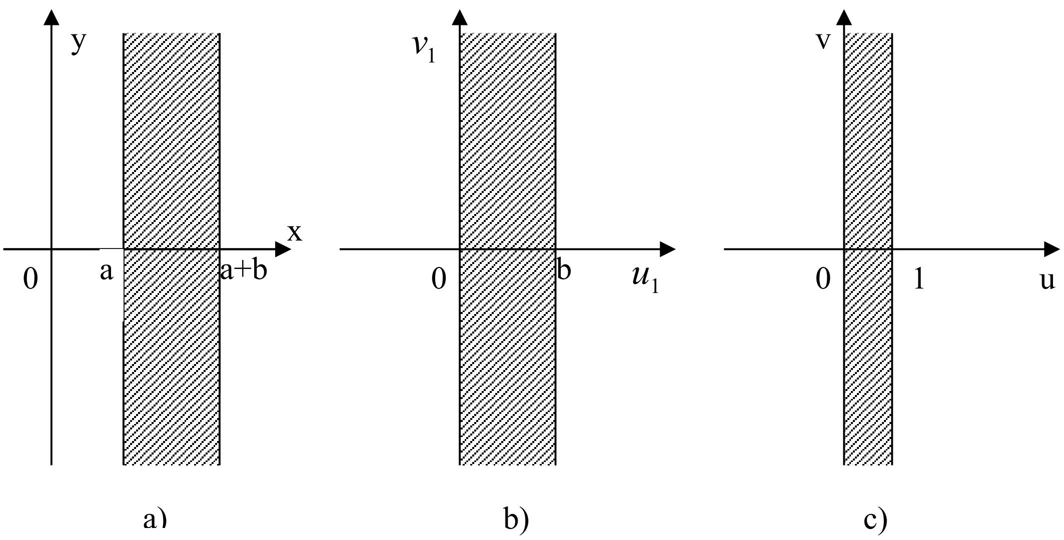
$$w = e^{-i\frac{\pi}{2}} z = \left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) z = -iz$$

Ikkinchi tomondan esa o'ng yarim tekislikni $b = (0,b)$ vektor bo'yicha parallel ko'chirsak va yana shuningdek o'xshashlik akslantirishi ham $m > 0$ bo'lganda yana o'ng yarim tekislik hosil bo'ladi Natijada yuqori yarim tekislikni o'ng yarim tekislikka akslantiruvchi akslantirishning umumiy ko'rinishini $w = -i(mz + b)$; $m, b \in R$ va $m > 0$ bo'ladi.

1.8-misol. $x = a$, $x = a + b$ to'g'ri chiziqlar orasidagi yo'lakni $\{0 < \operatorname{Re} w < 1\}$ birlik yo'lakka $w(a) = 0$ shartni qanoatlantiradigan qilib akslantiruvchi funksiyani toping.

Yechilishi. Masala shartidan kelib chiqib dastlab $x = a$ chiziqni $u_1 = 0$ chiziqqa akslantiramiz. Buning uchun parallel ko'chirishdan foydalanamiz. Ya'ni $w_1 = z - a$ deb olamiz. U holda $x = a$ va $x = a + b$ chiziqlar orasidagi soha (1.10 a- chizma) $u_1 = 0$ va $u_1 = b$ (1.10 b- chizma) to'g'ri chiziqlar orasiga akslanadi. Hosil bo'lgan akslantirishni o'xshashlik akslantirishi yordamida birlik yo'lakka akslantiramiz $w_2 = \frac{w_1}{b}$ (1.10 c-chizma).

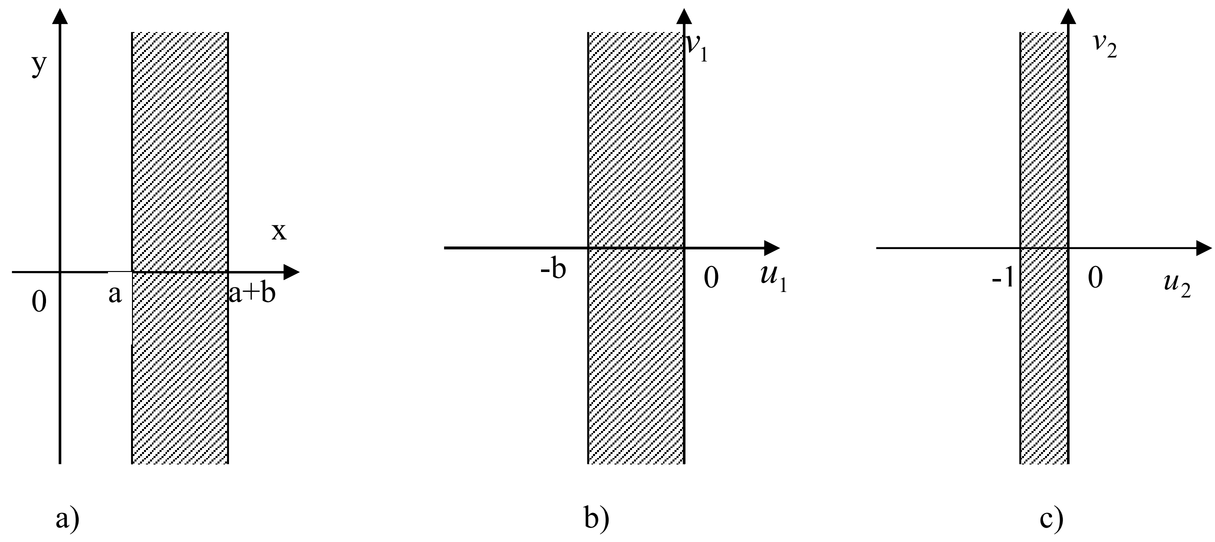
Ikkinchi tomondan esa yo'lakdagi har qanday nuqtani $c = (0, c)$ vector bo'yicha xarakatlantirsak yana yo'lakda qoladi. Lekin bizda shart borligi uchun akslantirishning ko'rinishi quyidagicha bo'ladi: $w = \frac{z - a}{b}$. (1.10-chizma)



1.10-chizma

1.9-misol. $x = a$, $x = a + b$ to'g'ri chiziqlar orasidagi yo'lakni $\{0 < \operatorname{Re} w < 1\}$ birlik yo'lakka $w(a + b) = 0$ shartni qanoatlantiradigan qilib akslantiruvchi funksiyani toping.

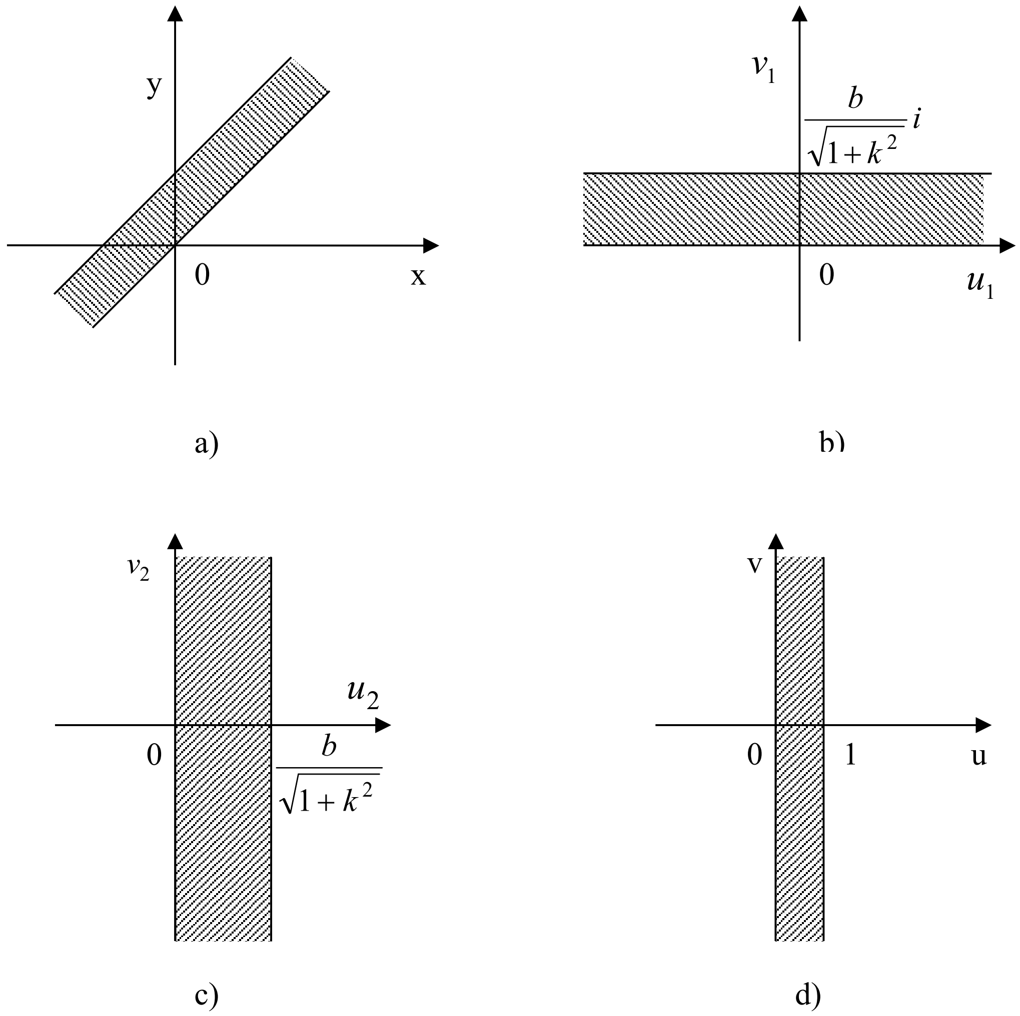
Yechilishi. Masala shartidan kelib chiqib dastlab $x = a + b$ chiziqni $u_1 = 0$ chiziqqa akslantiramiz. Buning uchun parallel ko'chirishdan foydalanamiz. Ya'ni $w_1 = z - a - b$ deb olamiz. U holda $x = a$, $x = a + b$ chiziqlar (1.11 a- chizma) $u_1 = -b$ va $u_1 = 0$ (1.11 b- chizma) to'g'ri chiziqlarga akslanadi. Soha esa chiziqlar orasidan iborat bo'ladi. Hosil bo'lgan sohani o'xshashlik akslantirishi yordamida chap birlik yo'lakka akslantiramiz $w_2 = \frac{w_1}{b}$ (1.11 c-chizma). $w_3 = -w_2$ akslantirish yordamida $\{0 < \operatorname{Re} w < 1\}$ sohaga akslantiramiz. (1.11 - chizma)



1.11- chizma

1.10-misol. $y = kx$, $y = kx + b$ to'g'ri chiziqlar orasidagi yo'lakni $\{0 < \operatorname{Re} w < 1\}$ birlik yo'lakka $w(0) = 0$ shartni qanoatlantiradigan qilib akslantiruvchi funksiyani toping.

Yechilishi. Masala shartidan kelib chiqib dastlab $y = kx$ chiziqni $v_1 = 0$ chiziqqa akslantiramiz. Buning uchun burishdan foydalanamiz. Ya'ni $w_1 = e^{-iarctgk} z$ deb olamiz. U holda $y = kx$, $y = kx + b$ chiziqlar (1.12-chizma, a)) mos ravishda $v_1 = 0$ va $v_1 = \frac{b}{\sqrt{1+k^2}}$ (1.12-chizma, b)) to'g'ri chiziqlarga akslanadi. Chunki, $y = kx$, $y = kx + b$ chiziqlar orasidagi masofa $h = \frac{b}{\sqrt{1+k^2}}$ ga teng. Soha esa chiziqlar orasi bo'ladi. Hosil bo'lgan yo'lakni yana $-\frac{\pi}{2}$ burchakka buramiz $w_2 = e^{-i\frac{\pi}{2}} w_1$. Natijada $u_2 = 0$ va $u_2 = \dfrac{b}{\sqrt{1+k^2}}$ (1.12 c- chizma) to’g’ri chiziqlar orasiga akslanadi. Hosil bo’lgan sohani o’xshashlik akslantirishi yordamida $\{0 < \operatorname{Re} w < 1\}$ birlik yo’lakka akslantiramiz: $w_3 = \dfrac{\sqrt{1+k^2}}{b} w_2$ (1.12- chizma d)). w larni o’rniga qaytarib qo’ysak, javob $w = \dfrac{\sqrt{1+k^2}}{b} e^{-i(\frac{\pi}{2}+arctgk)} z$ ko’rinishida bo’ladi. (1.12 – chizma).



1.12- chizma

Mustaqil yechish uchun mashqlar

1. Berilgan D sohaning $w = f(z)$ chiziqli funksiya yordamidagi aksini toping.

1. $D = \{|z-1| < 2\}, w = 1-2iz$. **2.** $D = \{|z-i| < 2\}, w = 1-2iz$.

3. $D = \{|z+1| < 2\}, w = 1-2iz$. **4.** $D = \{|z+i| < 2\}, w = 1-2iz$.

5. $D = \{|z-1| < 2\}, w = 1+2iz$. **6.** $D = \{|z-i| < 2\}, w = 1+2iz$.

2. Berilgan z_0 nuqtani qo'zg'almas qoldirib, z_1 nuqtani w_1 nuqtaga o'tkazadigan chiziqli akslantirishni toping.

1. $z_0 = 1+i$, $z_1 = i$, $w_1 = -i$.

2. $z_0 = 1-i$, $z_1 = i$, $w_1 = -i$.

3. $n \to \infty$

4. $z_0 = 1-i$, $z_1 = 1+i$, $w_1 = i$.

3. Quyidagi akslantirishlar uchun chekli qo'zg'almas nuqta z_0 (agar u mavjud bo'lsa), burilish burchagi φ va cho'zilish koeffitsienti k-ni toping. Akslantirishni $w - z_0 = \lambda(z - z_0)$ kanonik ko'rinishiga keltiring.

1. $w = z+1-2i$. **2.** $w = z+1+2i$. **3.** $w = z-1-2i$.

4. Berilgan D doirani G doiraga akslantiruvchi chiziqli funksiyani toping.

1. $D = \{|z-1+i| < 2\}$, $G = \{|w-i| < 4\}$.

2. $D = \{|z-1+i| < 2\}$, $G = \{|w+i| < 4\}$.

3. $D = \{|z+1-i| < 2\}$, $G = \{|w-i| < 4\}$.

2. KASR-CHIZIQLI FUNKSIYALAR VA ULAR YORDAMIDA BAJARILADIGAN AKSLANTIRISHLAR

Ushbu

$$w = \frac{az+b}{cz+d} \tag{2.1}$$

ko'rinishdagi funksiya *kasr-chiziqli funksiya* (*kasr-chiziqli akslantirish*) deyiladi, bunda a,b,c,d lar o'zgarmas kompleks sonlar va

$$ad - bc \neq 0 \tag{2.2}$$

(2.2) shartning bajarilmasligi w funksiyaning o'zgarmas bo'lib qolishiga olib keladi.

Haqiqatan ham, $ad - bc = 0$ bo'lsa, $\frac{a}{b} = \frac{c}{d} \quad (b \neq 0, d \neq 0)$ bo'lib,

$$w = \frac{az+b}{cz+d} = \frac{b\left(\frac{a}{b}z+1\right)}{d\left(\frac{c}{d}z+1\right)} = \frac{b}{d} = const$$

bo'ladi. Biz $c \neq 0$ bo'lganda

$$w(\infty) = \frac{a}{c}, \qquad w(-\frac{d}{c}) = \infty, \tag{2.3}$$

$c = 0$ bo'lganda esa $w(\infty) = \infty$ deb qaraymiz.

(2.1) munosabat z ga nisbatan yechish natijdasida berilgan kasr-chiziqli funksiyaga nisbatan teskari bo'lgan

$$z = \frac{-dw+b}{cw-a} \tag{2.4}$$

funksiyaga kelamiz. Bu yerda ham $c \neq 0$ bo'lganda, $z(\infty) = -\frac{d}{c}$, $z(\frac{a}{c}) = \infty$ va $c = 0$ bo'lganda, $z(\infty) = \infty$ deb qaraymiz.

Demak, $w = \dfrac{az+b}{cz+d}$ funksiya $\overline{C_z}$ to'plamda, $z = \dfrac{-dw+b}{cw-a}$ funksiya esa $w_0 + \Delta w$ to'plamda aniqlangan.

Ayni paytda (2.1) funksiya $\overline{C_z}$ to'plamning nuqtalarini $\overline{C_w}$ to'plam nuqtalariga o'zaro bir qiymatli akslantiradi.

Ravshanki,

$$w' = \left(\frac{az+b}{cz+d}\right)' = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

bo'lib, bu hosila

$$\overline{C_z} \setminus \left\{ z \in \overline{C_z} : z = -\frac{d}{c}, z = \infty \right\}$$

to'plamda chekli hamda (2.2) shartga binoan $w' \neq 0$.

Demak,

$$w = \frac{az+b}{cz+d}$$

akslantirish

$$\overline{C_z} \setminus \left\{ z \in \overline{C_z} : z = -\frac{d}{c}, z = \infty \right\}$$

to'plamda konform akslantirish bo'ladi.

Endi

$$w = \frac{az+b}{cz+d}$$

Akslantirishning $z = -\dfrac{d}{c}$ va $z = \infty$ nuqtalarda konform bo'lishini ko'rsatamiz.

1) $c \neq 0$ bo'lsin. Bu holda (2.1) akslantirishning $z = -\dfrac{d}{c}$ nuqtada konform bo'lishini ko'rsatish uchun $w = \dfrac{1}{w_1}$ ni qaraymiz. Ravshanki,

$$w_1 = \frac{az+b}{cz+d}, \quad w_1' = \frac{bc-ad}{(az+b)^2}$$

bo'lib,

$$w_1'(-\frac{d}{c}) = \frac{c^2}{bc-ad} \neq 0$$

bo'ladi. Demak, qaralayotgan akslantirish $z = -\frac{d}{c}$ nuqtada konform bo'ladi.

(2.1) akslantirish $z = \infty$ nuqtada konfom bo'lishini ko'rsatish uchun $z = \frac{1}{z_1}$ ni qaraymiz. Unda

$$w = \frac{az+b}{cz+d} = \frac{a+bz_1}{c+dz_1}, \quad w' = \frac{bc-ad}{(c-dz_1)^2}$$

bo'lib, $z_1 = 0$ bo'lganda

$$w' = \frac{bc-ad}{c^2} \neq 0$$

bo'ladi. Demak, (2.1) akslantirish $z = \infty$ nuqtada konform bo'ladi.

2) $c = 0$ bo'lsin. Bu holda

$$w = \frac{a}{d}z + \frac{b}{d}$$

bo'lib, $z = \infty$ nuqta $w = \infty$ nuqtaga akslanadi.

Agar $z = \frac{1}{z_1}$, $w = \frac{1}{w_1}$ deyilsa, unda

$$w_1 = \frac{dz_1}{a+bz_1}, \quad w_1' = \frac{ad}{(a+bz_1)^2}$$

bo'lib, $z_1 = 0$ nuqta $w_1' = \frac{d}{a} \neq 0$ bo'ladi. Demak, (2.1) akslantirish $z = \infty$ nuqtada konform bo'ladi.

Shunday qilib, $w = \frac{az+b}{cz+d}$ akslantirish $\overline{C_z}$ tekislik nuqtalarini $\overline{C_w}$ tekislik nuqtalariga konform akslantirar ekan.

Kasr-chiziqli akslantirish yordamida $\overline{C_z}$ dagi sohaning aksini aniqlash uchun avval (2.1) ning xususiy hollarini qaraymiz:

1. (2.1) da $a=0,\ b=1,\ c=1,\ d=0$ bo'lsin. Bu holda kasr-chiziqli funksiya ushbu $w=\dfrac{1}{z}$ ko'rinishga keladi.

Doiraviy xossasi.

Lemma-2.1. $w=\dfrac{1}{z}$ akslantirish $\overline{C_z}$ tekislikda aylana yoki to'g'ri chiziqni $\overline{C_w}$ tekislikda aylana yoki to'g'ri chiziqqa o'tkazadi.

Isboti. Ma'lumki, R^2 tekislikda

$$A(x^2+y^2)+2Bx+2Cy+D=0 \tag{2.5}$$

tenglama aylana ($A\neq 0,\ A^2+C^2-AD>0$ bo'lganda) yoki to'g'ri chiziqni ($A=0$ bo'lganda) ifodalaydi.

Agar

$$x^2+y^2=z\cdot\overline{z},\quad (z=x+iy)\quad x=\frac{1}{2}(z+\overline{z}),\quad y=-\frac{i}{2}(z-\overline{z})$$

bo'lishini e'tiborga olsak, unda (2.5) tenglama quyidagi

$$Az\overline{z}+\overline{E}z+E\overline{z}+D=0 \tag{2.6}$$

ko'rinishga kelishini topamiz, bunda $E=B+iC.$

Demak, kompleks tekislik $\overline{C_z}$ da aylana (yoki to'g'ri chiziq) kompleks o'zgaruvchi orqali (2.6) tenglama bilan ifodalanar ekan.

Endi, $w=\dfrac{1}{z}$ akslantirish yordamida $\overline{C_z}$ tekislikdagi

$$Az\overline{z}+\overline{E}z+E\overline{z}+D=0$$

aylananing (yoki to'g'ri chiziqning) $\overline{C_w}$ tekislikdagi aksini topamiz. Buning uchun (2.6) tenglamadagi z o'rniga $\dfrac{1}{w}$ ni, $\overline{z}$ o'rniga esa $\dfrac{1}{w}$ ni qo'yamiz.

Natijada

$$A\frac{1}{w}\frac{1}{\overline{w}}+\overline{E}\frac{1}{w}+E\frac{1}{\overline{w}}+D=0$$

bo'lib,

$$D\cdot w\cdot\overline{w}+E\cdot w+\overline{E}\cdot\overline{w}+A=0 \qquad (2.7)$$

bo'lishi kelib chiqadi. Bu (2.7) tenglamani yuqoridagi (2.6) tenglama bilan solishtirib, uning $\overline{C_w}$ tekislikda aylananing (yoki to'g'ri chiziq) tenglamasi ekanligini aniqlaymiz.

Demak, $w=\frac{1}{z}$ akslantirish aylanani (yoki to'g'ri chiziqni) aylanaga (yoki to'g'ri chiziqqa) akslantirar ekan.

Teorema-2.1. Ixtiyoriy kasr-chiziqli

$$w=\frac{az+b}{cz+d}, \quad ad-bc\neq 0$$

funksiya $\overline{C}_z$ tekislikdagi har qanday aylanani $\overline{C_w}$ tekislikdagi aylanaga akslantiradi, bunda $z=-\frac{d}{c}$ bo'lganda $w=\infty$, $w=\frac{a}{c}$ bo'lganda $z=\infty$ bo'ladi.

Isboti. $\overline{C}_z$ tekislikdagi aylananing umumiy tenglamasi

$$A(x^2+y^2)+Bx+Cy+D=0,\ A\neq 0,\ B^2+C^2-AD>0 \qquad (2.8)$$

Agar $A=0$ bo'lsa, $Bx+Cy+D=0$ to'g'ri chiziq tenglamasiga ega bo'lamiz. Ma'lumki,

$$x=\frac{z+\overline{z}}{2}, \qquad y=\frac{z-\overline{z}}{2i}, \quad z\cdot\overline{z}=x^2+y^2$$

Bularni (2.8) ga qo'ysak, z o'zgaruvchiga nisbatan aylana tenglamasiga ega bo'lamiz:

$$A\cdot z\cdot\overline{z}+\alpha\cdot z+\overline{\alpha}\cdot\overline{z}+D=0,$$

bunda $\alpha=\frac{1}{2}\cdot(B-C\cdot i)$.

Tenglamadagi z ning o'rniga $z=\frac{-dw+b}{cw-a}$ ni qo'ysak,

$$A'\cdot w\cdot\overline{w}+B'\cdot w+\overline{B}'\cdot\overline{w}+D'=0$$

tenglikka ega bo'lamiz. Bu esa aylananing $\overline{C_w}$ tekislikda tenglamasidir. Demak, aylananing aksi yana aylana bo'lar ekan. Teorema isbotlandi.

2. (2.1) da $c=0$ bo'lsin. U holda

$$w=\frac{az+b}{d}=\frac{a}{d}z+\frac{b}{d},\quad c=0,\; w=\frac{1}{z}$$

chiziqli akslantirishga kelamiz. Bunday akslantirish avvalgi mavzuda batafsil o'rganilgan.

Endi umumiy holda qaraymiz ($c\neq 0$). Bu holda kasr-chiziqli akslantirishni quyidagicha yozib olamiz:

$$w=\frac{az+b}{cz+d}=\frac{a}{c}+\frac{bc-ad}{c^2}\cdot\frac{1}{z+\frac{d}{c}}$$

Agar

$$w_1=z+\frac{d}{c}, \tag{2.9}$$

$$w_2=\frac{1}{w_1} \tag{2.10}$$

deyilsa, unda

$$w=\frac{a}{c}+\frac{bc-ad}{c^2}\cdot w_2 \tag{2.11}$$

bo'ladi. (2.9), (2.10) va (2.11) munosabatlardan $w=\frac{az+b}{cz+d}$ kasr-chiziqli akslantirish uchta: Chiziqli akslantirish $w_1=z+\frac{d}{c}$, so'ng yuqorida o'rganilgan $w_2=\frac{1}{w_1}$ akslantirish hamda $w=\frac{a}{c}+\frac{bc-ad}{c^2}\cdot w_2$ larning birin-ketin bajarilishi natijasidan iborat ekanligini ko'ramiz.

Biz yuqorida kasr-chiziqli akslantirish

$$w = \frac{az+b}{cz+d}$$

kengaytirilgan kompleks tekislik $\overline{C}_z$ da konform akslantirish ekanligini ko'rdik.

Teskari akslantirish. Aylanaga nisbatan simmetriya.

Kasr-chiziqli akslantirishga teskari bo'lgan akslantirish kasr-chiziqli bo'ladi.

Xossa-2.1. Kasr-chiziqli akslantirishga teskari akslantirish kasr-chiziqli bo'ladi.

(2.1) va (2.4) munosabatlardan (2.1) kasr chiziqli akslantirishga teskari akslantirish kasr-chiziqli bo'lishi kelib chiqadi.

Xossa-2.2. Kasr-chiziqli akslantirishlarning superpozitsiyasi kasr-chiziqli akslantirish bo'ladi.

Isboti. Aytaylik, ikkita

$$w_1 = w_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad (a_1 d_1 - b_1 c_1 \neq 0),$$

$$w = w(w_1) = \frac{a_2 w_1 + b_2}{c_2 w_2 + d_2} \quad (a_2 d_2 - b_2 c_2 \neq 0)$$

kasr-chiziqli akslantirishlar berilgan bo'lsin. Bu akslantirishlarning superpozitsiyasini qaraymiz:

$$w(w_1) = w(w_1(z)) = \frac{a_2 \dfrac{a_1 z + b_1}{c_1 z + d_1} - b_2}{c_2 \dfrac{a_1 z + b_1}{c_1 z + d_1} + d_2} = \frac{(a_1 a_2 + c_1 b_2) z + a_2 b_1 + b_2 d_1}{(a_1 c_2 + c_1 d_2) z + c_2 b_1 + d_1 d_2} = \frac{az+b}{cz+d}$$

bunda

$$a = a_1 a_2 + c_1 b_2, \quad b = a_2 b_1 + b_2 d_1, \quad c = a_1 c_2 + c_1 d_2, \quad d = c_1 b_1 + d_1 d_2$$

bo'lib

$$ad - bc = (a_1 a_2 + c_1 b_2)(c_2 b_1 + d_1 d_2) - (a_2 b_1 + b_2 d_1)(a_1 c_2 + c_1 d_2) =$$

$$= (a_1 d_1 - b_1 c_1)(a_2 b_2 - b_2 c_2) \neq 0$$

bo'ladi. Demak, $w(w_1(z))$ kasr-chiziqli akslantirish bo'ladi.

Xossa-2.2 dan foydalanib Teorema-2.1 ni quyidagicha isbotlash mumkin:

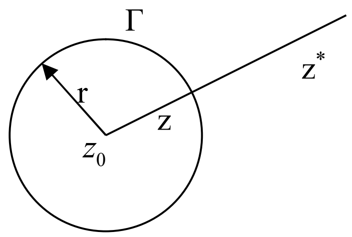
Ma'lumki, $w = az + b$ hamda $w = \frac{1}{z}$ ko'rinishdagi akslantirishlarning har biri $\overline{C_z}$ tekislikdagi aylanani yoki to'g'ri chiziqni $\overline{C_w}$ tekislikdagi aylanaga yoki to'g'ri chiziqqa akslantiradi. Kasr-chiziqli akslantirish esa chiziqli hamda $w = \frac{1}{z}$ ko'rinishdagi akslantirishlarning birin-ketin bajarilishidan iborat.

Faraz qilaylik, kompleks tekislikda

$$\Gamma = \{z \in C : |z - z_0| = r\}$$

aylana berilgan bo'lsin.

Ta'rif-2.1. Agar z va z^* nuqtalar Γ aylana markazidan chiqqan bitta nurda yotib, bu nuqtalardan aylana markazigacha bo'lgan masofalar ko'paytmasi aylana radiusi kvadratiga teng bo'lsa, z va z^* nuqtalar Γ aylanaga nisbatan simmetrik nuqtalar deyiladi (2.1-chizma).



