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AMALIY MATEMATIKA 1

TOSHKENT

O‘ZBEKISTON RESPUBLIKASI OLIY VA O‘RTA MAXSUS
TA’LIM VAZIRLIGI

TOSHKENT DAVLAT IQTISODIYOT UNIVERSITETI

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AMALIY MATEMATIKA 1

(Kredit-modul bo‘yicha)

O‘zbekiston Respublikasi Oliy va o‘rta maxsus ta’lim vazirligi
tomonidan o’quv qo’llanma sifatida tavsiya etilgan

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O’quv qo’llanmada matematikaning keyingi boblarini, shuningdek iqtisodiyot, statistika va biznes, menejment va axborot texnologiyalari sohasidagi umumiy nazariy maxsus fanlarni muvaffaqiyatli o‘zlash-tirish uchun zarur bo‘lgan «Oliy matematika» fanining asosiy bo‘limlari keltirilgan.

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Kirish

Ushbu o‘quv qo‘llanma o‘zbek tilida universitet va institutlarda iqtisodchi mutaxassisliklarida o‘qitiladigan oliy matematika fanining o‘quv dasturiga moslab yozilgan o‘quv darslik va qo‘llanmalarning kamligini hisobga olgan holda yozilgan. Qo‘llanma o‘z ichiga aniqlovchilar, matritsalar, chiziqli tenglamalar sistemasi, vektorlar, funksiyalar, limitlar, funksiyaning xosilali va differensiali, bir necha o‘zgaruvchining funksiyasining xosilasi va differensiali, aniqmas integral, aniq integral va qatorlarga doir qisqacha nazariy materiallarni, mashqlarni, misol va masalalarini qamrab olgan.

Xar bir bobda va mavzularda yechib ko‘rsatilgan misollarni qunt bilan takroran ishlab xar bir o‘quvchi mashqda berilgan misollarni mustaqil yyechish imkoniyatiga ega bo‘ladi.

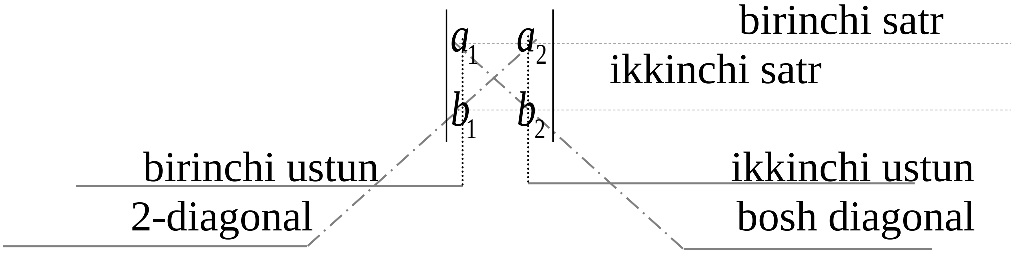
Bu qo‘llanmadan o‘quv dasturining xajmi va mazmuniga ko‘ra hamma turdagi iqtisodchi talabalar shuningdek, texnika, qishloq xo‘jaligi, pedagogika oliy o‘quv yurtlarining ba’zi fakultetlari talabalari qo‘shimcha o‘quv qo‘llanma sifatida to‘liq foydalanishlari mumkin.

MAVZU -1: MATRITSA VA ULAR USTIDA AMALLAR

1-§. Aniqlovchilar, ularni hisoblash usullari va asosiy hossalari

To‘rtta a_1, a_2, b_1, b_2 –haqiqiy sonlar berilgan bo‘lsin.

Ta’rif: $a_1b_2 - a_2b_1$ haqiqiy songa 2-tartibli aniqlovchi (yoki determinant) deyiladi va quyidagicha yoziladi. Ta’rifga binoan $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$ 2-tartibli aniqlovchida a_1, a_2, b_1, b_2 –sonlar uning elementlari deyiladi, undagi yo‘llar biri-biridan farqlaniladi va quyidagicha nomlanadi.



Aniqlovchilarning quyidagi asosiy hossalarini 2-tartibli aniqlovchi misolida osongina tekshirib ko‘rish mumkin. Aniqlovchilarning:

a) mos satrlari va ustunlari o‘zaro almashtirilganda qiymatlari o‘zgarmaydi:

$$(a_1b_2 - a_2b_1) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

b) ikki satrlari (yoki ustunlari) o‘rinlari o‘zaro almashtirilganda ishoralari qarama-qarshiga o‘zgaradi:

v) biror bir satri (yoki ustuni) har bir elementini k haqiqiy son marta orttirganda, aniqlovchining o‘zi ham k marta ortadi.

Boshqacha aytganda, satr (ustun) laridagi umumiy ko‘paytuv–chilarni aniqlovchi ishorasining tashqarisiga chiqarish mumkin.

g) biror satr (yoki ustun) elementlari faqat nollardan iborat bo‘lsa, aniqlovchining qiymati 0ga teng bo‘ladi.

d) ikki satr (yoki ustun) lari mos elementlari o‘zaro teng yoki proporsional bo‘lsa, aniqlovchining qiymati 0 ga teng bo‘ladi. Yuqoridagi xossalar, 3-tartibli va umuman, ixtiyoriy n tartibli aniqlovchilar uchun ham o‘rinlidir.

$3^2 = 9$ òà $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ - haqiqiy sonlar berilgan bo‘lsin.

Ta’rif: $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{32} = \Delta_3$ - haqiqiy songa 3-tartibli aniqlovchi deyiladi va quyidagicha yoziladi:

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |a_{ik}|$$

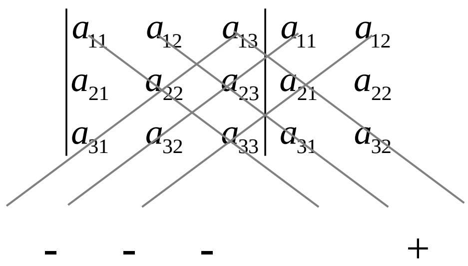
$a_{ik}(i,\ k = 1,\ 2,\ 3)$ sonlar uning elementlari deyiladi ($a_{ij} -$ element $i -$satr va $j -$ustun kesishmasidagi element).

Ta’rifga binoan:

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + \quad (1)$$
$$+a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

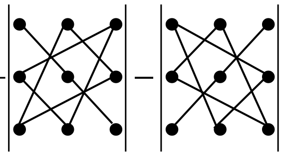
(1) ifodani Sarryus qoidasi yordamida osongina tuzish mumkin.

Buning uchun Sarryus jadvalini tuzamiz va bosh diagonal yo‘nalishidagi elementlarini o‘zaro ko‘paytirib, “+” ishora bilan, ikkilamchi diagonal yo‘nalishidagi elementlarini o‘zaro ko‘paytirib, “-” ishora bilan olib, yig‘indi ifoda hosil qilamiz



Agar har bir ko‘paytmada elementlarni satr nomerini o‘sish tartibida joylashtirsak, Sarryus jadvalidan (1) ifodaning aynan o‘zi yuzaga keladi. (1) ifodada ishorasi bilan birga har bir ko‘paytma aniqlovchining hadi deyiladi.

3-tartibli aniqlovchini “uchburchak usuli”da ham hisoblash mumkin:

$\Delta_3 = +$ 

(I) ifoda mos ravishda musbat va manfiy ishorali hadlari elementlarining ustun nomerlarining o‘zgarishi tartibi quyidagicha:

$$\left.\begin{matrix} 1,\ 2,\ 3 \\ 2,\ 3,\ 1 \\ 3,\ 1,\ 2 \end{matrix}\right\} \quad (2) \qquad \left.\begin{matrix} 3,\ 2,\ 1 \\ 1,\ 3,\ 2 \\ 2,\ 1,\ 3 \end{matrix}\right\} \quad (3)$$

Yuqoridagilar 1, 2 va 3 sonlar ustidagi o‘rin almashtirishlardir. (1, 2, 3) tartibga asosiy o‘rin almashtirish deyiladi.

Agar berilgan o‘rin almashtirishda uning ikki elementlari o‘rinli almashtirilsa (transpozitsiyalansa), natijada yangi o‘rin almashtirish yuzaga keladi. Masalan:

$$\left(1,\ \overset{\leftarrow}{\underset{\rightarrow}{2}},\ 3\right) \rightarrow (3,\ 2,\ 1)$$

Asosiy o‘rin almashtirishdan hosil qilish mumkin bo‘lgan ixtiyoriy tartib $j = (j_1,\ j_2,\ j_3)$ bo‘lsin, bu yerda $j_1,\ j_2,\ j_3$ – 1, 2, 3 sonlarning biri bo‘lib, har bir tartibda bir martadan uchraydi. Asosiy o‘rin almashtirish (1, 2, 3) dan $j = (j_1,\ j_2,\ j_3)$ ixtiyoriy o‘rin almashtirishni hosil qilish uchun kerak bo‘lgan o‘rin almashtirishlar (transpozitsiyalar) soni $t(j)$ bilan belgilaylik. Agar $t(j)$ juft son bo‘lsa, j o‘rin almashtirish juft tartibli va agar toq son bo‘lsa, toq tartibli o‘rin almashtirish deyiladi. Masalan, (2) o‘rin almashtirishning har biri juft tartibli, (3) o‘rin almashtirishlar esa toq tartiblidir. Yuqoridagi tushunchalardan foydalanib, 3-tartibli aniqlovchiga quyidagi umumiyroq ta’rifni berish mumkin:

3 – tartibli aniqlovchi (determinant) deb, quyidagi yig‘indiga teng bo‘lgan Δ_3 songa aytiladi (yig‘indida 31 ta had bor)

$$\Delta_3 = \sum_j (-1)^{t(j)} a_{1j_1} a_{2j_2} a_{3j_3}$$

bu yerda $j = (j_1,\ j_2,\ j_3)$ asosiy o‘rin almashtirish (1, 2, 3)dan yuzaga kelishi mukin bo‘lgan barcha o‘rin almashtirishlarning biridir:

h^2 ta $a_{ik}(i,k = 1,2,\dots,n)$– haqiqiy sonlar berilgan bo‘lsin.

Ta’rif: n– tartibli aniqlovchi (determinant) deb, quyidagi yig‘indiga teng bo‘lgan Δ_n songa aytiladi. $\Delta_n = \sum_j (-1)^{t(j)} a_{1j_1} a_{2j_2} a_{3j_3}$ va quyidagicha yoziladi (yig‘indida $n!$ ta had bor)

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2k} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ik} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nk} & \dots & a_{nn} \end{vmatrix} = |a_{ik}|, \quad (i,k = 1,\ 2,\ \dots,\ n)$$

bu yerda $j = (j_1,\ j_2,\ j_3,\ \dots,\ j_n)$ asosiy o‘rin almashtirish $(1,\ 2,\ \dots,\ n)$dan yuzaga kelishi mumkin bo‘lgan barcha o‘rin almashtirishlarning biridir, $t(j)-(1,\ 2,\ \dots,n)$dan j ga o‘tish uchun kerakli transpozitsiyalar soni.

Ta’rif: $n-$tartibli aniqlovchining a_{ik} elementi minori deb, shu element joylashgan $i-$satr va $k-$ustun o‘chirilgandan so‘ng qoladigan $n-1$- tartibli determinantga (aniqlovchiga) aytiladi va M_{ik} deb belgilanadi.

Ta’rif: a_{ik} elementining algebraik to‘ldiruvchisi (yoki ad’yunkti) deb, quyidagi A_{ik} songa aytiladi: $A_{ik} = (-1)^{i+k} M_{ik}$.

Yuqorida keltirilgan a), b), v), g) va d) xossalarga qo‘shimcha, yuqori tartibli aniqlovchilarni hisoblashda muhim ahamiyatga ega bo‘lgan xossalarni ham isbotsiz keltirib o‘tamiz:

ye) ixtiyoriy $n-$tartibli aniqlovchining qiymati biror satr (ustun) elementlarining o‘z algebraik to‘ldiruvchilariga ko‘paytmalarining yig‘indisi teng:

$$\Delta_n = \sum_{k=1}^{n} a_{ik} \cdot A_{ik} \quad (i=1,\ 2,\ \dots,n) \tag{4}$$

$$\Delta_n = \sum_{k=1}^{n} a_{ik} \cdot A_{ik} \quad (k=1,\,2,\,\ldots,n) \qquad (5)$$

(4) formula $n-$tartibli aniqlovchini $i-$satr elementlari bo‘yicha yoyish formulasi deyiladi, (5) esa $k-$ustun elementlari bo‘yicha yoyib hisoblash formulasi deyiladi.

ye) aniqlovchida uning ixtiyoriy satri (ustuni) elementlarining boshqa parallel satr (ustun) mos elementlari algebraik to‘ldiruvchilarga ko‘paytmalari yig‘indisi 0 ga teng:

$$\sum_{k=1}^{n} a_{ik} \cdot A_{ik} = 0 \quad (i,\, j = 1, 2, ..., n\ i \neq j)$$

$$\sum_{k=1}^{n} a_{ik} \cdot A_{ik} = 0 \quad (k,\, j = 1, 2, ..., n\ i \neq j)$$

j) aniqlovchining qiymati uning ixtiyoriy satri (ustuni) elementlariga boshqa bir parallel satr (ustun) mos elementlarini bir xil songa ko‘paytirib qo‘shganda o‘zgarmaydi.

Masalan:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} + k \cdot a_{11} & a_{13} \\ a_{21} & a_{22} + k \cdot a_{21} & a_{23} \\ a_{31} & a_{32} + k \cdot a_{31} & a_{33} \end{vmatrix}$$

z) agar aniqlovchining $k-$ustuni har bir elementi ikkita sonlar yig‘indisidan iborat bo‘lsa, ya’ni $a_{ik} = b_{ij} + c_{ik} \quad (i = 1, 2, ..., n)$,

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & ... & b_{1k} + c_{1k} & ... & a_{1n} \\ a_{21} & a_{22} & ... & b_{2k} + c_{2k} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & b_{ik} + c_{ik} & ... & a_{in} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & b_{nk} + c_{nk} & ... & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & ... & b_{1k} & ... & a_{1n} \\ a_{21} & a_{22} & ... & b_{2k} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & b_{ik} & ... & a_{in} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & b_{nk} & ... & a_{nn} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{11} & a_{12} & ... & c_{1k} & ... & a_{1n} \\ a_{21} & a_{22} & ... & c_{2k} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & c_{ik} & ... & a_{in} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & c_{nk} & ... & a_{nn} \end{vmatrix} = \Delta_n^{(1)} + \Delta_n^{(2)}$$

(Yuqoridagi qoida satrlar uchun ham o‘rinli).

i) ikki $n-$tartibli $\Delta_n^{(1)}=|d_{ik}|$ va $\Delta_n^{(2)}=|\beta_{ik}|$ aniqlovchilar ko‘paytmasi deb, elementlari quyidagicha hisoblanadigan $n-$tartibli $\Delta_n^{(3)}=|\gamma_{ik}|$ aniqlovchiga aytiladi:

$$|\gamma_{ik}|=\Delta_n^{(3)}=\Delta_n^{(1)}\cdot\Delta_n^{(2)}=|\alpha_{ik}|\cdot|\beta_{ik}|=\left|\sum_{j=1}^{n}\alpha_{ij}\cdot\beta_{ij}\right|,\quad (i,\ k=1,2,...,n)$$

Mashqlar

Aniqlovchilarga doir amaliy misol va masalalar yechimlari

1. Hisoblang:

a) $\begin{vmatrix} 7 & -5 \\ -3 & 2 \end{vmatrix}=7\cdot 2-(-5)\cdot(-3)=14-15=-1$

b)
$$\begin{vmatrix} \sqrt{a}+\sqrt{b} & \sqrt{a}-\sqrt{b} \\ \sqrt{a}-\sqrt{b} & \sqrt{a}+\sqrt{b} \end{vmatrix}=\left(\sqrt{a}+\sqrt{b}\right)^2-\left(\sqrt{a}-\sqrt{b}\right)^2=a+2\sqrt{ab}+b-a+2\sqrt{a\cdot b}-b=4\sqrt{ab}$$

v) $\begin{vmatrix} \sin 1^0 & \sin 91^0 \\ -\cos 1^0 & \cos 89^0 \end{vmatrix}=\begin{vmatrix} \sin 1^0 & \cos 1^0 \\ -\cos 1^0 & \sin 1^0 \end{vmatrix}=\sin^2 1^0-\left(-\cos^2 1^0\right)=\sin^2 1^0+\cos^2 1^0=1$

g) $\begin{vmatrix} \frac{x+y}{x} & \frac{2x}{x-y} \\ \frac{y-x}{x^2+y^2} & \frac{y-x}{x^2+y^2} \end{vmatrix}=\frac{y-x}{x^2+y^2}\cdot\begin{vmatrix} \frac{x+y}{x} & \frac{2x}{x-y} \\ 1 & 1 \end{vmatrix}=\frac{y-x}{x^2+y^2}\cdot\left(\frac{x+y}{x}-\frac{2x}{x-y}\right)=$

$$=\frac{-(x-y)}{x^2+y^2}\cdot\frac{x^2-y^2-2x^2}{x(x-y)}=-\frac{-\left(x^2+y^2\right)}{x}=\frac{1}{x},\qquad \begin{pmatrix} x\neq 0 \\ x\neq y \end{pmatrix}$$

2. Berilgan o‘rin almashtirishlarning juft yoki toqligini aniqlang.

a) $(2,\ 1,\ 3):\begin{pmatrix} \overset{\leftarrow}{1,\ 2},\ 3 \\ \underset{\rightarrow}{} \end{pmatrix}\rightarrow(2,\ 1,\ 3),\quad t=1$

Javob. toq

b) $(3,\ 1,\ 2):\begin{pmatrix} \overset{\longleftarrow}{1,\ 2,\ 3} \\ \underset{\longrightarrow}{} \end{pmatrix}\rightarrow\begin{pmatrix} 3,\ \overset{\leftarrow}{2,\ 1} \\ \underset{\rightarrow}{} \end{pmatrix}\rightarrow(3,\ 1,\ 2),\quad t=2$

Javob. juft

v) $(4,\ 2,\ 3,\ 1):\left(\overset{\longleftarrow}{\underset{\longrightarrow}{1,\ 2,\ 3,\ 4}}\right)\to(4,\ 2,\ 3,\ 1),\quad t=1$

Javob. toq

g) $(2,\ 1,\ 4,\ 3):\left(\overset{\leftarrow}{\underset{\rightarrow}{1,\ 2}},\ 3,\ 4\right)\to\left(2,\ 1,\ \overset{\leftarrow}{\underset{\rightarrow}{3,\ 4}}\right)\to(2,\ 1,\ 4,\ 3),\quad t=2$

Javob. juft

d) $(1,\ 2,\ 5,\ 4,\ 3):\left(1,\ 2,\ \overset{\longleftarrow}{\underset{\longrightarrow}{3,\ 4,\ 5}}\right)\to(1,\ 2,\ 5,\ 4,\ 3),\quad t=1$

Javob. toq

e) $(5,\ 1,\ 2,\ 3,\ 4):\left(1,\ 2,\ 3,\ \overset{\leftarrow}{\underset{\rightarrow}{4,\ 5}}\right)\to(1,\ 2,\ 3,\ 5,\ 4)\to\left(1,\ \overset{\leftarrow}{\underset{\rightarrow}{2,\ 5}},\ 3,\ 4\right)\to$

$\to(1,\ 5,\ 2,\ 3,\ 4)\to(1,\ 2,\ 3,\ 5,\ 4),\quad t=4$

Javob. juft

3. <u>Berilgan 3-tartibli aniqlovchilarini kamida 4 ta usulda hisoblang</u>

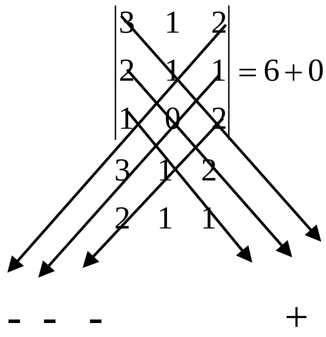
a) $\begin{vmatrix}3 & 1 & 2\\ 2 & 1 & 1\\ 1 & 0 & 2\end{vmatrix}=?$ b) $\begin{vmatrix}0 & x & 0\\ x & 1 & x\\ 2 & x & 2\end{vmatrix}=?$

1. a) Sarryus qoidasini qo‘llab hisoblaymiz (to‘g‘ridan – to‘g‘ri ta’rifdan foydalanish yoki uchburchak usulini qo‘llash ham mumkin edi):

$$\begin{vmatrix}3 & 1 & 2\\ 2 & 1 & 1\\ 1 & 0 & 2\end{vmatrix}\begin{matrix}3 & 1\\ 2 & 1\\ 1 & 0\end{matrix} = 3\cdot1\cdot2+1\cdot1\cdot1+2\cdot2\cdot0-2\cdot1\cdot1-3\cdot1\cdot0-1\cdot2\cdot2=1$$

\- - - + + +

yoki



$$\begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 6+0+1-2-0-4=7-6=1$$

2. a) 3 – satr elementlari bo‘yicha yoyib hisoblaymiz:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{k=1}^{3} a_{3k} \cdot A_{3k} = a_{31} \cdot A_{31} + a_{32} \cdot A_{32} + a_{33} \cdot A_{33} =$$

$$= (-1)^{3+1} \cdot a_{31} \cdot M_{31} + (-1)^{3+2} \cdot a_{32} \cdot M_{32} + (-1)^{3+3} a_{33} \cdot M_{33} = a_{31} \cdot M_{31} - a_{32} \cdot M_{32} + a_{33} \cdot M_{33}$$

Formulaga asosan:

$$\begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1(1-2) - 0 \cdot (3-4) + 2(3-2) = -1 - 0 + 2 = 1$$

3. a) 2 – ustun elementlari bo‘yicha yoyib hisoblaymiz:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12} \cdot M_{12} + a_{22} \cdot M_{22} - a_{32} \cdot M_{32}$$

$$\begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -(4-1) + (6-2) - 0 = -3 + 4 = 1$$

4. a) 2 – ustunda nollarni yig‘ib hisoblaymiz. j) hossaga binoan 2 – satr elementlariga 1 – satrning mos elementlarini (-1)ga ko‘paytirib qo‘shamiz:

$$\begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 2 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{vmatrix} = -1 \cdot \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 0 - 0 = -(2+1) = -1$$

1. b) Aniqlovchilarning d) xossasiga muvofiq, berilgan aniqlovchining 1 – ustuni va 3 – ustuni aynan bir xil bo‘lgani uchun, qiymati 0 ga teng.

2. b) Uchburchak usulini qo‘llab hisoblaymiz (sxemaga qaralsin):

$$\begin{vmatrix} 0 & x & 0 \\ x & 1 & x \\ 2 & x & 2 \end{vmatrix} = 0 + 2x^2 + 0 - 0 - 0 - 2x^2 = 0$$

3. b) Sarryus qoidasiga binoan:

$$\begin{vmatrix} 0 & x & 0 \\ x & 1 & x \\ 2 & x & 2 \end{vmatrix}\begin{matrix} 0 & x \\ x & 1 \\ 2 & x \end{matrix} = 0 + 2x^2 + 0 - 0 - 0 - 2x^2 = 0$$

4. b) 1 – satr 0 lar soni ko‘p bo‘lgani uchun, 1 – satr elementlari bo‘yicha yig‘ib hisoblaymiz:

$$\begin{vmatrix} 0 & x & 0 \\ x & 1 & x \\ 2 & x & 2 \end{vmatrix} = +0 - x\begin{vmatrix} x & x \\ 2 & 2 \end{vmatrix} + 0 = 0 - x \cdot 0 + 0 = 0$$

4. Quyidagi berilgan aniqlovchilarni eng qulay usulda hisoblang:

b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ -9 & -9 & -9 & -9 \\ 4 & 3 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix} = ?$

v) $A = \begin{vmatrix} 1 & -5 & 2 \\ -2 & 3 & 4 \\ 3 & 2 & 1 \end{vmatrix}$ $\quad B = \begin{vmatrix} 1 & 5 & 2 \\ -2 & -1 & 4 \\ 3 & -2 & 1 \end{vmatrix}$ $\quad A + B = ?$

a) $\begin{vmatrix} 1 & 2 & 0 & -3 \\ 3 & 1 & 0 & 4 \\ 1 & 5 & -1 & 7 \\ -2 & 1 & 0 & 1 \end{vmatrix} = (-1)^{3+3} \cdot (-1) \cdot \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ -2 & 1 & 1 \end{vmatrix}\begin{matrix} \\ \\ \\ 1 & 2 & -3 \\ 3 & 1 & 4 \end{matrix} = -(1 - 9 - 16 - 6 - 4 - 6) = 35$

b) v) hossadan foydalanib, -9 ni umumiy ko‘paytuvchi sifatida aniqlovchi ishorasidan tashqariga chiqaramiz, 4 – satrda 0 lar yig‘ib, shu satr elementlari bo‘yicha yoyib hisoblaymiz:

$$-9\begin{vmatrix}1&2&3&4\\1&1&1&1\\4&3&2&1\\1&0&1&0\end{vmatrix}=-9\begin{vmatrix}1&2&2&4\\1&1&0&1\\4&3&-2&1\\1&0&0&0\end{vmatrix}=-9\cdot(-1)^{4+1}\cdot 1\begin{vmatrix}2&2&4\\1&0&1\\3&-2&1\end{vmatrix}=9\cdot 2\begin{vmatrix}1&1&2\\1&0&1\\3&-2&1\end{vmatrix}=$$

$x(-1)$ +

$$=18\begin{vmatrix}1&1&1\\1&0&0\\3&-2&-2\end{vmatrix}=18(-1)^{2+1}\cdot 1\begin{vmatrix}1&1\\-2&-2\end{vmatrix}=0$$

v) $A+B=\begin{vmatrix}1&-5&2\\-2&3&4\\3&2&1\end{vmatrix}+\begin{vmatrix}1&5&2\\-2&-1&4\\3&-2&1\end{vmatrix}=\begin{vmatrix}1&0&2\\-2&2&4\\3&0&1\end{vmatrix}=2\cdot\begin{vmatrix}1&2\\3&1\end{vmatrix}=2\cdot(1-6)=-10$

(z) xossasiga muvofiq

5. Aniqlovchilarni ko‘paytirish qoidasidan foydalanib, berilgan $\Delta_2^{(1)}\cdot\Delta_2=\begin{vmatrix}5&7\\3&4\end{vmatrix}=-1$ va $\Delta_2^{(2)}=\begin{vmatrix}1&4\\2&-9\end{vmatrix}=-17$ aniqlovchilarni bir – biriga ko‘paytiring, $\Delta_2^{(1)}\cdot\Delta_2^{(2)}=17$ ekanini tekshirib ko‘ring.

$$\Delta_2^{(1)}\cdot\Delta_2^{(2)}=\begin{vmatrix}5&7\\3&4\end{vmatrix}\cdot\begin{vmatrix}1&4\\2&-9\end{vmatrix}=\begin{vmatrix}5\cdot1+7\cdot2&5\cdot4+7(-9)\\3\cdot1+4\cdot2&3\cdot4+4(-9)\end{vmatrix}=\begin{vmatrix}19&-43\\11&-24\end{vmatrix}=$$
$$=19\cdot(-24)-11\cdot(-43)=-456+473=17$$

Mashqlar

Aniqlovchilarni hisoblashga doir mustaqil ishlash uchun amaliy topshiriqlar:

1. Hisoblang:

a) $\begin{vmatrix}2&3\\5&8\end{vmatrix}$ j: 1;

b) $\begin{vmatrix} 1{,}5 & 2{,}25 \\ 2\frac{2}{3} & 6 \end{vmatrix}$ j: 2

v) $\begin{vmatrix} \sin 60^0 & \cos 45^0 \\ \sin 45^0 & tg30^0 \end{vmatrix}$ j: 0;

g) $\begin{vmatrix} tg\alpha & -1 \\ 4 & ctg\alpha \end{vmatrix}$ j: $5,\ \alpha \notin \left\{ \pi R; \frac{\pi}{2} + \pi n \right\}\ k, n \in z$

d) $\begin{vmatrix} 1+x & \frac{1}{x^2+y} \\ x-y & \frac{x}{x^2+y} \end{vmatrix}$ j: $1,\ x^2 + y \neq 0$

ye) $\begin{vmatrix} \frac{a-1}{2\sqrt{a}} & \frac{a+\sqrt{a}}{\sqrt{a}-1} \\ \frac{a\sqrt{a}-\sqrt{a}}{2a} & \frac{a-\sqrt{a}}{\sqrt{a}+1} \end{vmatrix}$ j: $-2\sqrt{a},\ a > 0,\ a \neq 1$

2. Tenglama va tengsizlikni yeching:

a) $\begin{vmatrix} x & 3 \\ 1 & 2x \end{vmatrix} + 3\begin{vmatrix} 0,(4) & x \\ 1 & 3 \end{vmatrix} = 0$ j: $x \in \left\{ \frac{1}{2};\ 1 \right\}$

b) $\begin{vmatrix} x & 1 \\ -4 & x \end{vmatrix} \leq \begin{vmatrix} 5 & 2 \\ 1 & x \end{vmatrix}$ j: $x \in [2;\ 3]$

3. Quyida berilgan o‘rin almashtirishlarning juft yoki toqligini aniqlang:

a) $(1,\ 3,\ 2)$ j: toq; b) $(2,\ 3,\ 1)$ j: juft;

v) $(1,\ 4, 3,\ 2)$ j: toq; g) $(3,\ 4,\ 1,\ 2)$ j: juft;

d) $(3,\ 5,\ 1,\ 4,\ 2)$ j: juft; ye) $(2,\ 5,\ 1,\ 4,\ 3)$ j: toq.

4. Berilgan 3-tartibli aniqlovchilarni kamida 2 ta usulda hisoblang:

a) $\begin{vmatrix} 1 & 2 & 3 \\ 8 & 1 & 4 \\ 2 & 1 & 1 \end{vmatrix}$ j: 15;

b) $\begin{vmatrix} 3 & -1 & -2 \\ 1 & 2 & 5 \\ -4 & 1 & 6 \end{vmatrix}$ j: 29;

v) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$ j: -7;

g) $\begin{vmatrix} \sin\alpha & \sin\beta & \sin\gamma \\ \cos\alpha & \cos\beta & \cos\gamma \\ 1 & 1 & 1 \end{vmatrix}$ j: $\begin{pmatrix} \sin(\alpha-\beta)+ \\ +\sin(\beta-\alpha)+ \\ +\sin(\lambda-\alpha) \end{pmatrix}$.

5. Quyida berilgan aniqlovchilarni eng qulay usulda hisoblang:

a) $\begin{vmatrix} 7 & 0 & 0 \\ -8 & 1 & -1 \\ 3 & 6 & -4 \end{vmatrix}$ j: 14;

b) $-0{,}125\cdot\begin{vmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ -3 & 5 & 1 \\ 26 & 26 & 26 \end{vmatrix}$ j: 1;

v) $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 6 \\ 0 & 5 & 4 \end{vmatrix}+\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & -2 \\ 0 & 5 & -4 \end{vmatrix}$ j: 10;

g) $\begin{vmatrix} 1 & 2 & -3 & 1 \\ 3 & 0 & 1 & -4 \\ 2 & 0 & 4 & 1 \\ 5 & 1 & 2 & 1 \end{vmatrix}$ j: -86.

6. Aniqlovchilarni ko‘paytirish qoidasidan foydalanib, berilgan $\Delta_2^{(1)} = \begin{vmatrix} 7 & 5 \\ 3 & 4 \end{vmatrix} = 13$ va $\Delta_2^{(2)} = \begin{vmatrix} 2 & 9 \\ 1 & 7 \end{vmatrix} = 5$ aniqlovchilarni bir – biriga ko‘paytiring, $\Delta_2^{(1)} \cdot \Delta_2^{(2)} = 65$ ekanini tekshirib ko‘ring.

2 - §. Matritsa determinanti

$n \cdot m$ ta $a_{ik}(i = 1, 2, ..., n,\ k = 1,\ 2,\ ...,\ m)$ haqiqiy sonlar berilgan bo‘lsin.

Ta’rif: Elementlari deb ataluvchi $a_{ik}(i = 1, 2, ..., n,\ k = 1,\ 2,\ ...,\ m)$ sonlar n ta satr va m ta ustunda joylashtirilgan quyidagi ko‘rinishdagi

$$(a_{ik}) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nm} \end{pmatrix} = \overline{\overline{A}} \quad \text{yoki} \quad \|a_{ik}\| = [a_{ik}] = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nm} \end{pmatrix}$$

jadvalga $n \times m$ o‘lchovli to‘g‘ri to‘rtburchakli matritsa deyiladi. Matritsada satrlar ustunlar sonidan ko‘p, unga teng va kam ya’ni $n > m,\ \ n = m,\ \ n < m$ bo‘lishi mumkin. $m = n$ bo‘lgan, ya’ni satrlar soni ustunlari soniga teng bo‘lgan matritsaga $n -$ tartibli kvadrat matritsa deyiladi. $n = 1$ da satr matritsa, $n - 1$ da ustun matritsa deyiladi. Agar ikki matritsaning satr va ustunlari soni mos ravishda teng bo‘lsa, bunday matritsalarga bir xil o‘lchovli matritsalar deyiladi. Mos elementlari o‘zaro teng bo‘lgan ikki bir xil o‘lchovli matritsalargina o‘zaro teng deyiladi.

Faqat kvadrat matritsaningina bosh va ikkilamchi diagonallari mavjud bo‘lib, uning quyidagi xususiy hollari bor:

$$\overline{\overline{B}}_1 = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ 0 & b_{22} & ... & b_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & ... & b_{nn} \end{pmatrix}, \qquad \overline{\overline{B}}_2 = \begin{pmatrix} b_{11} & 0 & ... & 0 \\ b_{21} & b_{22} & ... & 0 \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ b_{n1} & b_{n2} & ... & b_{nn} \end{pmatrix} -$$

uchburchakli

$$\overline{\overline{D}}=\begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix} \text{ - diagonal,} \quad \overline{\overline{E}}=\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \text{ - birlik.}$$

Kvadrat matritsalar quyidagi xarakteristikalarga ko‘ra o‘zaro taqqoslanadi: 1) aniqlovchisi (yoki determinanti). 2) normasi. 3) rangi.

1. Aniqlovchisi (determinant) kvadrat matritsaning sonli xarakteristikasidir, ya’ni $n-$tartibli aniqlovchi $n-$son, $n-$tartibli matritsa – jadvalning sonli ifodasi (xarakteristikasi)dir.

2. (a_{ik}), $(i=1,2,...,n)$ kvadrat matritsaning normasi deb, quyidagi N songa aytiladi:

$$N=\sqrt{\sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}^{2}}$$

3. Ixtiyoriy to‘g‘ri burchakli matritsaning rangi deb, 0 dan farqli matritsa osti minorlarning eng katta tartibiga aytiladi. Masalan,

$$\overline{\overline{A}}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

matritsaning rangi $r\left(\overline{\overline{A}}\right)=2$ bo‘lishi uchun

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \quad \left(\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\right) \text{ minorlarning}$$

hech bo‘lmaganda bittasi noldan farqli bo‘lishi kerak. Agar yuqoridagi minorlarning hammasi nolga teng bo‘lsa, berilgan matritsaning rangi $r\left(\overline{\overline{A}}\right)=1$ bo‘ladi. Bir xil o‘lchovli matritsalarni bir – biriga qo‘shish (ayirish) mumkin. Natijada o‘sha o‘lchovli, elementlari qo‘shilayotgan (ayrilayotgan) matritsalarning mos elementlari yig‘indisi (ayirmasi)ga teng bo‘lgan matritsa hosil bo‘ladi. Masalan:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} \end{pmatrix}$$

Matritsa songa ko‘paytirilganda, uning har bir elementi shu songa ko‘paytiriladi. Masalan:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot k = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$$

Matritsalarni songa ko‘paytirish va ularni qo‘shish (ayirish) quyidagi xossalarga bo‘ysinadi:

1. $(k \cdot \lambda) \cdot \overline{\overline{A}} = k \cdot \left(\lambda \cdot \overline{\overline{A}}\right), \quad k, \lambda \in R$
2. $\overline{\overline{A}} + \overline{\overline{B}} = \overline{\overline{B}} + \overline{\overline{A}}$
3. $\left(\overline{\overline{A}} + \overline{\overline{B}}\right) + \overline{\overline{C}} = \overline{\overline{A}} + \left(\overline{\overline{B}} + \overline{\overline{C}}\right)$
4. $(k + \lambda) \cdot \overline{\overline{A}} = k \cdot \overline{\overline{A}} + \lambda \overline{\overline{A}}$
5. $k \cdot \left(\overline{\overline{A}} + \overline{\overline{B}}\right) = k \cdot \overline{\overline{A}} + k \cdot \overline{\overline{B}}$

$(n \times m)$ o‘lchovli matritsani $(m \times p)$ o‘lchovli matritsagagina ko‘paytirish mumkin, ya’ni chapdan turib, ko‘paytuvchi matritsaning ustunlar soni o‘ngdan turib ko‘payuvchi matritsaning satrlar soniga teng bo‘lgandagina matritsalar ko‘paytmasi haqida masala qo‘yilishi mumkin. $(n \times m)$ o‘lchovli $(a_{ik}) = \overline{\overline{A}}, \ (i = 1, 2, ..., n, \ \ k = 1, 2, ..., m)$ matritsani $(m \times p)$ o‘lchovli $(a_{ik}) = \overline{\overline{B}}, \ (i = 1, 2, ..., m, \ \ j = 1, 2, ..., p)$ matritsaga ko‘paytirilgan, $(n \times p)$ o‘lchovli $\left(C_{ij}\right) = \overline{\overline{C}}, \ \overline{\overline{A}} \cdot \overline{\overline{B}}, \ (i = 1, 2, ..., n, \ \ k = 1, 2, ..., p)$ matritsa hosil bo‘lib, uning elementlari quyidagi qoida bo‘yicha topiladi:

$$C_{ij} = \sum_{k=1}^{m} a_{ik} \cdot b_{ik}; \ (i = 1, 2, ..., n, \ \ k = 1, 2, ..., p)$$

ya’ni, chapdagi $\overline{\overline{A}}$ matritsaning i satri elementlari o‘ngdagi $\overline{\overline{B}}$ matritsaning j ustuni mos elementlariga ko‘paytirilib, so‘ngra qo‘shiladi. Masalan:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{pmatrix}$$

Matritsalarni ko‘paytirish quyidagi xossalarga bo‘ysinadi:

1. $k\left(\overline{\overline{A}}\cdot\overline{\overline{B}}\right)=\left(k\cdot\overline{\overline{A}}\right), \quad k\in R$
2. $\left(\overline{\overline{A}}+\overline{\overline{B}}\right)\cdot\overline{\overline{C}}=\overline{\overline{A}}\cdot\overline{\overline{C}}+\overline{\overline{B}}\cdot\overline{\overline{C}}$
3. $\overline{\overline{C}}\cdot\left(\overline{\overline{A}}+\overline{\overline{B}}\right)=\overline{\overline{C}}\cdot\overline{\overline{A}}+\overline{\overline{C}}\cdot\overline{\overline{B}}$
4. $\left(\overline{\overline{A}}\cdot\overline{\overline{B}}\right)\cdot\overline{\overline{C}}=\overline{\overline{A}}\cdot\left(\overline{\overline{B}}\cdot\overline{\overline{C}}\right)$

Bir xil tartibli $\overline{\overline{A}}$ va $\overline{\overline{B}}$ kvadrat matritsalar uchun $\overline{\overline{A}}\cdot\overline{\overline{B}}=\overline{\overline{B}}\cdot\overline{\overline{A}}$ tenglik o‘rinli bo‘lavermaydi. Agar yuqoridagi tenglik o‘rinli bo‘lsa, $\overline{\overline{A}}$ va $\overline{\overline{B}}$ matritsalar ko‘paytmasiga nisbatan o‘zaro kommutativ (o‘rin almashuvchi) matritsalar deyiladi.

Teskari matritsa va uni topish

$\overline{\overline{A}}$ kvadrat, maxsusmas matritsa berilgan bo‘lsin $\left(\overline{\overline{A}}=(a_{ik}),\ (i,k=1,2,...,n)\right)$. Maxsusmas matritsa deb, parallel satr (ustun)lari mos elementlari proporsional bo‘lmagan, ya’ni determinanti (aniqlovchisi) 0 dan farqli bo‘lgan kvadrat matritsaga aytiladi (aks holda maxsus matritsa deyiladi).

Ta’rif: Berilgan $\overline{\overline{A}}$ maxsusmas matritsaning teskari matritsasi $\overline{\overline{A}}^{-1}$ deb, tartibi $\overline{\overline{A}}$ matritsaning tartibiga teng, (shunday bir kvadrat matritsaga aytiladiki) uni berilgan matritsaga chapdan yoki o‘ngdan ko‘paytirilganda birlik matritsani beradigan maxsusmas matritsaga aytiladi.

Ta’rifga binoan: $\overline{\overline{A}}^{-1}\cdot\overline{\overline{A}}=\overline{\overline{A}}\cdot\overline{\overline{A}}^{-1}=\overline{\overline{E}}$, bu yerda $\overline{\overline{E}}$ tartibi $\overline{\overline{A}}$ va $\overline{\overline{A}}^{-1}$ ning tartibiga teng bo‘lgan birlik matritsa. Faqatgina kvadrat, maxsusmas matritsaningina yagona teskari matritsasi mavjud.

Teskari matritsani topishning ikki usulini ko‘rib chiqamiz:

1. Klassik usul. Teskari matritsani topishning klassik usulida quyidagi to‘rt qadam shartlari bajariladi:

a) Berilgan $\overline{\overline{A}}$ matritsaning aniqlovchisi $\det\left(\overline{\overline{A}}\right)$ topiladi. Agar $\det\left(\overline{\overline{A}}\right)=0$ bo‘lsa, berilgan $\overline{\overline{A}}$ matritsa maxsus matritsa bo‘lib, teskari matritsaga ega emas. Agar $\det\left(\overline{\overline{A}}\right)\neq 0$ bo‘lsa, $\overline{\overline{A}}^{-1}$ matritsa mavjud, keyingi qadamga o‘tiladi;

b) Maxsusmas $\overline{\overline{A}}$ matritsaning har bir elementi algebraik to‘ldiruvchi (ad’yunkt)lari topiladi va algebraik to‘ldiruvchilar matritsasi $\tilde{\tilde{A}}_{ik}$ tuziladi:

v) Algebraik to‘ldiruvchilar matritsasi transponirlanib (mos satr va ustun o‘rinlari almashtirilib), transponirlangan algebraik to‘ldiruvchilar matritsasi $\overline{\overline{A}}_{ik}^{*}$ tuziladi;

g) Transponirlangan algebraik to‘ldiruvchilar matritsasi $\overline{\overline{A}}_{ik}^{*}$ni $\det\left(\overline{\overline{A}}\right) \neq 0$ songa bo‘lib, teskari matritsa $\overline{\overline{A}}^{-1}$ topiladi:

$$\overline{\overline{A}}^{-1} = \frac{1}{\det\left(\overline{\overline{A}}\right)} \cdot \overline{\overline{A}}_{ik}^{*}$$

Xususiy holda 2 – tartibli maxsusmas matritsa

$$\overline{\overline{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ning}$$

teskari matritsasi quyidagicha topiladi:

$$\overline{\overline{A}}^{-1} = \frac{1}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \cdot \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$$

o‘z navbatida, 3 – tartibli maxsusmas matritsa

$$\overline{\overline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ ning}$$

teskari matritsasini topish formulasi:

$$\overline{\overline{A}}^{-1} = \frac{1}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

va hokazo.