2.1-chizma

Ravshanki, bu holda

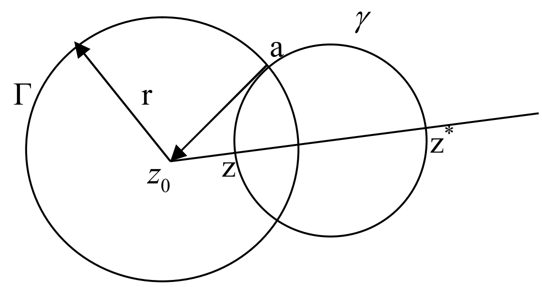
$$\arg(z^* - z_0) = \arg(z - z_0), \quad |z^* - z_0| \cdot |z - z_0| = r^2$$

bo'lib, $z^* - z_0 = \dfrac{r^2}{\overline{z - z_0}}$ bo'ladi.

Lemma-2.2. Berilgan z va z^* nuqtalarning Γ aylanaga nisbatan simmetrik bo'lishi uchun shu z va z^* nuqtalar orqali o'tuvchi har qanday γ aylananing $(\gamma \subset \overline{C_z})$ Γ aylanaga ortogonal bo'lishi zarur va yetarli.

Isboti. Zarurligi. Aytalik z va z^* nuqtalar Γ aylanaga nisbatan simmetrik bo'lsin. Demak,

$$\arg(z^* - z_0) = \arg(z - z_0), \quad |z^* - z_0| \cdot |z - z_0| = r^2 \qquad (2.12)$$



2.2-chizma

Shu z va z^* nuqtalar orqali o'tuvchi γ aylanani qaraymiz. z_0 nuqtadan γ aylanaga urinma o'tkazamiz. Urinish nuqtasi a bo'lsin. Unda geometriya kursidan ma'lum bo'lgan tasdiqqa ko'ra

$$|z_0 - a|^2 = |z_0 - z^*| \cdot |z_0 - z| \qquad (2.13)$$

bo'ladi. (2.12) va (2.13) munosabatlardan $|z_0 - a|^2 = r^2$ bo'lishi kelib chiqadi.

Demak, z_0 nuqtadan γ aylanaga o'tkazilgan urinmaning z_0a qismi Γ aylananing radiusi ekan (2.2-chizma):

$$|z_0 - a|^2 = r$$

Bu esa Γ va γ aylanalarning ortogonal bo'lishini bildiradi.

Yetarliligi. Faraz qilaylik z va z^* nuqtalardan o'tuvchi har qanday γ aylana $(\gamma \subset \overline{C_z})\Gamma$ aylanaga ortogonal bo'lsin. Xususan, bu aylana z va z^* nuqtalardan o'tuvchi $z \cdot z^*$ to'g'ri chiziq bo'lishi ham mumkin. Unda $z \cdot z^*$ to'g'ri chiziq Γ aylanaga ortogonal bo'lganligi uchun u z_0 nuqtadan o'tadi, ya'ni

$$\arg(z^* - z_0) = \arg(z - z_0)$$

bo'ladi.

Ikkinchi tomondan, geometriya kursidan ma'lum bo'lgan tasdiqqa ko'ra

$$\left|z^* - z_0\right| \cdot \left|z - z_0\right| = r^2$$

bo'ladi. Demak, z va z^* aylanaga nisbatan simmetrik bo'ladi. Lemma isbot bo'ldi.

Har qanday

$$w = \frac{az + b}{cz + d}$$

kasr-chiziqli akslantirish $\overline{C_z}$ tekislikda Γ aylanaga nisbatan simmetrik bo'lgan z va z^* nuqtalarni $\overline{C_w}$ tekslikdagi Γ aylananing aksi $w(\Gamma)$ aylanaga nisbatan simmetrik bo'lgan $w(z)$ va $w(z_0)$ nuqtalarga o'tkazadi.

Kasr-chiziqli akslantirishning yuqoridagi Teorema-2.1 ga ko'ra Γ aylananing $\overline{C_w}$ tekislikdagi aksi $w(\Gamma)$ xam aylana bo'ladi.

z va z^* nuqtalar orqali o'tuvchi ixtiyoriy γ aylanani olaylik. z va z^* nuqtalar Γ aylanaga nisbatan simmetrik nuqtalar bo'lgani uchun isbot etilgan lemmaga binoan Γ va γ aylanalar ortogonal bo'ladi.

Kasr-chiziqli akslantirish konform bo'lgani sababli $w(\Gamma)$ hamda $w(\gamma)$ aylanalar ortogonal bo'ladi. Unda lemmaga ko'ra $w(z)$ va $w(z^*)$ nuqtalar $w(\Gamma)$ aylanaga nisbatan simmaetrik bo'ladi. Xossa isbot bo'ldi.

Angarmonik nisbat.

Teorema-2.2. $\overline{C_z}$ tekislikda berilgan turli z_1, z_2, z_3 nuqtalarni $\overline{C_w}$ tekislikda berilgan turli w_1, w_2, w_3 nuqtalarga o'tuvchi kasr-chiziqli akslantirish mavjud va u yagonadir.

Isboti. Aytaylik $w = w(z) = \dfrac{az + b}{cz + d}$ kasr-chiziqli akslantirish z_1, z_2, z_3 tekislikdagi turli $\overline{C_z}$ nuqtalarni $\overline{C_w}$ tekislikdagi turli w_1, w_2, w_3 nuqtalarga akslantirsin. Unda

$$w_1 = w(z_1) = \frac{az_1 + b}{cz_1 + d}, \quad w_2 = w(z_2) = \frac{az_2 + b}{cz_2 + d}, \quad w_3 = w(z_3) = \frac{az_3 + b}{cz_3 + d}$$

bo'ladi. Quyidagi ayirmalarni hisoblaymiz:

$$w-w_1=\frac{az+b}{cz+d}-\frac{az_1+b}{cz_1+d}=\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)},$$

$$w_3-w_1=\frac{az_3+b}{cz_3+d}-\frac{az_1+b}{cz_1+d}=\frac{(ad-bc)(z_3-z_1)}{(cz_3+d)(cz_1+d)},$$

$$w-w_2=\frac{az+b}{cz+d}-\frac{az_2+b}{cz_2+d}=\frac{(ad-bc)(z-z_2)}{(cz+d)(cz_2+d)},$$

$$w_3-w_2=\frac{az_3+b}{cz_3+d}-\frac{az_2+b}{cz_2+d}=\frac{(ad-bc)(z_3-z_2)}{(cz+d)(cz_2+d)}$$

Bu ayirmalardan foydalanib topamiz

$$\frac{w-w_1}{w-w_2}:\frac{w_3-w_1}{w_3-w_2}=$$

$$=\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}\cdot\frac{(cz+d)(cz_2+d)}{(ad-bc)(z-z_2)}:\frac{(ad-bc)(z_3-z_1)}{(cz_3+d)(cz_1+d)}\cdot\frac{(cz_3+d)(cz_2+d)}{(ad-bc)(z_3-z_2)}=$$

$$=\frac{(cz_2+d)(z-z_1)}{(cz_1+d)(z-z_2)}:\frac{(cz_2+d)(z_3-z_1)}{(cz_1+d)(z_3-z_2)}=\frac{z-z_1}{z-z_2}:\frac{z_3-z_1}{z_3-z_2}.$$

Demak,

$$\frac{w-w_1}{w-w_2}:\frac{w_3-w_1}{w_3-w_2}=\frac{z-z_1}{z-z_2}:\frac{z_3-z_1}{z_3-z_2}. \quad (2.14)$$

Bu izlanayotagan kasr-chiziqli akslantirishdir. Teorema isbotlandi.

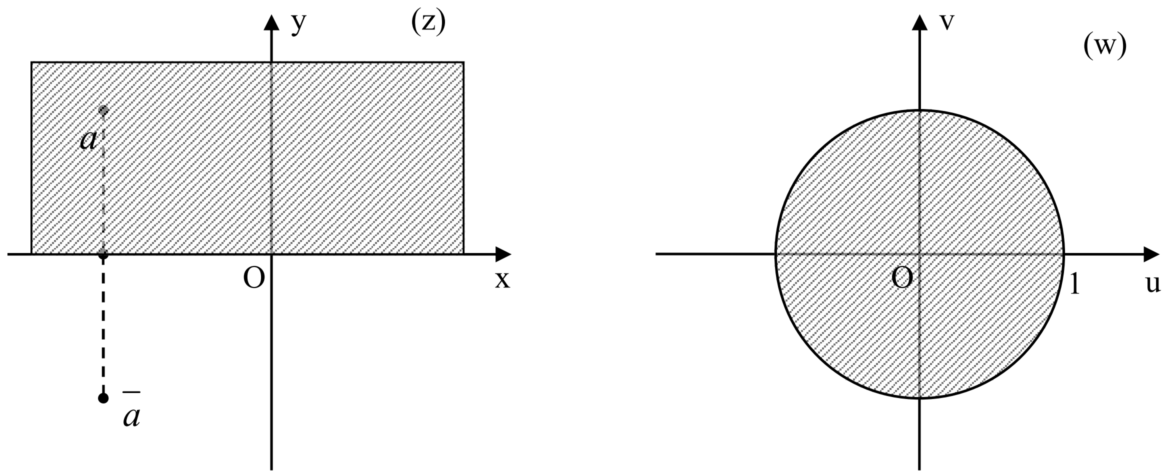
Na'munaviy misol va masalalar yechimi

2.1-Misol. Yuqori yarim tekislik $\{z \in C_z : \operatorname{Im} z > 0\}$ ni $\overline{C_w}$ tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga akslantiruvchi kasr-chiziqli funksiya ushbu

$$w = e^{i\varphi} \frac{z - a}{z - \overline{a}} \qquad (\varphi \in R; \operatorname{Im} a > 0)$$

ko'rinishda bo'ladi.

Yechilishi. Haqiqatan ham, yuqori yarim tekislikda a nuqtani $(a \in \{z \in C_z : \operatorname{Im} z > 0\})$ ni olaylik. Ravshanki bu a nuqtaga Ox o'qiga nisbatan simmetirik bo'lgan nuqta $\overline{a}$ bo'ladi. Izlanayotgan akslantirish $z = a$ nuqtani $\overline{C_w}$ tekislikdagi birlik doira markazi $w = 0$ nuqtaga o'tkazadigan bo'lsa, Teorema-2.1 ga ko'ra $z = \overline{a}$ nuqta $w = 0$ nuqtaga birlik aylanaga nisbatan simmetrik bo'lgan $w = \infty$ nuqtaga o'tkazishi lozim (2.3-chizma).



2.3-chizma

Demak, bunday akslantirishni bajaruvchi funksiya

$$w = \alpha \frac{z - a}{z - \overline{a}}$$

ko'rinishda bo'ladi. Ayni paytda bu akslantirish xaqiqiy o'qda joylashgan $z = x$ nuqtani C_w tekislikdagi $|w| = 1$ birlik aylana nuqtasiga o'tkazishi bo'ladi. Shunday qilib yuqori yarim tekislikning birlik doiraga akslantiruvchi kasr-chiziqli funksiya $w = e^{i\varphi}\dfrac{z-a}{z-\overline{a}}$ ko'rinishda bo'lar ekan.

2.2-Misol. Kompleks tekislik C_z dagi birlik doira $\{z \in C_z : |z| < 1\}$ ni C_w tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga akslantiruvchi kasr-chiziqli funksiya ushbu

$$w = e^{i\varphi}\frac{z-a}{1-\overline{a}z}\ , \ \ (|a| < 1)$$

ko'rinishda bo'ladi.

Yechilishi. Biror $a \in \{z \in C_z : |z| < 1\}$ nuqtani olib uni $a = re^{i\varphi}$ ko'rinishda ifodalaymiz. Unda a nuqtaga birlik aylanaga nisbatan simmetrik bo'lgan nuqta

$$a = \frac{1}{r}e^{i\varphi} = \frac{1}{re^{-i\varphi}} = \frac{1}{\overline{a}}$$

bo'ladi. Izlanayotgan akslantirish $z = a$ nuqtani C_w tekislikdagi birlik doira markazi $w = 0$ ga o'tkazadigan bo'lsa, Teorema-2.1 ga ko'ra $z = \dfrac{1}{\overline{a}}$ nuqta $w = 0$ nuqtaga $|w| = 1$ aylanaga nisbatan simmetrik bo'lgan $w = \infty$ nuqtaga o'tkazishi lozim. Demak, bunday akslantirishni bajaruvchi funksiya

$$w = \alpha_1\frac{z-a}{z-\dfrac{1}{\overline{a}}} = \alpha\frac{z-a}{1-\overline{a}z} \qquad (\alpha = -\overline{a}\alpha_1)$$

ko'rinishda bo'ladi.

Endi $|z| = 1$ bo'lganda $|w| = 1$ bo'lishidan foydalanib quyidagilarni topamiz:

$$|w| = |\alpha|\cdot\frac{|z-a|}{|1-\overline{a}z|},$$

$$1 = |\alpha|\cdot\frac{|z-a|}{\left|z(\overline{\dfrac{1}{z}-a})\right|} = |\alpha|\cdot\frac{|z-a|}{|\overline{z}-\overline{a}|}$$

(chunki $z = e^{i\varphi} \Rightarrow \dfrac{1}{z} = e^{-i\varphi} = \overline{z}$). Ravshanki, $|z-a| = |\overline{z}-\overline{a}|$.

Demak, $|\alpha| = 1$, ya'ni $\alpha = e^{i\varphi}$ bo'ladi.

Shunday qilib C_z tekislikdagi birlik doirani C_w tekislikdagi birlik doiraga akslantiruvchi kasr chiziqli funksiya ushbu

$$w = e^{i\varphi}\frac{z-a}{1-\overline{a}z}$$

ko'rinishda bo'ladi.

2.3-Misol. Kompleks tekislik C_z dagi yuqori yarim tekislik $\{z \in C_z : \operatorname{Im} z > 0\}$ ni C_w tekislikdagi yuqori yarim tekislik $\{w \in C_w : \operatorname{Im} w > 0\}$ga akslantiruvchi kasr-chiziqli funksiya ushbu

$$w = \frac{az+b}{cz+d}, \quad (a,b,c,d \in R, ad-bc > 0)$$

ko'rinishda bo'ladi.

Yechilishi. Faraz qilaylik kasr-chiziqli funksiya $\{\operatorname{Im} z > 0\}$ yarim tekislikni $\{\operatorname{Im} w > 0\}$ yarim tekislikka akslantirsin. z_1, z_2 va z_3 lar $\{\operatorname{Im} z = 0\}$ chiziqdagi xar hil sonlar bo'lsin, ya'ni ular xaqiqiy sonlar. Ularning akslari $w(z_1), w(z_2)$ va $w(z_3)$ lar ham xaqiqiy sonlar bo'ladi. Chunki ular ham $\{\operatorname{Im} z = 0\}$ chiziqda yotadi. U holda angarmonik nisbat (2.14) dan $w$ga nisbatan yechib $w = \frac{az+b}{cz+d}$ ni hosil qilishimiz mumkin. Agar $\{\operatorname{Im} z = 0\}$ chiziq $\{\operatorname{Im} z = 0\}$ chiziqqa akslantirsak va burilish burchagi o'zgarmasi ($\forall x \in R$ uchun $\arg w'(x) = 0$), u holda $\{\operatorname{Im} z > 0\}$ yarim tekislik $\{\operatorname{Im} w > 0\}$ yarim tekislikka konform akslanadi. $w'(x) = \frac{ad-bc}{(cx+d)^2} > 0$ ekanligidan $ad-bc > 0$ ekanligini kelib chiqadi.

2.4-Misol. Kompleks tekislik C_z dagi birlik doira $\{z \in C_z : |z| < 1\}$ ni C_w tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga $w(z_0) = w_0, \arg w'(z_0) = \alpha$ $(|w_0| < 1)$ shartlarni qanoatlantirib akslantiruvchi kasr-chiziqli funksiya ushbu

$$\frac{w-w_0}{1-\overline{w}_0 w} = \frac{z-z_0}{1-\overline{z}_0 z}e^{i\alpha}$$

ko'rinishda bo'ladi.

Yechilishi. $\zeta = g(z) = \frac{z - z_0}{1 - \bar{z}_0 z} e^{i\alpha}$ funksiya C_z dagi birlik doira $\{z \in C_z : |z| < 1\}$ ni C_ζ tekislikdagi birlik doira $(|\zeta| < 1)$ga akslantiradi va $g(z_0) = 0$, $\arg g'(z_0) = \alpha$ shartni qanoatlantiradi. Huddi shuningdek $\zeta = h(w) = \frac{w - w_0}{1 - \overline{w}_0 w}$ funksiya C_w tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ni C_ζ tekislikdagi birlik doira $(|\zeta| < 1)$ga akslantiradi va $h(w_0) = 0, \arg h'(w_0) = 0$ shartni qanoatlantiradi. Bundan kelib chiqadiki $\frac{w - w_0}{1 - \overline{w_0} w} = \frac{z - z_0}{1 - \bar{z}_0 z} e^{i\alpha}$ funksiya C_z dagi birlik doira $\{z \in C_z : |z| < 1\}$ ni C_w tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga $w(z_0) = w_0$, $\arg w'(z_0) = \alpha$ $(|w_0| < 1)$ shartlarni qanoatlantirib akslantiradi.

Eslatma 1. $w = e^{i\varphi} \frac{z - a}{1 - \overline{a} z}$ funksiyaning $z = z_0$ nuqtadagi burchak koeffisienti $\arg w'(z_0) = \arg \frac{1}{1 - |z_0|^2} e^{i\alpha} = \alpha$ ga teng bo'ladi.

2.5-Misol. Yuqori yarim tekislik $\{z \in C_z : \operatorname{Im} z > 0\}$ ni C_w tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga $w(z_0) = w_0, \arg w'(z_0) = \alpha$ $(\operatorname{Im} z_0 > 0, |w_0| < 1, \alpha \in R)$ shartlarni qanoatlantirib akslantiruvchi kasr-chiziqli funksiya ushbu

$$i \frac{w - w_0}{1 - \overline{w_0} w} = \frac{z - z_0}{z - \bar{z}_0} e^{i\alpha}$$

ko'rinishda bo'ladi.

Yechilishi. $\zeta = g(z) = \frac{z - z_0}{z - \bar{z}_0} e^{i\alpha}$ funksiya C_z dagi yarim tekislik $\{z \in C_z : \operatorname{Im} z > 0\}$ni C_ζ tekislikdagi birlik doira $(|\zeta| < 1)$ga akslantiradi va $g(z_0) = 0$, $\arg g'(z_0) = \alpha - \frac{\pi}{2}$ shartni qanoatlantiradi. Huddi shuningdek $\zeta = h(w) = i \frac{w - w_0}{1 - \overline{w}_0 w}$ funksiya C_w tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ni C_ζ

tekislikdagi birlik doira $(|\zeta|<1)$ ga akslantiradi va $h(w_0)=0, \arg h'(w_0)=\frac{\pi}{2}$ shartni qanoatlantiradi. Bundan kelib chiqadiki $i\frac{w-w_0}{1-\overline{w}_0 w}=\frac{z-z_0}{z-\overline{z}_0}e^{i\alpha}$ funksiya yuqori yarim tekislik $\{z\in C_z : \operatorname{Im} z>0\}$ ni C_w tekislikdagi birlik doira $\{w\in C_w : |w|<1\}$ ga $w(z_0)=w_0, \arg w'(z_0)=\alpha$ $(\operatorname{Im} z_0>0, |w_0|<1, \alpha\in R)$ shartlarni qanoatlantirib akslantiradi.

Eslatma 2. $w=e^{i\varphi}\frac{z-a}{z-\overline{a}}$ $(\varphi\in R; \operatorname{Im} a>0)$ funksiyaning $z=z_0$ nuqtadagi burchak koeffisienti $\arg w'(z_0)=\arg\frac{z_0-\overline{z}_0}{(z_0-\overline{z})^2}e^{i\varphi}=\arg\left(-i\frac{1}{\operatorname{Im} z_0}e^{i\varphi}\right)=\frac{\pi}{2}-\varphi$ ga teng bo'ladi.

2.6-Misol. Yuqori yarim tekislik $\{z\in C_z : \operatorname{Im} z>0\}$ ni C_w tekislikdagi yuqori yarim tekislik $\{w\in C_w : \operatorname{Im} w>0\}$ ga $w(z_0)=w_0, \arg w'(z_0)=\alpha$ shartlarni qanoatlantirib akslantiruvchi kasr-chiziqli funksiya ushbu

$$\frac{w-w_0}{w-\overline{w}_0}=\frac{z-z_0}{z-\overline{z}_0}e^{i\alpha}$$

ko'rinishda bo'ladi.

Yechilishi. $\zeta=g(z)=\frac{z-z_0}{z-\overline{z}_0}e^{i\alpha}$ funksiya C_z dagi yarim tekislik $\{z\in C_z : \operatorname{Im} z>0\}$ ni C_ζ tekislikdagi birlik doira $(|\zeta|<1)$ga akslantiradi va $g(z_0)=0,\quad \arg g'(z_0)=\alpha-\frac{\pi}{2}$ shartni qanoatlantiradi. Huddi shuningdek $\zeta=h(w)=\frac{w-w_0}{w-\overline{w}_0}$ funksiya $C_w$dagi yuqori yarim tekislik $\{w\in C_w : \operatorname{Im} w>0\}$ni C_ζ tekislikdagi birlik doira $(|\zeta|<1)$ga akslantiradi va $h(w_0)=0, \arg h'(w_0)=-\frac{\pi}{2}$ shartni qanoatlantiradi. Bundan kelib chiqadiki $\frac{w-w_0}{w-\overline{w}_0}=\frac{z-z_0}{z-\overline{z}_0}e^{i\alpha}$ funksiya yuqori yarim tekislik $\{z\in C_z : \operatorname{Im} z>0\}$ ni C_w tekislikdagi yuqori yarim tekislik

$\{w \in C_w : \operatorname{Im} w > 0\}$ ga $w(z_0) = w_0, \arg w'(z_0) = \alpha$ $\left(\operatorname{Im} z_0 > 0, |w_0| < 1, \alpha \in R\right)$ shartlarni qanoatlantirib akslantiradi.

2.7-Misol. C_z tekislikdagi 1, 1, -1 nuqtalarni mos ravishda C_w tekislikdagi -1,0,1 nuqtalarga akslantiruvchi kasr-chiziqli funksiyani toping.

Yechilishi. Teorema-2.2 da keltirilgan

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

tenglikda

$$z_1 = 1, \quad z_2 = i, \quad z_3 = -1 \quad w_1 = -1, \quad w_2 = 0, \quad w_3 = 1$$

deb topamiz:

$$\frac{w - (-1)}{w - 0} \cdot \frac{1 - 0}{1 - (-1)} = \frac{z - 1}{z - i} \cdot \frac{-1 - i}{-1 - 1} \Rightarrow w = \frac{z - i}{zi - 1}$$

Demak, izlanayotgan kasr - chiziqli funksiya

$$w = \frac{z - i}{zi - 1}$$

bo'ladi.

2.8-Misol. Kompleks tekislik C_z da $z_1 = 1 + i$ nuqta uchun ushbu $\{z \in C_z : |z| = 1\}$ aylanaga nisbatan simmetrik bo'lgshan nuqtani toping.

Yechilishi. Izlanayotgan nuqta z_1^* deylik bu nuqtani topishda

$$z_1^* - z_0 = \frac{r^2}{z_1 - z_0}$$

formuladan foydalanamiz. $z_0 = 0, \quad r - 1$ ekanligini e'tiborga olib,

$z_1^* = \dfrac{1}{\overline{z_1}}$ bo'lishini topamiz. Demak, $z_1^* = \dfrac{1}{\overline{z_1}} = \dfrac{1}{\overline{1+i}} = \dfrac{1}{1-i} = \dfrac{1}{2} + \dfrac{1}{2}i$ ekan.

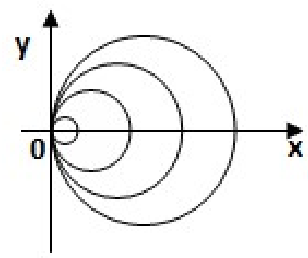
2.9-Misol. $x = 0$ to'g'ri chiziqni $w = \dfrac{1}{z}$ akslantirish yordamidagi aksini toping.

Yechilishi. Kasr-chiziqli funksiyaning doiraviylik xossasiga ko'ra bu to'g'ri chiziqning aksi $\overline{C_w}$ tekislikda aylana yoki to'g'ri chiziq bo'ladi. Ya'ni maxrajni nolga aylantiruvchi $z_0 = 0$ soni $x = 0$ chiziqda yotganligi uchun bu chiziqning aksi to'g'ri chiziqdan iborat bo'ladi. Ma'lumki, to'g'ri chiziq 2 ta nuqta orqali xarakterlanadi. Demak, $x = 0$ to'g'ri chiziqni akasini topish uchun unga tegishli bo'lgan 2 ta nuqtani aksini topsak yetarli ekan. $z_1 = i$ va $z_2 = -i$ bo'lsin.

U holda $w(z_1) = -i$ va $w(z_2) = i$ bo'ladi. Bu nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi esa $\operatorname{Re} w = 0$ dir.

2.10-Misol. $x^2 + y^2 < cx$ doiralar oilasini $w = \frac{1}{z}$ akslantirish yordamidagi aksini toping.

Yechilishi. Biz bu misolni chegaralarni akslantirish yo'lidan hal etamiz. $x^2 + y^2 < cx$ doiralar oilasini chegaralari aylanalardan iborat. Kasr-chiziqli funksiyaning doiraviylik xossasiga ko'ra bu aylanalarning aksi $\overline{C}_w$ tekislikda aylanalar yoki to'g'ri chiziqlar bo'ladi. $x^2 + y^2 < cx$ ifodani $\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$ ko'rinishida yozib olamiz.



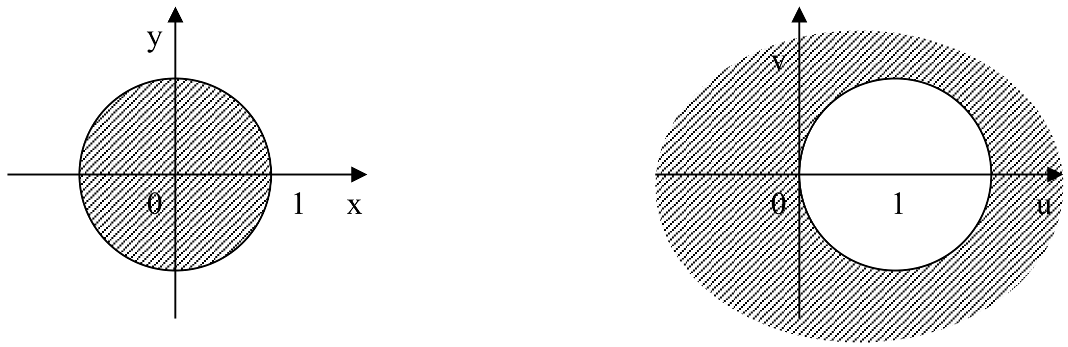
Maxrajni nolga aylantiruvchi $z_0 = 0$ soni (2.4-chizma).

$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$ aylanada yotganligi uchun bu chiziqning aksi to'g'ri chiziqdan iborat bo'ladi Yana shuningdek $z_1 = c$ va $z_2 = \frac{c}{2} + i\frac{c}{2}$ nuqtalar ham aylanada yotadi. $w(z_1) = \frac{1}{c}$ va $w(z_2) = \frac{1}{c} - i\frac{1}{c}$ ekanligini hisoblab, bu $w(z_1)$ va $w(z_2)$ nuqtalardam o'tuvchi to'g'ri chiziq tenglamasini $\operatorname{Re} w = \frac{1}{c}$ ekanligini topamiz. Hosil bo'lgan $\operatorname{Re} w = \frac{1}{c}$ bu $\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$ ayalanalar oilasini aksi bo'ladi (2.4 – chizma). $x^2 + y^2 < cx$ ni aksi esa $\operatorname{Re} w = \frac{1}{c}$ ning yuqori yoki quyi qismi. Buni aniqlash uchun aylana ichida yotgan ixtiyoriy nuqtani akslantirishga qo'yamiz.

$z_3 = \dfrac{c}{2}$ nuqtani olaylik $w(z_3) = \dfrac{2}{c}$ bo'ladi. Bundan $x^2 + y^2 < cx$ ning aksi $\operatorname{Re} w > \dfrac{1}{c}$ ekanligi kelib chiqadi.

2.11-Misol. $D = \{|z| < 1\}$ sohaning $w = \dfrac{z+1}{z}$ akslantirish yordamidagi aksini toping.