2. Jordano usuli. Berilgan $\overline{\overline{A}}$ kvadrat matritsaga o‘ng tomondan tartibli $\overline{\overline{A}}$ ning tartibiga teng $\overline{\overline{E}}$ birlik matritsa qo‘shilib, kengaytirilgan matritsa tuziladi:

$$\left(\overline{\overline{A}}\,\vdots\,\overline{\overline{E}}\right)=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & \vdots & 0 & 0 & \dots & 1 \end{pmatrix}$$

Paralell ravishda kengaytirilgan matritsaning chap va o‘ng qismlarida elementar almashtirishlar bajarilib, chap qismida birlik matritsa hosil qilishga erishiladi. Bunda o‘ng qismida hosil bo‘lgan matritsa berilgan matritsaning teskarisi bo‘ladi:

$$\left(\overline{\overline{A}}\,\vdots\,\overline{\overline{E}}\right)\rightarrow\left(\overline{\overline{E}}\,\vdots\,\overline{\overline{A}}^{-1}\right)$$

Mashqlar

Matritsalarga doir amaliy misol va masalalar yechimlari

1. Berilgan kvadrat matritsalarning determinantlari, normalari va ranglari topilsin:

a) $\overline{\overline{A}}_2=\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$ b) $\overline{\overline{A}}_3=\begin{pmatrix} -1 & 0 & 8 \\ 5 & 9 & 0 \\ 0 & 4 & 3 \end{pmatrix}$ v) $\overline{\overline{A}}_4=\begin{pmatrix} 2 & 3 & 4 & 0 \\ 1 & 5 & 7 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

a) $\det\left(\overline{\overline{A}}_2\right)=\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix}=0-2\cdot(-2)=4$,

$$N\left(\overline{\overline{A}}_2\right)=\sqrt{1^2+2^2+(-2)^2+0^2}=\sqrt{9}=3$$

$\overline{\overline{A}}_2$ kvadrat matritsaning rangi, uning o‘z tartibiga teng, chunki $\det\left(\overline{\overline{A}}_2\right)\neq 0$. Demak, $r\left(\overline{\overline{A}}_2\right)=2$.

b) $\det\left(\overline{\overline{A}}_3\right)=\begin{vmatrix}-1 & 0 & 8\\ 5 & 9 & 0\\ 0 & 4 & 3\end{vmatrix}\begin{matrix}-1 & 0\\ 5 & 9\\ 0 & 4\end{matrix}=-27+160=133\neq 0$ bo‘lgani uchun $r\left(\overline{\overline{A}}_3\right)=3$,

$$N\left(\overline{\overline{A}}_3\right)=\sqrt{(-1)^2+8^2+5^2+9^2+4^2+3^2}=\sqrt{196}=14$$

v)

$$\det\left(\overline{\overline{A}}_4\right)=\begin{vmatrix}2 & 3 & 4 & 0\\ 1 & 5 & 7 & 0\\ 3 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}=1\cdot(-1)^{4+4}\begin{vmatrix}2 & 3 & 4\\ 1 & 5 & 7\\ 3 & 1 & 1\\ 2 & 3 & 4\\ 1 & 5 & 7\end{vmatrix}=10+4+63-60-14-3=0$$

$\overline{\overline{A}}_4$ kvadrat matritsaning rangi o‘z tartibiga teng bo‘la olmaydi, chunki $\det\left(\overline{\overline{A}}_4\right)=0$. Shu bilan birga, bir qarashda barcha matritsa osti 3 – tartibli minorlar 0ga teng ko‘rinadi. Ammo diqqat bilan qaralganda barcha uchinchi tartibli $M_{tl}(t,\ l=1,\ 2,\ 3)$ matritsa osti minorlar 0 dan farqli. Demak, berilgan to‘rtinchi tartibli kvadrat matritsaning rangi $r\left(\overline{\overline{A}}_4\right)=3$.

2. Matritsalar ustida amallar bajaring:

a) $\overline{\overline{A}}=\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & 5\end{pmatrix}$, $\overline{\overline{B}}=\begin{pmatrix}0 & 3 & 2\\ -1 & 4 & 1\end{pmatrix}$, $\overline{\overline{C}}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$

1) $3\overline{\overline{A}}-2\overline{\overline{B}}=?$ 2) $\overline{\overline{A}}\cdot\overline{\overline{C}}=?$

1)

$$3\overline{\overline{A}}-2\overline{\overline{B}}=3\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & 5\end{pmatrix}-2\begin{pmatrix}0 & 3 & 2\\ -1 & 4 & 1\end{pmatrix}=\begin{pmatrix}3 & -3 & 9\\ 6 & 3 & 15\end{pmatrix}+\begin{pmatrix}0 & -6 & -4\\ 2 & -8 & -2\end{pmatrix}=\begin{pmatrix}3 & -9 & 5\\ 8 & -5 & 13\end{pmatrix}$$

2) $\overline{\overline{A}}\cdot\overline{\overline{C}}=\begin{pmatrix}1 & -1 & 3\\ 2 & 1 & 5\end{pmatrix}\cdot\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}=\begin{pmatrix}1\cdot 1+(-1)\cdot 2+3\cdot 3\\ 2\cdot 1+1\cdot 2+5\cdot 3\end{pmatrix}=\begin{pmatrix}8\\ 19\end{pmatrix}$

b) $\overline{\overline{A}}=\begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}$, $\overline{\overline{B}}=\begin{pmatrix}5 & 7\\ -1 & 2\end{pmatrix}$,

1) $\overline{\overline{A}}\cdot\overline{\overline{B}}=?$, 2) $\overline{\overline{B}}\cdot\overline{\overline{A}}=?$,

3) $\overline{\overline{A}}^{2}=?$, 4) $\overline{\overline{A}}^{2}-\overline{\overline{A}}\cdot\overline{\overline{B}}+2\overline{\overline{B}}\cdot\overline{\overline{A}}=?$

1) $$\overline{\overline{A}}\cdot\overline{\overline{B}}=\begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}\cdot\begin{pmatrix}5 & 7\\ -1 & 2\end{pmatrix}=\begin{pmatrix}4\cdot 5+(-1)\cdot 3 & 4\cdot 7+3\cdot 2\\ 2\cdot 5+1\cdot(-1) & 2\cdot 7+1\cdot 2\end{pmatrix}=\begin{pmatrix}17 & 34\\ 9 & 16\end{pmatrix}$$

2) $$\overline{\overline{B}}\cdot\overline{\overline{A}}=\begin{pmatrix}5 & 7\\ -1 & 2\end{pmatrix}\cdot\begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}=\begin{pmatrix}5\cdot 4+7\cdot 2 & 5\cdot 3+7\cdot 1\\ 1\cdot 4+2\cdot 2 & -1\cdot 3+2\cdot 1\end{pmatrix}=\begin{pmatrix}34 & 22\\ 0 & -1\end{pmatrix}$$

3) $$\overline{\overline{A}}^{2}=\begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}\cdot\begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}=\begin{pmatrix}4\cdot 4+3\cdot 2 & 4\cdot 3+3\cdot 1\\ 2\cdot 4+1\cdot 2 & 2\cdot 3+1\cdot 1\end{pmatrix}=\begin{pmatrix}22 & 15\\ 10 & 7\end{pmatrix}$$

4) $$\overline{\overline{A}}^{2}-\overline{\overline{A}}\cdot\overline{\overline{B}}+2\overline{\overline{B}}\cdot\overline{\overline{A}}=\begin{pmatrix}22 & 15\\ 10 & 7\end{pmatrix}-\begin{pmatrix}17 & 34\\ 9 & 16\end{pmatrix}+2\begin{pmatrix}34 & 22\\ 0 & -1\end{pmatrix}=$$
$$=\begin{pmatrix}22-17+2\cdot 34 & 15-34+2\cdot 22\\ 10-9+2\cdot 0 & 7-16+2\cdot(-1)\end{pmatrix}=\begin{pmatrix}73 & 25\\ 1 & -11\end{pmatrix}$$

v) $\overline{\overline{A}}=\begin{pmatrix}1 & -3 & 0\\ 2 & 5 & 1\end{pmatrix}$, $\overline{\overline{B}}=\begin{pmatrix}0 & -1 & 3\\ 3 & 5 & 2\\ 4 & -2 & 1\end{pmatrix}$, 1) $\overline{\overline{A}}\cdot\overline{\overline{B}}=?$ 2) $\overline{\overline{B}}^{2}=?$

1) $$\overline{\overline{A}}\cdot\overline{\overline{B}}=\begin{pmatrix}1 & -3 & 0\\ 2 & 5 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & -1 & 3\\ 3 & 5 & 2\\ 4 & -2 & 1\end{pmatrix}=$$
$$=\begin{pmatrix}1\cdot 0+(-3)\cdot +0\cdot 4 & 1\cdot(-1)+(-3)\cdot 5+0\cdot(-2) & 1\cdot 3+(-3)\cdot 2+0\cdot 1\\ 2\cdot 0+5\cdot 3+1\cdot 4 & 2\cdot(-1)+\cdot 5+1\cdot(-2) & 2\cdot 3+5\cdot 2+1\cdot 1\end{pmatrix}=$$
$$=\begin{pmatrix}-9 & -16 & -3\\ 19 & 21 & 17\end{pmatrix}$$

2) $$\overline{\overline{B}}^{2}=\begin{pmatrix}0 & -1 & 3\\ 3 & 5 & 2\\ 4 & -2 & 1\end{pmatrix}^{2}=\begin{pmatrix}0 & -1 & 3\\ 3 & 5 & 2\\ 4 & -2 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & -1 & 3\\ 3 & 5 & 2\\ 4 & -2 & 1\end{pmatrix}=$$

$$=\begin{pmatrix} 0-3+12 & 0-5-6 & 0-2+3 \\ 0+15+8 & -3+25-4 & 9+10+2 \\ 0-6+4 & -4-10-2 & 12-4+1 \end{pmatrix} = \begin{pmatrix} 9 & -11 & 1 \\ 23 & 18 & 21 \\ -2 & -16 & 9 \end{pmatrix}$$

3. Berilgan kvadrat matritsalarning teskari matritsasini 2 usulda toping:

a) $\overline{\overline{A}} = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$, b) $\overline{\overline{B}} = \begin{pmatrix} 1 & 5 & 7 \\ 3 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$, v) $\overline{\overline{C}} = \begin{pmatrix} 2 & -1 & 7 \\ 5 & 3 & 2 \\ 1 & 4 & -3 \end{pmatrix}$

a) 1 – chi usul (klassik metod): 4 – qadam shartlarini ketma – ket bajaramiz:

1) $\det\left(\overline{\overline{A}}\right) = \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = 10 \neq 0$

2) $A_{11} = (-1)^{1+1} \cdot 4 = 4$; $A_{12} = (-1)^{1+2} \cdot 3 = -3$

$A_{21} = (-1)^{2+1} \cdot (-2) = 2$ $A_{22} = (-1)^{2+2} \cdot 1 = 1$ $\tilde{\tilde{A}}_{ik} = \begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$

3) $\tilde{\tilde{A}}_{ik} = \begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix}$

4) $\overline{\overline{A}}^{-1} = \frac{1}{10} \cdot \begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ -0.3 & 0.1 \end{pmatrix}$

Berilgan matritsaning teskarisi to‘g‘ri topilganligini ta’rifdan foydalanib tekshirib ko‘ramiz:

$$\overline{\overline{A}} \cdot \overline{\overline{A}}^{-1} = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0.4 & 0.2 \\ -0.3 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.4+0.6 & 0.2-0.2 \\ 1.2-1.2 & 0.6+0.4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \overline{\overline{E}}$$

2 usulda (Jordano usuli): $\left(\overline{\overline{A}} \vdots \overline{\overline{E}}\right)$ kengaytirilgan matritsa tuzib, elementar almashtirishlar bajaramiz:

$$\underset{+}{(-3)}\begin{pmatrix} \boxed{1} & -2 & \vdots & 1 & 0 \\ 3 & 4 & \vdots & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & \vdots & 1 & 0 \\ 0 & \boxed{10} & \vdots & -3 & 1 \end{pmatrix}_{:10} \to \begin{pmatrix} 1 & -2 & \vdots & 1 & 0 \\ 0 & \boxed{1} & \vdots & 0.3 & 0.1 \end{pmatrix}_{\times 2} \to +$$

$$\to \begin{pmatrix} 1 & 0 & \vdots & 0.4 & 0.2 \\ 0 & 1 & \vdots & 0.3 & 0.1 \end{pmatrix}$$

$$\overline{\overline{A}}^{-1} = \begin{pmatrix} 0.4 & 0.2 \\ -0.3 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ -0.3 & 0.1 \end{pmatrix}$$

b)

$$\det\left(\overline{\overline{B}}\right) = \begin{vmatrix} 1 & 5 & 7 \\ 3 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 4 + 63 + 10 - 14 - 3 - 60 = 0$$

$$\begin{matrix} 1 & 5 & 7 \\ 3 & 1 & 1 \end{matrix}$$

bo‘lgani uchun

berilgan $\overline{\overline{B}}$ matritsa teskari matritsaga ega emas.

1 – usul: 1) $\det \overline{\overline{C}} = \begin{vmatrix} 2 & -1 & 7 \\ 5 & 3 & 2 \\ 1 & 4 & -3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 5 & 3 \\ 1 & 4 \end{matrix} = -18 - 2 + 140 - 21 - 16 - 15 = 68 \neq 0$

2) $A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix} = -17$, $A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17$,

$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 5 & 3 \\ 1 & 4 \end{vmatrix} = 17$

$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} -1 & 7 \\ 4 & -3 \end{vmatrix} = 25$, $A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 2 & 7 \\ 1 & -3 \end{vmatrix} = 13$,

$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = -9$

$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} -1 & 7 \\ 3 & 2 \end{vmatrix} = -23$, $A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 7 \\ 5 & 2 \end{vmatrix} = 31$, $A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = 11$

$$\tilde{\tilde{A}}_{ik} = \begin{pmatrix} -17 & 17 & 17 \\ 25 & -13 & -9 \\ -23 & 31 & 11 \end{pmatrix}$$

3) $\tilde{\tilde{A}}_{ik} = \begin{pmatrix} -17 & 25 & -23 \\ 17 & -13 & 31 \\ 17 & -9 & 11 \end{pmatrix}$

4) $\overline{\overline{A}}^{-1} = \frac{1}{68} \cdot \begin{pmatrix} -17 & 25 & -23 \\ 17 & -13 & 31 \\ 17 & -9 & 11 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{25}{68} & -\frac{23}{68} \\ \frac{1}{4} & -\frac{13}{68} & \frac{31}{68} \\ \frac{1}{4} & -\frac{9}{68} & \frac{11}{68} \end{pmatrix}$

2 – usul

$$\left(\begin{array}{ccc|ccc} \boxed{2} & -17 & 7 & 1 & 0 & 0 \\ 5 & 3 & 2 & 0 & 1 & 0 \\ 1 & 4 & -3 & 0 & 0 & 1 \end{array}\right)^{:2} \to \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} & 0 & 0 \\ 5 & 3 & 2 & 0 & 1 & 0 \\ 1 & 4 & -3 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \times(-5) \quad \times(-1) \\ + \end{matrix} \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \boxed{\frac{11}{2}} & -\frac{31}{2} & -\frac{5}{2} & 1 & 0 \\ 0 & \frac{9}{2} & -\frac{13}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \xrightarrow{\times\frac{2}{11}} \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{13}{2} & -\frac{5}{11} & \frac{2}{11} & 0 \\ 0 & \frac{9}{2} & -\frac{13}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{matrix} \times\frac{1}{2} \\ \times\left(-\frac{9}{2}\right) \\ + \end{matrix} \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{23}{11} & \frac{3}{11} & \frac{1}{11} & 0 \\ 0 & 1 & -\frac{31}{2} & -\frac{5}{2} & \frac{2}{11} & 0 \\ 0 & 0 & \frac{68}{11} & \frac{17}{11} & -\frac{9}{11} & 1 \end{array}\right) \xrightarrow{\times\left(\frac{23}{11}\right) \times\frac{31}{11}} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{23}{11} & \frac{3}{11} & \frac{1}{11} & 0 \\ 0 & 1 & -\frac{31}{11} & \frac{5}{11} & \frac{2}{11} & 0 \\ 0 & 0 & 1 & \frac{17}{68} & -\frac{9}{68} & \frac{11}{68} \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{17}{68} & \frac{25}{68} & -\frac{23}{68} \\ 0 & 1 & 0 & \frac{17}{68} & -\frac{13}{68} & \frac{31}{68} \\ 0 & 0 & 1 & \frac{17}{68} & -\frac{9}{68} & \frac{11}{68} \end{array}\right) + \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & \frac{25}{68} & -\frac{23}{68} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{13}{68} & \frac{31}{68} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{9}{68} & \frac{11}{68} \end{array}\right)$$

$$\overline{\overline{A}}^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{25}{68} & -\frac{23}{68} \\ \frac{1}{4} & -\frac{13}{68} & \frac{31}{68} \\ \frac{1}{4} & -\frac{9}{68} & \frac{11}{68} \end{pmatrix}$$

Mashqlar

Matritsalarga doir mustaqil ishlash uchun amaliy topshiriqlar

1. Berilgan kvadrat matritsalarning determinantlari normalari va ranglari topilsin? To‘g‘ri to‘rtburchakli matritsalarining ranglari topilsin?

a) $\overline{\overline{A}}_2 = \begin{pmatrix} 2 & 5 \\ -4 & 2 \end{pmatrix}$ $\left(\det\left(\overline{\overline{A}}_2\right) = 24, \quad N\left(\overline{\overline{A}}_2\right) = 7, \quad r = \left(\overline{\overline{A}}_2\right) = 2\right)$

b) $\overline{\overline{D}}_2 = \begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix}$ $\left(\det\left(\overline{\overline{D}}_2\right) = -12, \quad N\left(\overline{\overline{D}}_2\right) = 5, \quad r = \left(\overline{\overline{D}}_2\right) = 2\right)$

v) $\overline{\overline{A}}_3 = \begin{pmatrix} 1 & 0 & 5 \\ 4 & -2 & -1 \\ 2 & 1 & 3 \end{pmatrix}$ $\left(\det\left(\overline{\overline{A}}_3\right) = 35 \quad N\left(\overline{\overline{A}}_3\right) = \sqrt{61}, \quad r = \left(\overline{\overline{A}}_3\right) = 3\right)$

g) $\overline{\overline{B}}_3 = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}$ $\left(\det\left(\overline{\overline{B}}_3\right) = 0 \quad N\left(\overline{\overline{B}}_3\right) = \sqrt{29}, \quad r = \left(\overline{\overline{B}}_3\right) = 3\right)$

d) $\overline{\overline{A}}_4 = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \end{pmatrix}$ $\left(\det\left(\overline{\overline{A}}_4\right) = 0 \quad N\left(\overline{\overline{A}}_4\right) = 6, \quad r = \left(\overline{\overline{A}}_4\right) = 3\right)$

ye) $\overline{\overline{F}} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$ $\left(r = \left(\overline{\overline{F}}\right) = 2\right)$

yo) $\overline{\overline{L}} = \begin{pmatrix} 2 & 1 \\ 6 & 3 \\ 10 & 5 \end{pmatrix}$ $\left(r = \left(\overline{\overline{L}}\right) = 1\right)$

j) $\overline{\overline{\Phi}} = \begin{pmatrix} 1 & 6 & 4 & 0 \\ -1 & 0 & 3 & 2 \\ 2 & 1 & 7 & 5 \end{pmatrix}$ $\left(r = \left(\overline{\overline{\Phi}}\right) = 3\right)$

2. Berilgan matritsalar ustida talab qilingan amallarni bajaring:

a) $\overline{\overline{A}} = \begin{pmatrix} 1 & 5 \\ 2 & -4 \end{pmatrix}$, $\overline{\overline{B}} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$, $2\overline{\overline{A}} - \overline{\overline{B}} = ?$ $\left(\begin{pmatrix} -1 & 8 \\ 0 & -9 \end{pmatrix}\right)$

b) $\begin{pmatrix} 7 & 0 \\ 3 & 1 \\ -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 2 & \sqrt{21} \\ 1 & -1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & \sqrt{18} \\ 4 & -5 \\ 3 & 1 \end{pmatrix} = ?$ $\left(\begin{pmatrix} 2 & 0 \\ 4 & -1 \\ 5 & 3 \end{pmatrix}\right)$

v) $\overline{\overline{A}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\overline{\overline{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\overline{\overline{A}} \cdot \overline{\overline{E}} = ?$ $\overline{\overline{E}} \cdot \overline{\overline{A}} = ?$

$\left(\overline{\overline{A}} \cdot \overline{\overline{E}} = \overline{\overline{E}} \cdot \overline{\overline{A}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \overline{\overline{A}}\right)$

g) $\overline{\overline{C}} = (1 \quad 2 \quad 3)$, $\overline{\overline{F}} = \begin{pmatrix} 4 & -3 \\ 1 & 2 \\ 0 & 2 \end{pmatrix}$, $\overline{\overline{C}} \cdot \overline{\overline{F}} = ?$ $((6 \quad 7))$

d) $\overline{\overline{A}} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$, $\overline{\overline{F}} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 0 \end{pmatrix}$, $\overline{\overline{A}} \cdot \overline{\overline{F}} = ?$ $\left(\begin{pmatrix} 8 & 10 \\ 2 & 5 \\ 0 & 1 \end{pmatrix}\right)$

ye) $\overline{\overline{A}} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \\ -4 & 5 & 1 \end{pmatrix}$, $\overline{\overline{B}} = \begin{pmatrix} 3 & 4 & 1 \\ 0 & 2 & 5 \\ 1 & -1 & 4 \end{pmatrix}$, $\overline{\overline{A}} \cdot \overline{\overline{B}} = ?$ $\left(\begin{pmatrix} 5 & 0 & 4 \\ 10 & 10 & 33 \\ -11 & -3 & 25 \end{pmatrix}\right)$

yo) $\begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & -3 \\ 5 & 4 & 0 \end{pmatrix} + \begin{pmatrix} -10 & -9 & 7 \\ 1 & 5 & 8 \\ -1 & -4 & 4 \end{pmatrix} = ?$ $\left(\begin{pmatrix} 6 & 1 & 4 \\ 3 & 1 & 2 \\ -5 & -10 & 1 \end{pmatrix}\right)$

3. Berilgan matritsalarning teskarisini 2 usulda toping:

a) $\overline{\overline{A}} = \begin{pmatrix} -1 & 1 \\ 4 & -2 \end{pmatrix}$, $\left(\overline{\overline{A}}^{-1} = \begin{pmatrix} 1 & 0.5 \\ 2 & 0.5 \end{pmatrix}\right)$

b) $\overline{\overline{A}} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ ( )

v) $\overline{\overline{A}} = \begin{pmatrix} tg\alpha & 1 \\ 1 & ctg\alpha \end{pmatrix}$, $\left(\overline{\overline{A}}^{-1} = \begin{pmatrix} -ctg\alpha & 1 \\ 2 & -tg\alpha \end{pmatrix}\right)$

g) $\overline{\overline{A}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 2 \end{pmatrix}$, $\left(\overline{\overline{A}}^{-1} = \begin{pmatrix} -2 & -1 & 2 \\ 4 & 1 & -3 \\ 1 & 1 & -1 \end{pmatrix}\right)$

d) $\overline{\overline{A}} = \begin{pmatrix} 1 & -1 & 1 \\ -38 & 41 & -34 \\ 27 & -29 & 24 \end{pmatrix}$, $\left(\overline{\overline{A}}^{-1} = \begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{pmatrix}\right)$

3 - §. Chiziqli tenglamalar sistemasi

Ko‘pgina injenerlik va iqtisodiy masalalar tenglamalar sistemasini yyechishga keltiriladi.

Umumiy ko‘rinishda $n-$ta noma’lumli m ta chiziqlimas tenglamalar sistemasini quyidagi ko‘rinishda yozish mumkin.

$$\begin{cases} f_1(x_1, x_2, ..., x_n) = 0 \\ f_2(x_1, x_2, ..., x_n) = 0 \\ \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \\ f_m(x_1, x_2, ..., x_n) = 0 \end{cases} \qquad (1)$$

bu yerda $n-$ noma’lumlar soni, $m-$ tenglamalar soni.