Yechilishi. Bu misolni teskari akslantirish qurish yo'li bilan hal etamiz. Dastlab, $w = \dfrac{z+1}{z}$ ga teskari funksiya quramiz: $z = \dfrac{1}{w-1}$. $w = u + iv$ deylik. U holda $z = \dfrac{1}{u-1+iv} = \dfrac{u-1}{(u-1)^2 + v^2} - i\dfrac{v}{(u-1)^2 + v^2}$ bo'ladi. $D = \{|z| < 1\}$ sohaga olib kelsak $\left(\dfrac{u-1}{(u-1)^2 + v^2}\right)^2 + \left(\dfrac{v}{(u-1)^2 + v^2}\right)^2 < 1$ bo'lib, $(u-1)^2 + v^2 > 1$ tengsizlikka ega bo'lamiz. Bu esa $\overline{C}_w$ tekislikda aylananing tashqi qismini bildiradi. (2.5 - chizma)



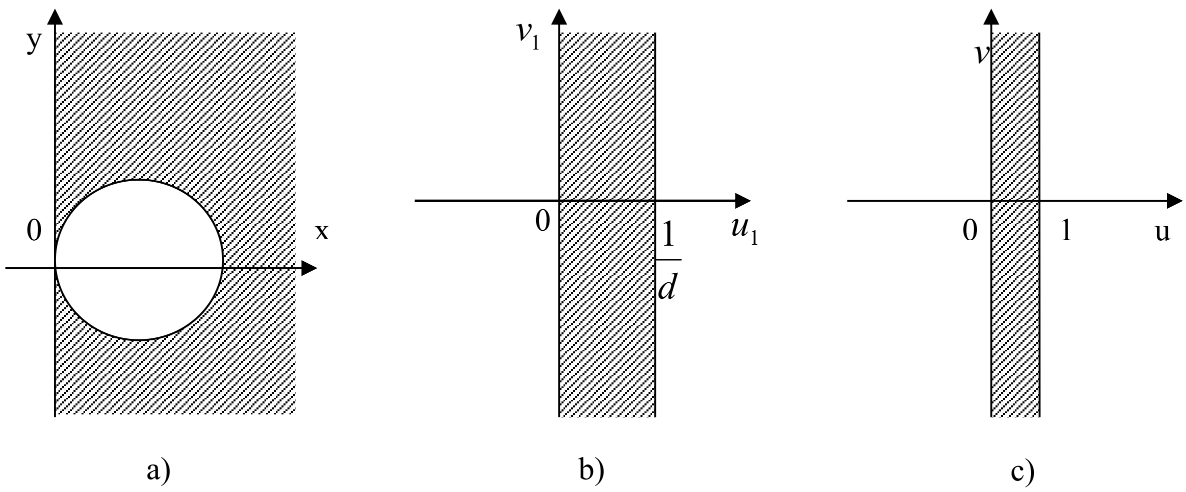
2.5- chizma

2.12-Misol. Kasr-chiziqli akslantirish yordamida quyidagi $D = \{z \in C : \operatorname{Re} z > 0, \left|z - \dfrac{d}{2}\right| > \dfrac{d}{2}\}$ sohani $G = \{w \in C : 0 < \operatorname{Re} w < 1\}$ sohaga akslantiruvchi akslantirish toping.

Yechilishi. D sohaning chegarasi 2 ta chiziqdan iborat. Ularni mos ravishda $l_1 : \operatorname{Re} z = 0$ va $l_2 : \left|z - \dfrac{d}{2}\right| = \dfrac{d}{2}$ (2.6 a-chizma). l_1 va l_2 chiziqlar keshishgan nuqtani aniqlaymiz. $z_0 = 0$ nuqta har 2 chegarada ham yotadi. $z_0 = 0$ nuqtani kasr-chiziqli akslantirishni maxrajiga qo'yamiz, chunki har 2 chegara ham faqat shundagina to'g'ri chiziqqa akslanadi. Ya'ni akslantirishni $w_1 = \frac{1}{z}$ ko'rinishida tanlaymiz. U holda l_1 va l_2 chiziqlar mos ravishda $u_1 = 0$ va $u_1 = \frac{1}{d}$ chiziqlarga o'tadi (2.6 b-chizma). Hosil bo'lgan sohani esa o'xshashlik akslantirishi yordamida $G = \{w \in C : 0 < \operatorname{Re} w < 1\}$ ga olib o'tamiz: $w_2 = dw_1$ (2.6 c-chizma). Ikkinchi tomondan esa birlik yo'lakni $b = (0, b)$ vektor bo'yicha parallel ko'chirsak yana birlik yo'lak paydo bo'ladi. Demak, izlanayotgan kasr chiziqli funksiyaning umumiy ko'rinishi

$$w = \frac{d}{z} + ib, \; b \in R$$

ekan (2.6 - chizma).



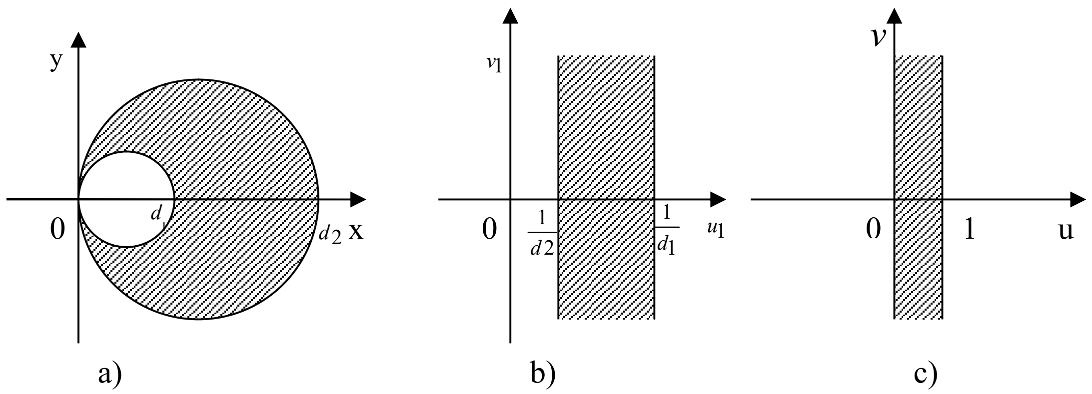
2.6- chizma

2.13-Misol. Kasr-chiziqli akslantirish yordamida quyidagi $D = \{z \in C : \left|z - \frac{d_1}{2}\right| > \frac{d_1}{2}, \left|z - \frac{d_2}{2}\right| < \frac{d_2}{2}, 0 < d_1 < d_2\}$ aylanalar orasidagi sohani $G = \{w \in C : 0 < \operatorname{Re} w < 1\}$ sohaga akslantiruvchi akslantirish toping.

Yechilishi. D sohaning chegarasi 2 ta chiziqdan iborat. Ularni mos ravishda $l_1 : \left|z - \frac{d_1}{2}\right| = \frac{d_1}{2}$ va $l_2 : \left|z - \frac{d_2}{2}\right| = \frac{d_2}{2}$ deylik. l_1 va l_2 chiziqlar keshishgan nuqtani aniqlaymiz. $z_0 = 0$ nuqta har 2 chegarada ham yotadi. $z_0 = 0$ nuqtani kasr-chiziqli akslantirishni maxrajiga qo'yamiz, chunki har 2 chegara ham faqat shundagina to'g'ri chiziqqa akslanadi. Ya'ni akslantirishni $w_1 = \frac{1}{z}$ ko'rinishida tanlaymiz. U holda l_1 va l_2 chiziqlar mos ravishda $u_1 = \frac{1}{d_2}$ va $u_1 = \frac{1}{d_1}$ chiziqlarga o'tadi (2.7 b-chizma). Hosil bo'lgan chiziqlarni parallel ko'chirish $w_2 = w_1 - \frac{1}{d_2}$ yordamida $u_2 = 0$ va $u_2 = \frac{1}{d_1} - \frac{1}{d_2}$ chiziqlarga olib o'tamiz. Keyin esa o'xshashlik akslantirishi $w_3 = \frac{d_2 d_1}{d_2 - d_1} w_2$ yordamida $u = 0$ va $u = 1$ chiziqlarga akslantiramiz (2.7 c-chizma). Soha esa chiziqlar orasi bo'ladi. Birlik yo'lakni $b = (0, b)$ vektor bo'yicha parallel ko'chirsak yana birlik yo'lak paydo bo'ladi. Demak, izlanayotgan kasr-chiziqli funksiyaning umumiy ko'rinishi

$$w = \frac{d_1}{d_2 - d_1}\left(\frac{d_2}{z} - 1\right) + ib, b \in R$$

ekan (2.7 - chizma).



2.7 - chizma

2.14-Misol. Kasr-chiziqli akslantirish yordamida quyidagi $D = \{z \notin C : \left|z + \frac{d_1}{2}\right| \le \frac{d_1}{2}, \left|z - \frac{d_2}{2}\right| \le \frac{d_2}{2}\}$ aylanalar orasidagi sohani $w(d_2) = 0$

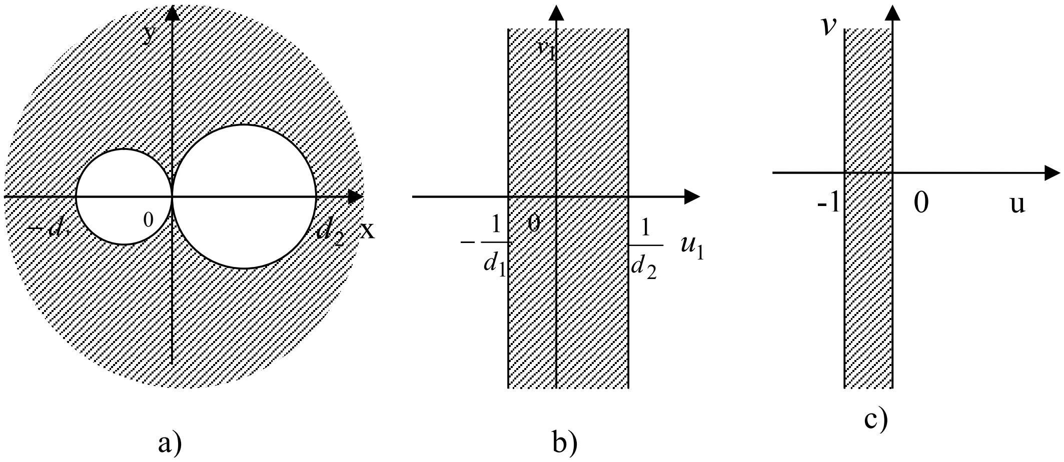
shartni qanoatlantiradigan qilib $G=\{w\in C:0<\operatorname{Re}w<1\}$ sohaga akslantiruvchi akslantirish toping.

Yechilishi. D sohaning chegarasi 2 ta chiziqdan iborat. Ularni mos ravishda $l_1:\left|z+\frac{d_1}{2}\right|=\frac{d_1}{2}$ va $l_2:\left|z-\frac{d_2}{2}\right|=\frac{d_2}{2}$ deylik. l_1 va l_2 chiziqlar keshishgan nuqtani topamiz (2.8 a-chizma). $z_0=0$ nuqta har 2 ta chiziqda ham yotadi. $z_0=0$ nuqtani kasr-chiziqli akslantirishni maxrajiga qo'yamiz, chunki har 2 chegara ham faqat shundagina to'g'ri chiziqqa akslanadi. Ya'ni akslantirishni $w_1=\frac{1}{z}$ ko'rinishida tanlaymiz. U holda l_1 va l_2 chiziqlar mos ravishda $u_1=-\frac{1}{d_1}$ va $u_1=\frac{1}{d_2}$ chiziqlarga o'tadi va sohani aniqlaymiz (2.8 b-chizma). Hosil bo'lgan sohani parallel ko'chirish $w_2=w_1-\frac{1}{d_2}$ yordamida $G_2=\{w\in C:-\frac{1}{d_1}-\frac{1}{d_2}<\operatorname{Re}w<0\}$ sohaga olib o'tamiz. Keyin esa o'xshashlik akslantirishi $w_3=\frac{d_2d_1}{d_2+d_1}w_2$ yordamida $G_3=\{w\in C:-1<\operatorname{Re}w<0\}$ (2.8 c-chizma) sohaga o'tadi. π burchakka burib $w_4=-w_3$ yordamida $G=\{w\in C:0<\operatorname{Re}w<1\}$ sohani xosil qilamiz. Demak, izlanayotgan kasr-chiziqli funksiyaning umumiy ko'rinishi

$$w=\frac{d_1}{z}\left(\frac{z-d_2}{d_1+d_2}\right)$$

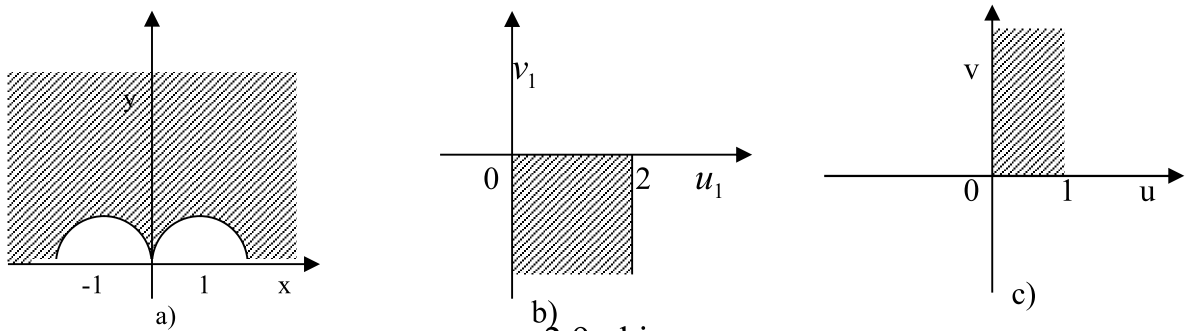
ekan (2.8 - chizma).

2.15-Misol. Kasr-chiziqli akslantirish yordamida quyidagi $D=\{z\in C:\left|z+1\right|>1,\left|z-1\right|>1,\operatorname{Im}z>0\}$ aylanalar aniqlagan sohani $G=\{w\in C:0<\operatorname{Re}w<1,\operatorname{Im}w>0\}$ sohaga akslantiruvchi akslantirish toping.



2.8 – chizma.

Yechilishi. D sohaning chegarasi 2 ta yarim aylana va to'g'ri chiziqdan iborat. Biz mos ravishda $|z+1|=1$, $|z-1|=1$ va $\operatorname{Im} z=0$ (2.9 a-chizma) tengliklarni birgalikda yechib olamiz. $z_0=0$ nuqta har 3 chegarada ham yotadi. $z_0=0$ nuqtani kasr-chiziqli akslantirishni maxrajiga qo'yamiz, chunki har bir chegara ham faqat shundagina to'g'ri chiziqqa akslanadi. Ya'ni akslantirishni $w_1=\frac{z+2}{z}$ ko'rinishida tanlaymiz. U holda $|z+1|=1$, $|z-1|=1$ va $\operatorname{Im} z=0$ chiziqlar mos ravishda $u_1=0$, $u_1=2$ va $v_1=0$ga akslanadi (2.9 b-chizma). O'xshashlik akslantirishi $w_2=\frac{w_1}{2}$ yordamida $G_2=\{w\in C: 0<\operatorname{Re} w<1, \operatorname{Im} w<0\}$ sohaga o'tadi. π burchakka burib $w_3=-(w_2-1)$ yordamida $G=\{w\in C: 0<\operatorname{Re} w<1, \operatorname{Im} w>0\}$ sohani xosil qilamiz. Demak, izlanayotgan kasr-chiziqli funksiyaning umumiy ko'rinishi $w=-\frac{z+2}{2z}$ ekan.



2.9-chizma.

2.16-Misol. $D=\{z\in C_z : 1<|z|<2\}$ sohani $w=\dfrac{2}{z-1}$ akslantirish yordamidagi aksini toping.

Yechilishi. D sohaning chegarasi 2 ta aylanadan iborat. Ularni mos ravishda $l_1 : |z|=1$ va $l_2 : |z|=2$ deylik. l_1 aylananing aksi kasr-chiziqli akslantirishning doiraviylik xossasiga ko'ra to'g'ri chiziqdan iborat bo'ladi. Chunki $z_0=1$ nuqta l_1 chiziqda yotadi. $l_1 : |z|=1$ chiziqqa tegishli 2 ta nuqta olaylik: $z_1=i$ va $z_2=-i$ bo'lsin. Bu nuqtalarning akslarini qaraylik: $w(i)=-1-i$ va $w(-i)=-1+i$ bo'ladi. Bu nuqtalarda o'tuvchi to'g'ri chiziq tenglamasi esa $\operatorname{Re} z=-1$. Ya'ni $l_1 : |z|=1$ chiziqni $w=\dfrac{2}{z-1}$ akslantirishdagi aksi $\operatorname{Re} z=-1$ chiziqdan iborat ekan. Endi $l_2 : |z|=2$ qaraylik. l_2 aylananing aksi kasr-chiziqli akslantirishning doiraviylik xossasiga aylanadan iborat bo'ladi. $w=\dfrac{2}{z-1}$ va $|z|=2$ dan $\left|\dfrac{2+w}{w}\right|=2$ ni hosil qilamiz. Bu yerdan $w_1=2$ va $w_2=0$ nuqtalar bu aylanaga nisbatan simmetrik nuqta ekanligi kelib chiqadi. Ma'lumki, $z_1$va z_2 aylanaga nisbatan simmetrik nuqtalar bo'lsa $\left|\dfrac{z-z_1}{z-z_2}\right|=k$ u holda aylana markazi va radiusi $z_0=\dfrac{z_1-z_2k^2}{1-k^2}$, $R=\dfrac{k}{|1-k^2|}|z_1-z_2|$ formula orqali topiladi. Agar biz bularni qo'llasak: $w_0=\dfrac{-2-0\cdot 4}{1-4}=\dfrac{2}{3}$, $R=2\left|\dfrac{-2-0}{1-4}\right|=\dfrac{4}{3}$ hosil qilamiz. U holda l_2 aylananing aksi $\left|w-\dfrac{2}{3}\right|=\dfrac{4}{3}$ aylanadan iborat ekan. l_1 va l_2 aylanalarni akslaridan foydalanib $D=\{z\in C_z : 1<|z|<2\}$ sohaning aksini $w(D)=\left\{w\in C_w : \left|w-\dfrac{2}{3}\right|>\dfrac{4}{3}, \operatorname{Re} w>-1\right\}$ ekanligini topamiz.

Eslatma 3. z_1 va z_2 $\left|\frac{z-z_1}{z-z_2}\right|=k$ aylanaga nisbatan simmetrik nuqtalar bo'lsa, u holda aylana markazi va radiusi $z_0=\frac{z_1-z_2k^2}{1-k^2}$, $R=\frac{k}{|1-k^2|}|z_1-z_2|$ formula orqali topiladi.

2.17-Misol. $D=\{z\in C_z:\operatorname{Re}z>1\}$ sohani $w=\frac{z}{z-3}$ akslantirish yordamidagi aksini toping.

Yechilishi. D sohaning chegarasi $l:\operatorname{Re}z=1$ to'g'ri chiziqdan iborat. l chiziqning $w=\frac{z}{z-3}$ yordamidagi aksi kasr-chiziqli akslantirishning doiraviylik xossasiga aylanadan iborat bo'ladi. $l:\operatorname{Re}z=1$ chiziqni $\left|\frac{z}{z-2}\right|=1$ ko'rinishida yozsak, $z_1=2$ va $z_2=0$ nuqtalar bu aylanaga nisbatan simmetrik nuqta ekanligi kelib chiqadi. $w=\frac{z}{z-3}$ ekanligidan $z=\frac{3w}{w-1}$ teskari akslantirish quramiz va $\left|\frac{z}{z-2}\right|=1$ tenglikka qo'ysak $\left|\frac{w}{w+2}\right|=\frac{1}{3}$ hosil bo'ladi. $z_1=2$ va $z_2=0$ nuqta va $z_0=\frac{z_1-z_2k^2}{1-k^2}=\frac{1}{4}$, $R=\frac{k}{|1-k^2|}|z_1-z_2|=\frac{3}{4}$ ekanligini topamiz. U holda $l:\operatorname{Re}z=1$ chiziqning aksi $\left|w-\frac{1}{4}\right|=\frac{3}{4}$ bo'ladi. $D=\{z\in C_z:\operatorname{Re}z>1\}$ sohadan nuqta olib bu sohaning aksi $\left|w-\frac{1}{4}\right|>\frac{3}{4}$ ekanligini hosil qilamiz.

2.18-Misol. $D=\{z\in C_z:1>\operatorname{Re}z>0\}$ sohani $w=\frac{z-1}{z-2}$ akslantirish yordamidagi aksini toping.

Yechilishi. D sohaning chegarasi 2 ta to'g'ri chiziqdan iborat. Ularni mos ravishda $l_1:\operatorname{Re}z=0$ va $l_2:\operatorname{Re}z=1$ deylik. l_1 va l_2 chiziqlarning aksi kasr-chiziqli akslantirishning doiraviylik xossasiga aylanadan iborat bo'ladi. $l_1:\operatorname{Re}z=0$

chiziqni $\left|\frac{z+1}{z-1}\right|=1$ ko'rinishida yozsak, $z_1=1$ va $z_2=-1$ nuqtalar bu aylanaga nisbatan simmetrik nuqta ekanligi kelib chiqadi. $w=\frac{z-1}{z-2}$ ekanligidan $z=\frac{2w-1}{w-1}$ teskari akslantirish quramiz va $\left|\frac{z+1}{z-1}\right|=1$ tenglikka qo'ysak $\left|\frac{3w-2}{w}\right|=1$ hosil bo'ladi. $z_1=1$ va $z_2=-1$ nuqta va $z_0=\frac{z_1-z_2k^2}{1-k^2}=\frac{3}{4}$, $R=\frac{k}{\left|1-k^2\right|}\left|z_1-z_2\right|=\frac{1}{4}$ ekanligini topamiz. U holda $l_1:\operatorname{Re}z=0$ aksi $\left|w-\frac{3}{4}\right|=\frac{1}{4}$ bo'ladi. $l_2:\operatorname{Re}z=1$ chiziqni $\left|\frac{2-z}{z}\right|=1$ ko'rinishida yozsak, $z_1=2$ va $z_2=0$ nuqtalar bu aylanaga nisbatan simmetrik nuqta ekanligi kelib chiqadi. $w=\frac{z-1}{z-2}$ ekanligidan $z=\frac{2w-1}{w-1}$ teskari akslantirish quramiz va $\left|\frac{2-z}{z}\right|=1$ tenglikka qo'ysak $\left|\frac{1}{2w-1}\right|=1$ hosil bo'ladi. U holda $l_2:\operatorname{Re}z=1$ aksi $\left|w-\frac{1}{2}\right|=\frac{1}{2}$ bo'ladi. l_1 va l_2 chiziqlarni akslaridan foydalanib $D=\{z\in C_z:1>\operatorname{Re}z>0\}$ sohaning aksini $w(D)=\left\{w\in C_w:\left|w-\frac{1}{2}\right|>\frac{1}{2},\left|w-\frac{3}{4}\right|>\frac{1}{4}\right\}$ ekanligini topamiz.

2.19-Misol. $x+y=1$ to'g'ri chiziqni $w=\frac{1}{z}$ akslantirish yordamidagi aksini toping.

Yechilishi. Kasr-chiziqli funksiyaning doiraviylik xossasiga ko'ra bu to'g'ri chiziqning aksi $\overline{C}_w$ tekislikda aylana bo'ladi. $z=x+iy$ va $w=u+iv$ deylik. U holda $x+iy=\frac{1}{u+iv}=\frac{u-iv}{u^2+v^2}$ bo'ladi. Ya'ni $x=\frac{u}{u^2+v^2}$, $y=\frac{-v}{u^2+v^2}$. $x+y=1$

ekanligidan $u^2+v^2-u+v=0$ni hosil qilamiz. Bundan esa $x+y=1$ chiziqning $w=\frac{1}{z}$ yordamidagi aksi $\left(u-\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\frac{1}{2}$ eanligini hosil qilamiz.

Ma'lumki, to'g'ri chiziq 2 ta nuqta orqali xarakterlanadi. Demak, $x=0$ to'g'ri chiziqni akasini topish uchun unga tegishli bo'lgan 2 ta nuqtani aksini topsak yetarli ekan. $z_1=i$ va $z_2=-i$ bo'lsin.

U holda $w(z_1)=-i$ va $w(z_2)=i$ bo'ladi. Bu nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi esa $\operatorname{Re} w=0$ dir.

2.20-Misol. $\{z\in C_z :|z|<1\}$ sohani $w=\frac{1}{z}$ akslantirish yordamidagi aksini toping.

Yechilishi . Kasr-chiziqli funksiyaning doiraviylik xossasiga ko'ra bu to'g'ri chiziqning aksi $\overline{C}_w$ tekislikda aylana bo'ladi. C_z tekislikdagi birlik aylana $|z|=1$ga tegishli 3ta nuqta olamiz. $z_1=1$, $z_2=-1$ va $z_3=i$. Bu nuqtalarning $w=\frac{1}{z}$ kasr-chiziqli akslantirish yordamidagi aksi $w(z_1)=1$, $w(z_2)=-1$va $w(z_3)=-i$ bo'ladi. Bu 3 nuqtadan o'tuvchi aylana tenglamasi esa $|w|=1$bo'ladi. Natijada $\{z\in C_z :|z|<1\}$ sohani $w=\frac{1}{z}$ akslantirish yordamidagi C_w tekislikdagi birlik doirani tashqi qismi$\{w\in C_w :|w|>1\}$ga akslantiradi.

Mustaqil yechish uchun mashqlar

1. Berilgan D sohaning kasr-chiziqli $w = f(z)$ akslantirish yordamida aksini toping.

a. $D = \{|z| > 1\}$, $w = \dfrac{z-1}{z+i}$. **b.** $D = \{x < 0, y < 0\}$, $w = \dfrac{1}{z}$.

c. $D = \{|z| < 1\}$, $w = \dfrac{z+i}{z+1}$.

2. Quyidagi shartlarni qanoatlantiruvchi kasr-chiziqli $w(z)$ akslantirishni toping.

a. $w(1) = 1$, $w(0) = -1$, $w(i) = i$.

b. $w(\frac{1}{2}) = \frac{1}{2}$, $w(2) = 2$, $w(\frac{5}{4} + \frac{3}{4}i) = \infty$.

c. $w(0) = 2$, $w(1+i) = 2+i$, $w(2i) = 0$.

3. D sohani G sohaga akslantiruvchi va quyidagi shartlarni qanoatlantiruvchi kasr-chiziqli $w(z)$ funksiyani toping.

a. $D = \{\operatorname{Im} z > 0\}$, $G = \{\operatorname{Im} w < 0\}$, $w(i) = i$, $\arg w'(i) = -\dfrac{\pi}{2}$.

b. $D = \{\operatorname{Im} z > 0\}$, $G = \{\operatorname{Im} w < 0\}$, $w(2i) = 2i$, $\arg w'(2i) = -\dfrac{\pi}{2}$.

c. $D = \{\operatorname{Im} z < 0\}$, $G = \{\operatorname{Im} w > 0\}$, $w(-i) = -i$, $\arg w'(-i) = \dfrac{\pi}{2}$.

3. DARAJALI FUNKSIYALAR VA ULAR YORDAMIDA BAJARILADIGAN AKSLANTIRISHLAR

Ushbu

$$w = z^n, \ \ n \in N \tag{3.1}$$

ko'rinishdagi *darajali funksiyani* o'rganamiz. Bu funksiya butun kompleks tekislik C_z da aniqlangan bo'lib, uning hosilasi $w' = n \cdot z^{n-1}$ ga teng.

Demak, (3.1) funksiya butun kompleks tekislik C_z da golomorf, $n > 1$ va $z \neq 0$ bo'lganda uning yordamida bajariladigan akslantirish $C \setminus \{0\}$ to'plamning har bir nuqtasida konform akslantirish bo'ladi.

C_z va C_w tekislikda qutb koordinatalarini kiritamiz:

$$z = re^{i\varphi}, \ \ (r = |z|, \varphi = \arg z) \quad \ w = \rho e^{i\psi}, \ (\rho = |w|, \psi = \arg w)$$

Natijada (3.1) akslantirish ushbu

$$\rho e^{i\psi} = r^n e^{i\varphi n}$$

ko'rinishga ega bo'ladi. Undan esa,

$$\rho = r^n, \ \ \psi = n\varphi$$

bo'lishi kelib chiqadi.

Demak, $w = z^n$ akslantirish qutb koordinatalar sistemasida ushbu

$$\begin{cases} \rho = r^n \\ \psi = n\varphi \end{cases} \tag{3.1*}$$

akslantirishga o'tadi. Binobarin, (3.1) akslantirishni o'rganish (3.1*) akslantirishni o'rganishga keladi.

(3.1*) akslantirishdan topamiz:

1) $r = const$ bo'lganda $\rho = const$ bo'ladi. Demak, (3.1) akslantirish C_z tekislikdagi markazi $z = 0$ nuqtada bo'lgan aylanalarni C_w tekislikdagi markazi $w = 0$ nuqtada bo'lgan aylanalarga akslantiradi.

2) $\varphi = const$ bo'lganda $\psi = const$ bo'ladi. Demak, (3.1) akslantirish C_z tekislikdagi $z = 0$ nuqtadan chiqqan nurlarni, C_w tekislikdagi $w = 0$ nuqtadan chiqqan nurlarga akslantiradi.