$f(x_1, x_2, ..., x_n),\ (i = 1, 2, ..., m)-$ umuman olganda, chiziqlimas funksiyalar va $m \neq n$

Agar tenglamalar soni (bir – biriga teng kuchli bo‘lmagan) m noma’lumlar soni n dan katta bo‘lsa, ya’ni $m > n$, sistema ortig‘i bilan aniqlangan va agar $m < n$ bo‘lsa kami bilan aniqlangan bo‘ladi deyiladi. Agar sistemadagi bir – biriga teng kuchli bo‘lmagan tenglamalar soni noma’lumlar soniga teng bo‘lsa, (1) sistemaga aniqlangan sistema deyiladi (asosan aniqlangan sistemalarni o‘rganamiz).

Sistemaning har bir tenglamasini qanoatlantiradigan (sonli ayniyatga aylantiradigan) n ta tartiblangan sonlardan iborat $\left(\alpha_1^{(i)}, \alpha_2^{(i)}, ..., \alpha_n^{(i)}\right)$ sistemalar to‘plamiga, (1) tenglamalar sistemasining yechimi (yechimlar to‘plami) deyiladi.

Kamida yagona yechimga ega bo‘lgan tenglamalar sistemasiga birgalikdagi tenglamalar sistemasi deyiladi. Agar tenglamalar sistemasi birorta ham yechimga ega bo‘lmasa, birgalikda bo‘lmagan tenglamalar sistemasi deb ataladi.

Aniqlangan (1) tenglamalar sistemasining xususiy holi n ta noma’lumli n ta chiziqli tenglamalar sistemasi hisoblanadi.

Chiziqlimas tenglamalar sistemasini eng sodda, chiziqli algebraning to‘la – to‘kis o‘rganilgan tarmog‘i chiziqli tenglamalar sistemasidir. n ta chiziqli tenglamalar sistemasining normal ko‘rinishi quyidagicha:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases} \qquad (2)$$

$a_{ik}(i, k = 1, 2, .., n)$ haqiqiy sonlarga sistemaning koeffitsientlari, $b_{ik}(i, k = 1, 2, .., n)$ haqiqiy sonlarga esa sistema tenglamalarini ozod hadlari deyiladi.

Agar sistema tenglamalaridagi barcha ozod hadlar 0ga teng bo‘lsa, (2) sistemaga bir jinsli chiziqli tenglamalar sistemasi (aniqlangan) deyiladi. Agar ozod hadlardan birortasi 0 dan farqli bo‘lsa, o‘z navbatida, bir jinslimas tenglamalar sistemasi deyiladi. O‘z – o‘zidan ma’lumki, bir jinsli chiziqli tenglamalar sistemasi yechimga ega, chunki n ta 0lardan iborat $(0, 0, ..., 0)$ sistema (2)ni qanoatlantiradi. Boshqacha aytganda, bir jinsli chiziqli tenglamalar sistemasi doimo birgalikdadir.

Bir jinslimas chiziqli tenglamalar sistemasi birgalikda yoki birgalikdamasligini quyidagi teorema ochib beradi (sistemaning aniqlangan bo‘lishi shart).

Kroneker – Kapelli teoremasi:

Bir jinslimas chiziqli tenglamalar sistemasi birgalikda bo‘lishi uchun (ya’ni yechimga ega bo‘lishi uchun) noma’lumlar oldidagi koeffitsientlaridan tuzilgan matritsa rangining ozod hadlar ustuni hisobiga kengaytirilgan matritsa rangiga teng bo‘lishi zarur va yetarlidir. (aks holda yechimga ega emas).

Ya’ni (2) sistema yechimga ega bo‘lishi uchun (birgalikda bo‘lishi uchun) quyidagi tenglik o‘rinli bo‘lishi kerak:

$$r = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} = r \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ a_{n1} & a_{2n} & ... & a_{nn} & b_n \end{pmatrix} \qquad (3)$$

$n-$ta noma'lumli $n-$ta chiziqli tenglamalar sistemasi yechimning yagonaligi masalasini Kramer teoremasi ochib beradi.

<u>Kramer teoremasi:</u> $n-$ta noma'lumli $n-$ta chiziqli tenglamalar sistemasi yagona yechimga ega bo'lishi uchun koeffitsientlar matritsasining determinanti 0 dan farqli bo'lishi zarur va yetarlidir. Yagona yechim quyidagi ko'rinishda bo'ladi.

$$\left(\frac{\det \overline{\overline{A_1}}}{\det \overline{\overline{A}}},\ \frac{\det \overline{\overline{A_2}}}{\det \overline{\overline{A}}},\ ...,\ \frac{\det \overline{\overline{A_j}}}{\det \overline{\overline{A}}},\ ...,\ \frac{\det \overline{\overline{A_n}}}{\det \overline{\overline{A}}}\right)$$

bu yerda

$$\det \overline{\overline{A}} = \begin{vmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{vmatrix}, \qquad \det \overline{\overline{A_j}} = \begin{vmatrix} a_{11} & a_{12} & ... & a_{1j-1}b_1 & a_{ij+1} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j-1}b_2 & a_{2j+1} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nj-1}b_n & a_{nj+1} & ... & a_{nn} \end{vmatrix}$$

ya'ni $\det \overline{\overline{A_j}}\,(j=1,2,..,n)\det \overline{\overline{A}}$ dan $j-$ustuni ozod hadlar ustuni bilan almashtirilganligi bilan farq qiladi.

Agar $\det \overline{\overline{A_j}}=0$ bo'lib, Kroneker-Kapelli teoremasi sharti (3) bajarilsa, (2) sistemasi cheksiz ko'p yechimga ega va agar $\det \overline{\overline{A_j}}=0$ bo'lib, Kroneker – Kapelli sharti bajarilmasa, ya'ni

$$r\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} < r\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n}\,b_1 \\ a_{21} & a_{22} & ... & a_{2n}\,b_2 \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn}\,b_n \end{pmatrix}$$

bo'lsa, (2) sistema yechimga ega bo'lmaydi.

Teskari matritsa metodi

Matritsa tushunchasi va matritsa ustida amallardan foydalanib, (2) sistemani quyidagicha yozish mumkin:

$$\overline{\overline{A}}\cdot\overline{\overline{X}}=\overline{\overline{B}}$$

bu yerda

$$\overline{\overline{A}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \overline{\overline{B}} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

noma'lumlar ustun matritsasi va ozod hadlar ustun matritsasi.

$\overline{\overline{A}}$ matritsa mahsusmas bo'lsin, ya'ni $\overline{\overline{A}}^{-1}$ teskari matritsasi mavjud bo'lsin. Oxirgi tenglamaning chap va o'ng tomonlarini chapdan teskari matritsa $\overline{\overline{A}}^{-1}$ga ko'paytiramiz. Natijada $\overline{\overline{A}}^{-1} \cdot \overline{\overline{A}} \cdot \overline{\overline{X}} = \overline{\overline{A}}^{-1} \cdot \overline{\overline{B}}$ $\overline{\overline{A}}^{-1} \cdot \overline{\overline{A}} = \overline{\overline{E}}$ ekanligi-ni hisobga olsak, $\overline{\overline{E}} \cdot \overline{\overline{X}} = \overline{\overline{A}}^{-1} \cdot \overline{\overline{B}}$ yoki $\overline{\overline{X}} = \overline{\overline{A}}^{-1} \cdot \overline{\overline{B}}$ formulani olamiz (ishlangan misollarga qarang).

Chiziqli tenglamalar sistemasi umumiy yechimini topishning Gauss usuli

Gauss usuli o'rta maktab elementar algebrasida aniqlangan tenglamalar sistemasini (chiziqli bo'lishi shart emas) yyechishga qo'llanilgan bo'lib, faqat noma'lumlarni ketma – ket yo'qotish usuli deb nomlanilardi. Umuman, bu usulning bir qancha hillari (modifikatsiyalari) mavjud. Quyida hozirga qadar uchragan va keyinchalik uchraydigan qoidalarni bir – biriga uzviy bog'lash uchun Gauss usulining "Jordano almashtirishlari" yordamida amalga oshiriladigan hili (modifikatsiyasi) bilan tanishamiz.

Ixtiyoriy $n-$ta noma'lumli $m-$ta chiziqli tenglamalar sistemasi berilgan bo'lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1j}x_j + \dots + a_{1n}x_n = b_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rj}x_j + \dots + a_{rn}x_n = b_r \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mj}x_j + \dots + a_{mn}x_n = b_m \end{cases} \qquad (4)$$

Yuqoridagi (4) sistemani jadval ko'rinishida ham yozish mumkin:

x_1	x_2	...	x_j	...	x_n	
a_{11}	a_{12}	...	a_{1j}	...	a_{1n}	b_1
...	...	...	...	...	...	...
a_{r1}	a_{r2}	...	a_{rj}	...	a_{rn}	b_r
...	...	...	...	...	...	...
a_{m1}	a_{m2}	...	a_{mj}	...	a_{mn}	b_m

(4`)

(4) sistemaning faqat bitta tenglamasidan $+x_j$ had qoldirilib, sistemaning qolgan barcha tenglamalarida shu x_j noma'lumli hadlar yo'qotilgan bo'lsa, bunday ko'rinishdagi chiziqli tenglamalar sistemasiga x_j noma'lumga nisbatan yechilgan tenglamalar sistemasi deyiladi. Chiziqli tenglamalar sistemasining har bir tenglamasi yechilgan noma'lumga ega bo'lgan ko'rinishiga yechilgan sistema deyiladi. Birgalikdagi chiziqli tenglamalar sistemasi (4)ning umumiy yechimi deb, unga teng kuchli bo'lgan yechilgan chiziqli tenglamalar sistemasiga aytiladi. Birgalikdagi sistemaning yechimlar to'plamini (barcha yechimlarini) topish uchun uning umumiy yechimini topish yetarlidir. Birgalikdagi umumiy yechimini topish usuliga Gauss usuli deyiladi.

Berilgan sistema (4)dan uning yechilgan sistema ko'rinishini olish uchun elementar ayniy almashtirishlar bajariladi. Elementar ayniy almashtirishlarga quyidagilar kiradi:

1) sistemaning biror tenglamasini ikkala tomonini 0 dan farqli songa ko'paytirish;

2) sistemaning biror tenglamasiga boshqa bir tenglamasini 0 dan farqli songa ko'paytirib, so'ngra qo'shish;

3) sistema tenglamalari o'rinlarini almashtirish;

4) agar biror tenglamaning barcha koeffitsientlari va ozod hadi 0 lardan iborat bo'lsa, uni o'chirish.

Masalan, (4) ((4`)) sistemani x_j noma'lumga nisbatan yyechish uchun (uning biror $r-$tenglamasida $+x_j$ had hosil qilib, qolgan tenglamalarida shu noma'lumli hadlarni yo'qotish uchun) 0 dan farqli bo'lgan a_{rj} element asosida Jordano almashtirishlari bajariladi, ya'ni

1) sistemaning $r-$tenglamasi (yoki xuddi shuning o‘zi, (4`) jadvalning $r-$satri) $\frac{1}{a_{rj}}$ songa ko‘paytiriladi;

2) so‘ngra, $r-$tenglama ($r-$satr) $-a_{1j}$ ga ko‘paytirilib, birinchi tenglamaga (satrga) qo‘shiladi, $-a_{2j}$ ga ko‘paytirilib, ikkinchi tenglamaga (satrga) qo‘shiladi va hokazo. Natijada, (4) sistema ((4`) jadval) quyidagi unga teng kuchli ko‘rinishni oladi:

$$\begin{cases} a'_{11}x_1 + ... + 0x_j + ... + a_{1n}x_n = b_1' \\ .... \\ a'_{r1}x_1 + ... + x_j + ... + a'_{rn}x_n = b_r' \\ .... \\ a'_{m1}x_1 + ... + 0x_j + ... + a'_{mn}x_n = b_n' \end{cases}$$

x_1	x_2	...	x_j	...	x_n	...
a'_{11}	a'_{12}	...	0	...	a'_{1n}	b_1'
...	...	...	...	...	...	...
a'_{r1}	a'_{r2}	...	1	...	a'_{rn}	b_r'
...	...	...	...	...	...	...
a'_{m1}	a'_{m2}	...	0	...	a'_{mn}	b_m'
		...		...		

Hosil qilingan sistema ustida, biror $a'_{st}(s \neq r,\ t \neq j)$ element asosida, yana Jordano almashtirishlari bajarilib (ikkinchi qadam) x_j va x_t noma’lumlarga nisbatan yechilgan teng kuchli sistema olinadi. Agar Jordano almashtirishlari yuqoridagi tartibda ($m-$qadamga qadar) davom ettirilsa, oxirida, (4)ning yechilgan sistema ko‘rinishi yuzaga keladi (ishlangan misollarga qarang). Bir sistemadan ekvivalent ikkinchi sistemaga o‘tayotganda:

a) barcha koeffitsientlari va ozod hadi 0 ga teng tenglama (trivial tenglama) uchiriladi;

b) agar biror bir qarama – qarshi tenglama (yechimga ega bo‘lmagan) hosil bo‘lsa, sistema ham yechimga ega emas, ya’ni birgalikda bo‘lmagan sistema bo‘ladi;

v) yechilgan sistemaning yechilgan noma’lumlari jumlasiga kirmagan noma’lumlari erkin noma’lumlar deyilib, har biri ozod hadlar tomonga o‘tkazilib yoziladi va ular har qanday haqiqiy son qiymatni qabul qilish mumkin (ishlangan amaliy misollarga qaralsin). Chiziqli tenglamalar sistemasini yuqoridagi Gauss usulida yechganda, noma’lumlarni yozmasdan, matritsa (jadval) shaklida bajargan ma’qul (misollarga qarang).

Gauss – Jordano usuli

Bu usul asosida teskari matritsa topishning Jordano algoritmi va ketma – ket yo‘qotish Gauss usuli yotadi. Aniqlangan chiziqli tenglamalar sistemasi (2) ning yechimi $\overline{\overline{X}} = \overline{\overline{A}}^{-1} \cdot \overline{\overline{B}}$ ni topish uchun, teskari matritsa $\overline{\overline{A}}^{-1}$ alohida oshkora qidirilmaydi, balki birdaniga teskari matritsa $\overline{\overline{A}}^{-1}$ ning ozod hadlari ustun matritsasi $\overline{\overline{B}}$ ga ko‘paytmasi $\overline{\overline{A}}^{-1} \cdot \overline{\overline{B}}$ topiladi. Shuning uchun Jordano algoritmi $\left(\overline{\overline{A}} \vdots \overline{\overline{B}}\right)$ kengaytirilgan matritsaga qo‘llaniladi.

Elementar (oddiy) almashtirishlardan so‘ng, chapda birlik matritsa, o‘ngda esa noma’lumlar ustun matritsasi yuzaga keladi $\left(\overline{\overline{A}} \vdots \overline{\overline{B}}\right) \to \left(\overline{\overline{E}} \vdots \overline{\overline{X}}\right)$, ya’ni

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & \vdots & b_n \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \dots & 0 & \vdots & \tilde{b}_1 \\ 0 & 1 & \dots & 0 & \vdots & \tilde{b}_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ 0 & 0 & \dots & 1 & \vdots & \tilde{b}_n \end{pmatrix}$$

yoki xuddi shuning o‘zi

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \Leftrightarrow \begin{cases} x_1 = \tilde{b} \\ x_2 = \tilde{b}_2 \\ \dots \quad \dots \\ \dots \quad \dots \\ x_n = \tilde{b}_n \end{cases}$$

Mashqlar

Chiziqli tenglamalar sistemasini yyechishga doir misol – masalalardan namunalar

Berilgan chiziqli tenglamalar sistemalarining birgalikda yoki birgalikda emasliklarini tekshiring, birgalikdagi yagona yechimga har bir sistemani kamida 4 usulida yeching:

a) $\begin{cases} 4x_1 + x_2 = 6 \\ 2x_1 + 3x_2 = -1 \end{cases}$ b) $\begin{cases} 4x_1 - 6x_2 = 2 \\ 2x_1 - 3x_2 = 1 \end{cases}$

v) $\begin{cases} 4x_1 - 6x_2 = 1 \\ 2x_1 - 3x_2 = 2 \end{cases}$ g) $\begin{cases} x_1 + 2x_2 - 4x_3 = 8 \\ 3x_1 - x_2 + x_3 = 4 \\ 2x_1 + x_2 + 5x_3 = 0 \end{cases}$

d) $\begin{cases} 2x_1 - x_2 + 3x_3 = 5 \\ x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 - 4x_2 + 2x_3 = 3 \end{cases}$ ye) $\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 + x_2 - 2x_3 = 3 \\ 2x_1 + 2x_2 - 4x_3 = 6 \end{cases}$

yo) $\begin{cases} x_1 + 3x_2 + 5x_3 = 0 \\ x_1 - 4x_2 - 2x_3 = 0 \\ 4x_1 - x_2 + 7x_3 = 0 \end{cases}$ j) $\begin{cases} x_1 + 3x_2 + 5x_3 - 2x_4 = 3 \\ x_1 + 4x_2 - 2x_3 + 3x_4 = 2 \\ -x_1 - 2x_2 - 12x_3 + 7x_4 = -4 \\ 3x_1 + 11x_2 + x_3 + 4x_4 = 7 \end{cases}$

z) $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 4x_2 - 3x_3 = 7 \end{cases}$ i) $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 4x_2 - 2x_3 = 2 \end{cases}$

y) $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 4x_2 - 2x_3 = 5 \end{cases}$ k) $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 4x_2 - 2x_3 = 5 \end{cases}$

a) $\begin{cases} 4x_1 + x_2 = 6 \\ 2x_1 + 3x_2 = -1 \end{cases}$ sistema birgalikda, chunki

$$r\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = 2 = 2 = r\begin{pmatrix} 4 & 1 & 6 \\ 2 & 3 & 1 \end{pmatrix}$$

1 – usul: Gaussning ixcham sistemasi bo‘yicha yechamiz:

$$\begin{cases}4x_1+x_2=6\\2x_1+3x_2=-1\end{cases}\Leftrightarrow\begin{cases}4x_1+x_2=6\\0+\frac{5}{2}x_2=-4\end{cases}\Leftrightarrow\begin{cases}4x_1+\left(-\frac{8}{5}\right)x_2=6\\x_2=-\frac{8}{5}\end{cases}\Leftrightarrow\begin{cases}x_1=1.9\\x_2=-1.6\end{cases}$$

2 – usul: Kramer formulalari yordamida yechamiz: $\begin{vmatrix}4&1\\2&3\end{vmatrix}\neq 0$ demak, yechim yagona:

$$x_1=\frac{\begin{vmatrix}6&1\\-1&3\end{vmatrix}}{\begin{vmatrix}4&1\\2&3\end{vmatrix}}=\frac{18+1}{12-2}=1.9,\qquad x_1=\frac{\begin{vmatrix}4&6\\2&-1\end{vmatrix}}{\begin{vmatrix}4&1\\2&3\end{vmatrix}}=\frac{-4-12}{12-2}=-1.6$$

Javob: (1.9; -1.6)

3 – usul: Teskari matritsa metodini qo‘llaymiz:

$\overline{\overline{A}}=\begin{pmatrix}4&1\\2&3\end{pmatrix}$ matritsa mahsusmas bo‘lgani uchun $\overline{\overline{A}}^{-1}$ teskari matritsaga ega va

$$\overline{\overline{A}}^{-1}=\begin{pmatrix}0.3&-0.1\\-0.2&0.4\end{pmatrix}\qquad \overline{\overline{X}}=\overline{\overline{A}}^{-1}\cdot\overline{\overline{B}}$$

formuladan foydalanib,

$$X=\begin{pmatrix}0.3&-0.1\\-0.2&0.4\end{pmatrix}\begin{pmatrix}6\\-1\end{pmatrix}=\begin{pmatrix}1.8+0.1\\-1.2-0.4\end{pmatrix}=\begin{pmatrix}1.9\\-1.6\end{pmatrix}$$

Javob: (1.9; -1.6)

4 – usul. Gauss – Jordano usulini qo‘llaymiz:

$$\begin{pmatrix}\boxed{4}&1&:&6\\2&3&:&-1\end{pmatrix}^{x\frac{1}{4}}\to\begin{pmatrix}\boxed{1}&\frac{1}{4}&:&\frac{3}{2}\\2&3&:&-1\end{pmatrix}^{x(-2)}+\to\begin{pmatrix}1&\frac{1}{4}&:&\frac{3}{2}\\0&\boxed{\frac{5}{2}}&:&-4\end{pmatrix}\to$$

$$\to\begin{pmatrix}1&\frac{1}{4}&:&\frac{3}{2}\\0&\boxed{1}&:&-\frac{8}{5}\end{pmatrix}+_{x\left(-\frac{1}{4}\right)}\to\begin{pmatrix}1&0&:&\frac{19}{10}\\0&1&:&-\frac{8}{5}\end{pmatrix}$$

Javob: (1.9; -1.6)

b) $\begin{cases} 4x_1 - 6x_2 = 2 \\ 2x_1 - 3x_2 = 1 \end{cases}$ sistema birgalikda, chunki:

$$r\begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix} = 1 = 1 = r\begin{pmatrix} 4 & -6 & 2 \\ 2 & -3 & 1 \end{pmatrix}$$

$\begin{vmatrix} 4 & -6 \\ 2 & -3 \end{vmatrix} = 0$ bo‘lgani uchun, cheksiz ko‘p yechimga ega. Umumiy yechimini topamiz:

$$\begin{cases} 4x_1 - 6x_2 = 2 \\ 2x_1 - 3x_2 = 1 \end{cases} \overset{x\left(\frac{1}{2}\right)}{\Leftrightarrow} \begin{cases} 2x_1 - 3x_2 = 1 \\ 2x_1 - 3x_2 = 1 \end{cases} \Leftrightarrow 2x_1 - 3x_2 = 1 \Leftrightarrow x_2 = \frac{2}{3}x_1 - \frac{1}{3}, \quad (x_1 \in R)$$

Javob: $\left(x_2 = \frac{2}{3}x_1 - \frac{1}{3}\right), \quad x_1 \in R$

v) $\begin{cases} 4x_1 - 6x_2 = 1 \\ 2x_1 - 3x_2 = 2 \end{cases}$ sistema birgalikda emas, chunki:

$$r\begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix} = 1 \neq 2 = r\begin{pmatrix} 4 & -6 & 1 \\ 2 & -3 & 2 \end{pmatrix}$$

Javob: Berilgan sistema yechimga ega emas.

g) $\begin{cases} x_1 + 2x_2 - 4x_3 = 8 \\ 3x_1 - x_2 + x_3 = 4 \\ 2x_1 + x_2 + 5x_3 = 0 \end{cases}$ berilgan sistema birgalikda, chunki:

$$r\begin{pmatrix} 1 & 2 & -4 \\ 3 & -1 & 1 \\ 2 & 1 & 5 \end{pmatrix} = 3 = 3 \cdot r\begin{pmatrix} 1 & 2 & -4 & 8 \\ 3 & -1 & 1 & 4 \\ 2 & 1 & 5 & 0 \end{pmatrix} \qquad \begin{vmatrix} 1 & 2 & -4 \\ 3 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = 52 \neq 0$$

bo‘lgani uchun ham sistema yagona yechimga ega.

1. Gaussning ixcham sistemasi bo‘yicha yechamiz:

$$\begin{cases} x_1+2x_2-4x_3=8 \\ 3x_1-x_2+x_3=4 \\ 2x_1+x_2+5x_3=0 \end{cases} \begin{matrix} x(-3) \\ x(-2) \end{matrix} \Leftrightarrow \begin{cases} x_1+2x_2-4x_3=8 \\ 0-7x_2+13x_3=-20 \\ 0-3x_2+13_3=-16 \end{cases} x(-1) \Leftrightarrow \begin{cases} x_1+2x_2-4x_3=8 \\ -7x_2+13x_3=-20 \\ 4x_2=4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1+2\cdot1-4x_3=8 \\ x_2+x_3=4 \\ -7\cdot1+13x_3=-20 \end{cases} \Leftrightarrow \begin{cases} x_1=6+4\cdot(-1) \\ x_2=1 \\ x_3=-1 \end{cases} \Leftrightarrow \begin{cases} x_1=2 \\ x_2=1 \\ x_3=-1 \end{cases}$$

Javob: (2; 1; -1)

2. Kramer formulasini qo‘llaymiz:

$$x_1=\frac{\begin{vmatrix} 8 & 2 & -4 \\ 4 & -1 & 1 \\ 0 & 1 & 5 \end{vmatrix}\begin{matrix} 8 & 2 \\ 4 & -1 \\ 0 & 1 \end{matrix}}{-52}=\frac{-40-16-8-40}{-52}=2$$

$$x_2=\frac{\begin{vmatrix} 1 & 8 & -4 \\ 3 & 4 & 1 \\ 2 & 0 & 5 \end{vmatrix}}{-52}=\frac{20+16+32-120}{-52}=1$$

$$x_3=\frac{\begin{vmatrix} 1 & 2 & 8 \\ 3 & -1 & 4 \\ 2 & 1 & 0 \end{vmatrix}\begin{matrix} 1 & 2 \\ 3 & -1 \\ 2 & 1 \end{matrix}}{-52}=\frac{16+24+16-4}{-52}=-1$$

Javob: (2; 1; -1)

3. Teskari matritsa metodi. Berilgan aniqlangan sistemaning koeffitsientlari matritsasining teskarisini topamiz:

$$\begin{pmatrix} \boxed{1} & 2 & -4 & : & 1 & 0 & 0 \\ 3 & -1 & 1 & : & 0 & 1 & 0 \\ 2 & 1 & 5 & : & 0 & 0 & 1 \end{pmatrix} \begin{matrix} x(-3) \\ x(-2) \end{matrix} \to \begin{pmatrix} 1 & 2 & -4 & : & 1 & 0 & 0 \\ 0 & \boxed{-7} & 13 & : & -3 & 1 & 0 \\ 0 & -3 & 13 & : & -2 & 0 & 1 \end{pmatrix} x\left(\frac{1}{7}\right) \to$$

$$\to \begin{pmatrix} 1 & 2 & -4 & : & 1 & 0 & 0 \\ 0 & 1 & -\frac{13}{7} & : & \frac{3}{7} & -\frac{1}{7} & 0 \\ 0 & -3 & 13 & : & -2 & 0 & 1 \end{pmatrix} \begin{matrix} x(-2) \\ x(3) \end{matrix} \to \begin{pmatrix} 1 & 0 & -\frac{2}{7} & : & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 1 & -\frac{13}{7} & : & -\frac{3}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & \boxed{\frac{52}{7}} & : & -\frac{5}{7} & -\frac{3}{7} & 1 \end{pmatrix} x\frac{7}{52} \to$$

$$\to \begin{pmatrix} 1 & 0 & -\frac{2}{7} & : & \frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 1 & -\frac{13}{7} & : & \frac{3}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & : & -\frac{5}{52} & -\frac{3}{52} & \frac{7}{52} \end{pmatrix} \begin{matrix} x\frac{2}{7} & x\frac{13}{7} \end{matrix} \to \begin{pmatrix} 1 & 0 & 0 & : & \frac{3}{26} & \frac{7}{26} & \frac{1}{26} \\ 0 & 1 & 0 & : & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & : & -\frac{5}{52} & -\frac{3}{52} & \frac{7}{52} \end{pmatrix}$$

$$\overline{\overline{X}} = \begin{pmatrix} \frac{3}{26} & \frac{7}{26} & \frac{1}{26} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{5}{52} & -\frac{3}{52} & \frac{7}{52} \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{24}{26} + \frac{28}{26} \\ 2-1 \\ -\frac{40}{52} - \frac{12}{52} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Javob: (2; 1; -1)

4. Gauss – Jordano usuli yuqoridagi teskari matritsa usuliga qaraganda ratsionalroq ekanligini quyida payqash qiyin emas:

$$\left(\begin{array}{cccc} \boxed{1} & 2 & -4 & : & 8 \\ 3 & -1 & 1 & : & 4 \\ 2 & 1 & 5 & : & 0 \end{array}\right) \begin{matrix} x(-3) & x(-2) \\ + & \\ & + \end{matrix} \to \left(\begin{array}{ccccc} 1 & 2 & -4 & : & 8 \\ 0 & \boxed{-7} & 13 & : & -20 \\ 0 & -3 & 13 & : & -16 \end{array}\right) x\left(-\frac{1}{7}\right) \to$$

$$\to \left(\begin{array}{ccccc} 1 & 2 & -4 & : & 8 \\ 0 & 1 & -\frac{13}{7} & : & \frac{20}{7} \\ 0 & -3 & 13 & : & -16 \end{array}\right) \begin{matrix} + \\ x(-2) \\ x(-3) \\ + \end{matrix} \to \left(\begin{array}{ccccc} 1 & 0 & -\frac{2}{7} & : & \frac{16}{7} \\ 0 & 1 & -\frac{13}{7} & : & \frac{20}{7} \\ 0 & 0 & \frac{52}{7} & : & -\frac{52}{7} \end{array}\right) x\frac{7}{52} \to$$

$$\to \left(\begin{array}{ccccc} 1 & 0 & -\frac{2}{7} & : & \frac{16}{7} \\ 0 & 1 & -\frac{13}{7} & : & \frac{20}{7} \\ 0 & 0 & 1 & : & -1 \end{array}\right) \begin{matrix} x\frac{2}{7} & x\frac{13}{7} \\ & + \end{matrix} \to \left(\begin{array}{ccccc} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -1 \end{array}\right)$$

$x_1 = 2; \quad x_2 = 1; \quad x_3 = -1$

J: (2; 1; -1)

d) Berilgan aniqlangan chiziqli tenglamalar sistemasi birgalikda emas, chunki:

$$r\begin{pmatrix} 2 & -1 & 3 \\ 1 & 3 & 5 \\ 3 & -4 & 2 \end{pmatrix} = 2 \neq 3 = r\begin{pmatrix} 2 & -1 & 3 & 5 \\ 1 & 3 & 5 & -1 \\ 3 & -4 & 2 & 3 \end{pmatrix}$$

Sistemaning yechimga ega emasligini Gauss usulidan foydalanib, umumiy yechimga ega emasligidan ham xulosa qilish mumkin:

$$\begin{pmatrix} 2 & -1 & 3 & 5 \\ \boxed{1} & 3 & 5 & -1 \\ 3 & -4 & 2 & 3 \end{pmatrix} \begin{matrix} \times(-2) \\ \\ \times(3) \end{matrix} \to \begin{pmatrix} 0 & -7 & -7 & 7 \\ 1 & 3 & 5 & -1 \\ 0 & -13 & -13 & 6 \end{pmatrix} \times\left(-\frac{1}{7}\right) \to \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 3 & 5 & -1 \\ 0 & -13 & -13 & 6 \end{pmatrix} \times(-3) \ \times 13 \to$$

$$\to \begin{pmatrix} 0 & 1 & 1 & -7 \\ 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

x

+

Jadval (sistemasi)ning oxirgi satri (tenglamasi) qarama – qarshidir, chunki $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -7$ bo‘lishi mumkin emas. Demak, berilgan sistema yechimga ega emas.

y) Berilgan sistema birgalikda, chunki:

$$r\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & 2 & 4 \end{pmatrix} = 2 = 2r\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -2 & 3 \\ 2 & 2 & -4 & 6 \end{pmatrix}$$

Shu bilan birga $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & 2 & -4 \end{vmatrix}$ bo‘lgani uchun cheksiz ko‘p yechimga ega.

Umumiy yechimini Gauss usuli yordamida topamiz:

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -2 & 3 \\ 2 & 2 & -4 & 6 \end{pmatrix} \times(-2) \to \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -2 & 3 \end{pmatrix} \times(-1) \to$$

$$\to \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & \boxed{2} & -3 & 2 \end{pmatrix} \times\frac{1}{2} \to \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 2 \\ 0 & 1 & -\frac{3}{2} & 1 \end{pmatrix}$$

yoki boshqacha

$$\begin{cases} x_1 - x_2 + x_3 = 1 & \times(-1) \\ x_1 + x_2 - 2x_3 = 3 \end{cases} \to \left[r\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \quad \text{á¢ëãàíè ó÷óí} \right] \Leftrightarrow \begin{cases} x_1 - x_2 + x_3 = 1 \\ 0 + 2x_2 - 3x_3 = 2 & \times\frac{1}{2} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 - x_2 + x_3 = 1 \\ 0 + x_2 - \frac{3}{2}x_3 = 1 \end{cases} \Leftrightarrow \begin{cases} x_1 + -\frac{1}{2}x_3 = 2 \\ 0 + x_2 - \frac{3}{2}x_3 = 1 \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{1}{2}x_3 + 2 \\ x_2 = \frac{3}{2}x_3 + 1 \end{cases}$$

J: $\left(\frac{1}{2}x_3+2;\ \ \frac{3}{2}x_3+1;\ x_3\right),\quad x_3\in R$

yo) Berilgan bir jinsli tenglamalar sistemasi, trivial yechimdan tashqari yana cheksiz ko‘p yechimga ega, chunki:

$$\begin{vmatrix} 1 & 3 & 5 \\ 1 & -4 & -2 \\ 4 & -1 & 7 \end{vmatrix} = 0$$

Umumiy yechimni Gauss usulida qidiramiz:

$$\begin{pmatrix} 1 & 3 & 5 & 0 \\ 1 & -4 & -2 & 0 \\ 4 & -1 & 7 & 0 \end{pmatrix} \begin{matrix} \times(-1)\ \times(-4) \end{matrix} \to \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & -13 & -13 & 0 \end{pmatrix} \begin{matrix} \\ \times\left(-\frac{1}{7}\right) \\ \times\left(-\frac{1}{13}\right) \end{matrix} \to \left[\begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}\right] \times(-1) \to$$

$$\to \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \times(-3) \to \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \end{cases}$$

J: $(-2x_3,\ -x_3;\ x_3),\ x_3 \in R$

j) Berilgan aniqlangan tenglamalar sistemasi birgalikda, chunki:

$$r\begin{pmatrix} 1 & 3 & 5 & -2 \\ 1 & 4 & -2 & 3 \\ -1 & -2 & -12 & 7 \\ 3 & 11 & 1 & 4 \end{pmatrix} = 3 = 3 = r\begin{pmatrix} 1 & 3 & 5 & -2 & 3 \\ 1 & 4 & -2 & 3 & 2 \\ -1 & -2 & -12 & 7 & -4 \\ 3 & 11 & 1 & 4 & 7 \end{pmatrix}$$

$\det \overline{\overline{A}} = 0$ bo‘lgani uchun berilgan cheksiz ko‘p yechimga ega. Umumiy yechimini Gauss usulida topamiz:

$$\begin{pmatrix} \boxed{1} & 3 & 5 & -2 & 3 \\ 1 & 4 & -2 & 3 & 2 \\ -1 & -2 & -12 & 7 & -4 \\ 3 & 11 & 1 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 & -2 & 3 \\ 0 & 1 & -7 & 5 & -1 \\ 0 & 1 & 7 & 5 & -1 \\ 0 & 2 & -14 & 10 & -2 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 3 & 5 & -2 & 3 \\ 0 & \boxed{1} & -7 & 5 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 26 & -17 & 6 \\ 0 & 1 & -7 & 5 & -1 \end{pmatrix}$$

$$\begin{cases} x_1 + 26x_3 - 17x_4 = 16 \\ x_2 - 7x_3 + 5x_4 = -1 \end{cases} \Leftrightarrow \begin{cases} x_1 = 6 - 26x_3 + 17x_4 \\ x_2 = -1 + 7x_3 - 5x_4 \end{cases}$$

Javob: $(6 - 26x_3 + 17x_4;\ \ -1 + 7x_3 - 5x_4;\ \ x_3;\ \ x_4),\quad x_3,\ x_4 \in R$

z) Berilgan kami bilan aniqlangan chiziqli tenglamalar sistemasi birgalikda, chunki:

$$r\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -3 \end{pmatrix} = 2 = 2 = r\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 7 \end{pmatrix}$$

$r\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} = 2$ bo‘lgani uchun

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 4x_2 - 3x_3 = 7 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_3 = 1 - 2x_2 \\ 2x_1 - 3x_3 = 7 - 4x_2 \end{cases}$$

Oxirgi sistemani Kramer formulalari yordamida yechamiz (albatta Gauss usuli yordamida yechgan ma’qulroq):

$$x_1 = \frac{\begin{vmatrix} 1 - 2x_2 & -1 \\ 7 - 4x_2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}} = \frac{-3 + 6x_2 + 7 - 4x_2}{-3 + 2} = -2x_2 - 4$$

$$x_3 = \frac{\begin{vmatrix} 1 & 1 - 2x_2 \\ 2 & 7 - 4x_2 \end{vmatrix}}{-1} = \frac{7 - 4x_2 - 2 + 4x_2}{-1} = -5$$

Javob: $(-2x_2 - 4;\ \ x_2;\ \ -5),\ \ x_2 \in R$

i) Berilgan kami bilan aniqlangan sistema birgalikda:

$$r\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} = 1 = 1 = r\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \end{pmatrix}$$

Demak,

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 4x_2 - 2x_3 = 2 \end{cases} \Leftrightarrow x_1 + 2x_2 - x_3 \Leftrightarrow x_1 = 1 - 2x_2 + x_3$$

Javob: $(1 - 2x_2 + x_3; \; x_x; \; x_3), \; x_{2,3} \in R$

y) Berilgan kami bilan aniqlangan chiziqli tenglamalar sistemasi birgalikda, chunki:

$$r\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} = 1 \neq 2 = r\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 5 \end{pmatrix}$$

Javob: Yechimga ega emas.

Gauss usulida yechamiz:

$$\begin{pmatrix} \boxed{2} & -1 & 3 \\ 3 & -5 & 1 \\ 4 & -7 & 1 \end{pmatrix}^{\times\frac{1}{2}} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 3 & -5 & 1 \\ 4 & -7 & 1 \end{pmatrix}^{\times(-3)\;\times(-4)} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{5}{2} & -\frac{7}{2} \\ 0 & -5 & -5 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 \end{pmatrix}_{\times\frac{1}{2}} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$$

Javob: (2; 1)

Mashqlar

Chiziqli tenglamalar sistemasini mustaqil yyechish uchun amaliy misol – masalalar

Berilgan chiziqli tenglamalar sistemalarining birgalikda yoki birgalikda emasliklarini tekshiring, birgalikdagi yagona yechimga ega har bir sistemani kamida 2 usulda yeching:

1. $\begin{cases} x+y=3 \\ x-y=-5 \end{cases}$ (-1; 4)

2. $\begin{cases} x_1-3x_2=7 \\ 2x_1+x_2=0 \end{cases}$ (1; -2)

3. $\begin{pmatrix} x_1+2x_2=2 \\ 2x_1-8x_2=-5 \end{pmatrix}$ $\left(\frac{1}{2};\ \frac{3}{4}\right)$

4. $\begin{cases} 3x_1+x_2=0 \\ 7x_1-x_2=4 \end{cases}$ (0.4; -1.2)

5. $\begin{cases} x_1-2x_2=1 \\ 3x_1-6x_2=4 \end{cases}$ (yechimga ega emas)

6. $\begin{cases} x_1-3x_2=0.5 \\ 2x_1-6x_2=1 \end{cases}$ $(3x_2+0.5;\ x_2),\ x_2\in R$

7. $\begin{cases} x+y-3z=-1 \\ 2x-y+z=2 \\ 3x+2y-4z=1 \end{cases}$ (1; 1; 1)

8. $\begin{cases} x_1+3x_2-4x_3=-1 \\ x_1-5x_2+x_3=7 \\ 2x_1+x_2-3x_3=3 \end{cases}$ (2; -1; 0)

9. $\begin{cases} 3x_1+x_2-x_3=2 \\ 2x_1-3x_2+x_3=-1 \\ x_1-x_2+2x_3=5 \end{cases}$ (1; 2; 3)

10. $\begin{cases} x_1+2x_2+3x_3=0 \\ 2x_1-3x_2-x_3=1 \\ 3x_1+x_2+4x_3=-1 \end{cases}$ (yechimga ega emas)

11. $\begin{cases} x_1 - 2x_2 - 5x_3 = 1 \\ 4x_1 + x_2 - 2x_3 = -3 \\ -x_1 + 3x_2 + 7x_2 = 2 \end{cases}$ (yechimga ega emas)

12. $\begin{cases} x_1 - x_2 + 3x_3 = 3 \\ 2x_1 + 3x_2 - 4x_3 = -1 \\ 3x_1 + 2x_2 - x_3 = 2 \end{cases}$ $\left(\frac{8}{5} - x_3;\ 2x_3 - \frac{7}{5};\ x_3\right),\ \ x_3 \in R$

13. $\begin{cases} -x_1 + 2x_2 - 3x_3 = 4 \\ 3x_1 + x_2 - 2x_3 = 1 \\ 4x_1 - x_2 + x_3 = -3 \end{cases}$ $\left(\frac{1}{7}x_3 - \frac{2}{7};\ \frac{11}{7}x_3 + \frac{13}{7};\ x_3\right),\ \ x_3 \in R$

14. $\begin{cases} x_1 + 4x_2 - x_3 = 0 \\ 3x_1 - 5x_2 + x_3 = 0 \\ 2x_1 - x_2 + 6x_3 = 0 \end{cases}$ (0; 0; 0)

15. $\begin{cases} 5x_1 - x_2 + 4x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 - 3x_2 - x_3 = 0 \end{cases}$ $(-x_3;\ -x_3;\ x_3),\ \ x_3 \in R$

16. $\begin{cases} x_1 - 3x_2 + 2x_3 + x_4 = 2 \\ 2x_1 + x_2 + 4x_3 + 3x_4 = 1 \\ x_1 + 5x_2 - x_3 + x_4 = -4 \\ 3x_1 - x_2 + 6x_3 + 5x_4 = 0 \end{cases}$ (1; 0; 2; -3)

17. $\begin{cases} 2x_1 + x_2 + 3x_3 - 4x_4 = 3 \\ x_1 - 2x_2 + x_3 - 3x_4 = -1 \\ 3x_1 + 4x_2 - 5x_3 + x_4 = 4 \\ 2x_1 - 4x_2 + 2x_3 - 6x_4 = 5 \end{cases}$ (yechimga ega emas)

18. $\begin{cases} x_1 - 2x_2 + x_3 = 4 \\ x_1 + 3x_2 + x_3 = 0 \end{cases}$ $\left(\frac{12}{5} - x_3;\ -\frac{4}{5};\ x_3\right),\ \ x_3 \in R$

19. $\begin{cases} x_1 - x_2 - 3x_3 = 6 \\ -2x_1 + 2x_2 + 6x_3 = -9 \end{cases}$ (yechimga ega emas)

20. $\begin{cases} x_1 - 3x_2 = -5 \\ -x_1 + x_2 = 1 \\ 4x_1 - x_2 = 2 \end{cases}$ (1; 2)

4 - §. Vektorlar. Vektorlar sistemasi

O‘rta maktab geometriyasi kursida real mavjud uch o‘lchovli fazodagi geometrik vektorlar (boshi va oxiri tartiblangan yo‘nalishli kesmalar), ular ustida amallar va ikki vektorning okalyar ko‘paytmasi kabi ma’lumotlar berilgan. Geometrik tasviri yo‘nalishi kesma bo‘lgan bir, ikki, uch o‘lchovli vektorlar ustidagi tushunchalarini geometrik tasvirlash mumkin bo‘lmagan, faqat arifmetik izohlash mumkin bo‘lgan ixtiyoriy $n(n \geq 4)$ o‘lchovli vektorlar uchun umumlashtiramiz.

$n-$ o‘lchovli vektorlar va ular ustida chiziqli amallar

n ta $x_1, x_2, ..., x_n$ haqiqiy sonlarning tartiblangan $(x_1, x_2, ..., x_n)$ sistemasiga n o‘lchovli x vektor deyiladi: $x = (x_1, x_2, ..., x_n)$ x_1 son x vektorining birinchi koordinatasi (yoki komponenti), x_2 son uning ikkinchi koordinatasi (yoki komponenti) deyiladi va hokazo. Vektordagi koordinatalar soniga uning o‘lchovli deyiladi.