Ayni paytda (3.1) akslantirish $\varphi = 0$ nurni (haqiqiy musbat yo'nalish bo'yicha olingan nurni), $\psi = 0$ nurga, C_z tekislikdagi $\varphi = \alpha$ nurni esa, C_w tekislikdagi $\psi = n \cdot \alpha$ nurga akslantiradi.

Yuqorida keltirilgan tasdiqlardan $w = z^n$ akslantirish C_z tekislikdagi

$$D = \{z \in C_2 : 0 < \arg z < \alpha\} \ \left(\alpha < \frac{2\pi}{n}\right)$$

sohani (uchi $z = 0$ nuqtada bo'lgan burchakni-sektorni) C_w tekislikdagi

$$w(D) = \{w \in C_w : 0 < \arg w < n\alpha\}$$

sohaga (uchi $w = 0$ nuqtada bo'lgan burchakka-sektorga) akslantirishi kelib chiqadi.

Darajali funksiya yordamida bajariladigan akslantirishda $z = 0$ nuqtada burchak n marta oshganligi sababli $z = 0$ nuqtada $w = z^n$ akslantirish $(n > 1)$ konform bo'lmaydi.

Xususan, $w = z^n$ akslantirish yordamida C_z tekislikdagi

$$\left\{z \in C_z : 0 < \arg z < \frac{\pi}{n}\right\}$$

soha (burchak-sektor), C_w tekislikdagi

$$\{w \in C_w : 0 < \arg w < \pi\}$$

sohaga (yuqori yarim tekislikka) o'tadi.

Demak, $w = z^n$ funksiya $\left\{z \in C_z : 0 < \arg z < \frac{\pi}{n}\right\}$ sohani $\{w \in C_w : 0 < \arg w < \pi\}$ sohaga konform akslantiradi.

Endi C_z tekislikda ushbu

$$D = \left\{ z \in C_z : 0 < \arg z < \frac{2\pi}{n} \right\}$$

sohani (uchi $z = 0$ nuqtada, tomonlari $\arg z = 0, \arg z = \dfrac{2\pi}{n}$ nurlardan iborat burchakni - sektorni) olamiz. Ravshanki, $w = z^n$ funksiya yordamida bu soha C_w tekislikdagi

$$\{ w \in C_w : 0 < \arg w < 2\pi \}$$

sohaga akslanadi.

Butun C_z tekislik bilan o'zaro bir qiymatli moslik o'rnatish maqsadida $w = z^n$ funksiyaning bir varaqlilik sohasini topamiz. Buning uchun (z) tekislikda

$$z_1 = z_2, \quad z_1 = |z_1| e^{i\varphi_1}, \quad z_2 = |z_2| e^{i\varphi_2}$$

nuqtalarni olamiz. Ularning C_w tekislikdagi aksi

$$w_1 - w_2 = |z_1|^n \left(e^{in\varphi_1} - e^{in\varphi_2} \right) = 0 .$$

Bundan

$$\begin{cases} \cos n\varphi_1 - \cos n\varphi_2 = 0 \\ \sin n\varphi_1 - \sin n\varphi_2 = 0 \end{cases}$$

Birinchi tenglamadan

$$-2\sin \frac{n\varphi_1 + n\varphi_2}{2} \sin \frac{n\varphi_1 - n\varphi_2}{2} = 0 \Rightarrow$$

$$\Rightarrow \frac{n\varphi_1 - n\varphi_2}{2} = \pi k \Rightarrow \varphi_1 = \frac{2\pi k}{n} + \varphi_2$$

Demak, (3.1) akslantirish uchi $z = 0$ nuqtada bo'lgan har bir $\dfrac{2\pi k}{n}$ burchak ichida bir varaqli ekan. (z) tekislikni n ta sohaga bo'lamiz:

$$D_k = \left\{ z : \frac{2\pi k}{n} < \arg z < \frac{2(k+1)\pi}{n} \right\}, \quad k = \overline{0, n-1}.$$

(3.1) yordamida bajariladigan akslantirishda $\varphi=\arg z$ burchak n marta kattalashadi, ya'ni (3.1) akslantirish natijasida

$$\varphi \overset{(13.1)}{\rightarrow} n\varphi .$$

Endi, $w'=nz^{n-1}\neq 0,\ z\in D_k$ bo'lgani uchun D_k soha (3.1) akslantirish yordamida E sohaga konform akslanadi, bunda E soha $w\geq 0$ nurning nuqtalari kirmaydigan butun tekislik ($E=C\setminus\{w\geq 0\}-$kesilgan tekislik) va $\arg z=\dfrac{2\pi k}{n}$ nur kesilgan tekislikning yuqori, $\arg z=\dfrac{2(k+1)\pi}{n}$ nur esa quyi chegarasiga akslanadi.

$k=0$ da

$$0<\varphi<\frac{2\pi}{n}\overset{(13.1)}{\rightarrow} 0<\theta<2\pi,$$

ya'ni $\varphi=0\overset{(13.1)}{\rightarrow}\theta=0$, $\varphi=\dfrac{2\pi}{n}\overset{(13.1)}{\rightarrow}\theta=2\pi$, lekin $\theta=0$ va $\theta=2\pi$ lar Ou o'qning musbat tomonida yotadi, ya'ni o'zaro bir qiymatli akslantirish buziladi. Bundan qutulish uchun Ou o'qning musbat tomonini O nuqtadan boshlab «kesamiz».

Darajali funksiyaga teskari funksiya. Riman sirti.

(3.1) ga teskari funksiyani ko'raylik:

$$z=\sqrt[n]{w}. \tag{3.2}$$

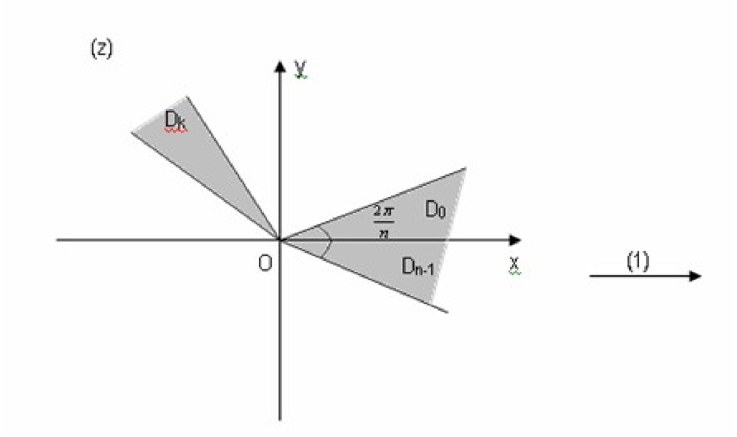
Bu funksiya ko'p qiymatli bo'lib, w ning bir qiymatiga z ning n ta qiymati mos keladi.

D_k sohada $z=\sqrt[n]{w}$ teskari funksiyani $z_k=\left(\sqrt[n]{w}\right)_k$ deb belgilaymiz.

$$z_k=\left(\sqrt[n]{w}\right)_k=\sqrt[n]{\rho}\left(\cos\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)\right),$$

bunda

$$\rho=|w|,\ \ \theta=\arg w,\ \ k=0,1,2,....,n-1.$$



3.1-chizma.

Har bir z ga alohida funksiya sifatida qarash maqsadga muvofiq bo'lmaydi, chunki, masalan,

$D: \frac{\pi}{n} < \arg z < \frac{2\pi}{n}$ sohada $z = w^{\frac{1}{n}}$ teskari funksiya $\frac{\pi}{n} < \arg z < \frac{2\pi}{n}$ da z_0 bilan, $\frac{2\pi}{n} < \arg z < \frac{3\pi}{n}$ da esa z_1 bilan ustma-ust tushadi. Shuning uchun $z_k,\ k = \overline{0, n-1}$ larni ko'p qiymatli $z = w^{\frac{1}{n}}$ funksiyaning tarmoqlari deyiladi. E sohada har bir z_k tarmoq analitikdir va

$$\frac{dz_k}{dw} = \frac{1}{nz^{n-1}} = \frac{1}{n} w^{\frac{1}{n}-1}.$$

(3.1) akslantirish vositasida D_0 soha $(w) \setminus \{w \geq 0\}$ ga akslanadi, ya'ni butun tekislikni qoplayapti. Qolgan $D_i,\ i = \overline{1, n-1}$ larga joy qolmadi. Bundan qutulish uchun, yani o'zaro bir qiymatli akslantirish hosil qilish uchun quyidagicha yo'l tutamiz. Har bir D_k soha yuqoridagi singari E_k tekislikka akslansin: $E_0, E_1, ..., E_{n-1}$. Natijada n ta tekislik hosil bo'ldi. Ularni ustma-ust qo'yamiz. Endi E_0 tekislikning quyi qirg'og'i bilan E_1 tekislikning yuqori qirg'og'ini, E_1 ning quyi qirg'og'ini E_2 ning yuqori qirg'og'i (bilan) va hokazo, E_{n-1} ning quyi qirg'og'i bilan E_0 ning yuqori qirg'og'ini yelimlaymiz (ya'ni yopishtiramiz). Hosil bo'lgan n varaqli soha $z = w^{\frac{1}{n}}$ funksiyaning Riman sirti deyiladi.

Xulosa qilib aytganda, (3.1) akslantirish kengaytirilgan $\overline{C_z}$ tekislik bilan $z = 0,\ \ z = \infty$ nuqtalardan boshqa nuqtalarda konform akslantirish bajaruvchi

$z = w^{\frac{1}{n}}$ funksiyaning Riman sirti o’rtasida o’zaro bir qiymatli akslantirish bajaradi. Bunda $z = 0$, $z = \infty$ nuqtalarning aksi $w = 0, w = \infty$ nuqtalar quyidagi xossaga ega: z nuqta $|z| = r = \sqrt[n]{\rho}$ aylana bo’ylab $z = 0$ nuqta atrofida bir marta aylanib chiqsa, w nuqta $w = 0$ nuqta atrofida $|w| = \rho$ aylana bo’ylab n marta aylanib chiqadi, ya’ni Riman sirtida w nuqta bir varaqdan ikkinchisiga, undan uchinchisiga va hokazo, so’nggi varaqqa o’tadi.

Xuddi shunga o’xshash, z nuqta $z = \infty$ nuqta atrofida bir marta aylanganda w nuqta $w = \infty$ nuqta atrofida n marta aylanadi. Natijada $z = w^{\frac{1}{n}}$ funksiyaning tarmoqlari paydo bo’ladi.

Mana shunday xossaga ega bo’lgan $w = 0, w = \infty$ nuqtalar n-tartibli tarmoqlanish nuqtalari deyiladi.

Eslatma-3.1. $z = w^{\frac{1}{n}}$ funksiyaning boshqa tarmoqlash nuqtalari yo’q.

Na'munaviy misol va masalalar yechimi

3.1-Misol. Ushbu $w = z^3$ darajali funksiya yordamida C_z tekislikdagi $E = \left\{ z \in C_z : 0 < \arg z = \frac{\pi}{4} \right\}$ to'plamning C_w tekislikdagi aksini toping.

Yechilishi. Berilgan E to'plamni

$$E = \left\{ z \in C_z : \arg z = \frac{\pi}{4} \right\} = \left\{ \varphi = \frac{\pi}{4}, 0 < r < +\infty \right\}$$

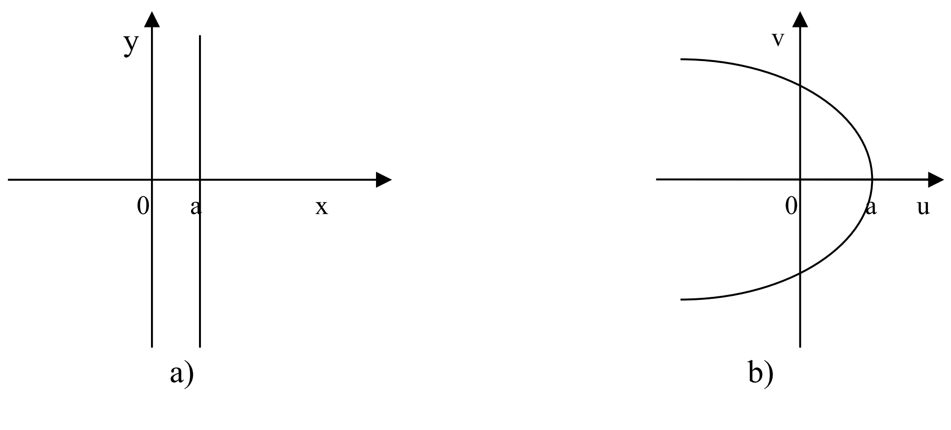
deb,

$$w(E) = \left\{ w \in C_w : \psi = 3 \cdot \frac{\pi}{4}, 0 < \rho < +\infty \right\} = \left\{ w \in C_w : \arg w = 3 \cdot \frac{\pi}{4} \right\}$$

bo'lishini topamiz.

3.2-Misol. $\operatorname{Re} z = a, \ (a > 0)$ chiziqni $w = z^2$ akslantirish yordamidagi aksini toping.

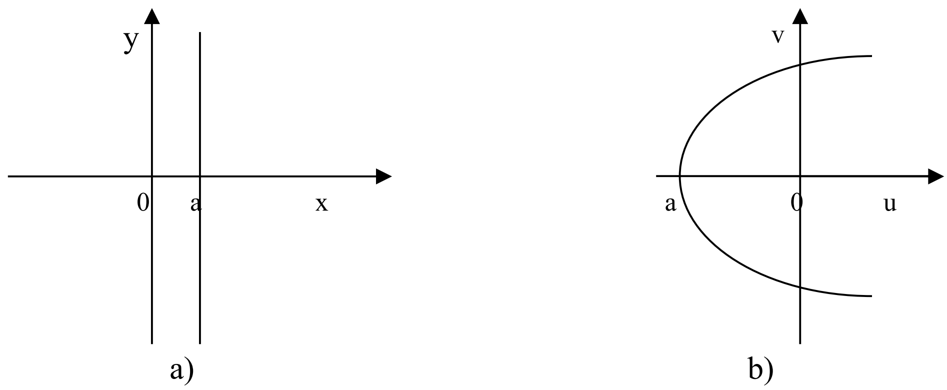
Yechilishi. $z = x + iy$ deylik. U holda $\operatorname{Re} z = a$ chiziqdan $x = a$ ekanligini topamiz. $w = u + iv$ desak $w = z^2$ ekanligidan $u = x^2 - y^2$ va $v = 2xy$ kelib chiqadi. $x = a$ ni v ga qo'ysak $y = \frac{v}{2a}$ va mos ravishda u ga qo'ysak $u = a^2 - \frac{v}{4a^2}^2$ hosil bo'ladi. Bu esa C_w tekislikdagi parabola aksini beradi (3.2-chizma). Demak, $w(E) = \left\{ z \in C : \operatorname{Re} w = a^2 - \frac{(\operatorname{Im} w)^2}{4a^2} \right\}$



3.2-chizma

3.3-Misol. $\operatorname{Im} z = a, \ (a > 0)$ chiziqni $w = z^2$ akslantirish yordamidagi aksini toping.

Yechilishi. $z = x + iy$ deylik. U holda $\operatorname{Im} z = a$ chiziqdan $y = a$ ekanligini topamiz. $w = u + iv$ desak $w = z^2$ ekanligidan $u = x^2 - y^2$ va $v = 2xy$ kelib chiqadi. $y = a$ ni v ga qo'ysak $x = \frac{v}{2a}$ va mos ravishda u ga qo'ysak $u = \frac{v}{4a^2}^2 - a^2$ hosil bo'ladi. Bu esa C_w tekislikdagi parabola aksini beradi (3.3-chizma). Demak, $w(E) = \left\{ z \in C : \operatorname{Re} w = \frac{(\operatorname{Im} w)^2}{4a^2} - a^2 \right\}$



3.3-chizma

3.4-Misol. $E = \left\{ z \in C : |z| > \frac{1}{2}, \operatorname{Re} z > 0 \right\}$ sohani $w = z^2$ akslantirish yordamidagi aksini toping.

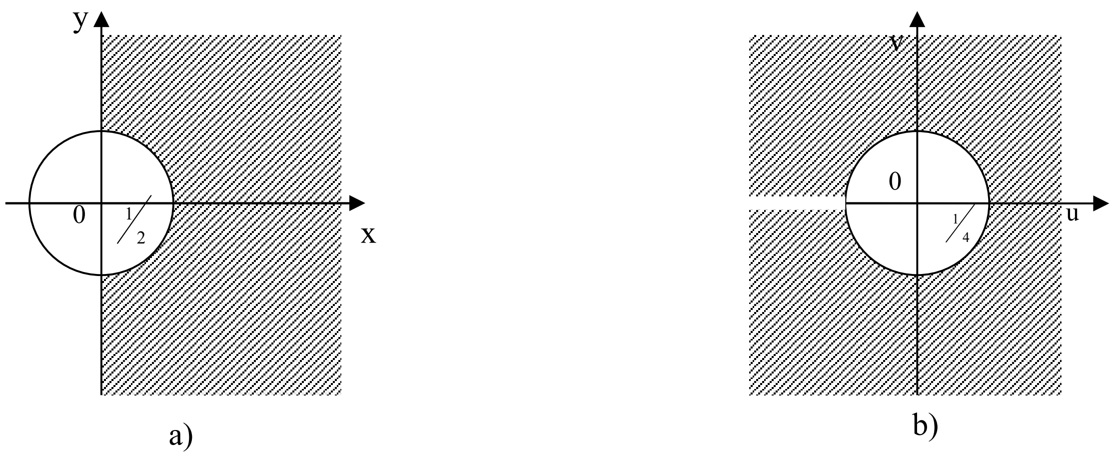
Yechilishi. Berilgan E to'plamni

$$E = \left\{ z \in C : |z| > \frac{1}{2}, \operatorname{Re} z > 0 \right\} = \left\{ z \in C : -\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \frac{1}{2} < r < +\infty \right\}$$

deb,

$$w(E) = \left\{ w \in C_w : -\pi < \arg z < \pi, \frac{1}{4} < r < +\infty \right\}$$

bo'lishini topamiz (3.4 - chizma).



3.4- chizma

3.5-Misol. $E = \left\{ z \in C : |z| < 1, 0 < \arg z < \frac{\pi}{3} \right\}$ sohani $w = z^3$ akslantirish yordamidagi aksini toping.

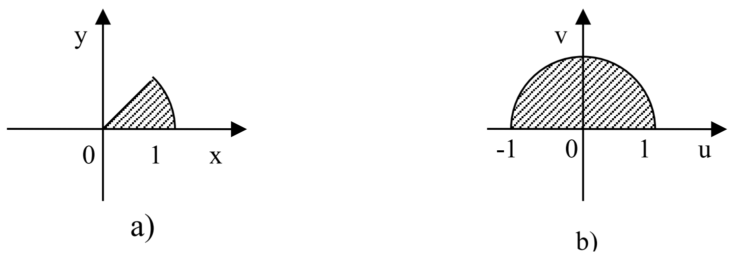
Yechilishi. Berilgan E to'plamni

$$E = \left\{ z \in C : |z| < 1, 0 < \arg z < \frac{\pi}{3} \right\} = \left\{ z \in C : 0 < \arg z < \frac{\pi}{3}, 0 < r < 1 \right\}$$

deb,

$$w(E) = \{ w \in C_w : 0 < \arg z < \pi, 0 < r < 1 \}$$

bo'lishini topamiz (3.5 - chizma).



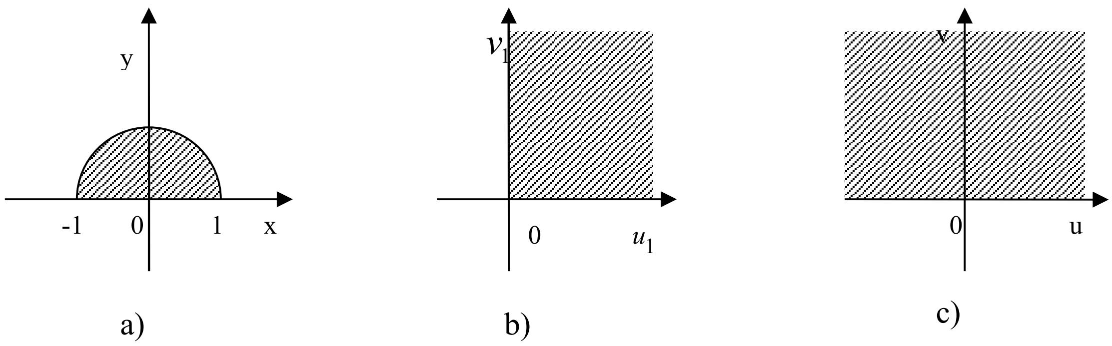
3.5 –chizma.

3.6-Misol. $E = \{ z \in C : |z| < 1, \operatorname{Im} z > 0 \}$ sohani $G = \{ z \in C : \operatorname{Im} w > 0 \}$ sohaga $w(-1) = 0,\ w(0) = 1$ va $w(1) = \infty$ shartni qanoatlantiradigan akslantirish quring.

Yechilishi. Teorema-2.2 da keltirilgan

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

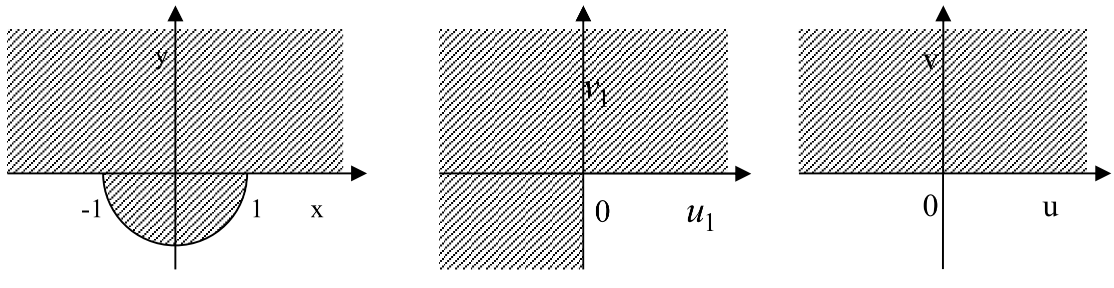
tenglikdan $w_1 = \dfrac{z+1}{z-1}$ ekanligini topamiz. w_1 akslantirish $|z|=1$ aylanani $u_1 = 0$ to'g'ri chiziqqa, $\operatorname{Im} z = 0$ chiziqni esa $v_1 = 0$ chiziqqa akslantiradi. $w_2 = w_1^2$ akslantirish yordamida esa $G = \{z \in C : \operatorname{Im} w > 0\}$ sohaga akslanadi (3.6 -chizma)



3.6 -chizma

3.7-Misol. $E = \{z \in C : |z| < 1 \vee \operatorname{Im} z > 0\}$ sohani $G = \{z \in C : \operatorname{Im} w > 0\}$ sohaga akslantiradigan akslantirish quring.

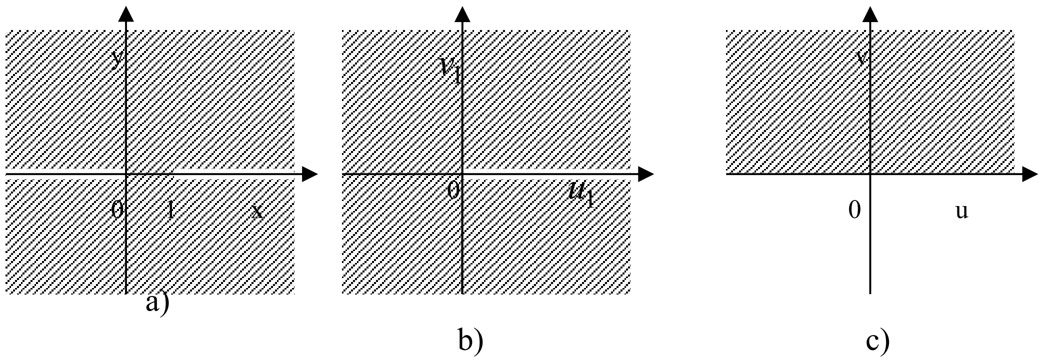
Yechilishi. D sohaning chegarasi 2 ta chiziqdan iborat. Ularni mos ravishda $l_1 : |z| = 1$ va $l_2 : \operatorname{Im} z = 0$ deylik. l_1 va l_2 chiziqlar kesishishi nuqtasi $z_1 = -1$ va $z_2 = 1$ nuqtalar. Biz bu nuqtalarni mos ravishda kasr-chiziqli akslantirishning surat va maxrajiga qo'yamiz. Ya'ni $w_1 = \dfrac{z-1}{z+1}$. U holda mos ravishda l_1 va l_2 chiziqlar $\arg w_1 = \dfrac{3\pi}{2}$ va $\arg w_1 = 0$ chiziqlarga akslanadi. Soha esa $G_1 = \left\{ w \in C_w : 0 < \arg w < \dfrac{3\pi}{2} \right\}$ bo'ladi. Bu sohani $w_2 = w_1^{2/3}$ akslantirish esa $G = \{z \in C : \operatorname{Im} w > 0\}$ sohaga akslantiradi. Demak, $w = \left(\dfrac{z-1}{z+1} \right)^{2/3}$ ekan.



3.7-chizma.

3.8-Misol. $E=\{z\in C: z\notin R\setminus[0;1]\}$ sohani $G=\{z\in C:\operatorname{Im} w>0\}$ sohaga akslantiradigan akslantirish quring.

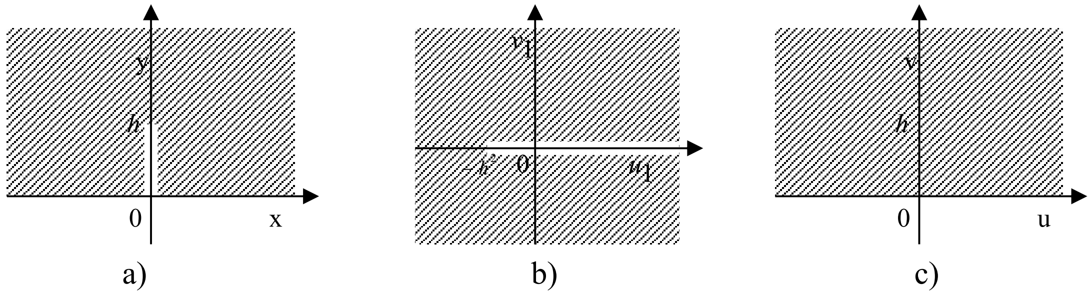
Yechilishi. D sohaning chegarasi 2 ta chiziqdan iborat. Ularni mos ravishda $l_1:(-\infty;0]$ va $l_2:[1;\infty)$ deylik. $w_1=\dfrac{z}{z-1}$ fuksiya l_1 va l_2 chiziqlarni $\arg w_1=0$ chiziqqa akslantiradi. Soha esa $G_1=\{w\in C_w: 0<\arg w<2\pi\}$ bo'ladi. Bu sohani $w_2=\sqrt{w_1}$ aksalantirish esa $G=\{z\in C:\operatorname{Im} w>0\}$ sohaga akslantiradi. Demak, $w=\sqrt{\dfrac{z}{z-1}}$ ekan (3.8-chizma).



3.8-chizma.

3.9-Misol. $E=\{z\in C:\operatorname{Im} z>0, z\notin[0;ih]\}$ sohani $G=\{z\in C:\operatorname{Im} w>0\}$ sohaga akslantiradigan akslantirish quring.

Yechilishi. D sohaning chegarasi 2 ta chiziqdan iborat. Ularni mos ravishda $l_1:\operatorname{Im} z=0$ va $l_2:[0;ih]$ deylik. $w_1=z^2$ fuksiya l_1 va l_2 chiziqlarni mos ravishda $\arg w_1=0$ va $w_1=\lfloor -h^2;0\rfloor$ chiziqlarga akslantiradi. Soha esa $G_1=\{w\in C_w: w\notin\lfloor h^2;\infty)\}$ bo'ladi. Bu sohani $w_2=w_1+h^2$ akslantirish esa $G_2=\{w\in C_w: 0<\arg w<2\pi\}$ sohaga akslantiradi. $w_3=\sqrt{w_2}$ akslantirish esa $G_2=\{w\in C_w: 0<\arg w<2\pi\}$ sohani $G=\{z\in C:\operatorname{Im} w>0\}$ sohaga akslantiradi. Demak, $w=\sqrt{z^2+h^2}$ ekan (3.9-chizma).



3.9-chizma

3.10-Misol. $E = \left\{ z \in C : |z| < 2, 0 < \arg z < \dfrac{\pi}{3} \right\}$ sohani $G = \{ z \in C : \operatorname{Im} w > 0 \}$ sohaga akslantiradigan akslantirish quring.