Barcha koordinatalari 0 lardan iborat vektorga nolinchi θ vektor deyiladi: $\theta = (0; 0; ..., 0)$. Ikki bir xil o‘lchovli $x = (x_1, x_2, ..., x_n)$ va $y = (y_1, y_2, ..., y_n)$ vektorlar berilgan bo‘lsin. Berilgan vektorlar o‘zaro teng bo‘lishi uchun, ularning barcha mos komponentlari o‘zaro teng bo‘lishlari kerak, ya’ni $x_i = y_i (i = 1, 2, ..., n)$ bo‘lgandagina $x = y$. Agar o‘zaro mos komponentlarining aqali bir juft teng bo‘lmasa, u holda $x \neq y$. Bir xil n o‘lchovli berilgan x va y vektorlarning yig‘indisi deb, quyidagi n o‘lchovli vektorga aytiladi:

$$x + y = (x_1 + y_1;\ x_2 + y_2; ...,\ x_n + y_n)$$

Berilgan x vektorning λ songa ko‘paytmasi quyidagicha amalga oshiriladi: $x \cdot \lambda = \lambda \cdot x = (\lambda \cdot x_1,\ \lambda \cdot x_2, ..., \lambda \cdot x_n)$. $(-1) \cdot x = -x$ vektorga x vektorning qarama – qarshi vektori deyiladi. Vektorlar ayirmasi quyidagicha aniqlanadi:

$$x - y = x + (-y) = (x_1 - y_1;\ x_2 - y_2;\ ...,\ x_n - y_n)$$

Vektorlar ustida bajariladigan yuqoridagi amallarga chiziqli amallar deb ataladi va ular quyidagi xossalarga bo‘ysinadi:

1. $x + y = y + x$, 2. $(x + y) + z = x + (y + z)$

3. $(x + y)\cdot\lambda = \lambda x + \lambda y$ 4. $x(\lambda_1 + \lambda_2) = \lambda_1 x + \lambda_2 x$

Skalyar ko‘paytma. Vektor uzunligi. Vektorlar orasidagi burchak

Ikki n o‘lchovli x va y vektorlarning skalyar ko‘paytmasi deb, quyidagi yig‘indiga teng bo‘lgan $(x\cdot y)$ songa aytiladi:

$$x\cdot y = (x_1\cdot y_1 + x_2\cdot y_2 + \ldots + x_n\cdot y_n) = \sum_{j=1}^{n} x_j\cdot y_j$$

Skalyar ko‘paytma quyidagi xossalarga ega:

1. $(x\cdot y) = (y\cdot x)$, 2. $(x\cdot(y+z)) = (x\cdot y) + (x\cdot z)$

3. $\lambda(x\cdot y) = ((\lambda\cdot x)\cdot y)$, 4. $(x\cdot x) = x^2 \geq 0$

Berilgan n o‘lchovli $x = (x_1, x_2, \ldots, x_n)$ vektorning uzunligi (moduli yoki normasi) deb, $|x| = \sqrt{x^2} = \sqrt{x_1^2 + x_2^2 + x_n^2} = \sqrt{\sum_{j=1}^{n} x^2}$ nomanfiy $|x|$ songa aytiladi.

Ikki n o‘lchovli x va y vektorlarning skalyar ko‘paytmasi va ularning modullari orasida quyidagi Koshi – Bunyakovskiy tengsizligi o‘rinli:

$$|(x\cdot y)| \leq |x|\cdot|y| \qquad \text{yoki} \qquad \left|\sum_{j=1}^{n} x_j\cdot y_j\right| \leq \sqrt{\sum_{j=1}^{n} x_j^2}\cdot\sqrt{\sum_{j=1}^{n} y_j^2}$$

Uchburchak tengsizligi deb ataluvchi tengsizlik esa quyidagi ko‘rinishga ega:

$$|(x+y)| \le |x| + |y| \quad \text{yoki} \quad \sqrt{\sum_{j=1}^{n}(x_j + y_j)^2} \le \sqrt{\sum_{j=1}^{n} x_j^2} + \sqrt{\sum_{j=1}^{n} y_j^2}$$

Ikki x va y, n o‘lchovli vektorlarning skalyar ko‘paytmasi formulasidan shu vektorlar orasidagi burchak kattaligi kosinusini aniqlash mumkin:

$$\cos\left(x \hat{:} y\right) = \frac{(x \cdot y)}{|x| \cdot |y|} = \frac{\sum_{j=1}^{n} x_j \cdot y_j}{\sqrt{\sum_{j=1}^{n} x_j^2} \cdot \sqrt{\sum_{j=1}^{n} y_j^2}}$$

So‘ngra natijaning musbat yoki manfiyligiga qarab, o‘tkir yoki o‘tmas burchak kattaligi topiladi.

Vektorlarning chiziqli kombinatsiyasi. Vektorni vektorlar sistemasi orqali yoyish. Vektorlarning chiziqli bog‘liqligi

k ta n o‘lchovli vektorlardan iborat vektorlar sistemasi berilgan bo‘lsin:

$$\begin{cases} a^{(1)} = (a_{11};\ a_{21};\ \ldots;\ a_{n1}) \\ a^{(2)} = (a_{12};\ a_{22};\ \ldots;\ a_{n2}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^{(k)} = (a_{1k};\ a_{2k};\ \ldots;\ a_{nk}) \end{cases} \qquad (1)$$

Har qanday $\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} = \sum_{i=1}^{n} \lambda_i a^{(i)}$ ko‘rinishdagi vektorga $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ vektorning $\lambda_1, \lambda_2, \ldots, \lambda_k$ haqiqiy son koeffitsientli chiziqli kombinatsiyasi deyiladi. Barcha chiziqli koeffitsientlari 0 lardan iborat $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ vektorlarning chiziqli kombinatsiyasiga trivial chiziqli kombinatsiyasi deyiladi.

O‘z – o‘zidan ma’lumki vektorlarning ixtiyoriy trivial chiziqli kombinatsiyasi 0 vektorga teng sonli koeffitsientlarning birortasi 0 dan farqli bo‘lsa, vektorlarning chiziqli kombinatsiyasiga notrivial chiziqli kombinatsiyasi deyiladi. Berilgan vektorlar sistemasining har bir vektor koordinatalarda berilgan bo‘lgani uchun, ularning ixtiyoriy $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_k$ koeffitsientli chiziqli koordinatalarda topish mumkin:

$$\sum_{i=1}^{n}\lambda_i a^{(1)} = \lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \cdots + \lambda_k a^{(k)} = \lambda_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \ldots \\ \ldots \\ a_{n1} \end{pmatrix} + \lambda_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \ldots \\ \ldots \\ a_{n2} \end{pmatrix} + \ldots + \lambda_k \begin{pmatrix} a_{1k} \\ a_{2k} \\ \ldots \\ \ldots \\ a_{nk} \end{pmatrix} =$$

$$= \begin{pmatrix} \sum_{i=1}^{k}\lambda_i a_{1k} \\ \ldots\ldots\ldots \\ \sum_{i=1}^{k}\lambda_i a_{nk} \end{pmatrix}$$

Vektorlar ustida chiziqli amallardan foydalanib, berilgan

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1k}x_k = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2k}x_k = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nk}x_k = b_n \end{cases}$$

chiziqli tenglamalar sistemasini

$$a^{(1)} = \begin{pmatrix} a_{11} \\ a_{21} \\ \ldots \\ \ldots \\ a_{n1} \end{pmatrix},\quad a^{(2)} = \begin{pmatrix} a_{12} \\ a_{22} \\ \ldots \\ \ldots \\ a_{n2} \end{pmatrix},\quad \ldots,\quad a^{(k)} = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \ldots \\ \ldots \\ a_{nk} \end{pmatrix},\quad b^{(0)} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_n \end{pmatrix}$$

vektorlar orqali quyidagicha yozish mumkin:

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ \dots \\ a_{n1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \dots \\ \dots \\ a_{n2} \end{pmatrix} x_2 + \dots \begin{pmatrix} a_{1k} \\ a_{2k} \\ \dots \\ \dots \\ a_{nk} \end{pmatrix} x_k = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ \dots \\ b_n \end{pmatrix}$$

yoki xuddi shuning o‘zi

$$a^{(1)}x_1 + a^{(2)}x_2 + \ldots + a^{(k)}x_k = b^{(0)}.$$

k ta sonlarning tartiblangan $\lambda_1, \lambda_2, \ldots, \lambda_k$ sistemasi yuqoridagi chiziqli tenglamalar sistemasining yechimi bo‘lishi uchun $\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} = b^{(0)}$ vektor tenglikning bajarilishi yetarli. Ta’rifga binoan k ta haqiqiy sonlarning shunday bir notrival tartibi – $\lambda_1, \lambda_2, \ldots, \lambda_k$ sistemasini ko‘rsatish mumkin bo‘lsa, n o‘lchovi $b^{(0)}$ vektorni n o‘lchovli $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ vektorlar sistemasi orqali yoyish mumkin deyiladi. Bunda $\lambda_1, \lambda_2, \ldots, \lambda_k$ sonlarga yoyilma koeffitsientlari deyiladi. Berilgan $b^{(0)}$ vektorni berilgan $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ vektorlar sistemasi orqali yoyish uchun yuqoridagi chiziqli tenglamalar sistemasining biror bir yechimini topish kifoya. Agar chiziqli tenglamalar sistemasi yechimga ega bo‘lmasa, o‘z navbatida $b^{(0)}$ vektorni $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ vektorlar orqali yoyish mumkin emas.

Berilgan vektorlar sistemasi $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ uchun $\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} = \theta$ tenglikni qanoatlantiradigan va $\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_k^2 \neq 0$, ya’ni hech bo‘lmaganda bittasi 0 dan farqli bo‘lgan $\lambda_1, \lambda_2, \ldots, \lambda_k$ son koeffitsientlar tanlash mumkin bo‘lsa, $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ vektorlar sistemasiga chiziqli bog‘liq vektorlar sistemasi deyiladi. Boshqacha aytganda, vektorlar sistemasining biror bir notrivial chiziqli kombinatsiyasi 0 vektorga teng bo‘lsa, berilgan vektorlar sistemasi chiziqli bog‘liq deyiladi.

Agar $\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)}$ vektorli tenglikdan faqat $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$ ekanligi kelib chiqsa (yoki faqat trivial koeffitsientlardagina berilgan vektorlarning chiziqli kombinatsiyasi 0 vektorga teng bo‘lishi mumkin bo‘lsa), $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ vektorlar sistemasiga chiziqli bog‘liq bo‘lmagan vektorlar sistemasi deyiladi.

Masalan, ixtiyoriy ikki koleniar vektorlar chiziqli bog‘liq sistemani tashkil etsa, o‘z navbatida ixtiyoriy ikki o‘zaro koleniar bo‘lmagan vektorlar chiziqli bog‘liq bo‘lmagan sistemani tashkil qiladi. Uchta o‘zaro komplanar vektorlar chiziqli bog‘liq bo‘lgan sistema bo‘lishsa, aksi uchta o‘zaro komplanar bo‘lmagan vektorlar chiziqli bog‘liq bo‘lmagan sistemadir. Har qanday to‘rtta geometrik vektordan iborat sistema chiziqli bog‘liq sistemadir va hokazo.

Yuqoridagi k ta n o‘lchovli $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ vektorlar sistemasi chiziqli bog‘liq bo‘lishi uchun $\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} = \theta$ bir jinsli chiziqli tenglamalar sistemasi notrival yechimga ega bo‘lishi shart. Agar bir jinsli chiziqli tenglamalar sistemasi faqat trivial yechimgagina ega bo‘lsa, berilgan vektorlar sistemasi chiziqli bog‘liq vektorlar sistemasi hisoblanadi. Ta’rifga binoan, agar $\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} = \theta$ tenglik bajarilib, koeffitsientlarning hammasi 0 ga teng bo‘lmasa, ya’ni hech bo‘lmaganda biror bir koeffitsient 0 dan farqli bo‘lsa, berilgan vektorlar sistemasi chiziqli bog‘liq sistemadir. Masalan, $\lambda_k \neq 0$ bo‘lsin. U holda $a^{(k)} = -\frac{\lambda_1}{\lambda_k} a^{(1)} - \frac{\lambda_2}{\lambda_k} a^{(2)} - \ldots - \frac{\lambda_{k-1}}{\lambda_k} a^{(k-1)}$, ya’ni vektorlar sistemasi chiziqli bog‘liq, chunki sistemaning bir vektori qolganlarining chiziqli kombinatsiyasiga teng.

$a^{(1)}, a^{(2)}, \ldots, a^{(i)}, \ldots, a^{(k)}$ vektorlar sistemasi chiziqli bog‘lanmagan bo‘lsa, sistemaning ixtiyoriy qism sistemasi ham chiziqli bog‘lanmagan bo‘ladi.

Agar $a^{(1)}, a^{(2)}, \ldots, a^{(i)}$ vektorlar sistemasi chiziqli bog‘liq bo‘lsa, ixtiyoriy to‘ldirilgan sistema ham chiziqli bog‘liq bo‘ladi. Agar vektorlar sistemaning vektorlar sistemasining biror vektori θ vektor bo‘lsa, sistema chiziqli bog‘liq bo‘ladi, ya’ni θ vektorni o‘z ichiga olgan ixtiyoriy vektorlar sistemasi chiziqli bog‘liq sistemadir.

Elementlari (1) sistema vektorlarining koordinatalaridan iborat quyidagi $\overline{\overline{A}}$ matritsa tuzamiz:

$$\overline{\overline{A}} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nk} \end{pmatrix}$$

Teorema: Agar $\overline{\overline{A}}$ matritsa rangi $r\left(\overline{\overline{A}}\right)$ sistemadagi vektorlar soni k ga teng bo‘lsa, ya’ni $r\left(\overline{\overline{A}}\right)=k$, u holda vektorlar sistemasi chiziqli bog‘liq bo‘lmagan sistema bo‘ladi. Agar $r\left(\overline{\overline{A}}\right)>k$ bo‘lsa, vektorlar sistemasi chiziqli bog‘liq sistema bo‘ladi.

Teoremani isbotlashdan oldin misollarda asoslaymiz:

Misol. Ma’lumki, berilgan ikki o‘lchovli ikkita $a^{(1)}=(a_{11},a_{21})$ va $a^{(2)}=(a_{12},a_{22})$ vektorlar nokollenear (chiziqli bog‘liq emas) bo‘lishi sharti

$$\det\overline{\overline{A}}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\neq 0$$

dan iborat. Agar

$$\det\overline{\overline{A}}\neq 0$$

bo‘lsa, o‘z navbatida $r\left(\overline{\overline{A}}\right)=2=2=k$ ekanligi kelib chiqadi. Agar vektorlar kollenear bo‘lishsa (chiziqli bog‘liq). $\det\overline{\overline{A}}=0$ bo‘lib, $r\left(\overline{\overline{A}}\right)=1<2=k$ kelib chiqadi. Haqiqatdan ham, $\lambda_1 a^{(1)}+\lambda_2 a^{(2)}=\theta$ vektor tenglama koordinatalarda quyidagi bir jinsli chiziqli tenglamalar sistemasi ko‘rinishini oladi:

$$\begin{cases} \lambda_1 a_{11}+\lambda_2 a_{22}=0 \\ \lambda_1 a_{21}+\lambda_2 a_{22}=0 \end{cases}$$

Kramer teoremasidan ma’lumki, $\det\overline{\overline{A}}\neq 0$ (ya’ni $r\left(\overline{\overline{A}}\right)=k$) shart sistemaning yagona trivial

$$\begin{cases} \lambda_1=0 \\ \lambda_2=0 \end{cases}$$

yechimga ega bo‘lishi zarur va yetarlidir. Bu degan so‘z, berilgan vektorlar sistemasi chiziqli bog‘liq emas. Agar $\det\overline{\overline{A}}=0$ bo‘lsa, (ya’ni $r\left(\overline{\overline{A}}\right)<k$), ma’lumki birgalikdaligi tufayli, bir jinsli chiziqli tenglamalar sistemasi cheksiz ko‘p (jumladan notrivaial) yechimlarga ega.

Buning ma’nosi, berilgan vektorlar sistemasi chiziqli bog‘liq (kollenear)dirlar.

Misol. Ikki o‘lchovli $a^{(1)} = (a_{11}, a_{21}, a_{31})$ va $a^{(2)} = (a_{12}, a_{22}, a_{32})$ vektorlar berilgan bo‘lsin. Quyidagi sistemani tuzamiz:

$$\begin{cases} \lambda_1 a_{11} + \lambda_2 a_{22} = 0 \\ \lambda_1 a_{21} + \lambda_2 a_{22} = 0 \\ \lambda_1 a_{31} + \lambda_2 a_{32} = 0 \end{cases}$$

Ma’lumki, bir jinsli chiziqli tenglamalar sistemasining koeffitsientlar matritsasining rangi $r(\overline{\overline{A}})$ noma’lumlar soniga teng bo‘lgandagina yagona trivial yechimga ega. Demak, $r(\overline{\overline{A}}) = k$ bo‘lganda berilgan geometrik vektorlar chiziqli bog‘liq bo‘lmagan (nokollenear) sistemani tashkil qiladi. Agar bir jinsli chiziqli tenglamalar sistemasida koeffitsientlar matritsasining rangi $r(\overline{\overline{A}})$ noma’lumlar sonidan kam bo‘lsa, notrivial yechimlarga ham ega. Demak, $r(\overline{\overline{A}}) < k$ bo‘lganda, berilgan geometrik vektorlar chiziqli bog‘liq (kollenear) sistemani tashkil etadi.

Xuddi shu tartibda, uchta o‘zaro komplanar vektorlar chiziqli bog‘liq, uchta komplanar vektorlar esa chiziqli bog‘liq, bo‘lmagan har qanday to‘rtta geometrik vektorlardan iborat sistema chiziqli bog‘liq sistema ekanliklarini asoslash mumkin va hokazo.

k ta n o‘lchovli $a^{(1)} = (a_{11}, a_{21}, \ldots, a_{n1})$, $a^{(2)} = (a_{12}, a_{22}, \ldots, a_{n2})$, …, $a^{(k)} = (a_{1k}, a_{2k}, \ldots, a_{nk})$ vektorlar sistemasi uchun $\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} = \theta$ vektor tenglama koordinatalarda quyidagicha ifodalanadi:

$$\begin{cases} a_{11}\lambda_1 + a_{12}\lambda_2 + \ldots + a_{1k}\lambda_k = 0 \\ a_{21}\lambda_1 + a_{22}\lambda_2 + \ldots + a_{2k}\lambda_k = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\lambda_1 + a_{n2}\lambda_2 + \ldots + a_{nk}\lambda_k = 0 \end{cases}$$

$r(\overline{\overline{A}}) = k$ shart bajarilganda yuqoridagi bir jinsli chiziqli tenglamalar sistemasi yagona trivial yechimga ega bo‘lib, berilgan vektorlar sistemasi chiziqli bog‘liq bo‘lmagan sistema bo‘ladi.

$r(\overline{\overline{A}}) < k$ shart bajarilganda esa notrivial yechimlarga ham ega bo‘lib, berilgan vektorlar chiziqli bog‘liq sistemani tashkil etadi.

Vektorlar sistemasining bazisi va rangi. Ortogonal vektorlar sistemasi. Ortonormallangan vektorlar sistemasi

$a^{(1)},\ a^{(2)},\ \ldots, a^{(k)},\ a^{(s)}$ vektorlar sistemasi berilgan bo‘lsin. Berilgan vektorlar sistemasining bazisi deb, uning chiziqli bog‘liq bo‘lmagan shunday bir qismga aytiladiki (aytaylik $a^{(1)},\ a^{(2)},\ \ldots, a^{(k)}$) bunda berilgan sistemaning har bir vektori bazis vektorlari orqali yoyilishi mumkin bo‘ladi.

$a^{(1)},\ a^{(2)},\ \ldots, a^{(k)},\ a^{(s)}$ sistemaning bir necha vektordan iborat biror qism sistemasi chiziqli bog‘lanmagan bo‘lsa, uni bazisgacha to‘lg‘azish mumkin. Berilgan sistemaning har bir vektori shu sistemaning bazis vektorlari orqali faqat bir xil ko‘rinishda ifodalanishi mumkin. Berilgan vektorlar sistemasining bazisini topish uchun $\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} + \ldots + \lambda_s a^{(s)} = \theta$ tenglamalar sistemasining umumiy yechimi topiladi va yechilgan sistemada yechilgan noma’lumlar oldidagi koeffitsient vektorlardan sistema tuziladi. Tuzilgan vektorlar sistemasi berilgan vektorlar sistemasining bazisi bo‘ladi (misollarga qarang).

Berilgan vektorlar sistemasining rangi deb, uning bazisida vektorlar soniga aytiladi. Agar berilgan sistemaning rangi k ga teng bo‘lsa, sistemada k ta chiziqli bog‘liq bo‘lmagan vektorlar qism sistemasi, berilgan vektorlar sistemasining bazisi bo‘ladi. Agar ikki n o‘lchovli $a^{(i)} = (a_{1i}, a_{2i}, \ldots, a_{ni})$ va $a^{(j)} = (a_{1j}, a_{2j}, \ldots, a_{nj})$vektorlarning skalyar ko‘paytmasi 0 ga teng bo‘lsa, ya’ni $\left(a^{(i)} \cdot a^{(j)}\right) = \sum_{t=1}^{n} a_{ti} \cdot a_{tj} = 0$ berilgan vektorlar o‘zaro ortogonal vektorlar deyiladi. Vektorlar sistemasida vektorning ixtiyoriy har bir jufti o‘zaro ortogonal bo‘lsa, bunday vektorlar sistemasiga ortogonal vektorlar sistemasi deyiladi.

Chiziqli bog‘liq bo‘lmagan $a^{(1)},\ a^{(2)},\ \ldots, a^{(k)},\ a^{(k+1)}$ vektorlar sistemasidan ortogonal sistemaga o‘tish quyidagicha amalga oshiriladi:

$$b^{(1)} = a^{(1)}$$

$$b^{(2)} = -\frac{b^{(1)} \cdot a^{(2)}}{b^{(1)} \cdot b^{(1)}} \cdot b^{(1)} + a^{(2)}$$

....

....

....

$$b^{(k+1)} = -\frac{b^{(1)} \cdot a^{(k+1)}}{b^{(1)} \cdot b^{(1)}} \cdot b^{(1)} - \frac{b^{(2)} \cdot a^{(k+1)}}{b^{(2)} \cdot b^{(2)}} \cdot b^{(2)} - \ldots - \frac{b^{(k)} \cdot a^{(k+1)}}{b^{(k)} \cdot b^{(k)}} \cdot b^{(k)} + a^{(k+1)}$$

Yuqoridagi chiziqli bog‘lanmagan sistemadan ortogonal sistemaga o‘tish yo‘li ortogonallash jarayoni deyiladi.

Har bir vektori birlik vektor bo‘lgan ortogonal vektorlar sistemasiga ortonormal vektorlar sistemasi deyiladi.

Agar $b^{(1)}, b^{(2)}, \ldots, b^{(k)}$ sistema ortogonal vektorlar sistemasi bo‘lsa, u holda

$$\frac{b^{(1)}}{\left|b^{(1)}\right|}, \frac{b^{(2)}}{\left|b^{(2)}\right|}, \ldots, \frac{b^{(k)}}{\left|b^{(k)}\right|}$$

Sistema ortonormallangan vektorlar sistemasi bo‘lib hisoblanadi.