Yechilishi. D sohaning chegarasi 3 ta chiziqdan iborat. Ularni mos ravishda $l_1 : |z| = 2$, $l_2 : \arg z = 0$ va $l_3 : \arg z = \dfrac{\pi}{3}$ deylik. $w_1 = z^3$ fuksiya l_1, l_2 va l_3 chiziqlarni mos ravishda $|w_1| = 8$, $\arg w_1 = 0$ va $\arg w_1 = \pi$ chiziqlarga akslantiradi. Soha esa $G_1 = \{ w \in C_w : |w| < 8; 0 < \arg w < \pi \}$ bo'ladi. Bu sohani $w_2 = \dfrac{w_1 - 8}{w_1 + 8}$ akslantirish esa $G_2 = \left\{ w \in C_w : \dfrac{\pi}{2} < \arg w < \pi \right\}$ sohaga akslantiradi. $w_3 = -w_2$ akslantirish esa $G_2 = \left\{ w \in C_w : \dfrac{\pi}{2} < \arg w < \pi \right\}$ sohani $G_3 = \left\{ w \in C_w : 0 < \arg w < \dfrac{\pi}{2} \right\}$ sohaga akslantiradi. $w_4 = w_3^2$ esa $G_3 = \left\{ w \in C_w : 0 < \arg w < \dfrac{\pi}{2} \right\}$ sohani $G = \{ z \in C : \operatorname{Im} w > 0 \}$ sohaga o'tkazadi.

Demak, $w = -\left(\dfrac{z^3 - 8}{z^3 + 8} \right)^2$ ekan.

Mustaqil yechish uchun mashqlar

Quyidagi D to'plamning berilgan akslantirish yordamidagi aksini toping.

1. $D = \{\mathrm{Re}\, z = 2\}$ $\qquad w = z^2.$

2. $D = \{\mathrm{Im}\, z = 3\}$ $\qquad w = z^2.$

3. $D = \{\arg z = \frac{\pi}{3}\}$ $\qquad w = z^4.$

4. $D = \{|z| = 2, \frac{\pi}{3} < \arg z < \frac{2\pi}{3}\}$ $\qquad w = z^2.$

5. $D = \{\mathrm{Im}\, z > 1\}$ $\qquad w = z^2.$

6. $D = \{\mathrm{Re}\, z > 1\}$ $\qquad w = z^2.$

4. KO`RSATKICHLI FUNKSIYALAR VA ULAR YORDAMIDA BAJARILADIGAN AKSLANTIRISHLAR

Masala-4.1. Quyidagi Koshi masalasini qanoatlantiruvchi

$$\begin{cases} \dfrac{df(z)}{dz} = f(z) \\ f(0) = 1 \end{cases} \tag{4.1}$$

analitik $f(z) = u(x,y) + iv(x,y)$ funksiya topilsin.

Yechilishi. Hosilani hisoblash fomulasiga ko'ra

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Bundan

$$\frac{df(z)}{dz} = f(z)$$

bo'lgani uchun

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = u + iv = f(z).$$

Bundan esa $u_x^{'} = v_y^{'} = u(x,y)$, $v_x^{'} = -u_y^{'} = v(x,y)$ kelib chiqadi. Boshlang'ich shart $f(0) = 1$ dan $u(0,0) = 1$, $v(0,0) = 0$ ga egamiz.

Quyidagi

$$\begin{cases} u_x^{'} = u(x,y), \\ u(0,0) = 1 \end{cases}$$

Koshi masalasining yechimi

$$u(x,y) = e^x \lambda(y)$$

ko'rinishga ega ekanligi differensial tenglamalar kursidan bizga ma'lum, bu yerda $\lambda(y)$-differensiallanuvchi funksiya, $\lambda(0) = 1$.

Xuddi shunga o'xshash

$$\begin{cases} v_x' = v(x,y), \\ v(0,0) = 0 \end{cases}$$

masalaning yechimi

$$v(x,y) = e^x \mu(y)$$

ko'rinishga ega, bunda $\mu(y)-$differensiallanuvchi funksiya va $\mu(0)=0$. Demak,

$$v_y' = e^x \mu'(y) = u(x,y) = e^x \lambda(y) \Rightarrow \mu'(y) = \lambda(y), \tag{4.2}$$

$$-u_y' = -e^x \lambda'(y) = v(x,y) = e^x \mu(y) \Rightarrow -\lambda'(y) = \mu(y). \tag{4.3}$$

(4.2) dan y bo'yicha hosila olamiz:

$$\mu''(y) - \lambda'(y) = 0.$$

Bundan (4.3) ga ko'ra ushbu

$$\mu''(y) + \mu(y) = 0 \tag{4.4}$$

tenglamaga ega bo'lamiz:

Endi (4.3) dan y bo'yicha hosila olamiz. U holda

$$-\lambda''(y) - \mu'(y) = 0$$

yoki (4.2-4.4) ga ko'ra

$$\lambda''(y) + \lambda(y) = 0. \tag{4.5}$$

(4.4) va (4.5) tenglmalarning yechimi mos ravishda quyidagi ko'rinishlarga ega:

$$\mu(y) = c_1 \cos y + c_2 \sin y,$$

$$\lambda(y) = c_3 \cos y + c_4 \sin y$$

Endi $\mu(0)=0$ ekanligidan

$$\mu(y) = c \sin y,$$

ya'ni $c_1 = 0$, $c_2 = c$, $\lambda(y) = \mu'(y) = c\cos y$, lekin $\lambda(0)=1$ edi. Bundan $c=1$. Demak,

$$\lambda(y)=\cos y, \quad \mu(y)=\sin y.$$

Bundan (14.1) masalaning yechimi quyidagicha

$$f(z)=u(x,y)+iv(x,y)=e^x(\cos y+i\sin y)=e^xe^{iy}=e^z \qquad (4.6)$$

funksiya ekanligi kelib chiqadi.

Ta'rif-4.1. (4.1) masalaning yechimi $f(z)=e^z$ funksiya *ko'rsatkichli funksiya* deyiladi.

Eslatma-4.1. $f(z)=e^z$ ko'rsatkichli funksiyani $e^z=\lim\limits_{n\to\infty}\left(1+\dfrac{z}{n}\right)^n$ kabi ham kiritiladi ([4] ga qarang).

Ko'rsatkichli funksiyaning xossalari.

Endi $w=e^z$ funksiyaning asosiy xossalarini keltiramiz.

Xossa-4.1. Ko'rsatkichli $w=e^z$ funksiya butun kompleks tekislikda golomorf funksiya bo'ladi.

Isboti. Haqiqatan ham, $w=e^z=u+iv$ deb, (4.6) munosabatdan foydalanib $u=e^x\cos y,\ v=e^x\sin y$ bo'lishini topamiz. Ravshanki, bu funksiyalar R^2 ma'noda differentsiallanuvchi. Ayni paytda bu funksiyalar uchun

$$\frac{\partial u}{\partial x}=e^x\cos y,\ \frac{\partial u}{\partial y}=-e^x\sin y,\ \frac{\partial v}{\partial x}=e^x\sin y,\ \frac{\partial v}{\partial y}=e^x\cos y$$

bo'lib,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

bo'ladi (Koshi-Riman sharti bajariladi). Demak, $w=e^z$ funksiya C da golomorf bo'ladi.

Xossa-4.2. $w=e^z$ funksiya kompleks tekislik C ning har bir nuqtasida hosilaga ega va $w'=(e^z)'=e^z$ bo'ladi.

Isboti. Hosilani topish formulasiga ko'ra topamiz:

$$(e^z)' = \frac{\partial}{\partial x}\left[e^x(\cos y + i\sin y)\right] = e^z(\cos y + i\sin y) = e^z$$

Xossa-4.3. Ko'rsatkichli funksiya uchun ushbu $e^{z_1+z_2} = e^{z_1}\cdot e^{z_2}$ formula o'rinli bo'ladi.

Isboti. Aytaylik, $z_1 = x_1 + iy_1,\ \ z_2 = x_2 + iy_2$ bo'lsin. Unda

$$e^{z_1}e^{x_1}(\cos y_1 + \sin y_1),\ \ e^{z_2} = e^{x_2}(\cos y_2 + \sin y_2)$$

bo'ladi. Kompleks sonlarni ko'paytirish qoidasidan foydalanib topamiz:

$$e^{z_1}\cdot e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1)\cdot\ e^{x_2}(\cos y_2 + i\sin y_2) =$$

$$= e^{x_1}\cdot e^{x_2}\left[(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2)\right] =$$

$$= e^{x_1+x_2}\left[(\cos y_1\cdot\cos y_2 - \sin y_1\cdot\sin y_2) + i(\sin y_1\cdot\cos y_2 + \cos y_1\cdot\sin y_2)\right] =$$

$$= e^{x_1+x_2}\left[\cos(y_1+y_2) + i\sin(y_1+y_2)\right] = e^{z_1+z_2}\ .$$

Demak,

$$e^{z_1}\cdot e^{z_2} = e^{z_1+z_2}$$

Xossa-4.4. Ko'rsatkichli funksiya $w(z) = e^z$ davriy funksiya bo'lib, uning davri $T = 2\pi i$ ga teng.

Isboti. Eyler formulasiga ko'ra $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$ bo'lishini e'tiborga olib hamda ko'rsatkichli funksiyaning 14.3-xossasidan foydalanib $\forall z\in C$ uchun

$$w(z+2\pi i) = e^{z+2\pi i} = e^z$$

bo'lishini topamiz. Demak,

$$w(z+2\pi i) = w(z).$$

Bu esa $w(z) = e^z$ funksiyaning davriy funksiya ekanini, uning davri $T = 2\pi i$ ga tengligini bildiradi.

Endi (4.6) funksiya yordamida bajariladigan akslantirishni o'rganamiz. (4.6) bir qiymatli, analitik va

$$w' = f'(z) = (e^z)' = e^z \neq 0\ .$$

Demak, (14.6) funksiya konform akslantirish bajaradi.

Ko'rsatkichli, logarifmik funksiyalar va Riman sirti.

Kompleks sonning trigonometrik ko'rinishiga ko'ra $w = e^z$ dan ushbu

$$w = \rho(\cos\theta + i\sin\theta) = e^x(\cos y + i\sin y) \Rightarrow \rho = e^x = \left|e^z\right|,\ \theta = y$$

tengliklarga egamiz. Bundan (4.6) funksiya mavhum o'qqa parallel bo'lgan $x = a$ chiziqlar oilasini markazi O nuqtada joylashgan $\rho = e^\alpha$ aylanalar oilasiga, $y = b$ chiziqlarni esa O nuqtadan chiquvchi $\theta = b$ nurlar oilasiga va demak, $0 \le y < b$ ni $0 \le \theta < b$ ga akslantirar ekan.

Birvaraqlilik sohasi topamiz:

$$z_1 = z_2 \Rightarrow e^{z_1} = e^{z_2} \Rightarrow z_1 = z_2 + 2\pi ki$$

Bundan birvaraqlilik sohasi haqiqiy o'qqa parallel bo'lgan ixtiyoriy 2π kenglikdagi yo'lak ekanligi kelib chiqadi.

Demak, $w = e^z$ funksiya C_z tekislikdagi biror sohada o'zaro bir qiymatli funksiya bo'lishi uchun shu sohaga tegishli bo'lgan turli z_1 va z_2 nuqtalarda $z_1 - z_2 = 2\pi ki$ shartni bajarmasligi zarur va yetarli.

C_z tekislikni cheksiz ko'p D_k sohalarga bo'lamiz.

$$D_k = \{z : 2\pi k < \operatorname{Im} z < 2\pi(k+1),\ k = 0,1,2,...\}.$$

(4.6) akslantirishda $y = 2\pi \xrightarrow{(14.6)} \theta = 2\pi$, ya'ni $0 \le y < 2\pi \xrightarrow{(14.6)} 0 \le \theta < 2\pi$.

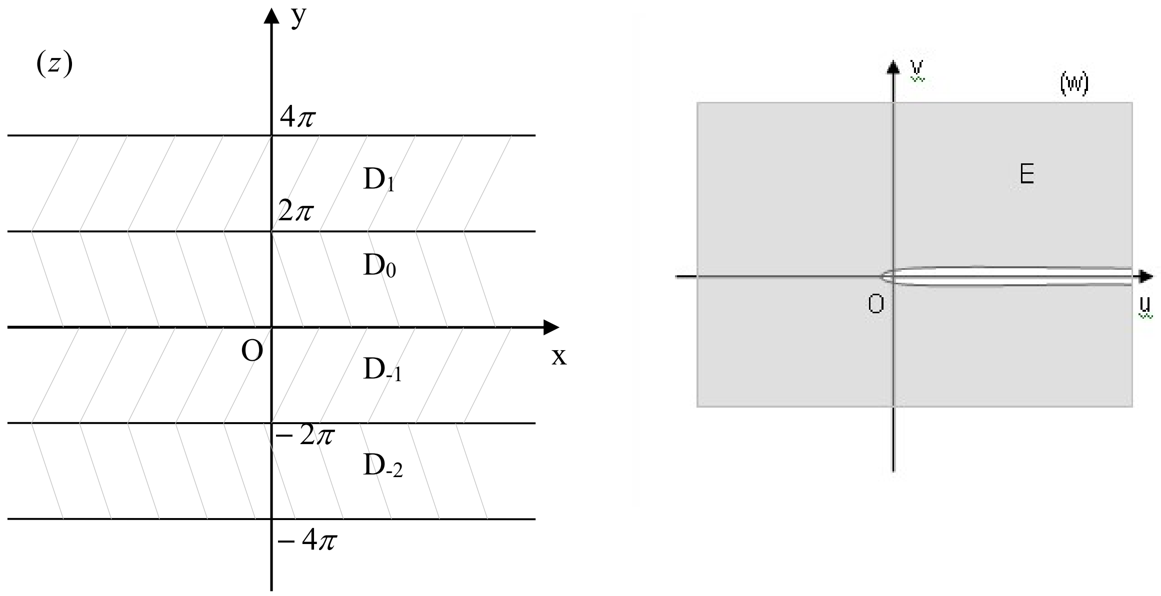
Yuqoridagi singari (4.6) funksiya D_k sohani E kesilgan tekislikka akslantiradi. Bunda $\operatorname{Im} z = 2\pi k$ E ning yuqori qirg'og'iga, $\operatorname{Im} z = 2(k+1)\pi$ esa quyi qirg'ogiga akslanadi (4.1-chizma). Bu yerda ham o'zaro bir qiymatli moslik o'rnatish masalasi hal qilmoq kerak, buning uchun (4.6) ga teskari bo'lgan funksiyani ko'ramiz. (4.6) tenglik quyidagi ikki tengliklarga ekvivalent:

$$|w| = e^{\operatorname{Re} z},\quad \arg w = \operatorname{Im} z + 2\pi k,\quad k \in Z$$

Bundan D_k sohada teskari funksiya

$$z_k = (Lnw)_k = \ln|w| + i(\varphi + 2\pi k), \quad \varphi = \arg w$$

ko'rinishga ega. Shu sababdan $z = f^{-1}(w)$ teskari funksiya logarifmik funksiya, z_k lar esa funksiyaning tarmoqlari deyiladi. z_k tarmoqlar har xil bo'lgani uchun $z = Lnw$ funksiya cheksiz qiymatli, yoki $w = e^z$ funksiya cheksiz varaqli funksiya deyiladi.



4.1-chizma.

Cheksiz ko'p (w) tekisliklar olamiz va ularga D_k larni akslantiramiz. Natijada cheksiz ko'p varaqlar $E_k,\ k \in Z$ hosil bo'ladi. Ularni ustma-ust qo'yib, $z = w^{\frac{1}{n}}$ funksiya uchun Riman sirtini qanday qurgan bo'lsak, o'sha usulda $z = Lnw$ funksiyaning Riman sirtini quramiz.

(4.6) funksiya (z) tekislikni 0 nuqta olib tashlangan Riman sirtiga konform akslantiradi. Bunda $w = 0,\ 2 = \infty$ nuqtalar $z = Lnw$ funksiyaning tarmoqlanish nuqtalaridir va bu nuqtalar barcha varaqlar uchun umumiydir.

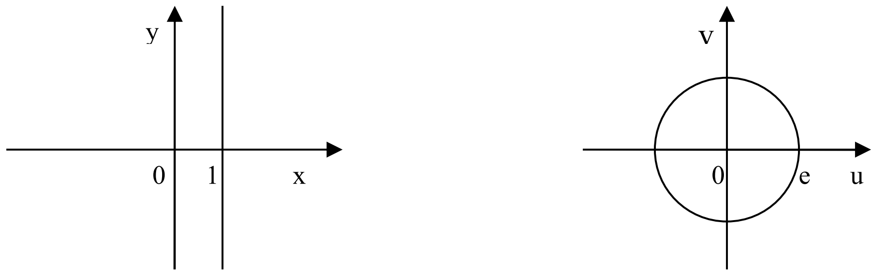
Na'munaviy misol va masalalar yechimi

4.1-Misol. Ushbu $w=e^z$ funksiya yordamida C_z tekislikdagi $D=\{z\in C:\operatorname{Re} z=1\}$ sohaning C_w tekislikdagi aksini toping.

Yechilishi. Agar $z=x+iy,\ w=\rho e^{i\varphi}$ deyilsa, unda

$$D=\{(x,y)\in R^2 : x=1, y\in R\}$$

bo'lib, $\rho=e,\ 0<\phi\le 2\pi$ bo'ladi. Demak, $w(D)=\{w\in C_w : |w|=e\}$ bo'ladi (4.2 - chizma).



a) b)

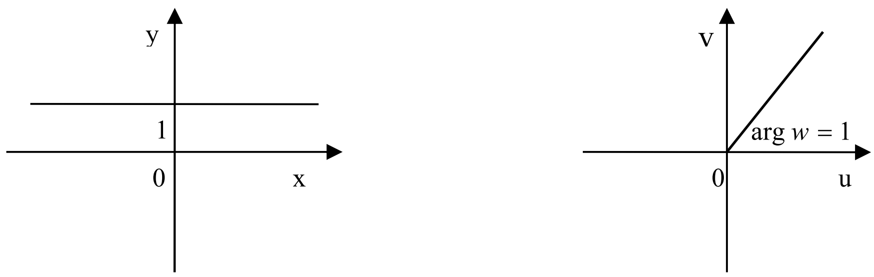
4.2 - chizma

4.2-Misol. Ushbu $w=e^z$ funksiya yordamida C_z tekislikdagi $D=\{z\in C:\operatorname{Im} z=1\}$ sohaning C_w tekislikdagi aksini toping.

Yechilishi. Agar $z=x+iy,\ w=\rho e^{i\varphi}$ deyilsa, unda

$$D=\{(x,y)\in R^2 : x=1, y\in R\}$$

bo'lib, $0\le\rho<+\infty,\ \phi=1$ bo'ladi. Demak, $w(D)=\{w\in C_w : \arg w=1\}$ bo'ladi (4.3 - chizma).



a) b)

4.3 - chizma

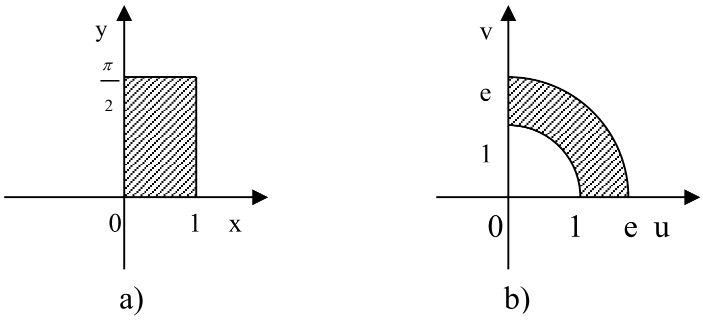
4.3-Misol. Ko'rsatkichli $w = e^z$ funksiya C_z tekislikdagi

$$D = \left\{ z \in C_z : 0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < \frac{\pi}{2} \right\}$$

sohani C_w tekislikdagi qanday sohaga akslantiradi?

Yechilishi. $z = x + iy$, $w = \rho e^{i\varphi}$ deb olaylik. Unda $\rho \cdot e^{i\varphi} = e^{x+iy}$ bo'lib, D sohada $e^0 < \rho < e^1$, $0 < \varphi < \frac{\pi}{2}$ bo'ladi (4.4 – chizma). Shularni e'tiborga olib topamiz:

$$w(D) = \left\{ w \in C_w : w = \rho e^{i\phi}; 1 < \rho < e, 0 < \phi < \frac{\pi}{2} \right\}.$$



4.4 - chizma

4.4-Misol. Ushbu $w = e^z$ funksiya yordamida C_z tekislikdagi

$$D = \{ z \in C_z : \operatorname{Re} z > 0, -\pi < \operatorname{Im} z < \pi \}$$

sohaning C_w tekislikdagi aksini toping.

Yechilishi. Agar $z = x + iy$, $w = \rho e^{i\varphi}$ deyilsa, unda

$$D = \left\{ (x, y) \in R^2 : x > 0,\ -\pi < y < \pi \right\}$$

bo'lib, $\rho > 1,\ -\pi < \varphi < \pi$ bo'ladi. Demak,

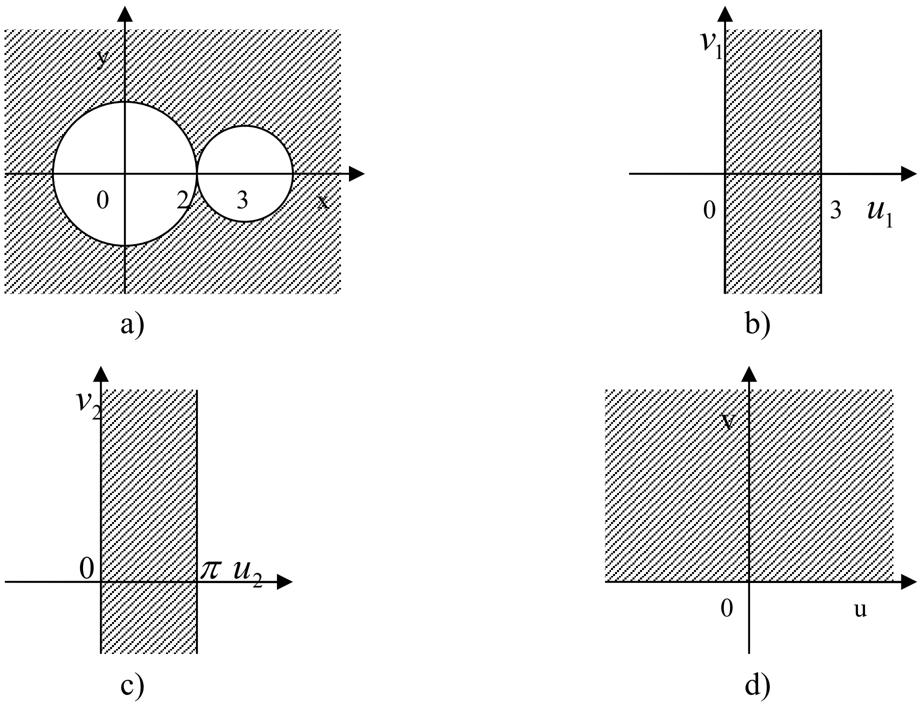
$$w(D) = \left\{ w \in C_w : w = \rho e^{i\varphi} :\ \rho > 1,\ -\pi < \varphi < \pi \right\} =$$

$$= \left\{ w \in C_w : |w| > 1 \right\} \Big| \left\{ w \in C_w : w \in (-\infty, -1) \right\}.$$

4.5-Misol. Ushbu C_z tekislikdagi $D=\{z\in C:|z|>2,|z-3|>1\}$ sohani C_w tekislikdagi $G=\{w\in C:\operatorname{Im}w>0\}$ sohaga akslantiruvchi akslantirish toping.

Yechilishi. Biz dastlab $|z|=2$ va $|z-3|=1$ (4.5 a-chizma) tengliklarni birgalikda yechib olamiz. $z_0=2$ nuqta har 2 chegarada ham yotadi. $z_0=2$ nuqtani kasr-chiziqli akslantirishni maxrajiga qo'yamiz, chunki har 2 chegara ham faqat shundagina to'g'ri chiziqqa akslanadi. Kasr-chiziqli akslantirishni suratiga esa $z_1=-2$ nuqtani tanlaymiz, chunki $|z|=2$ aylana shunda koordinata boshidan o'tadigan to'g'ri chiziqqa akslanadi. Ya'ni akslantirishni $w_1=\dfrac{z+2}{z-2}$ ko'rinishida tanlaymiz. U holda $|z|=2$ va $|z-3|=1$ chiziqlar mos ravishda $u_1=0$ va $u_1=3$ chiziqlarga o'tadi (4.5 b-chizma). Hosil bo'lgan sohani o'xshashlik akslantirishi $w_2=\dfrac{\pi}{3}w_1$ yordamida $G_2=\{w\in C:0<\operatorname{Re}w<\pi\}$sohaga olib o'tamiz (4.5 c-chizma). $w_3=e^{w_2}$ G_2 soha $G=\{w\in C:\operatorname{Im}w>0\}$sohaga akslanadi (4.5 chizma).

Demak, akslantirishning umumiy ko'rinishi $w=e^{\frac{\pi}{3}\frac{z+2}{z-2}}$ ekan.

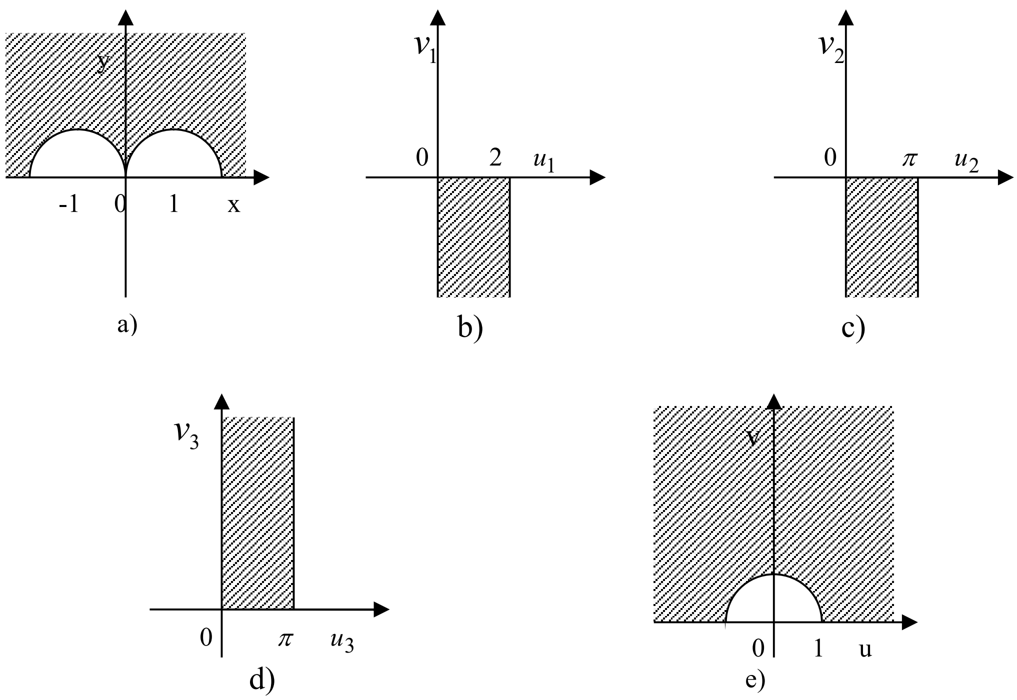


4.5 – chizma.

4.6-Misol. Ushbu $D=\{z\in C:|z+1|>1,|z-1|>1,\operatorname{Im} z>0\}$ aylanalar aniqlagan sohani $G=\{w\in C:|w|>1,\operatorname{Im} w>0\}$ sohaga akslantiruvchi akslantirish toping.

Yechilishi. Dastlab $|z+1|=1$, $|z-1|=1$ va $\operatorname{Im} z=0$ (4.6 a-chizma) tengliklarni birgalikda yechib olamiz. $z_0=0$ nuqta har 3 chegarada ham yotadi. $z_0=0$ nuqtani kasr-chiziqli akslantirishni maxrajiga qo'yamiz, chunki har bir chegara ham faqat shundagina to'g'ri chiziqqa akslanadi. Ya'ni akslantirishni $w_1=\dfrac{z+2}{z}$ ko'rinishida tanlaymiz. U holda $|z+1|=1$, $|z-1|=1$ va $\operatorname{Im} z=0$ chiziqlar mos ravishda $u_1=0$, $u_1=2$ va $v_1=0$ (4.6 b-chizma) ga akslanadi.

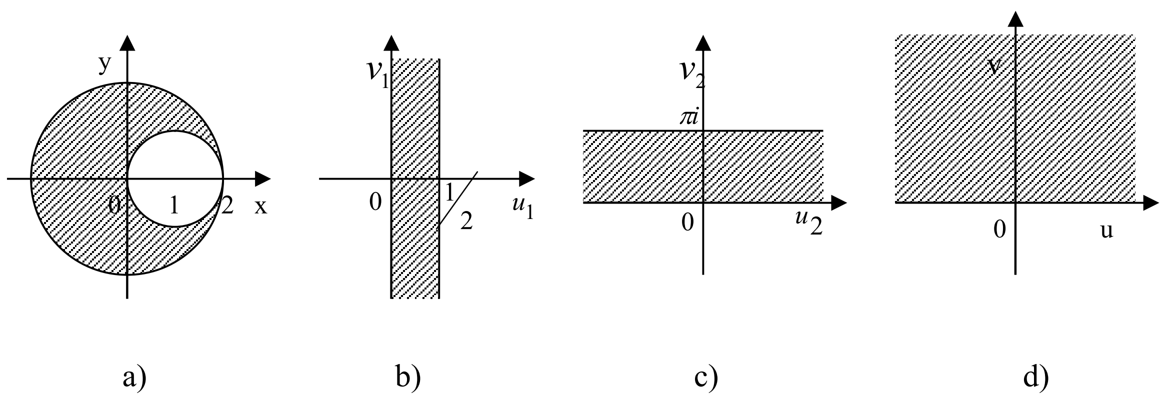
O'xshashlik akslantirishi $w_2=\dfrac{\pi}{2}w_1$ yordamida $G_2=\{w\in C:0<\operatorname{Re} w<\pi,\operatorname{Im} w<0\}$ sohaga o'tadi (4.6 c-chizma). π burchakka burib $w_3=-w_2+\pi$ yordamida $G_3=\{w\in C:0<\operatorname{Re} w<\pi,\operatorname{Im} w>0\}$ sohani xosil qilamiz (4.6 d-chizma). Hosil bolgan G_3 sohani $w=e^{w_3}$ funksiya $G=\{w\in C:|w|>1,\operatorname{Im} w>0\}$ sohaga akslantiradi (4.6 - chizma). Demak izlanayotgan funksiyaning ko'rinishi: $w=e^{-\frac{\pi(z+2)}{2z}+\pi}$ ekan.



4.6- chizma

4.7-Misol. Ushbu $D=\{z\in C:|z|<2,|z-1|>1\}$ aylanalar aniqlagan sohani $G=\{w\in C:\operatorname{Im}w>0\}$ sohaga akslantiruvchi akslantirish toping.

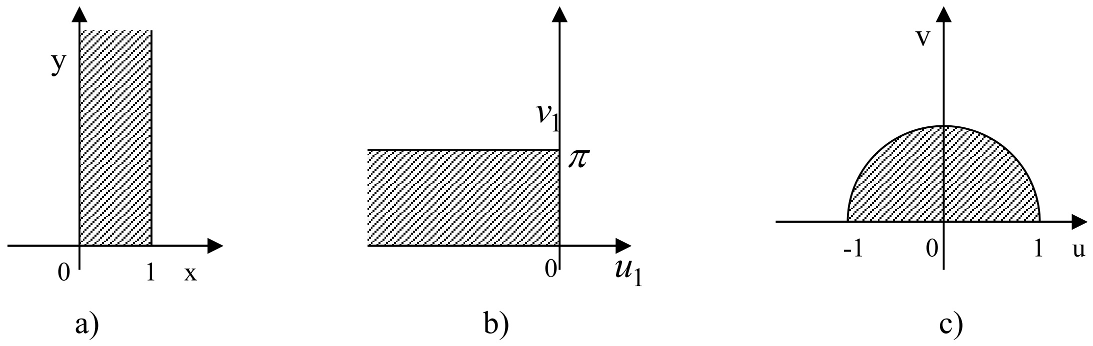
Yechilishi. Dastlab $|z|=2$ va $|z-1|=1$ (4.7 a-chizma) tengliklarni birgalikda yechib olamiz. $z_0=2$ nuqta har 2 chegarada ham yotadi. $z_0=2$ nuqtani kasr-chiziqli akslantirishni maxrajiga qo'yamiz, chunki har 2 chegara ham faqat shundagina to'g'ri chiziqqa akslanadi. Kasr-chiziqli akslantirishni suratiga esa $z_1=0$ nuqtani tanlaymiz, chunki $|z-1|=1$ aylana shunda koordinata boshidan o'tadigan to'g'ri chiziqqa akslanadi. Ya'ni akslantirishni $w_1=\frac{z}{z-2}$ ko'rinishida tanlaymiz. U holda $|z|=2$ va $|z-1|=1$ chiziqlar mos ravishda $u_1=0$ va $u_1=\frac{1}{2}$ chiziqlarga o'tadi (4.7 b-chizma). Hosil bo'lgan sohani o'xshashlik akslantirishi $w_2=2\pi i w_1$ yordamida $G_2=\{w\in C:0<\operatorname{Im}w<\pi\}$ sohaga olib o'tamiz (4.7 c-chizma). $w_3=e^{w_2}$ G_2 soha $G=\{w\in C:\operatorname{Im}w>0\}$sohaga akslanadi (4.7 chizma).



4.7- chizma

4.8-Misol. Ushbu $D=\{z\in C:0<\operatorname{Re}z<1,\operatorname{Im}z>0\}$ sohani $G=\{w\in C:|w|<1,\operatorname{Im}w>0\}$ sohaga akslantiruvchi akslantirish toping.

Yechilishi. D sohaning chegarasi 2 ta to'g'ri chiziqdan iborat. Ularni mos ravishda $l_1:\operatorname{Re}z=0$ va $l_2:\operatorname{Re}z=1$ deylik. $w_1=\pi e^{\frac{\pi}{2}i}z=\pi i z$ chiziqli aksalantirish yordamida l_1 va l_2 chiziqlarning aksi $v_1=0$ va $v_1=\pi$ chiziqlarga o'tadi (4.8 b-chizma). Hosil bo'lgan sohani $w_2 = e^{w_1}$ akslantirish orqali $G = \{w \in C : |w| < 0, \operatorname{Im} w > 0\}$ sohaga olib o'tamiz (4.8 c-chizma). Demak, $D = \{z \in C : 0 < \operatorname{Re} z < 1, \operatorname{Im} z > 0\}$ sohani $w = e^{\pi i z}$ akslantirish $G = \{w \in C : |w| < 1, \operatorname{Im} w > 0\}$ sohaga o'tkazadi (4.8 chizma).

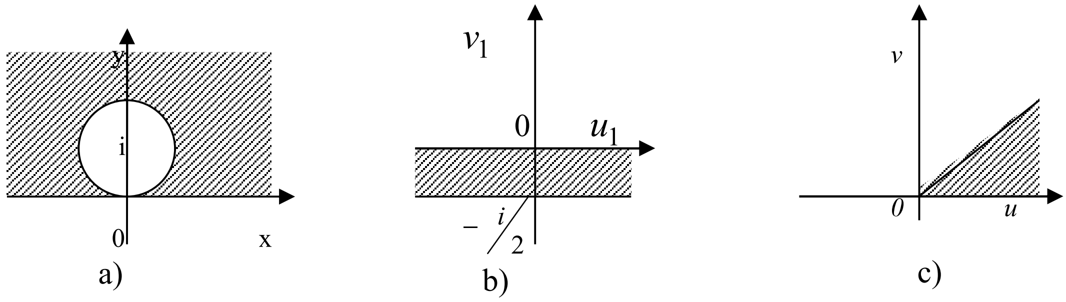


4.8 chizma

4.9-Misol. Ushbu $D = \{z \in C : |z - i| > 1, \operatorname{Im} z > 0\}$ sohani $G = \{w \in C : \operatorname{Im} w > 0\}$ sohaga akslantiruvchi akslantirish toping.

Yechilishi. D sohaning chegarasi 2 ta chiziqdan iborat. Ularni mos ravishda $l_1 : \operatorname{Im} z = 0$ va $l_2 : |z - i| = 1$ deylik. $w_1 = \frac{1}{z}$ aksalantirish yordamida l_1 va l_2 chiziqlarning aksi $v_1 = 0$ va $v_1 = -\frac{1}{2}$ chiziqlarga o'tadi (4.9 b-chizma). Hosil bo'lgan chiziqlarni $w_2 = w_1 + \frac{i}{2}$ akslantirish orqali $v_2 = \frac{i}{2}$ $v_2 = 0$ chiziqlarga olib o'tamiz. $w_3 = e^{w_2}$ akslantirish yordamida $\arg w_3 = 0$ va $\arg w_3 = \frac{1}{2}$ chiziqlarga olib o'tamiz (4.9 c-chizma). $w = w_3^{2\pi}$ akslantirish esa $\arg w = 0$ va $\arg w = \pi$ ga olib o'tadi.

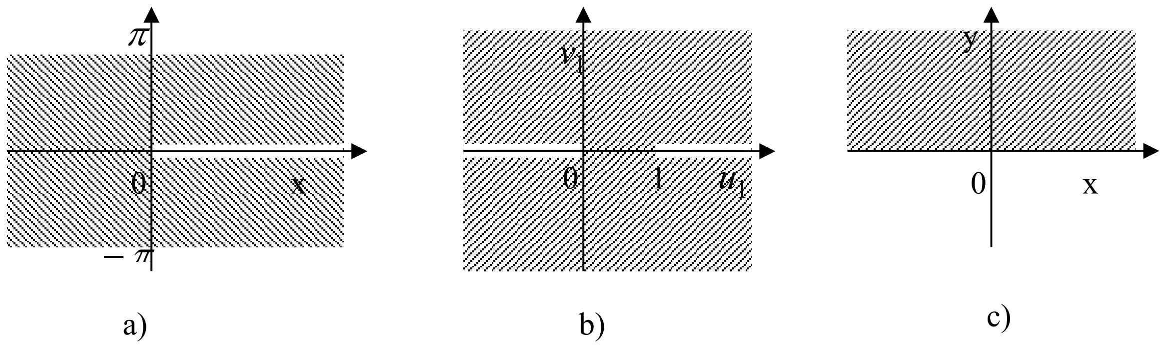
Demak, $D = \{z \in C : |z - i| > 1, \operatorname{Im} z > 0\}$ sohani $w = e^{-\frac{2\pi}{z}}$ akslantirish $G = \{w \in C : \operatorname{Im} w > 0\}$ sohaga o'tkazadi (4.9 chizma).



4.9 chizma

4.10-Misol. $E=\{z\in C:-\pi<\operatorname{Im}z<\pi, z\notin[0;\infty)\}$ sohani $G=\{z\in C:\operatorname{Im}w>0\}$ sohaga akslantiradigan akslantirish quring.

Yechilishi. D sohaning chegarasi 2 ta chiziqdan iborat. Ularni mos ravishda $l_1:\operatorname{Im}z=0$ va $l_2:[0;\infty)$ deylik. $w_1=e^z$ fuksiya l_1 va l_2 chiziqlarni mos ravishda $\arg w_1=\pi$ va $w_1=[1;\infty)$ chiziqlarga akslantiradi. Soha esa $G_1=\{w\in C_w: w\notin[1;\infty), \arg w\neq\pi\}$ bo'ladi. Bu sohani $w_2=\sqrt{\dfrac{w_1}{w_1-1}}$ akslantirish $G=\{z\in C:\operatorname{Im}w>0\}$sohaga akslantiradi. Demak, $w=\sqrt{\dfrac{e^z}{e^z-1}}$ ekan (3.10-chizma).



3.10-chizma

Mustaqil yechish uchun mashqlar

1. Quyidagi to'plamlarning e^z akslantirish yordamidagi aksini toping.

a. $0 < \operatorname{Re} z < \pi$, $\operatorname{Im} z < 0$. **c.** $\operatorname{Re} z > 0$, $\frac{\pi}{2} < \operatorname{Im} z < \pi$.

b. $-\pi < \operatorname{Re} z < 0$, $\operatorname{Im} z > 0$. **d.** $\operatorname{Re} z < 0$, $-\frac{\pi}{2} < \operatorname{Im} z < 0$.

2. Berilgan D sohani $G = \{\operatorname{Im} w > 0\}$ yuqori yarim tekislikka konform akslantiruvchi birorta $w(z)$ funksiyani toping.

a. $D = \{|z| < 1 \;\; , \;\; \operatorname{Im} z > 0\}$.

b. $D = \{|z| < 1 \;\; , \;\; \operatorname{Im} z < 0\}$.

c. $D = \{|z| < 1 \;\; , \;\; \operatorname{Re} z > 0\}$.

3. Quyidagi ifodalarning barcha qiymatlarini toping.

1. $Ln5$. **2.** $Ln(-1)$. **3.** Lni.

4. $\ln(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})$. **5.** $\ln i$. **6.** $Ln(1 + i\sqrt{3})$.

5. JUKOVSKIY FUNKSIYASI VA UNING YORDAMIDA BAJARILADIGAN AKSLANTIRISHLAR

Ta'rif-15.1. Ushbu

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{5.1}$$

ko'rinishdagi funksiya **Jukovskiy funksiyasi** deyiladi.

Bu funksiya $C \setminus \{0\}$ to'plamda aniqlangan. Uning hosilasi

$$w' = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

bo'ladi.

Demak, (5.1) funksiya $C \setminus \{0\}$ sohada golomorf, uning yordamida bajariladigan akslantirish $C_z \setminus \{0; -1; 1\}$ to'plamning har bir nuqtasida konform akslantirish bo'ladi.

z va w o'zgaruvchilarni

$$z = re^{i\varphi}, \ (z \in C_z), \quad w = u + iv \ (z \in C_w)$$

deb olamiz. Unda (5.1) akslantirish ushbu

$$u + iv = \frac{1}{2}\left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right)$$

ko'rinishga keladi. Ravshanki,

$$re^{i\varphi} + \frac{1}{r}e^{-i\varphi} = \left(r + \frac{1}{r}\right)\cos\varphi + i\left(r - \frac{1}{r}\right)\sin\varphi$$

Natijada

$$u + iv = \frac{1}{2}\left[\left(r + \frac{1}{r}\right)\cos\varphi + i\left(r - \frac{1}{r}\right)\sin\varphi\right]$$

bo'lib, undan

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\varphi, \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\varphi$$

bo'lishi kelib chiqadi. Demak,

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

akslantirish ushbu

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\varphi, \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\varphi \tag{5.2}$$

akslantirishga keladi. (5.2) munosabatdan foydalanib quyidagilarni topamiz:

1) C_z tekislikda radiusi ρ ga $(0<\rho<1)$ teng bo'lgan

$$z=\rho e^{i\varphi},\ (0\le\varphi\le 2\pi)$$

aylanani olaylik. Jukovskiy funksiyasi yordamida bu aylana C_w tekislikdagi

$$u=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\cos\varphi,\quad v=\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)\sin\varphi\ \ (0\le\varphi\le 2\pi)$$

chiziqqa akslanadi. Agar

$$\grave{a}_\rho=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right),\ b_\rho=\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)$$

deyilsa, unda $u=a_p\cos\varphi,\quad v=b_p\sin\varphi$ bo'lib,

$$\frac{u^2}{a_\rho^2}+\frac{v^2}{b_\rho^2}=1$$

bo'ladi. Bu C_w tekislikda fokuslari ±1 nuqtada, yarim o'qlari a_ρ va b_ρ bo'lgan ellipsni ifodalaydi. Demak, Jukovskiy funksiyasi

$$w=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

C_z tekislikdagi radiusi $\rho(0<\rho<1)$ bo'lgan

$$z=\rho e^{i\varphi} \tag{5.3}$$

aylanani, C_w tekislikdagi

$$\frac{u^2}{a_\rho^2}+\frac{v^2}{b_\rho^2}=1 \tag{5.4}$$

ellipsga akslantiradi.

2) C_z tekislikda radiusi ρ ga $(\rho>1)$ teng bo'lgan ushbu

$$z=\rho\cdot e^{i\varphi},\ (0\le\varphi\le 2\pi) \tag{5.5}$$

aylanani olaylik. Jukovskiy funksiyasi

$$w=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

bu aylanani ham

$$\frac{u^2}{a_\rho^2}+\frac{v^2}{b_\rho^2}=1 \tag{5.6}$$

ellipsga akslantiradi.

3) C_z tekislikda radiusi $\rho=1$ bo'lgan

$$z=e^{i\varphi},\ (0\le\varphi\le 2\pi)$$

aylanani olaylik. Jukovskiy yunktsiyasi yordamida bu aylana C_z tekislikdagi

$$\begin{cases} u=\cos\varphi \\ v=0 \end{cases},\qquad 0\le\varphi\le 2\pi$$

chiziqqa akslanadi.

Ravshanki, bu chiziq C_w tekislikdagi $[-1,1]\in R$ kesmani ifodalaydi.

Demak, Jukovskiy funksiyasi C_z tekislikdagi $\rho=1$ radiusli aylanani C_w tekislikdagi $[-1,1]\in R$ kesmaga akslantiradi.

Nurlarni akslantirish.

C_z tekislikda $z=\rho e^{ia},\ (0\le\rho\le+\infty)$ ya'ni, $\{z\in C_z : \arg z=\alpha\}$ nurni olaylik. (15.1) akslantirish bu nurni C_w tekislikdagi

$$\begin{aligned} u&=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\cos\alpha, \\ v&=\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)\sin\alpha, \end{aligned}\qquad (0\le\rho\le+\infty) \tag{5.7}$$

chiziqqa akslantiradi. (15.7) munosabatdan topamiz:

$$\frac{u^2}{\cos^2\alpha}-\frac{v^2}{\sin^2\alpha}=1\quad\left(\alpha\ne\frac{k\pi}{2}, k=0,\pm1,...\right) \tag{5.8}$$

Ravshanki, bu chiziq fokuslari ±1 nuqtada bo'lgan giperbola bo'lib, $0<\alpha<\frac{\pi}{2}$ bo'lganda (5.8) chiziq giperbolaning o'ng tarmog'ining yuqori qismini, $\frac{\pi}{2}<\alpha<\pi$ bo'lganda esa chap tarmog'ining yuqori qismini ifodalaydi.

Demak, Jukovskiy funksiyasi

$$w=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

C_z tekislikdagi $\{z\in C_z : \arg z=\alpha\}$ nurni C_w tekislikdagi

$$\frac{u^2}{\cos^2\alpha}-\frac{v^2}{\sin^2\alpha}=1$$

giperbolaning qismiga akslantiradi.

Jumladan, $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ funksiya yordamida quyidagi akslantirishlar

$$\left\{z\in C_z:\arg z=\frac{\pi}{2}\right\}\to\{w\in C_w:\operatorname{Re}w=0\},$$

$$\left\{z\in C_z:\arg z=\frac{3\pi}{2}\right\}\to\{w\in C_w:\operatorname{Re}w=0\},$$

$$\left\{z\in C_z:\arg z=0\right\}\to\{w\in C_w:\operatorname{Re}w=[1,+\infty]\},$$

$$\left\{z\in C_z:\arg z=\pi\right\}\to\{w\in C_w:\operatorname{Re}w=[-\infty,-1]\}.$$

bajariladi.

Faraz qilaylik, Jukovskiy funksiyasi

$$w=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

C_z tekislikdagi turli z_1 va z_2 nuqtalarni C_w tekislikdagi bitta nuqtaga akslantirsin. Unda

$$z_1+\frac{1}{z_1}=z_2+\frac{1}{z_2}(z_1\neq z_2)$$

ya'ni

$$z_1+\frac{1}{z_1}-\left(z_2+\frac{1}{z_2}\right)=(z_1-z_2)\left(1-\frac{1}{z_1\cdot z_2}\right)=0$$

bo'ladi. Keyingi tenglikdan

$$z_1\cdot z_2=1 \qquad (5.9)$$

bo'lishi kelib chiqadi.

Demak, Jukovskiy funksiyasi biror sohada o'zaro bir qiymatli bo'lishi uchun shu sohaning ixtiyoriy turli ikki z_1 va z_2 nuqtalarida $z_1\cdot z_2=1$ shartning bajarmasligi zarur va yetarli bo'ladi.

Quyidagi

$$D_1=\{z\in C_z:|z|>1\},\ D_2=\{z\in C_z:|z|<1\},$$
$$D_3=\{z\in C_z:\operatorname{Im}z>0\},\ D_4=\{z\in C_z:\operatorname{Im}z<0\}$$

sohalarning har biridan olingan ixtiyoriy ikkita turli z_1 va z_2 nuqtalar (5.9) shartning bajarilmasligini ko'rsatish qiyin emas. Binobarin, Jukovskiy funksiyasi bu sohalarda o'zaro bir qiymatli funksiya bo'ladi.

Endi C_z tekislikdagi bu D_1,D_2,D_3,D_4 sohalarni Jukovskiy funksiyasi C_w tekislikdagi qanday sohalarga akslantirishini topamiz.

1) Aytaylik, C_z tekislikda $D_1 = \{z \in C_z : |z| > 1\}$ soha-birlik doiraning tashqarisi berilgan bo'lsin.

Ravshanki, Jukovskiy funksiyasi bu sohada konform akslantirish bo'ladi. D_1 sohada ixtiyoriy

$$\{z \in C_z : |z| = \rho, \rho > 1\}$$

aylanani olaylik. Ma'lumki, bunday aylana Jukovskiy funksiyasi

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

yordamida C_w tekislikdagi ellipsga akslanadi.

Agar aylana radiusi $\rho(1, +\infty)$ oraliqda o'zgara borsa, ularga mos ellipslar C_w tekislikdagi $C_w \setminus \{w \in C_w : w \in [-1,1]\}$ sohani hosil qiladi.

Demak, (5.1) akslantirish C_z tekislikdagi birlik doira tashqarisi D_1 sohani C_w tekislikdagi

$$C_w \setminus \{w \in C_w : w \in [-1,1]\}$$

sohaga konform akslantiradi:

$$w(D_1) = C_w \setminus \{w \in C_w : w \in [-1,1]\}.$$

2) C_z tekislikda ushbu $D_2 = \{z \in C_w : |z| < 1\}$ sohani – birlik doirani olaylik. Yuqoridagidek ko'rsatish mumkinki, (5.1) akslantirish D_2 sohani C_w tekislikdagi $C_w \setminus \{w \in C_w : w \in [-1,1]\}$ sohaga konform akslantiradi:

$$w(D_2) = C_w \setminus \{w \in C_w : w \in [-1,1]\}$$

3) C_z tekislikda $D_3 = \{w \in C_z : \operatorname{Im} z > 0\}$ sohani - yuqori yarim tekislikni qaraylik. (5.1) akslantirish bu D_3 sohani C_w tekislikdagi

$$C_w \setminus \{w \in C_w : w \in [-\infty, -1] \cup [1, +\infty]\}$$

sohaga konform akslantiradi:

$$w(D_3) = C_w \setminus \{w \in C_w : w \in [-\infty, -1] \cup [1, +\infty]\}$$

4) C_z tekislikda $D_4 = \{w \in C_z : \operatorname{Im} z < 0\}$ pastki yarim tekislikni qaraylik. (5.1) akslantirish bu D_4 sohani C_w tekislikdagi

$$C_w \setminus \{w \in C_w : w \in [-\infty, -1] \cup [1, +\infty]\}$$

sohaga konform akslantiradi:

$$w(D_4) = C_w \setminus \{w \in C_w : w \in [-\infty, -1] \cup [1, +\infty]\}.$$

Na'munaviy misol va masalalar yechimi

5.1-Misol. Jukovskiy funksiyasi yordamida C_z tekislikdagi

$$l=\left\{z\in C_z :|z|=1, \frac{5\pi}{4}<\arg z<\frac{7\pi}{4}\right\}$$

yoyning aksini toping.

Yechilishi. Ravshanki,

$$l=\left\{z\in C_z :|z|=1, \frac{5\pi}{4}<\arg z<\frac{7\pi}{4}\right\}=\left\{r=1,\ \frac{5\pi}{4}<\varphi<\frac{7\pi}{4}\right\}$$

(5.2) munosabatga ko'ra

$$u=\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\varphi=\cos\varphi,\quad v=\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\varphi=0$$

bo'ladi. Agar $\frac{5\pi}{4}<\varphi<\frac{7\pi}{4}$ bo'lganda

$$-\frac{\sqrt{2}}{2}<\cos\varphi<\frac{\sqrt{2}}{2}$$

bo'lishini e'tiborga olsak,

$$w(l)=\left\{-\frac{\sqrt{2}}{2}<u<\frac{\sqrt{2}}{2}, v=0\right\}=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

ekanini topamiz.

5.2-Misol. Ushbu $w=\frac{z}{z^2+1}$ akslantirish yordamida C_z tekislikdagi

$$D=\{z\in C_z :|z|<1\}$$

sohaning (birlik doiraning) C_w tekislikdagi aksini toping.

Yechilishi. Avvalo berilgan $w=\frac{z}{z^2+1}$ funksiyani quyidagi

$$w=\frac{1}{2\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

ko'rinishda yozib olamiz. Agar

$$w_1=\frac{1}{2}\cdot\left(z+\frac{1}{z}\right)$$

desak, unda $w=\frac{1}{2\cdot w_1}$ bo'ladi.

Ma’lumki, $w_1 = \frac{1}{2} \cdot \left(z + \frac{1}{z} \right)$ Jukovskiy funksiyasi birlik doira

$$D = \{z \in C_z : |z| < 1\}$$

ni $C_{w_1} \setminus \{w_1 \in C_{w_1} : w_1 \in [-1,1]\}$ (to’plamga) sohaga ($[-1,1]$ kesmaning tashqarisiga) akslantiradi. Kasr chiziqli

$$w = \frac{1}{2 \cdot w_1}$$

funksiya [0,1] kesmani $\left[\frac{1}{2}, +\infty\right)$ nurga [-1,0] kesmani esa $\left(-\infty, -\frac{1}{2}\right)$ nurga akslantiradi.

Demak, berilgan D sohaning aksi

$$w(D) = C_w \setminus \left\{ w \in C_w : w \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2} + \infty\right) \right\}$$

bo’ladi.

5.3-Misol. Ushbu $w = \frac{1}{2}\left(z + \frac{1}{z} \right)$ akslantirish yordamida C_z tekislikdagi $D = \left\{ z \in C_z : \frac{\pi}{4} < \arg z < \frac{3\pi}{4} \right\}$ sohaning C_w tekislikdagi aksini toping.

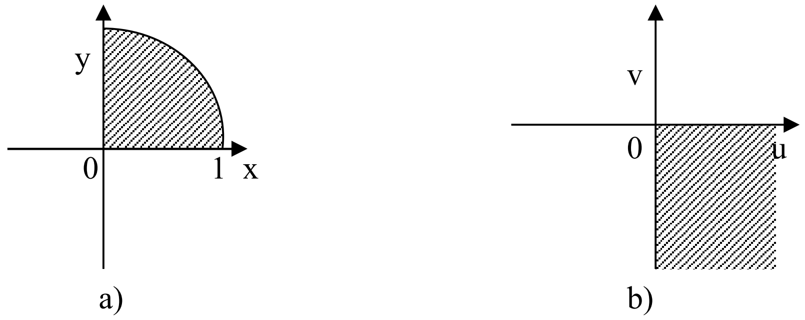
Yechilishi. Ravshanki, $D = \left\{ \frac{\pi}{4} < \arg z < \frac{3\pi}{4} \right\}$ sohaning chegarasi $l_1 : \arg z = \frac{\pi}{4}$ va $l_2 : \arg z = \frac{3\pi}{4}$ nurlardan iborat. Ma’lumki Jukovskiy funksiyasi C_z tekislikdagi $\{z \in C_z : \arg z = \alpha\}$ nurni C_w tekislikdagi $\frac{u^2}{\cos^2 \alpha} - \frac{v^2}{\sin^2 \alpha} = 1$ giperbolaning qismiga akslantiradi. Bundan kelib chiqadiki $l_1 : \arg z = \frac{\pi}{4}$ va $l_2 : \arg z = \frac{3\pi}{4}$ nurlar butun boshli $u^2 - v^2 = \frac{1}{2}$ giperbolaga akslanadi.

5.4-Misol. Ushbu $w = \frac{1}{2}\left(z + \frac{1}{z} \right)$ akslantirish yordamida C_z tekislikdagi

$D = \left\{ z \in C_z : |z| < 1; 0 < \arg z < \frac{\pi}{2} \right\}$ sohaning C_w tekislikdagi aksini toping.

Yechilishi. $D = \left\{ z \in C_z : |z| < 1; 0 < \arg z < \frac{\pi}{2} \right\}$ sohaning chegarasi $l_1 : \arg z = 0$, $l_2 : \arg z = \frac{\pi}{2}$ va $l_3 : \left\{ z \in C_z : \rho = 1; 0 < \phi < \frac{\pi}{2} \right\}$ chiziqlardan iborat.

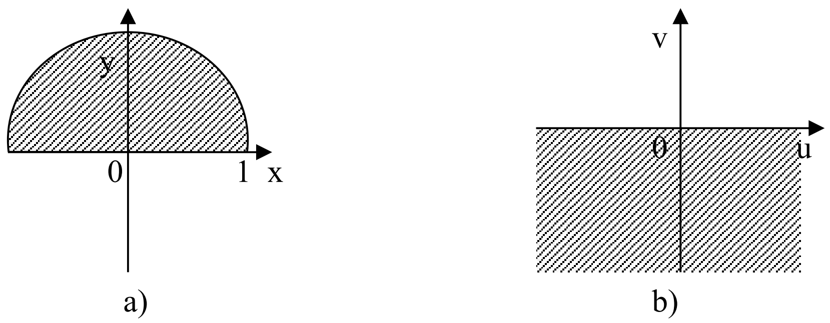
Ma’lumki Jukovskiy funksiyasi C_z tekislikdagi $l_1 : \arg z = 0$ nurni C_w tekislikdagi $\operatorname{Re} w = [1;\infty)$ bo’lgan nurga, $l_2 : \arg z = \dfrac{\pi}{2}$ nurni esa $u = 0$ to’g’ri chiziqqa akslantiradi. $l_3 : \left\{ z \in C_z : \rho = 1; 0 < \phi < \dfrac{\pi}{2} \right\}$ yoyni esa $\{ w \in C_w : 0 \le u \le 1; v = 0 \}$ chiziqqa akslantiradi. Bundan kelib chiqadiki $w(D) = \left\{ w \in C_w : \dfrac{3\pi}{2} < \arg w < 2\pi \right\}$.