Mashqlar

Vektorlar va vektorlar sistemalariga doir misol – masalalarning yechimlaridan namunalari

1. Quyida a va b vektorlar berilgan. Berilgan vektorlar modullarini, ularning chiziqli kombinatsiyasi c vektor koordinatalarini va uzunligini, a va b vektorlarning skalyar ko‘paytmasini, ular orasidagi burchak kattaligini, o‘zaro ortogonallarini aniqlang:

a) $\vec{a} = \left(0;\ \frac{\sqrt{2}}{2};\ \frac{\sqrt{2}}{2}\right)$ b) $\vec{a} = (1;\ 2;\ 3)$ v) $\vec{a} = (0;\ 0;\ -1;\ 1)$

$\vec{b} = (-1;\ 2;\ -2)$ $\vec{b} = (-5;\ 3;\ 2)$ $\vec{b} = (1;\ 1;\ 1;\ 1)$

$\vec{c} = 3\sqrt{2}\vec{a} - \vec{b}$ $\vec{c} = 4\vec{a} + 3\vec{b}$ $\vec{c} = 2\vec{a} + b$

a) $|\vec{a}|=\sqrt{0^2+\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}=1$ (bir)

$|\vec{b}|=\sqrt{(-1)^2+2^2+(-2)^2}=1$ (bir)

$$\vec{c}=3\sqrt{2}\vec{a}-\vec{b}=2\sqrt{2}\begin{pmatrix}0\\ \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}-\begin{pmatrix}-1\\ 2\\ -2\end{pmatrix}=\begin{pmatrix}1\\ 1\\ 5\end{pmatrix}$$, ya'ni $\vec{c}=(1;\ 1;\ 5)$

$|\vec{c}|=\sqrt{1^2+1^2+5^2}=\sqrt{27}=3\sqrt{3}$ (bir)

$(\vec{a}:\vec{b})=0\cdot(-1)+\frac{\sqrt{2}}{2}\cdot 2+\frac{\sqrt{2}}{2}\cdot(-2)=0$ bo'lgani uchun berilgan $\vec{a}$ va $\vec{b}$ vektorlar ortogonal, ya'ni ular orasidagi burchak kattaligi: $\left(\vec{a}\cdot\hat{}\vec{b}\right)=90^0\left(\frac{\pi}{2}\right)$

b) $\vec{a}=\sqrt{1^2+2^2+3^2}=\sqrt{14}$ (bir)

$|\vec{b}|=\sqrt{(-5)^2+3^2+2^2}=\sqrt{38}$ (bir)

$$|\vec{c}|=-4\vec{a}+3\vec{b}=-4\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}+3\begin{pmatrix}-5\\ 3\\ 2\end{pmatrix}=\begin{pmatrix}-19\\ 1\\ -6\end{pmatrix}$$, ya'ni $\vec{c}=(-19;1;-6)$

$|\vec{c}|=\sqrt{(19)^2+1^2+(-6)^2}=\sqrt{398}$ (bir)

$\left(\vec{a}\cdot\hat{}\vec{b}\right)=1\cdot(-5)+2\cdot 3+3\cdot 2=7\neq 0$ bo'lgani uchun berilgan $\vec{a}$ va $\vec{b}$ vektorlar o'zaro ortogonal emas.

$$\cos\left(\vec{a}\cdot\hat{}\vec{b}\right)=\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|\cdot|\vec{b}|}=\frac{7}{\sqrt{14}\cdot\sqrt{38}}=\frac{\sqrt{133}}{38},\qquad \left(\vec{a}:\hat{}\vec{b}\right)=\arccos\frac{\sqrt{133}}{38}$$

v) $\vec{a} = \sqrt{0^2 + 0^2 + (-1)^2 + 1^2} = \sqrt{2}$

$$\left|\vec{b}\right| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$$

$$\vec{c} = 2\vec{a} + \vec{b} = 2\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \qquad \vec{c} = (1;\ 1;\ -1;\ 3)$$

$$\vec{c} = \sqrt{1^2 + 1^2 + (-1)^2 + 3^2} = \sqrt{12} = 2\sqrt{3}$$

$\left(\vec{a} \hat{\cdot} \vec{b}\right) = 0 \cdot 1 + 0 \cdot 1 + (1) \cdot 1 + 1 \cdot 1 = 0$ bo‘lgani uchun berilgan $\vec{a}$ va $\vec{b}$ vektorlar ortogonal.

2. Quyidagi $b^{(0)}$ vektorni $a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}$ vektorlar sistemasining chiziqli kombinatsiyasi ko‘rinishida yoyish mumkin yoki mumkin emasligini ko‘rsating:

$$b^{(0)} = (8;\ -3;\ -10;\ 10)$$

$$a^{(1)} = (1;\ 0;\ 4;\ 3); \qquad a^{(2)} = (-2;\ 3;\ 1;\ 4); \qquad a^{(3)} = (1;\ 1; -4;\ 5); \qquad a^{(4)} = (1;\ -2;\ 0;\ 3)$$

$a^{(1)}x_1 + a^{(2)}x_2 + a^{(3)}x_3 + a^{(4)}x_4 = b^{(0)}$ vektor tenglamani koordinatalarda chiziqli tenglamalar sistemasi ko‘rinishida yozib olamiz va Gauss usulida yechamiz:

$$\begin{pmatrix} 1 & -2 & 1 & 1 & 8 \\ 0 & 3 & 1 & -2 & -3 \\ 4 & 1 & -4 & 0 & -10 \\ 3 & 4 & 5 & 3 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 & 1 & 8 \\ 0 & 3 & 1 & -2 & -3 \\ 0 & 9 & -8 & -4 & -42 \\ 0 & 10 & 2 & 0 & -14 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & -5 & 0 & 3 & 11 \\ 0 & 3 & 1 & -2 & -3 \\ 0 & 33 & 0 & -20 & -66 \\ 0 & 4 & 0 & 4 & -8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -5 & 0 & 3 & 11 \\ 0 & 3 & 1 & -2 & -3 \\ 0 & 33 & 0 & -20 & -66 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 8 & 1 \\ 0 & 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & -53 & 0 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} : \begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 3 \\ x_4 = 0 \end{cases}$$

yagona yechim

Demak, $b^{(0)}$ vektor berilgan $a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}$ vektorlar sistemasi orqali yagona usulda yoyilishi mumkin:

$$b^{(0)} = a^{(1)} - 2a^{(2)} + 3a^{(3)} + 0a^{(4)}$$

3. Quyidagi berilgan vektorlar sistemalarining chiziqli bog'liq yoki chiziqli bog'liq emasligini aniqlang:

a) $\vec{a} = (1;\ -3)$, $\vec{b} = (2;\ -6)$

b) $\vec{a} = (1;\ -3;\ 2)$, $\vec{b} = (2;\ -6;\ 5)$

v) $\vec{a} = (-2;\ 4;\ 6)$, $\vec{b} = (-3;\ 6;\ 9)$

g) $\vec{a} = (-1;\ 1;\ 3)$, $\vec{b} = (2;\ -1;\ 1)$, $\vec{c} = (1;\ 4;\ 5)$

d) $\vec{a} = (1;\ 2;\ 4)$, $\vec{b} = (3;\ -1;\ -2)$, $\vec{c} = (4;\ 1;\ 2)$

ye) $a^{(1)} = (1;\ 1;\ 1;\ 1)$, $a^{(2)} = (3;\ 2;\ 4;\ 1)$, $a^{(3)} = (2;\ -1;\ 2;\ -1)$, $a^{(4)} = (1;\ 0;\ 0;\ 1)$

Berilgan ixtiyoriy vektorlar sistemasining chiziqli bog‘liq yoki bog‘liq emasligini aniqlashning eng qulay yo‘li yuqoridagi teorema shartlarini tekshirib ko‘rishdir.

a) Ikki $\vec{a}$ va $\vec{b}$ vektorlarini o‘z ichiga olgan sistema chiziqli bog‘liq, chunki:

$$r\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix} = 1 < 2 = k$$

b) $r\begin{pmatrix} 1 & 2 \\ -3 & -6 \\ 2 & 5 \end{pmatrix} = 2 = 2 = k$ bo‘lgani uchun berilgan sistema chiziqli bog‘liq emas.

v) $r\begin{pmatrix} -2 & -3 \\ 4 & 6 \\ 6 & 9 \end{pmatrix} = 1 < 2 = k$ tengsizlik bajarilgani uchun berilgan vektorlar sistemasi chiziqli bog‘liq.

g) $r\begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 4 \\ 3 & 1 & 5 \end{pmatrix} = 3 = 3 = k$ tenglik bajarilgani uchun berilgan vektorlar sistemasi chiziqli bog‘liq bo‘lmagan sistemadir.

d) $r\begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 4 & -2 & 2 \end{pmatrix} = 2 < 3 = k$ tengsizlik bajarilgani uchun vektorlar sistemasi chiziqli bog‘liq sistemani tashkil etadi.

ye) $r\begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & 4 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix} = 3 < 4 = k$ tengsizlik bajarilgani uchun ham berilgan vektorlar sistemasi chiziqli bog‘liq.

Izoh. Katta o‘lchamlarga ega matritsalarning rangini hisoblashning eng qulay usuli esa matritsa ustida elementar almashtirishlar bajarib, nollar yig‘ib, noldan farqli eng katta minor tartibini aniqlashdan iborat.

4. Quyida berilgan vektorlar sistemasining bazisi va rangi topilsin:

$$a^{(1)}=(1;\ 2;\ -1;\ 3)$$
$$a^{(2)}=(0;\ 3;\ 4;\ 1)$$
$$a^{(3)}=(-2;\ -1;\ 6;\ -5)$$
$$a^{(4)}=(5;\ 1;\ 2;\ -4)$$

$$\begin{pmatrix} 1 & 0 & -2 & 5 & 0 \\ 2 & 3 & -1 & 1 & 0 \\ -1 & 4 & 6 & 2 & 0 \\ 3 & 1 & -5 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & 5 & 0 \\ 0 & 3 & 3 & -9 & 0 \\ 0 & 4 & 4 & 7 & 0 \\ 0 & 1 & 1 & -19 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 0 & -2 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -19 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Yechilgan sistemadan $x_1, x_2, x_4 -$yechilgan noma’lumlar. x_3 erkin noma’lum ekanligi ko‘rinib turibdi. Demak, berilgan vektorlar sistemasining bazasi $a^{(1)}, a^{(2)}$ va $a^{(4)}$ vektorlar sistemasi bo‘lib, sistemaning rangi bazisdagi vektorlar soni 3 ga teng.

5. Quyida berilgan chiziqli bog‘liq bo‘lmagan vektorlar sistemasidan ortogonal va ortonormallangan vektorlar sistemalariga o‘tilsin:

$$a^{(1)}=(1;\ -2;\ 4)$$
$$a^{(2)}=(0;\ 1;\ 3)$$

Yuqorida keltirilgan ortogonallashgan jarayoni formulalarini qo‘llaymiz:

$$b^{(1)}=a^{(1)}=(1;\ -2;\ 4)$$

$$b^{(2)}=-\frac{(1;-2;\ 4)\begin{pmatrix}0\\1\\3\end{pmatrix}}{(1;-2;\ 4)\begin{pmatrix}1\\-2\\4\end{pmatrix}}(1;\ -2;\ 4)+(0;\ 1;\ 3)=\frac{0-2+12}{1+4+16}(1;\ -2;\ 4)+$$

$$+(0;\ 1;\ 3)=-\frac{10}{21}(1;\ -2;\ 4)+(0;\ 1;\ 3)=\left(-\frac{10}{21};\ \frac{20}{21};\ -\frac{40}{21}\right)+$$

$$+(0;\ 1;\ 3)=\left(-\frac{10}{21};\ \frac{41}{21};\ \frac{23}{21}\right)$$

Har bir vektorni ortonormallab,

$$\frac{b^{(1)}}{\left|b^{(1)}\right|}=\frac{1}{\sqrt{1^2+(-2)^2+4^2}}(1;-2;\ 4)=\left(\frac{1}{\sqrt{21}};-\frac{2}{\sqrt{21}};\ \frac{4}{\sqrt{21}}\right)$$

$$\frac{b^{(2)}}{\left|b^{(2)}\right|}=\frac{\left(-\frac{10}{21};\ \frac{41}{21};\ \frac{23}{21}\right)}{\sqrt{\frac{100}{441}+\frac{1681}{441}+\frac{529}{441}}}=\left(-\frac{\sqrt{2310}}{2310};\ \frac{41\sqrt{2310}}{2310};\ \frac{23\sqrt{2310}}{2310}\right)$$

Mashqlar

Mustaqil ishlash uchun amaliy misol va masalalar

1. Quyida a va b vektorlar berilgan. Berilgan vektorlar modullarini, ularning chiziqli kombinatsiyasini c vektor koordinatalarini va uzunligini, a va b vektorlarning skalyar ko‘paytmasini, ular orasidagi burchak kattaligini, o‘zaro ortogonallarini aniqlang:

a)

$$\vec{a}=(-2;\ 0;\ 1) \qquad \vec{b}=(2;\ 0;\ 4) \qquad \vec{c}=2\vec{a}-3\vec{b}$$

$$\left(\begin{array}{ll} |\vec{a}|=\sqrt{5}\ (бир), & |\vec{b}|=2\sqrt{5}\ (бир), \\ \vec{c}=(-10;\ 0;\ -10), & |\vec{c}|=10\sqrt{2}\ (бир), \\ (\vec{a}\cdot\vec{b})=0,\ (\vec{a}\cdot\vec{b})=\frac{\pi}{2},\ ортогонал & \end{array}\right)$$

b)

$\vec{a} = (1;\ 3;\ 0)$
$\vec{b} = (2;\ -2;\ -4)$
$\vec{c} = \vec{a} + \vec{b}$

$$\left(\begin{array}{ll} |\vec{a}| = \sqrt{10} \quad (bir), & |\vec{b}| = 2\sqrt{6} \quad (bir), \\ \vec{c} = (3;\ \ 1;\ \ -4), & |\vec{c}| = \sqrt{26} \quad (bir), \\ (\vec{a} \cdot \vec{b}) = -4, \quad \left(\vec{a} \hat{\cdot} \vec{b}\right) = \pi - \arccos \dfrac{\sqrt{15}}{15} & \end{array}\right)$$

ortogonal emas.

v) $\vec{a} = (3;\ 1;\ -5;\ 1)$
$\vec{b} = (4;\ 2;3;\ 1)$
$\vec{c} = -\vec{a} + \vec{b}$

$$\left(\begin{array}{ll} |\vec{a}| = 6, & |\vec{b}| = \sqrt{30}, \\ \vec{c} = (1;\ \ 1;\ \ 8;\ \ 0) & |\vec{c}| = \sqrt{66}, \\ (\vec{a} \cdot \vec{b}) = 0, \quad (\vec{a} \cdot \vec{b}) = \dfrac{\pi}{2}, & ortogonal \end{array}\right)$$

2. Quyida berilgan vektorlarni berilgan vektorlar sistemalarining chiziqli kombinatsiyalari ko‘rinishida yoyish mumkin yoki mumkin emasligini ko‘rsating:

a) $\vec{b} = (-4;\ 9)$, $\vec{a}^{(1)} = (1;\ -3)$,

$\vec{a}^{(2)} = (2;\ -5)$ $\left(\vec{b} = 2\vec{a}^{(1)} - 3\vec{a}^{(2)}\right)$

b) $\vec{b} = (2;\ 5)$ $\vec{a}^{(1)} = (1;\ 2)$

$\vec{a}^{(2)} = (3;\ 6)$ (yoyish mumkin emas)

v) $\vec{b} = (-5;\ -3;\ 2)$ $\vec{a}^{(1)} = (1;\ 2;\ 3)$

$\vec{a}^{(2)} = (0;\ 1;\ -1)$ $\vec{a}^{(3)} = (3;\ 4;\ -1)$ $\left(\vec{b} = \vec{a}^{(1)} + 3\vec{a}^{(2)} - 2\vec{a}^{(3)}\right)$

g) $b^{(0)} = (5;\ 1;\ -1;\ 4)$ $\vec{a}^{(3)} = (2;\ 5;\ -4;\ -1)$

$\vec{a}^{(1)} = (1;\ 2;\ 0;\ 3)$ $\vec{a}^{(4)} = (1;\ 6;\ -1;\ 3)$

$\vec{a}^{(2)} = (4;\ 3;\ -2;\ 1)$ $\left(\vec{b}^{(0)} = 2\vec{a}^{(1)} + \vec{a}^{(2)} - 0 \cdot \vec{a}^{(3)} - a^{(4)}\right)$

3. Quyida berilgan vektorlar sistemalarining chiziqli bog‘liq yoki chiziqli bog‘liq emasligini aniqlang:

a) $\vec{a}=(6;\,-15)$, $\vec{b}=(4;\,-10)$ (chiziqli bog‘liq)

b) $\vec{a}=(-3;\,-2)$, $\vec{b}=(11;\;7)$ (chiziqli bog‘liq emas)

v) $\vec{a}=(3;\,5;\,-2)$, $\vec{b}=(6;\,10;\,-4)$ (chiziqli bog‘liq)

g) $\vec{a}=(1;\,-4;\,3)$, $\vec{b}=(2;\,-7;\,6)$ (chiziqli bog‘liq emas)

d) $\vec{a}=(1;\,-3;\,-4)$, $\vec{b}=(2;\,-1;\,0)$, $\vec{c}=(-4;\,5;\,3)$ (chiziqli bog‘liq emas)

e) $\vec{a}=(1;\,-2;\,-3)$, $\vec{b}=(2;\,1;\,-1)$, $\vec{c}=(3;\,7;\,4)$ (chiziqli bog‘liq)

y) $a^{(1)}=(1;\,3;\,1;\,0)$
$a^{(2)}=(-2;\,1;\,-3;-1)$
$a^{(3)}=(4;\,0;\,5;\,1)$ (chiziqli bog‘liq)
$a^{(4)}=(3;\,2;\,-1;\,-4)$

j) $a^{(1)}=(1;\,3;\,1;\,0)$
$a^{(2)}=(3;\,8;\,-1;5)$
$a^{(3)}=(1;\,0;\,-2;\,4)$ (chiziqli bog‘liq emas)
$a^{(4)}=(-1;\,0;\,1;\,-3)$

4. Quyida berilgan vektorlar sistemalarining bazislari va ranglari topilsin:

a) $a^{(1)} = (1;\ -2;\ 5)$

$a^{(2)} = (3;\ 4;\ -1)$

$a^{(3)} = (2;\ -5;\ 0)$

(bazisi: $a^{(1)}, a^{(2)}, a^{(3)}$, rangi: 3)

b) $a^{(1)} = (1;\ 1;\ -1;\ -2)$

$a^{(2)} = (3;\ 4;\ -1;\ 2)$

$a^{(3)} = (4;\ 1;\ -2;\ 3)$

$a^{(4)} = (5;\ 2;\ -3;\ 1)$

(bazisi: $a^{(1)}, a^{(2)}, a^{(3)},\ a^{(4)}$, rangi: 3)

5. Quyida berilgan chiziqli bog‘liq bo‘lmagan vektorlar sistemasidan ortogonal va ortonormallangan sistemalariga o‘tilsin:

$a^{(1)} = (1;\ 1;\ 1;\ )$

$b^{(2)} = (0;\ 1;\ 1;\ 1)$

$a^{(3)} = (0;\ 0;\ 1;\ 1)$

$$\left(\begin{array}{ll} b^{(1)} = (1;\ 1;\ 1;\ 0), & \dfrac{b^{(1)}}{\left|b^{(1)}\right|} = \left(\dfrac{1}{\sqrt{3}};\ \dfrac{1}{\sqrt{3}};\ \dfrac{1}{\sqrt{3}};0\right) \\ b^{(2)} = \left(-\dfrac{2}{3};\ \dfrac{1}{3};\ \dfrac{1}{3};1\right), & \dfrac{b^{(2)}}{\left|b^{(2)}\right|} = \left(-\dfrac{2\sqrt{3}}{3\sqrt{5}};\ \dfrac{\sqrt{3}}{3\sqrt{5}};\ \dfrac{\sqrt{3}}{3\sqrt{5}};\ \dfrac{\sqrt{3}}{\sqrt{5}}\right) \\ b^{(3)} = \left(\dfrac{1}{5};\ -\dfrac{3}{5};\ \dfrac{2}{5};\ \dfrac{1}{5}\right), & \dfrac{b^{(3)}}{\left|b^{(3)}\right|} = \left(\dfrac{\sqrt{5}}{5\sqrt{3}};\ \dfrac{3\sqrt{5}}{5\sqrt{3}};\ \dfrac{2\sqrt{5}}{5\sqrt{3}};\dfrac{\sqrt{5}}{5\sqrt{3}}\right) \end{array}\right)$$

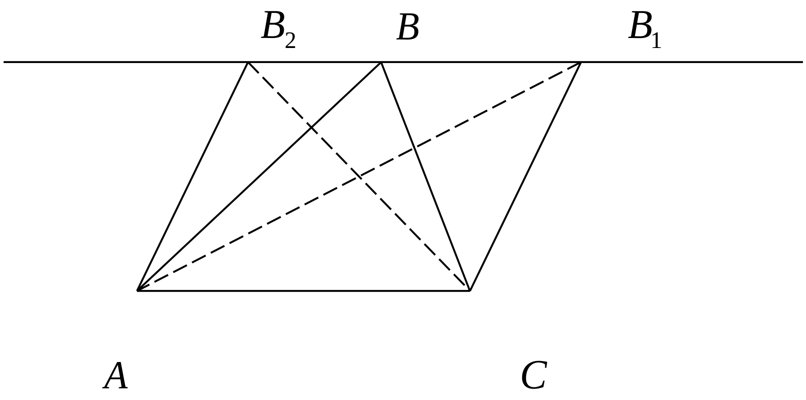
II - BOB

Funksiyalar

1 - §. Funksional bog‘lanish. Funksiya tushunchasi

Turli protsesslarni kuzatish shuni ko‘rsatadiki, bu protsessda qatnashayotgan miqdorlar ikki xil bo‘ladi. Ayrimlari o‘z qiymatlarini bu protsess davomida o‘zgartirib turadilar. Shunga asosan miqdorlarni o‘zgarmas va o‘zgaruvchi miqdorlarga bo‘linadi.

Masalan: 1. Samolyotning uchish davridagi miqdorlar.
2. Uchburchakdagi miqdorlar



O‘zgaruvchi miqdorlar bir – biriga bog‘langan holda o‘zgaradi, ya’ni bir miqdorning o‘zgarishiga ikkinchi o‘zgaruvchi miqdor sabab bo‘ladi.

Masalan: 1. Boyl – Mariott qonuni $V = \frac{C}{P}$. 2. $S = \pi R^2$

Bu o‘zgaruvchi miqdorlardan birini erkli, ikkinchisini erksiz deb qabul qilinadi. Erkli o‘zgaruvchi miqdorning o‘zgarishi bilan erksiz miqdor ham qiymatini o‘zgartiradi.