5.1-chizma.

5.5-Misol. Ushbu $w = \dfrac{1}{2}\left(z + \dfrac{1}{z} \right)$ akslantirish yordamida C_z tekislikdagi $D = \{ z \in C_z : |z| < 1; 0 < \arg z < \pi \}$ sohaning C_w tekislikdagi aksini toping.

Yechilishi. $D = \{ z \in C_z : |z| < 1; 0 < \arg z < \pi \}$ sohaning chegarasi $l_1 : \arg z = 0$, $l_2 : \arg z = \pi$ va $l_3 : \{ z \in C_z : \rho = 1; 0 < \phi < \pi \}$ chiziqlardan iborat. Ma’lumki, Jukovskiy funksiyasi C_z tekislikdagi $l_1 : \arg z = 0$ nurni C_w tekislikdagi $\operatorname{Re} w = [1;\infty)$ bo’lgan nurga, $l_2 : \arg z = \pi$ nurni esa $\operatorname{Re} w = (-\infty; -1]$ bo’lgan nurga akslantiradi. $l_3 : \{ z \in C_z : \rho = 1; 0 < \phi < \pi \}$ yoyni esa $\{ w \in C_w : -1 \le u \le 1; v = 0 \}$ chiziqqa akslantiradi. Bundan kelib chiqadiki $w(D) = \{ w \in C_w : \pi < \arg w < 2\pi \}$.



5.2-chizma.

Mustaqil yechish uchun mashqlar

1. Jukovskiy funksiyasidan foydalanib quyidagi to'plamlarning aksini toping.

1) $|z| = \frac{1}{2}$, $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$.

2) $|z| = 2$, $\frac{3\pi}{4} < \arg z < \frac{5\pi}{4}$.

3) $|z| > 2$, $z \notin [2, +\infty)$.

4) $|z| < \frac{1}{2}$, $z \notin [-\frac{1}{2}; 0]$.

5) $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$, $z \notin [i, +i\infty)$.

6) $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$, $z \notin [0, 4i]$.

6. TRIGONOMETRIK FUNKSIYALAR VA ULAR YORDAMIDA BAJARILADIGAN AKSLANTIRISHLAR

Ushbu

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i},$$

$$tgz = \frac{\sin z}{\cos z} = -i\cdot\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}, \qquad ctgz = \frac{\cos z}{\sin z} = i\cdot\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$$

ko'rinishdagi funksiyalar *trigonometrik funksiyalar* deyiladi.

$w=\sin z,\ w=\cos z$ funksiyalarning aniqlanish sohasi butun C kompleks tekislik, $w=tgz$ funksiya

$$C\setminus\left\{z\in C: z=k\pi+\frac{\pi}{2}, k\in Z\right\}$$

to'plamda, $w=\tilde{n}tgz$ funksiya esa

$$C\setminus\{z\in C: z=k\pi, k\in Z\}$$

to'plamda aniqlangan.

1) $\cos z$ va $\sin z$ funksiyalar C kompleks tekislikda golomorf va ularning hosilalari

$$(\cos z)' = -\sin z, \quad (\sin z)' = \cos z$$

bo'ladi.

2) tgz funksiya

$$\left\{z\in C; \quad z\neq\frac{\pi}{2}+k\pi, \quad k=0,\ \pm1,\ \pm2,\ .\ .\ .\right\}$$

to'plamda, ctgz funksiya esa

$$\{z\in C; \quad z\neq k\pi, \quad k=0,\ \pm1,\ \pm2,\ .\ .\ .\}$$

to'plamda golomorf bo'ladi.

GIPERBOLIK FUNKSIYALAR

Quyidagicha

$$chz = \frac{e^z + e^{-z}}{2}, \qquad shz = \frac{e^z - e^{-z}}{2}, \qquad thz = \frac{shz}{chz}, \quad cthz = \frac{chz}{shz}$$

aniqlangan funksiyalar giperbolik funksiyalar deyiladi.

Xossa-6.1. Quyidagi tengliklar o'rinli:

$$\cos z = ch(iz) \; \sin z = -ish(iz) \quad thz = -itg(iz)$$

$$chz = \cos(iz),\; shz = -i\sin(iz),\; cthz = ictg(iz)$$

Isboti. Masalan, $shz = -i\sin(iz)$ ekanligini isbotlaymiz:

$$\sin(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{i^2 z} - e^{-i^2 z}}{2i} = -\frac{e^z - e^{-z}}{2i} = -\frac{1}{i}\frac{e^z - e^{-z}}{2} = -\frac{1}{i} shz \,.$$

Qolgan xossalarni ham shu usulda isbotlash mumkin.

Izoh-6.1. Trigonometrik funksiyalar ko'rsatkichli funksiya orqali ta'riflanganidan, ularning ko'rsatkichli funksiyalar xossalariga o'xshash xossalarga ega bo'lishi kelib chiqadi. Ayni paytdi trigonometrik funksiyalar orasida haqiqiy argumentli trigonometrik funksiyalar orasida munosabatlar kabi formulalar o'rinli bo'ladi.

Xossa-6.2. Quyidagi formulalar o'rinli:

1) $\sin^2 z + \cos^2 z = 1;$

2) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2;$

3) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2;$

4) $\sin\left(z + \frac{\pi}{2}\right) = \cos z, \qquad \cos\left(z + \frac{\pi}{2}\right) = -\sin z;$

5) $\sin(z + \pi) = -\sin z \qquad \cos(z + \pi) = -\cos z.$

Isboti. Birinchi formulani isbotlaymiz:

$$\sin^2 z+\cos^2 z=\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2+\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2=$$

$$\frac{e^{2iz}-2+e^{-2iz}}{-4}+\frac{e^{2iz}+2+e^{-2iz}}{4}=1.$$

3. $w=\sin z$ toq funksiya, $w=\cos z$ funksiya juft funksiya.

4. Trigonomerik funksiyalar davriy bo'lib, $w=\sin z$, $w=\cos z$ funksiyalarning davri 2π ga, $w=tgz$, $w=ñtgz$ funksiyalarning davri esa π ga teng.

Haqiqatan, $w=\sin z$ funksiya ta'rifi hamda

$$e^{2\pi i}=1$$

bo'lishini e'tiborga olib topamiz:

$$\sin(z+2\pi)=\frac{e^{i(z+2\pi)}-e^{-i(z+2\pi)}}{2i}=\frac{1}{2i}\left(e^{iz}\cdot e^{2\pi i}-e^{-iz}\cdot\frac{1}{e^{2\pi i}}\right)=\frac{e^{iz}-e^{-iz}}{2i}=\sin z.$$

Demak,

$$\sin(z+2\pi)=\sin z.$$

$w=tgz$ funksiyaning ta'rifidan foydalanib, ushbu

$$tg(z+\pi)=-i\cdot\frac{e^{i(z+\pi)}-e^{-i(z+\pi)}}{e^{i(z+\pi)}+e^{-i(z+\pi)}}=\frac{e^{i\pi}(e^{iz}-e^{-iz}\cdot e^{-2\pi i})}{e^{i\pi}(e^{iz}+e^{-iz}\cdot e^{-2\pi i})}=-i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}=tgz$$

tenglikka kelamiz. Demak, $tg(z+\pi)=tgz$.

5. $w=\sin z$, $w=\cos z$ funksiyalar yuqorida aytilganidek $\forall z\in C$ larda hosilaga ega va

$$(\sin z)'=\cos z,\ (\cos z)'=-\sin z.$$

$w=tgz$ funksiya $\forall z\in C\setminus\left\{z\in C: z=k\pi+\frac{\pi}{2}, k\in Z\right\}$ to'plamda hosilaga ega bo'lib,

$$(tgz)'=\frac{1}{\cos^2 z}$$

bo'ladi.

$w = ñtgz$ funksiya $\forall z \in C \setminus \{z \in C : z = k\pi, k \in Z\}$ to'plamda hosilaga ega bo'lib,

$$(ctgz)' = -\frac{1}{\sin^2 z}$$

bo'ladi.

Haqiqatan ham,

$$(\sin z)' = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)' = \frac{1}{2i}\left(ie^{iz} - ie^{-iz}\right) = \frac{e^{iz} + e^{-iz}}{2} = \cos z,$$

$$(\cos z)' = \left(\frac{e^{iz} + e^{-iz}}{2}\right)' = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) = \frac{i}{2}\left(e^{iz} - e^{-iz}\right) = -\frac{e^{iz} - e^{-iz}}{2i} = \sin z.$$

Izoh-6.2. Haqiqiy argumentli $\sin x$, $\cos x$ funksiyalarinig qiymatlari $[-1, 1]$ kesmada bo'lishini bilamiz. Kompleks argumentli $w = \sin z$, $w = \cos z$ funksiyalarinig qiymatlari modul jihatidan birdan katta ham bo'lishi mumkin:

Isboti. Ma'lumki,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Bu tenglikda $z = iy$ deb olamiz. Unda

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2}$$

bo'ladi. Ravshanki,

$$\lim_{y \to +\infty} \frac{e^{-y} + e^{y}}{2} = \infty$$

Bu esa $w = \cos z$ funksiyaning C da chegaralanmaganligini bildiradi.

TESKARI TRIGONOMETRIK FUNKSIYALAR

Kompleks o’zgaruvchili funksiyalar nazariyasida teskari funksiya tushunchasi haqiqiy o’zgaruvchili funksiyalar sinfidagi kabi kiritiladi.

Masalan,

$$w = Arc\cos z$$

funksiya $z = \cos w$ tenglamani qanoatlantiruvchi barcha w larning qiymatlari to’plamidan iborat, ya’ni $\cos z$ funksiyaga teskari funksiyadir.

$$Arc\sin z\ , \quad Arctgz\ , \quad Arcctgz$$

va boshqa funksiyalar ham shunga o’xshash aniqlanadi.

Ta’rifdan foydalanib

$$Arc\cos z = -iLn(z + \sqrt{z^2 - 1}) \tag{6.1}$$

tenglikning o’rinli ekanligini ko’rsatish qiyin emas. Bu erda ildizning barcha qiymatlari olinadi.

(6.1)-tenglikdan ko’rinib turibdiki, logarifmik funksiya kabi $Arc\cos z$ funksiya ham bir qiymatli emas. $Arc\cos z$ funksiyaning bosh qiymati $w = \arccos z$ deb olinadi va ushbu

$$w = \arccos z = -i\ln(z + \sqrt{z^2 - 1}) \tag{6.2}$$

tenglik yordamida aniqlanadi.

$w = Arc\cos z$ funksiya $\{z : \operatorname{Im} z > 0\}$ yuqori yarim tekislikda cheksiz ko’p qiymatli bo’lib, (6.2)–tenglikdan foydalanib uning bir qiymatli tarmoqlarini ajratish mumkin. Ular

$$(Arc\cos z)_k = -i(Ln(z + \sqrt{z^2 - 1}))_k \qquad k = 0, \pm 1, \pm 2, ...$$

tenglik yordamida aniqlanadi. Masalan, $k = 0$ bo’lsa,

$$(Arc\cos z)_0 = \arccos z = -i\ln(z + \sqrt{z^2 - 1})$$

funksiya

$$\{z : \operatorname{Im} z > 0\}$$

sohani

$$\{w : 0 < \operatorname{Re} w < \pi, \quad \operatorname{Im} z < 0\}$$

yarim yo’lakka konform akslantiradi.

Na'munaviy misol va masalalar yechimi

6.1-misol. Yig'indini hisoblang:

$$S = 1 + C_n^1 \cos\alpha + C_n^2 \cos 2\alpha + ... + C_n^n \cos n\alpha;$$

Yechilishi. Yana bir yig'indini olaylik:

$$T = C_n^1 \sin\alpha + C_n^2 \sin 2\alpha + ... + C_n^n \sin n\alpha.$$

U holda S quyidagi yig'indining haqiqiy qismidir:

$$S + iT = 1 + C_n^1(\cos\alpha + i\sin\alpha) + C_n^2(\cos 2\alpha + i\sin 2\alpha) + ... + C_n^n(\cos n\alpha + i\sin n\alpha).$$

Bundan quyidagini hosil qilamiz:

$$S + iT = 1 + C_n^1 e^{i\alpha} + C_n^2 e^{2i\alpha} + ... + C_n^n e^{in\alpha} = (1 + e^{i\alpha})^n = \left[e^{\frac{i\alpha}{2}}\left(e^{\frac{i\alpha}{2}} + e^{-\frac{i\alpha}{2}}\right)\right]^n =$$

$$= \left(e^{\frac{i\alpha}{2}} 2\cos\frac{\alpha}{2}\right)^n = 2^n \cos^n\frac{\alpha}{2} e^{ni\frac{\alpha}{2}} = 2^n \cos^n\frac{\alpha}{2}\left(\cos\frac{n\alpha}{2} + i\sin\frac{n\alpha}{2}\right) =$$

$$= 2^n \cos^n\frac{\alpha}{2}\cos\frac{n\alpha}{2} + i2^n \cos^n\frac{\alpha}{2}\sin\frac{n\alpha}{2}.$$

Shuning uchun

$$S = \operatorname{Re}(S + iT) = 2^n \cos^n\frac{\alpha}{2}\cos\frac{n\alpha}{2}.$$

6.2-misol. Yig'indini hisoblang:

$$S = 1 + a\cos\alpha + a^2\cos 2\alpha + a^3\cos 3\alpha + ... + a^k\cos k\alpha;$$

Yechilishi. Yig'indini hisoblash uchun ushbu yordamchi yig'indidan foydalanamiz:

$$T = a\sin\alpha + a^2\sin 2\alpha + a^3\sin 3\alpha + ... + a^k\sin k\alpha.$$

Bulardan

$$S + iT = 1 + a(\cos\alpha + i\sin\alpha) + a^2(\cos 2\alpha + i\sin 2\alpha) + ... + a^k(\cos k\alpha + i\sin k\alpha) =$$

$$= 1 + ae^{i\alpha} + a^2e^{2i\alpha} + ... + a^ke^{ki\alpha} = \frac{(ae^{i\alpha})^{k+1} - 1}{ae^{i\alpha} - 1} = \frac{(a^{k+1}e^{i\alpha(k+1)} - 1)(ae^{-i\alpha} - 1)}{(ae^{i\alpha} - 1)(ae^{-i\alpha} - 1)} =$$

$$=\frac{a^{k+2}e^{i\alpha k}-a^{k+1}e^{i\alpha(k+1)}-ae^{-i\alpha}+1}{a^2-a(e^{i\alpha}+e^{-i\alpha})+1}=$$

$$=\frac{a^{k+2}(\cos\alpha k+i\sin\alpha k)-a^{k+1}[\cos(k+1)\alpha+i\sin(k+1)\alpha]-a(\cos\alpha-i\sin\alpha)+1}{a^2-2a\cos\alpha+1}.$$

Bundan

$$S=\mathrm{Re}(S+iT)=\frac{a^{k+2}\cos k\alpha-a^{k+1}\cos(k+1)\alpha-a\cos\alpha+1}{a^2-2a\cos\alpha+1}$$

$$T=\mathrm{Im}(S+iT)=\frac{a^{k+2}\sin k\alpha-a^{k+1}\sin(k+1)\alpha+a\sin\alpha}{a^2-2a\cos\alpha+1}.$$

6.3-misol. Quyidagi $Arc\sin 2$ ifodaning barcha qiymatlarini toping.

Yechilishi. Bu tipdagi masalalarni yechishda isbotlash qiyin bo'lmagan quyidagi tengliklardan foydalaniladi.

1) $Arc\sin z=-iLni(z+\sqrt{z^2-1})$. 2) $Arc\cos z=-iLn(z+\sqrt{z^2-1})$.

3) $Arctgz=\frac{1}{2}Ln\frac{i+z}{i-z}=\frac{1}{2i}Ln\frac{1+iz}{1-iz}$. 4) $Arcctgz=\frac{i}{2}Ln\frac{z-i}{z+i}$

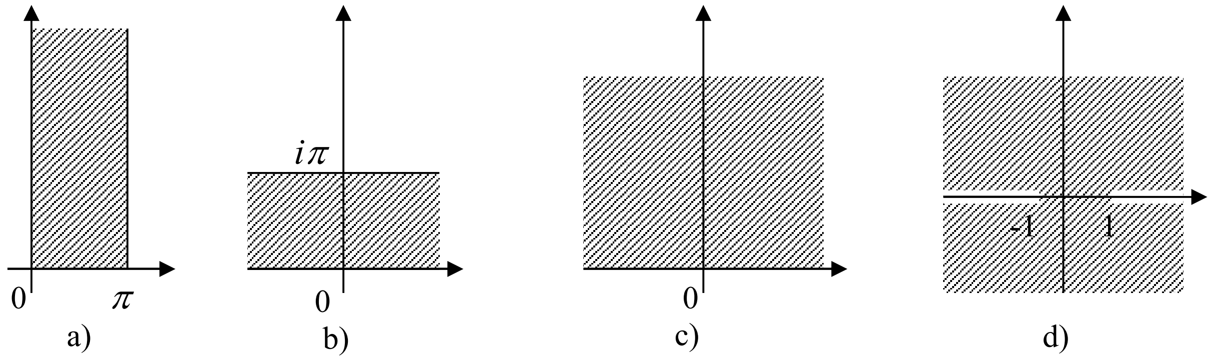
Bu tengliklarda ildizning barcha qiymatlari olingan. Biz 1)-tenglik va $Ln\,z$ ning formuladan foydalanamiz:

$$Arc\sin 2=-iLni(2+\sqrt{2^2-1})=-iLn(2\pm\sqrt{3})i=$$
$$=-i[\ln(2\pm\sqrt{3})+i\cdot\frac{\pi}{2}+2k\pi i]=\frac{\pi}{2}+2k\pi-i\ln(2\pm\sqrt{3})=$$
$$=\frac{(4k+1)\pi}{2}-i\cdot\ln(2\pm\sqrt{3}),\quad k\in Z.$$

6.4-misol. Quyidagi $D_1=\{z\in C_z:0<\mathrm{Re}\,z<\pi\}$ sohani $w=\cos z$ akslantirish yordamidagi aksini toping.

Yechilishi. $w=\cos z$ akslantirishni quyidagi akslantirishlar superpozitsiyasi ko'rinishida qarash mumkin: $w_1=iz$, $w_2=e^{w_1}$ va $w=\frac{1}{2}\left(w_2+\frac{1}{w_2}\right)$. Ya'ni $w=\cos z$ akslantirish dastlab burish, keyin ko'rsatkichli va Jukovskiy akslantirishlarini ketma-ket bajarishdan iborat ekan. $D_1=\{z\in C_z:0<\mathrm{Re}\,z<\pi\}$

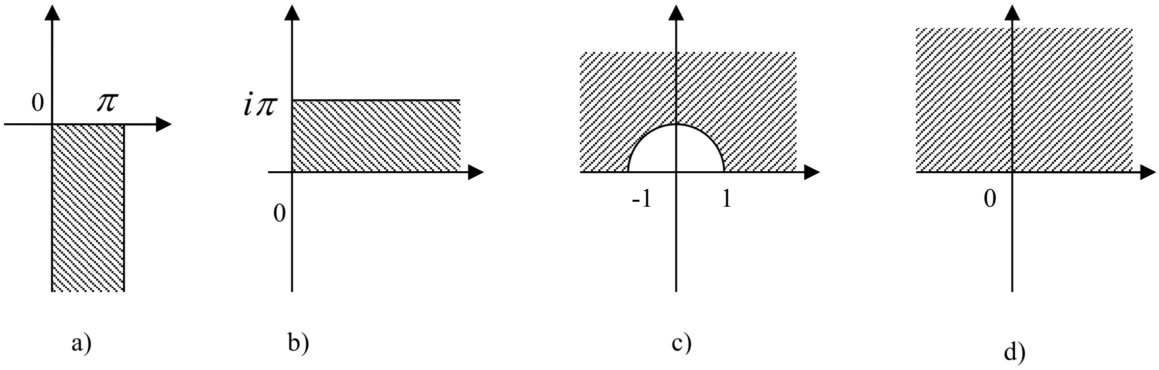
$w_1 = iz$ akslantirish yordamida $w_1(D_1) = \{w_1 \in C_w : 0 < \operatorname{Im} w_1 < \pi\}$ sohaga akslanadi. $D_2 = \{w \in C_w : 0 < \operatorname{Im} w_1 < \pi\}$ soha $w_2 = e^{w_1}$ yordamida $w_2(D_2) = \{w_2 \in C_w : \operatorname{Im} w_2 > 0\}$ sohaga akslanadi. $D_3 = \{w \in C_w : \operatorname{Im} w_2 > 0\}$ soha $w = \frac{1}{2}\left(w_2 + \frac{1}{w_2}\right)$ Jukovskiy akslantirishi orqali $w(D_3) = C_w \setminus \{w \in C_w : w \in [-\infty, -1] \cup [1, +\infty]\}$ sohaga akslanadi. Ya'ni $D = \{z \in C_z : 0 < \operatorname{Re} z < \pi\}$ sohani $w = \cos z$ akslantirish yordamidagi aksi $w(D_3) = C_w \setminus \{w \in C_w : w \in [-\infty, -1] \cup [1, +\infty]\}$ iborat ekan (6.1-chizma).



6.1-chizma

6.5 -misol. Quyidagi $D = \{z \in C_z : 0 < \operatorname{Re} z < \pi, \operatorname{Im} z < 0\}$ sohani $w = \cos z$ akslantirish yordamidagi aksini toping.

Yechilishi. $w = \cos z$ akslantirishni quyidagi akslantirishlar superpozitsiyasi ko'rinishida qarash mumkin: $w_1 = iz$, $w_2 = e^{w_1}$ va $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$. $w_1 = iz$ funksiya yordamida $D = \{z \in C_z : 0 < \operatorname{Re} z < \pi, \operatorname{Im} z < 0\}$ sohaning aksi $w_1(D) = \{w_1 \in C_w : 0 < \operatorname{Im} w_1 < \pi, \operatorname{Re} w_1 > 0\}$ bo'ladi. $w_2 = e^{w_1}$ funksiya orqali $D_1 = \{w \in C_w : 0 < \operatorname{Im} w_1 < \pi, \operatorname{Re} w_1 > 0\}$ sohaning aksi $w_2(D_1) = \{w_2 \in C_w : \operatorname{Im} w_2 > 0, |w_2| > 1\}$. $D_2 = \{w_2 \in C_w : \operatorname{Im} w_2 > 0, |w_2| > 1\}$ sohaning $w = \frac{1}{2}\left(w_2 + \frac{1}{w_2}\right)$ funksiya yordamida aksi $w(D_2) = \{w \in C_w : \operatorname{Im} w > 0\}$ bo'ladi. Ya'ni $D = \{z \in C_z : 0 < \operatorname{Re} z < \pi, \operatorname{Im} z < 0\}$ sohani $w = \cos z$ akslantirish yordamidagi aksi $w(D)=\{w\in C_w : \operatorname{Im} w>0\}$ yuqori yarim tekislikdan iborat ekan (6.2-chizma).

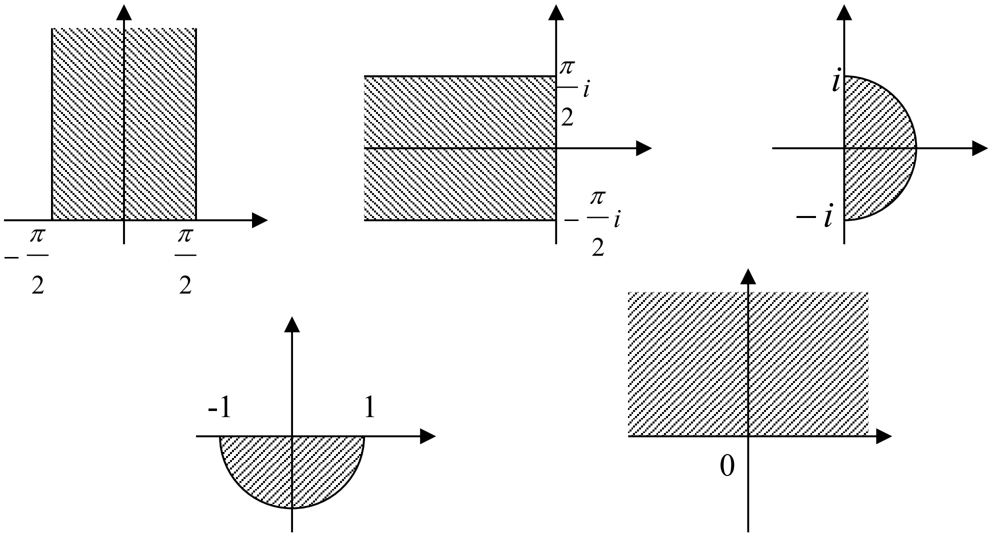


6.2-chizma.

6.5 -misol. Quyidagi $D=\left\{z\in C_z : -\frac{\pi}{2}<\operatorname{Re} z<\frac{\pi}{2}, \operatorname{Im} z>0\right\}$ sohani $w=\sin z$ akslantirish yordamidagi aksini toping.

Yechilishi. $w=\sin z$ akslantirishni quyidagi akslantirishlar superpozitsiyasi ko'rinishida qarash mumkin: $w_1=iz$, $w_2=e^{w_1}$, $w_3=-iw_2$ va $w=\frac{1}{2}\left(w_3+\frac{1}{w_3}\right)$.

$w_1=iz$ funksiya yordamida $D=\left\{z\in C_z : -\frac{\pi}{2}<\operatorname{Re} z<\frac{\pi}{2}, \operatorname{Im} z>0\right\}$ sohani aksi $w_1(D)=\left\{w_1\in C_w : -\frac{\pi}{2}<\operatorname{Im} w_1<\frac{\pi}{2}, \operatorname{Re} w_1<0\right\}$ bo'ladi. $w_2=e^{w_1}$ funksiya orqali $D_1=\left\{w_1\in C_w : -\frac{\pi}{2}<\operatorname{Im} w_1<\frac{\pi}{2}, \operatorname{Re} w_1<0\right\}$ sohaning aksi $w_2(D_1)=\{w_2\in C_w : \operatorname{Im} w_2>0, |w_2|<1\}$. $D_2=\{w_2\in C_w : \operatorname{Im} w_2>0, |w_2|<1\}$ sohaning $w_3=-iw_2$ funksiya yordamida aksi $w_3(D_2)=\{w_3\in C_w : \pi<\arg w_3<2\pi, |w_3|<1\}$ bo'ladi. Hosil bo'lgan sohaning $w=\frac{1}{2}\left(w_3+\frac{1}{w_3}\right)$ orqali aksi $w(D_3)=\{w\in C_w : \operatorname{Im} w>0\}$.Ya'ni $D=\left\{z\in C_z : -\frac{\pi}{2}<\operatorname{Re} z<\frac{\pi}{2}, \operatorname{Im} z>0\right\}$ sohani $w=\sin z$ akslantirish yordamidagi aksi $w(D)=\{w\in C_w : \operatorname{Im} w>0\}$ yuqori yarim tekislikdan iborat ekan (6.3-chizma).



6.3-chizma

6.6 -misol. Quyidagi $D=\{z\in C_z: 0<\operatorname{Re} z<1\}$ sohani $w=tgz$ akslantirish yordamidagi aksini toping.

Yechilishi. $w=tgz$ akslantirishni quyidagi akslantirishlar superpozitsiyasi ko'rinishida qarash mumkin: $w_1=2iz$, $w_2=e^{w_1}$, $w_3=\dfrac{w_2-1}{w_2+1}$ va $w=-iw_3$.

$w_1=iz$ funksiya yordamida $D=\{z\in C_z: 0<\operatorname{Re} z<1\}$ sohani aksi $w_1(D)=\{w_1\in C_w: 0<\operatorname{Im} w_1<2\pi\}$ bo'ladi. $w_2=e^{w_1}$ funksiya orqali $D_1=\{w_1\in C_w: 0<\operatorname{Im} w_1<2\pi\}$ sohaning aksi $w_2(D_1)=\{w_2\in C_w: w_2\notin[0;1]\}$.

$D_2=\{w_2\in C_w: w_2\notin[0;1]\}$ sohaning $w_3=\dfrac{w_2-1}{w_2+1}$ funksiya yordamida aksi

$w_3(D_2)=\{w_3\in C_w: w_3\notin[-1;1]\}$. $D_3=\{w_3\in C_w: w_3\notin[-1;1]\}$ $w=-iw_3$ funksiya orqali aksi $w(D_3)=\{w\in C_w: w\notin[-i;i]\}$. Demak, $D=\{z\in C_z: 0<\operatorname{Re} z<1\}$ $w=tgz$ funksiya orqali aksi $w(D)=C_w\setminus[-i;i]$ ekan .

6.7 -misol. Quyidagi $D=\left\{z\in C_z: -\dfrac{\pi}{4}<\operatorname{Re} z<\dfrac{\pi}{4}\right\}$ sohani $w=tgz$ akslantirish yordamidagi aksini toping.

Yechilishi. $w=tgz$ akslantirishni quyidagi akslantirishlar superpozitsiyasi ko'rinishida qarash mumkin: $w_1=2iz$, $w_2=e^{w_1}$, $w_3=\dfrac{w_2-1}{w_2+1}$ va $w=-iw_3$.

$w_1 = iz$ funksiya yordamida $D = \left\{ z \in C_z : -\frac{\pi}{4} < \operatorname{Re} z < \frac{\pi}{4} \right\}$ sohani aksi $w_1(D) = \left\{ w_1 \in C_w : -\frac{\pi}{2} < \operatorname{Im} w_1 < \frac{\pi}{2} \right\}$ bo'ladi. $w_2 = e^{w_1}$ funksiya orqali $D_1 = \left\{ w_1 \in C_w : -\frac{\pi}{2} < \operatorname{Im} w_1 < \frac{\pi}{2} \right\}$ sohaning aksi $w_2(D_1) = \{ w_2 \in C_w : \operatorname{Re} w_2 > 0 \}$. sohaning $w_3 = \frac{w_2 - 1}{w_2 + 1}$ funksiya yordamida aksi

$w_3(D_2) = \{ w_3 \in C_w : |w_3| < 1 \}$. $D_3 = \{ w_3 \in C_w : |w_3| < 1 \}$ $w = -iw_3$ funksiya orqali aksi $w(D_3) = \{ w \in C_w : |w| < 1 \}$. Demak, $D = \left\{ z \in C_z : -\frac{\pi}{4} < \operatorname{Re} z < \frac{\pi}{4} \right\}$ $w = tgz$ funksiya orqali aksi $w(D) = \{ w \in C_w : |w| < 1 \}$ ekan .

Mustaqil yechish uchun mashqlar

1. Quyidagi D to'plamning berilgan $w = f(z)$ akslantirish yordamidagi aksini toping.

a. $D = \{\operatorname{Re} z = 2\}$, $w = \cos z$.

b. $D = \{0 < \operatorname{Re} z < \frac{\pi}{2} , \operatorname{Im} z < 0\}$, $w = \sin z$

c. $D = \{0 < \operatorname{Re} z < 1 , \operatorname{Im} z > 0\}$, $w = tg\pi z$

2. Berilgan D sohani $G = \{\operatorname{Im} w > 0\}$ yuqori yarim tekislikka konform akslantiruvchi birorta $w(z)$ funksiyani toping.

a. $D = \{|z| < 1 \ , \ \operatorname{Im} z > 0\}$. **c.** $D = \{|z| < 1 \ , \ \operatorname{Re} z > 0\}$.

b. $D = \{|z| < 1 \ , \ \operatorname{Im} z < 0\}$. d. $D = \{|z| < 1 \ , \ \operatorname{Re} z < 0\}$.

7. Ilovalar

I. Chiziqli funksiya

1. $w = z + b$ akslantirish ixtiyoriy D sohani har bir nuqtasini b vektor bo'ylab siljitadi. Bunday akslantirishni parallel ko'chirish deyiladi.

2. $w = e^{ia}z\ (\alpha \in R)$ akslantirish ixtiyoriy D sohani har bir nuqtasini α burchakka buradi.

3. $w = mz$ akslantirish D soxga tegishli z ning radius vektorini r uzunligini m marta cho'zadi. Bunday akslantirishni o'xshashlik akslantirishi deyiladi.

4. $w = az + b$ akslantirish $t_1 = e^{i\alpha} \cdot z$ burish, $t_2 = mt_1$ o'xshashlik va $w = t_2 + b$ parallel ko'chirish akslantirishlarining birlashmasidir.

5. $w = az + b$ chiziqli akslantirishni kanonik ko'rinishi $w - z_0 = a(z - z_0)$

II. Kasr-chiziqli funksiya

1. $w = \dfrac{1}{z}$ akslantirish $\overline{C_z}$ tekislikda aylana yoki to'g'ri chiziqni $\overline{C_w}$ tekislikda aylana yoki to'g'ri chiziqqa o'tkazadi.

2. Har qanday kasr-chiziqli akslantirish $\overline{C_z}$ tekislikda Γ aylanaga nisbatan simmetrik bo'lgan z va z^* nuqtalarni $\overline{C_w}$ tekslikdagi Γ aylananing aksi $w(\Gamma)$ aylanaga nisbatan simmetrik bo'lgan $w(z)$ va $w(z_0)$ nuqtalarga o'tkazadi.

3. $\overline{C_z}$ tekislikda berilgan turli z_1, z_2, z_3 nuqtalarni $\overline{C_w}$ tekislikda berilgan turli w_1, w_2, w_3 nuqtalarga o'tuvchi kasr-chiziqli akslantirishni quyidagicha:

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

4. C_z yuqori yarim tekislik $\{z \in C_z : \operatorname{Im} z > 0\}$ ni $w = e^{i\varphi}\dfrac{z - a}{z - \overline{a}}\quad (\varphi \in R; \operatorname{Im} a > 0)$ kasr-chiziqli akslantirish $\overline{C_w}$ tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga akslantiradi.

5. C_z tekislikdagi birlik doira $\{z \in C_z : |z| < 1\}$ ni $w = e^{i\varphi}\dfrac{z-a}{1-\overline{a}z}$, $(|a| < 1)$ kasr-chiziqli akslantirish $\overline{C_w}$ tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga akslantiradi.

6. C_z tekislikdagi yuqori yarim tekislik $\{z \in C_z : \operatorname{Im} z > 0\}$ni $w = \dfrac{az+b}{cz+d}$, $(ad - bc > 0;\ a,b,c,d \in R)$ kasr-chiziqli akslantirish $\overline{C_w}$ tekislikdagi yuqori yarim tekislik $\{w \in C_w : \operatorname{Im} w > 0\}$ga akslantiradi.

7. C_z tekislikdagi birlik doira $\{z \in C_z : |z| < 1\}$ ni $\dfrac{w-w_0}{1-\overline{w}_0 w} = \dfrac{z-z_0}{1-\overline{z}_0 z}e^{i\alpha}$ kasr-chiziqli akslantirish C_w tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga $w(z_0) = w_0$, $(|w_0| < 1)$ $\arg w'(z_0) = \alpha$ shartlarni qanoatlantirib akslantiradi.

Eslatma 1. $w = e^{i\varphi}\dfrac{z-a}{z-\overline{a}}$ $(\varphi \in R; \operatorname{Im} a > 0)$ fuksiyaning $z = z_0$ nuqtadagi burchak koeffisienti $\arg w'(z_0) = \arg\dfrac{z_0 - \overline{z}_0}{(z_0 - \overline{z})^2}e^{i\varphi} = \arg\left(-i\dfrac{1}{\operatorname{Im} z_0}e^{i\varphi}\right) = \dfrac{\pi}{2} - \varphi$ ga teng bo'ladi.

8. C_z tekislikdagi yuqori yarim tekislik $\{z \in C_z : \operatorname{Im} z > 0\}$ ni

$$i\frac{w-w_0}{1-\overline{w}_0 w} = \frac{z-z_0}{z-\overline{z}_0}e^{i\alpha}$$

kasr-chiziqli akslantirish C_w tekislikdagi birlik doira $\{w \in C_w : |w| < 1\}$ ga $w(z_0) = w_0$, $\arg w'(z_0) = \alpha$ $(\operatorname{Im} z_0 > 0, |w_0| < 1, \alpha \in R)$ shartlarni qanoatlantirib akslantiradi.

Eslatma 2. $w = e^{i\varphi}\dfrac{z-a}{1-\overline{a}z}$ fuksiyaning $z = z_0$ nuqtadagi burchak koeffisienti $\arg w'(z_0) = \arg\dfrac{1}{1-|z_0|^2}e^{i\alpha} = \alpha$ ga teng bo'ladi.

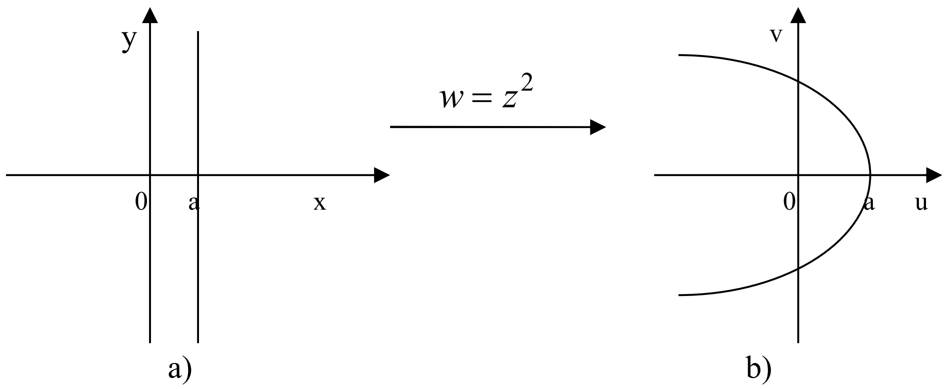
9. C_z tekislikdagi yuqori yarim tekislik $\{z \in C_z : \operatorname{Im} z > 0\}$ ni $\dfrac{w - w_0}{w - \overline{w}_0} = \dfrac{z - z_0}{z - \bar{z}_0} e^{i\alpha}$ kasr-chiziqli akslantirish C_w tekislikdagi yuqori yarim tekislik $\{w \in C_w : \operatorname{Im} w > 0\}$ga $w(z_0) = w_0, \arg w'(z_0) = \alpha$ shartlarni qanoatlantiradi.

Eslatma 3. z_1 va z_2 $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ aylanaga nisbatan simmetrik nuqtalar bo'lsa, u holda aylana markazi va radiusi $z_0 = \dfrac{z_1 - z_2 k^2}{1 - k^2}$, $R = \dfrac{k}{|1 - k^2|}|z_1 - z_2|$ formula orqali topiladi.

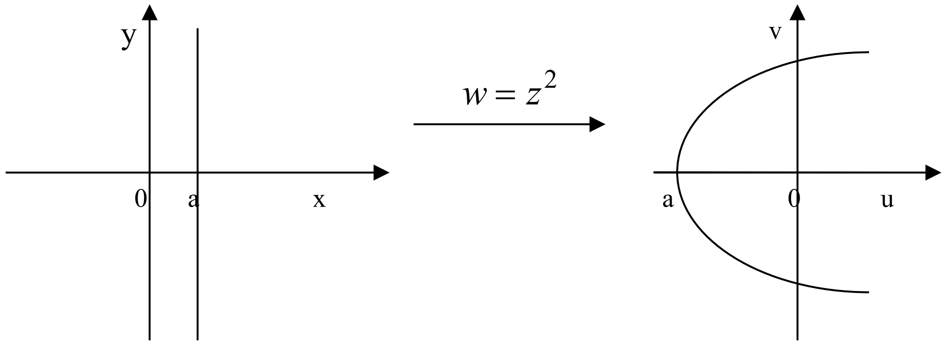
10. C_z tekislikdagi birlik doira $\{z \in C_z : |z| < 1\}$ ni $w = \dfrac{1}{z}$ kasr-chiziqli akslantirish C_w tekislikdagi birlik doirani tashqi qismi $\{w \in C_w : |w| > 1\}$ga akslantiradi.

III. Darajali funksiya.

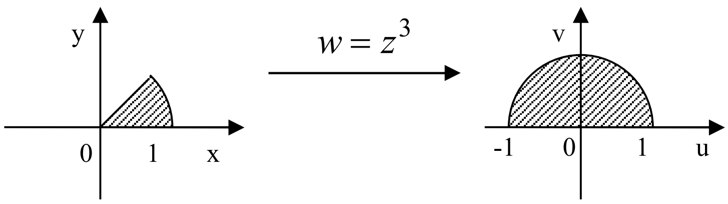
1. $w = z^2$ va $D = \{z \in C : \operatorname{Re} z = a\}$ $(a > 0)$ bo'lsa u holda $w(D) = \left\{z \in C : \operatorname{Re} w = a^2 - \dfrac{(\operatorname{Im} w)^2}{4a^2}\right\}$



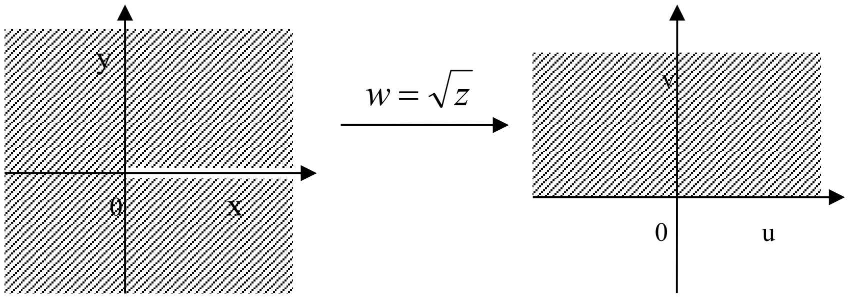
2. $w = z^2$ va $D = \{z \in C : \operatorname{Im} z = a\}$ $(a > 0)$ bo'lsa u holda $w(D) = \left\{z \in C : \operatorname{Re} w = \dfrac{(\operatorname{Im} w)^2}{4a^2} - a^2\right\}$.



3. $w=z^3$ va $E=\left\{z\in C:|z|<1, 0<\arg z<\dfrac{\pi}{3}\right\}$ bo'lsa, u holda $w(E)=\{w\in C_w : 0<\arg z<\pi, 0<r<1\}$.

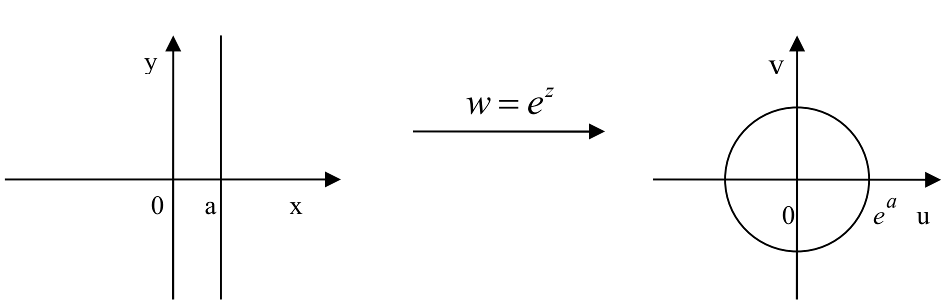


4. $w=\sqrt{w}$ va $G=\{w\in C_w : 0<\arg w<2\pi\}$ bo'lsa u holda $w(G)=\{z\in C:\operatorname{Im} w>0\}$

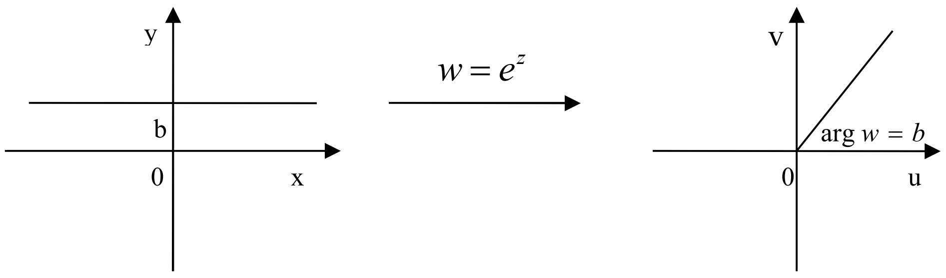


IV. Ko'rsatkichli funksiya.

1. $w=e^z$ va $D=\{z\in C:\operatorname{Re} z=a\}$ bo'lsa u holda $w(D)=\{w\in C_w : |w|=e^a\}$.

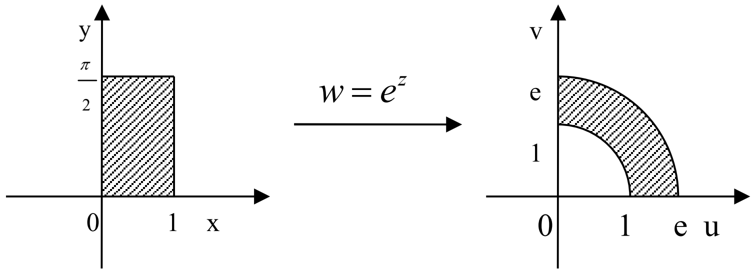


2. $w = e^z$ va $D = \{z \in C : \operatorname{Im} z = b\}$ bo'lsa u holda $w(D) = \{w \in C_w : \arg w = b\}$.

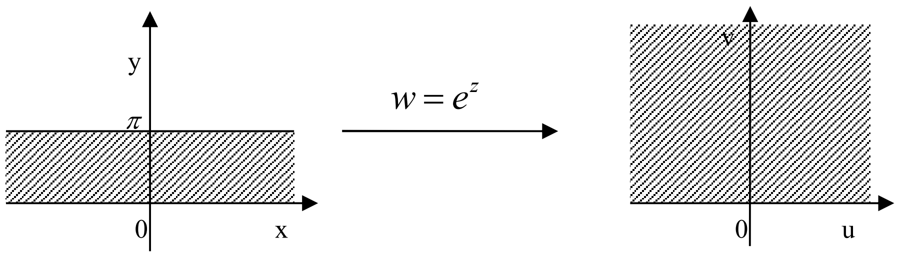


a) b)

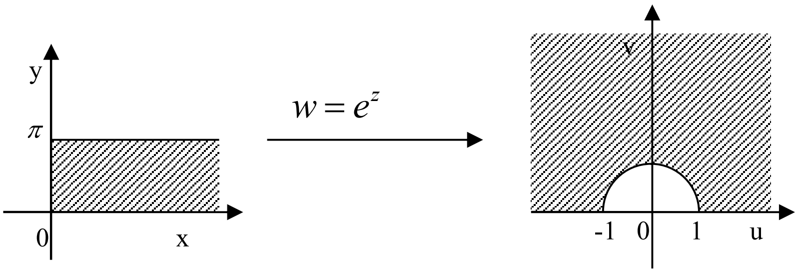
3. $w = e^z$ va $D = \left\{z \in C_z : 0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < \dfrac{\pi}{2}\right\}$ bo'lsa u holda $w(D) = \left\{w \in C_w : w = \rho e^{i\phi}; 1 < \rho < e, 0 < \phi < \dfrac{\pi}{2}\right\}$.



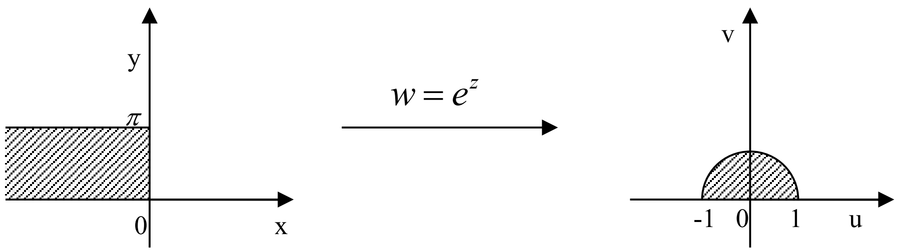
4. $w = e^z$ va $D = \{z \in C : 0 < \operatorname{Im} z < \pi\}$ bo'lsa u holda $w(D) = \{w \in C_w : \operatorname{Im} w > 0\}$.



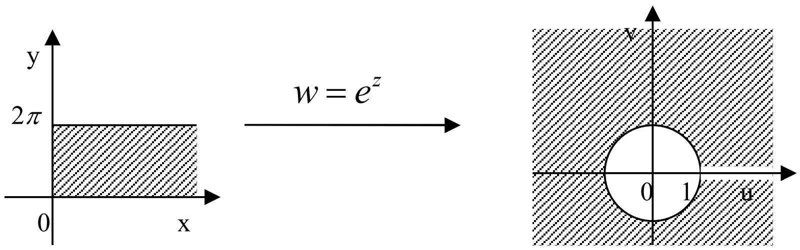
5. $w = e^z$ va $D = \{z \in C : 0 < \operatorname{Im} z < \pi, \operatorname{Re} z > 0\}$ bo'lsa u holda $w(D) = \{w \in C_w : |w| > 1, \operatorname{Im} w > 0\}$.



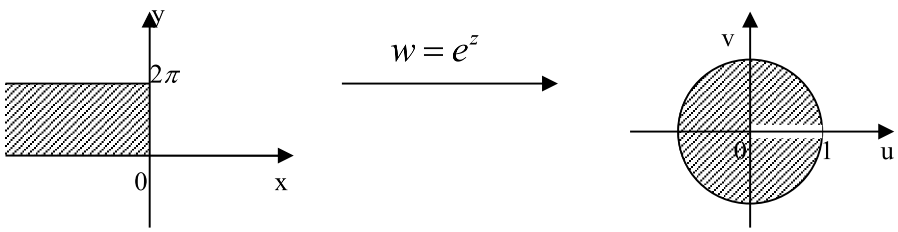
6. $w=e^z$ va $D=\{z\in C:0<\operatorname{Im}z<\pi,\operatorname{Re}z<0\}$ bo'lsa u holda $w(D)=\{w\in C_w:|w|<1,\operatorname{Im}w>0\}$.



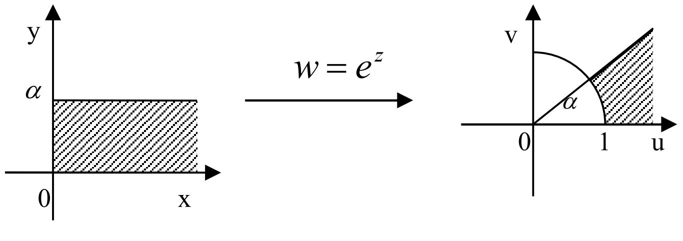
7. $w=e^z$ va $D=\{z\in C:0<\operatorname{Im}z<2\pi,\operatorname{Re}z>0\}$ bo'lsa u holda $w(D)=\{w\in C_w:|w|>1,\arg w\neq 0\}$.



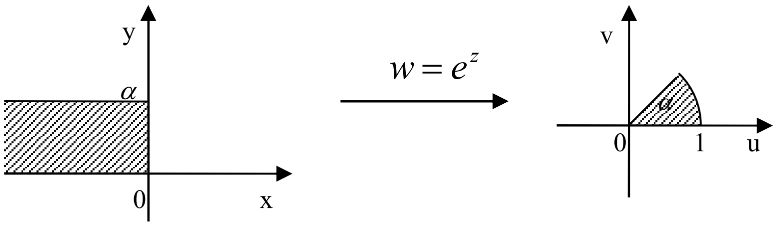
8. $w=e^z$ va $D=\{z\in C:0<\operatorname{Im}z<2\pi,\operatorname{Re}z<0\}$ bo'lsa u holda $w(D)=\{w\in C_w:|w|<1,\arg w\neq 0\}$.



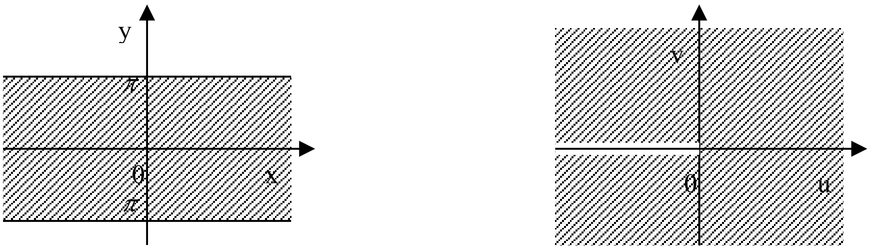
9. $w=e^z$ va $D=\{z\in C:0<\operatorname{Im}z<\alpha,\operatorname{Re}z>0\}$ $(0<\alpha<2\pi)$ bo'lsa u holda $w(D)=\{w\in C_w:|w|>1;0<\arg w<\alpha\}$.



10. $w=e^{z}$ va $D=\{z\in C: 0<\operatorname{Im}z<\alpha, \operatorname{Re}z<0\}$ $(0<\alpha<2\pi)$ bo'lsa u holda $w(D)=\{w\in C_w: |w|<1; 0<\arg w<\alpha\}$.



11. $w=e^{z}$ va $D=\{z\in C: -\pi<\operatorname{Im}z<\pi\}$ bo'lsa u holda $w(D)=\{w\in C_w: \arg w\neq\pi\}$.



V. Jukovskiy funksiyasi

1. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ Jukovskiy funksiyasi $w=\frac{1+j}{1-j}$, $j=h^{2}$ va $h=\frac{z-1}{z+1}$ funksiyalarning kombinatsiyasidan iborat.

2. $z=re^{i\varphi}$, $(z\in C_z)$, $w=u+iv$ $(z\in C_w)$ deyilsa u holda $u=\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\varphi$, $v=\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\varphi$ bo'ladi.

3. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ va $D=\{z\in C_z: z=\rho e^{i\varphi}\}(0<\rho<1)$ aylana bo'lsa, u holda $w(D)=\frac{u^{2}}{a_{\rho}^{2}}+\frac{v^{2}}{b_{\rho}^{2}}=1$, $\grave{a}_{\rho}=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)$, $b_{\rho}=\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)$ ellips bo'ladi.

4. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ va $D=\{z\in C_z: z=\rho e^{i\varphi}\}(\rho>1)$ aylana bo'lsa, u holda $w(D)=\frac{u^{2}}{a_{\rho}^{2}}+\frac{v^{2}}{b_{\rho}^{2}}=1$, $\grave{a}_{\rho}=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)$, $b_{\rho}=\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)$ ellips bo'ladi.

5. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ va $D=\{z\in C_z: |z|=1\}$ bo'lsa $w(D)=[-1;1]$ bo'ladi.

6. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ bo'lsa koordinata o'qlari quyidagicha bo'ladi:

$$\left\{z\in C_z : \arg z=\frac{\pi}{2}\right\}\to\{w\in C_w : \operatorname{Re} w=0\},$$

$$\left\{z\in C_z : \arg z=\frac{3\pi}{2}\right\}\to\{w\in C_w : \operatorname{Re} w=0\},$$

$$\left\{z\in C_z : \arg z=0\right\}\to\{w\in C_w : \operatorname{Re} w=[1,+\infty]\},$$

$$\left\{z\in C_z : \arg z=\pi\right\}\to\{w\in C_w : \operatorname{Re} w=[-\infty,-1]\}.$$

7. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ bo'lsa C_z tekislikdagi $\{z\in C_z : \arg z=\alpha\}$ nurni C_w tekislikdagi $\frac{u^2}{\cos^2\alpha}-\frac{v^2}{\sin^2\alpha}=1$ giperbolaga akslantiradi.

8. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ va $D_1=\{z\in C_z :|z|>1\}$ bo'lsa $w(D_1)=C_w\setminus\{w\in C_w : w\in[-1,1]\}$.

9. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ va $D_2=\{z\in C_w :|z|<1\}$ bo'lsa $w(D_2)=C_w\setminus\{w\in C_w : w\in[-1,1]\}$.

10. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ va $D_3=\{w\in C_z : \operatorname{Im} z>0\}$ bo'lsa u holda: $w(D_3)=C_w\setminus\{w\in C_w : w\in[-\infty,-1]\cup[1,+\infty]\}$.

11. $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ va $D_4=\{w\in C_z : \operatorname{Im} z<0\}$ bo'lsa u holda: $w(D_4)=C_w\setminus\{w\in C_w : w\in[-\infty,-1]\cup[1,+\infty]\}$.

TRIGONOMETRIK FUNKSIYALAR.

1. $w = \cos z$ akslantirishni quyidagi akslantirishlar superpozitsiyasi ko'rinishida qarash mumkin: $w_1 = iz$, $w_2 = e^{w_1}$ va $w = \frac{1}{2}\left(w_2 + \frac{1}{w_2}\right)$.

2. $w = \sin z$ akslantirishni quyidagi akslantirishlar superpozitsiyasi ko'rinishida qarash mumkin: $w_1 = iz$, $w_2 = e^{w_1}$, $w_3 = -iw_2$ va $w = \frac{1}{2}\left(w_3 + \frac{1}{w_3}\right)$.

3. $w = tgz$ akslantirishni quyidagi akslantirishlar superpozitsiyasi ko'rinishida qarash mumkin: $w_1 = 2iz$, $w_2 = e^{w_1}$, $w_3 = \frac{w_2 - 1}{w_2 + 1}$ va $w = -iw_3$.

FOYDALANILGAN ADABIYOTLAR

1. **Xudoyberganov G., Vorisov A., Mansurov X.** *Kompleks analiz.* T. “Universitet”, 1998.
2. **Sadullaev A., Xudoyberganov G., Mansurov X., Vorisov A., Tuychiev T.** *Matematitk analiz kursidan misol va masalalar to`plami (kompleks analiz) 3 qism.* T. “O`zbekiston”, 2000.
3. **Ahlfors L.** *Complex analysis.* McGraw-Hill Education, 1979.
4. **Шабат Б. В.** *Введение в комплексный анализ.* M. URSS, 2015.
5. **Волковыский Л. И., Лунц Г. А., Араманович И. Г.** *Сборник задач по теории функций комплексного переменного.* М. «ФИЗМАТЛИТ», 2002.
6. **Palka B. P.** *Complex analysis.* Springer, Germany, 1995.
7. **Бицадзе А. В.** *Основы теории аналитических функций комплексного переменного.* М. «Наука», 1972.
8. **Xakimov R., Maxammadaliyev M.** *Kompleks o`zgaruvchili funksiyalar nazariyasi*. O`UM. Namangan. Universitet, 2019.
9. **Xakimov R., Sattarov I.** *Kompleks o`zgaruvchili funksiyalar nazariyasidan mashqlar.* Uslubiy qo`llanma. N. Universitet, 2017.