Ta’rif: Agar o‘zgaruvchi miqdor $x-$ning olishi mumkin bo‘lgan har bir qiymatiga boshqa o‘zgaruvchi miqdor $y-$ning to‘la aniqlangan bir qiymati mos kelsa u o‘zgaruvchi miqdorni, x o‘zgaruvchi miqdorning funksiyasi deyiladi. ($x-$argument, $y-$esa x ning funksiyasi bo‘ladi). Bu bog‘lanishni odatda $y = f(x)$, $y = \varphi(x)$ kabi ifoda qilinadi. f, φ funksiya xarakteristika bo‘lib, $x-$argument ustida qanday amallar bajarilishi lozimligini ko‘rsatadi. $x = a$ bo‘lgandagi funksiyaning xususiy qiymatlari $f(a)$ shaklida ko‘rsatiladi.

Masalan: $f(x) = x^2 + 1$

1. $f(1) = 1^2 + 1 = 2$, $\quad f(0) = 0^2 + 1 = 1$

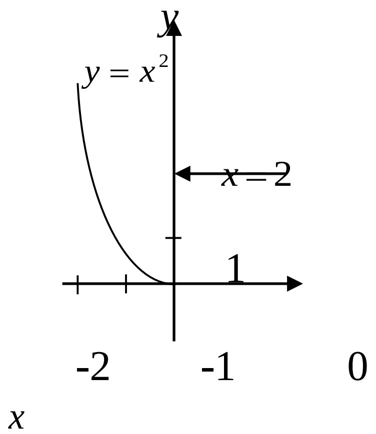
$f(a^2) = (a^2)^2 + 1 = a^4 + 1$

2. $f(x) = \sin x$, $\quad f(a) = \sin a$, $\quad f(0) = \sin 0 = 0$

$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

2 - §. Funksiyaning berilish usullari

1. Analitik usul.
2. Grafik usul.
3. Jadval usuli.

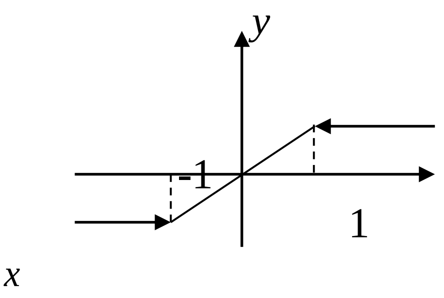


x	y
-1	1
0	0
-2	4
3	9

Misollar:

1. $y = \begin{cases} x^2, & x \le 0 \\ 2, & x > 0 \end{cases}$

2. $y = \begin{cases} -1, & x < -1 \\ x, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$



x	y
$-\frac{1}{2}$	$-\frac{1}{2}$
0	0
1	1

Demak, funksiyalar faqat bir formula bilan emas, balki ikki va undan ortiq formula yordami bilan berilishi mumkin ekan.

3 - §. Funksiyaning aniqlanish sohasi

Funksiyani o‘rganish davomida argumentning olishi mumkin bo‘lgan sonlar to‘plami va funksiyaning qabul qiladigan qiymatlari to‘plami bilan ish ko‘riladi.

Masalan: $y=2^x$ ko‘rsatkichli argumentga ixtiyoriy haqiqiy qiymatni bera oladi. Ammo funksiyaning qiymati esa, faqat musbat sonlardan iborat bo‘ladi.

Ta’rif. $y=f(x)$ funksiyaning argumenti qabul qilishi mumkin bo‘lgan barcha qiymatlar to‘plami, funksiyaning aniqlanish sohasi deyiladi, u funksiyaning qabul qilgan qiymatlar to‘plami esa funksiyaning o‘zgarish sohasi deyiladi.

Masalan: $y=x^2-3x$ funksiyasining aniqlanish va o‘zgarish sohasi haqiqiy sonlar to‘plamidan iborat. Funksiyaning aniqlanish sohasini tekshirishda quyidagi tengsizliklardan foydalanamiz.

1. a va b sonlar oralig‘idagi sonlar (a va b dan tashqari) ochiq interval deyilib, (a, b) shaklida yoki $a<x<b$ shaklida ko‘rsatiladi.

2. a va b sonlar oralig‘idagi sonlar (a va b ham qo‘yiladi), aniq interval deyilib $[a, b]$ shaklida yoki $a\le x\le b$ shaklida ko‘rsatiladi.

Agar o‘zgaruvchi miqdor ixtiyoriy haqiqiy qiymatlar qabul qilsa uni quyidagicha ko‘rsatiladi:

$$-\infty<x<+\infty$$

Misollar:

1. $y=\arcsin x$ funksiyaning aniqlanish sohasi $-1\le x\le 1$ aniq intervaldan iborat.

2. $y=\dfrac{1}{x^2-1}$ funksiya $x^2=1$ bo‘lganda ma’noga ega bo‘lmaydi. Shuning uchun bu funksiyaning aniqlanish sohasi ± 1 dan boshqa haqiqiy sonlar to‘plamidan iborat.

$$-\infty < x < -1 \cup -1 < x < 1 \cup 1 < x < +\infty$$

3. $y = \sqrt{2-x}$ funksiyada $2 - x \geq 0$ bo‘lishi kerak.
Bundan $x \leq 2$. Demak, funksiyaning aniqlanish sohasi $x \leq 2$.

4. $y = \dfrac{\sqrt{5-x}}{\lg(x-1)}$ Aniqlanish sohasi

$$\begin{cases} 5 - x \geq 0 \\ x - 1 > 0 \\ x - 1 \neq 0 \end{cases}$$

$1 < x < 2$

$2 < x < 5$

Funksiyanng aniqlanish sohasi tekshirganda quyidagilarni e’tiborga olish kerak:

1. Nolga, bo‘lish mumkin emas.
2. Manfiy sondan juft ko‘rsatkichli ildiz chiqarib bo‘lmaydi.
3. Manfiy sonning va nolning logarifmi bo‘lmaydi.
4. arcsin$x$, arccos$x$ da $|x| \leq 1$ bo‘lishi va
5. $arctgx$ da $x \neq \dfrac{\pi}{2} + k\pi$, $arcctgx$ da $x \neq k\pi$

4 - §. Funksiyaning ayrim hossalari

1. Juft va toq funksiyalar:

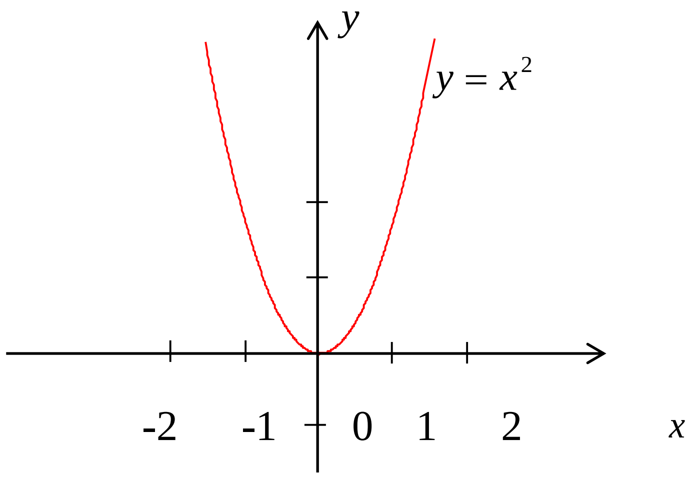
Agar $y = f(x)$ funksiyaning aniqlanish sohasida olingan hamma qiymatlarida $f(-x) = f(x)$ bo‘lsa, bu funksiyani juft funksiya deyiladi.

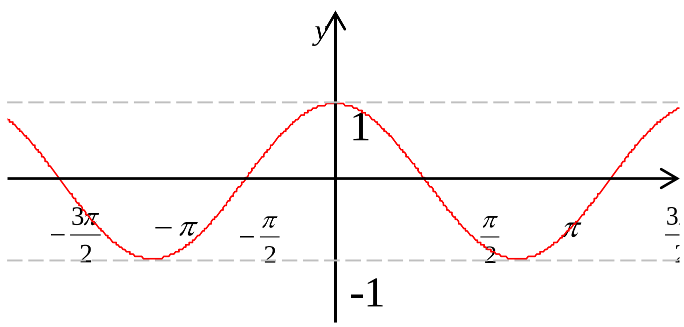
Masalan: $y = x^2$, $y = \cos x$

$y = f(x) = x^2$, $f(-x) = (-x)^2 = x^2$, $f(x) = \cos x$,

$f(-x) = \cos(-x) = \cos x = f(x)$

Juft funksiyalar grafigi ordinata o‘qiga nisbatan simmetrik bo‘ladi.



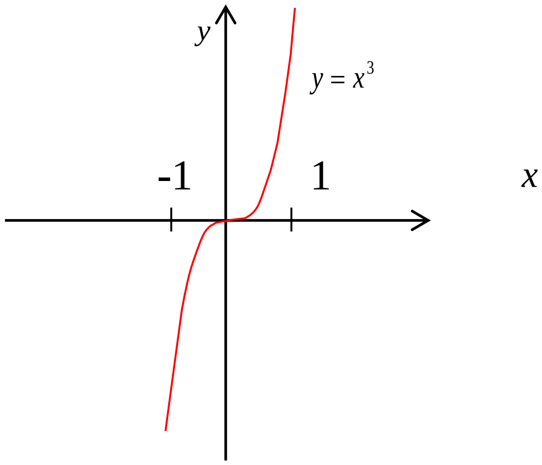


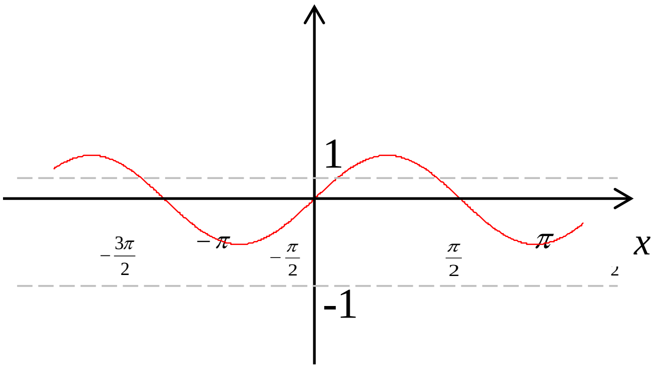
Agar $y = f(x)$ funksiyaning aniqlanish sohasida olingan hamma qiymatlarida $f(-x) = -f(x)$ bo‘lsa, bu funksiyani toq funksiya deyiladi.

Masalan: $y = x^3$, $y = \sin x$

$$y = f(x) = x^3,\ f(-x) = (-x)^3 = -x^3 = -f(x),\ f(-x) = -f(x)$$

$$y = \sin x,\ f(-x) = \sin(-x) = -\sin x = -f(x)$$
$$f(-x) = -f(x)$$





Mashqlar

Funksiyaning aniqlanish sohasi topilsin.
(Javoblari qavs ichida berilgan)

1. $y = \sqrt{1 + x}$ $\qquad ([-1; +\infty))$

2. $y = \sqrt{4 - x}$ $\qquad ([-4; 4])$

3. $y = 1 - \lg x$ $\qquad (0, \infty)$

4. $y = \lg(x + 3)$ $\qquad (3, +\infty)$

5. $y = \sqrt{5 - 2x}$ $\qquad \left(\left[\frac{5}{2}, \infty\right)\right)$

6. $y = \frac{1}{x^2+1}$ $\left((-\infty, -1)\cup(-1, 1)\cup(1, +\infty)\right)$

7. $y = \frac{1}{\sqrt{9-x^2}}$ $(-3, 3)$

8. $y = \sqrt{x^2-4x+3}$ $\left((-\infty, 1]\cup[3, \infty)\right)$

9. $y = \arcsin(x-2)$ $([1, 3])$

10. $y = \arccos(1-2x)$ $([0, 1])$

2. Funksiyaning chizmasi chizilsin:

1. $y = x^2+1$

2. $y = 2x-1$

3. $y = \frac{1}{x}$

4. $y = \begin{cases} -2 & x < 0 \\ x & x \geq 0 \end{cases}$

5. $y = |x|$

6. $y = \sqrt{x}$

7. $y = x^3+1$

8. $y = 2^x$

9. $y = \frac{1}{x^2}$

10. $y = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$

3. Funksiya berilgan:

1. $f(x)=x^3-1$

Topilsin: $f(0),\ f(1),\ f(-1),\ f(2),\ f(a),$

$f\left(\frac{1}{a}\right),\ f(a+1),\ \frac{1}{f(a)},\ \frac{1}{f\left(\frac{1}{a}\right)}$

Javobi:

$-1,\ 0,\ -2,\ 7,\ \ a^3-1,\ \ \frac{1-a^3}{a^3},\ a^3+3a^2+3a,\ \frac{1}{a^3-1},\ \ \frac{a^3}{1-a^3}$

2. Funksiya berilgan

$f(x)=2^{x-2}, \qquad \varphi(x)=2^{x+2}$

Topilsin: $f(0),\ f(2),\ f(-1),\ \varphi\left(\frac{1}{2}\right)$

$f(x)+\varphi(x),\quad f(2)+\varphi(-2),\quad f(0)+\varphi(0)$

Javobi: $\frac{1}{4},\ 1,\ \frac{1}{8},\ 4\sqrt{2},\quad 2,\ \frac{17}{4}$

3. Funksiya berilgan:

$f(x)=\sin x, \qquad \varphi(x)=\cos x$

Topilsin: $f\left(\frac{\pi}{2}\right),\ \varphi(\pi),\ \ f\left(\frac{\pi}{3}\right),\ \ \varphi\left(\frac{\pi}{4}\right)$

$\varphi\left(\frac{\pi}{4}\right)\cdot f\left(\frac{\pi}{4}\right),\ \ 2f(x)\cdot\varphi(x),\ \ \frac{f(x)}{\varphi(x)}$

$f\left(\frac{\pi}{2}\right)-\varphi\left(\frac{\pi}{2}\right),\quad f^2(x)+\varphi^2(x)$

Javobi: $1,\ \ -1,\ \ \frac{\sqrt{3}}{2},\ \frac{\sqrt{2}}{2},\ \frac{1}{2},\ \ \sin 2x,\ \ tgx,\ \ 1,\ \ 1$

4. Juft yoki toq funksiyalarni aniqlang:

$$f(x)=\frac{\sin x}{x}, \qquad f(x)=\frac{x^3}{tgx}$$

$$f(x)=\frac{e^x+1}{e^x-1}, \qquad f(x)=\frac{a^x+a^{-x}}{2}$$

$$f(x)=x^4-4x^2, \qquad f(x)=\frac{x}{a^x-1}$$

III - BOB

FUNKSIYANING LIMITI, HOSILASI VA DIFFERENSIALI

1-§. Limitlar

Aniq tartib bo‘yicha birining ketidan ikkinchisi keluvchi sonlar to‘plami

$$x_1,\ x_2,\ ...,\ x_n \qquad (1)$$

sonlar ketma – ketligi deyiladi. Shuning uchun sonlar ketma – ketligining umumiy hadi x_n natural argumentining funksiyasi sifatida beriladi. Ya’ni

$$f(n) = x_n$$

Masalan, agar qandaydir ketma – ketligining umumiy hadi $x_n = \frac{1}{n}$ formula bilan berilgan bo‘lsa, u holda n ga natural sonlar qatoridagi qiymatlarini, ya’ni $1, 2, 3, ...$ berib, quyidagi sonlar ketma – ketligini hosil qilish mumkin:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}$$

Ta’rif. Agar har qanday kichik musbat son ε uchun shunday N nomerni topish mumkin bo‘lsaki, $n > N$ uchun quyidagi $|x_n - a| < \varepsilon$ tengsizlik o‘rinli bo‘lsa, u holda a soni (1) sonli ketma – ketlikning limiti deyiladi.

Agar a soni (1) ketma – ketlikning limiti bo‘lsa, u holda

$$\lim_{n\to\infty} x_n = a$$

deb yoziladi.

Agar ketma – ketlik limitga ega bo‘lsa, u holda bu ketma – ketlik uchun quyidagi teoremalar o‘rinlidir.

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty} x_n \cdot y_n = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\frac{\lim\limits_{n\to\infty}x_n}{\lim\limits_{n\to\infty}y_n}\left(\lim_{n\to\infty}y_n\neq 0\right)$$

Agar (1) ketma – ketlikning limiti nolga teng bo‘lsa, u holda (1) ketma – ketlik cheksiz kichik deyiladi. Ya’ni

$$\lim_{n\to\infty}x_n=0$$

Boshqacha aytganda har qanday $\varepsilon<0$ uchun shunday N nomer topish mumkin bo‘lsaki, $n>N$ uchun quyidagi

$$|x_n-0|=|x_n|<\varepsilon\,.$$

tengsizlik o‘rinli bo‘ladi.

Agar (1) ketma – ketlikning limiti cheksizlikka teng bo‘lsa, u holda (1) ketma – ketlik cheksiz katta miqdor deyiladi. Ya’ni,

$$\lim_{n\to\infty}x_n=\infty$$

Boshqacha aytganda, har qanday $\varepsilon>0$ uchun shunday nomer N topish mumkinki, unda $n>N$ bo‘lganda quyidagi

$$|x_n|>\varepsilon$$

tengsizlik o‘rinli bo‘ladi.

Cheksiz kichik ketma – ketliklarga misol qilib, quyidagi umumiy hadlarni berilgan ketma – ketliklarni keltirish mumkin.

$$x_n=\frac{1}{n},\ x_n=\frac{1}{n^2},\ x_n=\frac{1}{2^n}$$ va boshqalar.

Cheksiz katta va cheksiz kichik miqdorlar o‘zaro bog‘liqdirlar. Cheksiz katta miqdordagi teskari miqdor, cheksiz kichik miqdordir va aksincha.

Endi ixtiyoriy argumentli funksiyani ko‘ramiz. Faraz qilaylik, $f(x)$ funksiya biror a nuqtani atrofida (a nuqtadan boshqa, nuqtani o‘zida ham) aniqlangan bo‘lsin.

Agar har qanday kichik musbat son ε uchun, shunday kichik δ sonni topish mumkin bo‘lsaki, x ning quyidagi

$$|x_n-a|<\delta$$

tengsizlikni qanoatlantiruvchi barcha qiymatlari uchun

$$|f(x)-A|<\varepsilon$$

tengsizlik o‘rinli bo‘lsa, u holda A soni $f(x)$ funksiyaning $x\to a$ intilgandagi limiti deyiladi va bu quyidagicha yoziladi.

$$\lim_{x\to a} f(x) = A$$

Shunga o‘xshash $\lim_{x\to a} f(x) = A$, $|x| > N$ qiymatlarida $|f(x) - A| < \varepsilon$ tengsizlik o‘rinli bo‘lsa, shartli ravishda $\lim_{x\to\infty} f(x) = \infty$ deb yoziladi, $|f(x) - A| > M$, $|x - a| < \delta$ bo‘lganda M ixtiyoriy musbat son.

Bu holda $f(x)$ funksiya $x \to a$ cheksiz katta miqdor deyiladi.

Agar $\lim_{x\to a}\varphi(x) = 0$ bo‘lsa, unda $\varphi(x)$ funksiya $x \to a$ cheksiz miqdor deyiladi $x < a$ va $x \to a$ u holda shartli $x \to a - 0$ deb yoziladi.

$x > a$ va $x \to a$ unda $x \to a + 0$ deb yoziladi.

$f(a-0) = \lim_{x\to a-0} f(x)$ songa $f(x)$ funksiyaning a nuqtada chap tomondan limiti, $f(a+0) = \lim_{x\to a+0} f(x)$ songa $f(x)$ funksiyani a nuqtada o‘ng tomondan limiti deyiladi. $\lim_{x\to a} f(x)$ - mavjudligi uchun $f(a-0) = f(a+0)$ bo‘lishi zaruriy va yetarli shartdir.

Misol:

$$f(x) = \frac{1}{x + 4^{\frac{1}{x-3}}}$$

funksiyaning chap va o‘ng limitini toping. Agar $x \to 3 - 0$ unda $\frac{1}{x-3} \to -\infty$ va $4^{\frac{1}{x-3}}$ unda

$$\lim_{x\to 3-0} \frac{1}{x + 4^{\frac{1}{x-3}}} = \frac{1}{3}$$

Agar $x \to 3 + 0$ unda $\frac{1}{x-3} \to +\infty$, $4^{\frac{1}{x-3}} \to +\infty$ unda

$$\lim_{x\to 3+0} \frac{1}{x + 4^{\frac{1}{x-3}}} = 0$$

Funksiyalarning limitlarini hisoblash uchun quyidagi hossalarni bilish zarur.

$\lim C = C$, $\qquad C -$ o‘zgarmas son

$\lim_{x\to a} Cf(x) = C\lim_{x\to a} f(x)$, $\qquad$ bunda $C -$ o‘zgarmas

Agar $\lim_{x\to a} f(x)$ va $\lim_{x\to a}\varphi(x)$ mavjud bo‘lsa, u holda quyidagi tengliklar o‘rinlidir.

$$\lim_{x\to a}\left[f(x)\pm\varphi(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a}\varphi(x)$$

$$\lim_{x\to a} f(x)\cdot\varphi(x)=\lim_{x\to a} f(x)\cdot\lim_{x\to a}\varphi(x)$$

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a}\varphi(x)},\quad \left(\lim_{x\to a}\varphi(x)\neq 0\right),$$

$$\left[\lim_{x\to a} f(x)\right]^{\varphi(x)}=\left[\lim_{x\to a} f(x)\right]^{\lim_{x\to a}\varphi(x)}$$

Misol: Quyidagi limit hisoblansin. $\lim_{x\to 2}\frac{x^2+4}{x+3}$

$\lim_{x\to 2} x^2=2^2=4$

$\lim_{x\to 2} x=2$

$\lim 4=4$, $\lim 3=3$

Shunday qilib, $\lim_{x\to 2}\frac{x^2+4}{x+3}=\frac{2^2+4}{2+3}=\frac{8}{5}$

Funksiyalarning limitlarini hisoblaganda, ya’ni ko‘pincha x ni o‘rniga intilgan sonini qo‘yganda aniqmasliklarga duch kelinadi, ya’ni aniq javobga ega bo‘linmaydi.

Masalan: $\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{4-4}{2-2}=\frac{0}{0}$, ya’ni javob aniq emas. Bu holda $\frac{0}{0}$ tipidagi aniqmaslikka ega bo‘lindi deyiladi.

Bu aniqmaslikni topish uchun, ya’ni berilgan kasrning limitini hisoblash uchun, kasrning surat va mahrajida ayrim shakl o‘zgartirishlar qilishadi. Masalan, $x^2-4=(x+2)(x-2)$. Bu holda $\frac{x^2-4}{x-2}$ kasr quyidagi ko‘rinishni oladi:

$$\frac{(x+2)(x-2)}{x-2}$$

Buni surat va mahrajini $(x-2)$ga qisqartirib, $(x+2)$ga ega bo‘lamiz.

Shunday qilib,

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

Yana quyida bir necha misollarni ko‘rib o‘tamiz.

1. $\lim\limits_{x\to 2}\frac{x^2-5x+6}{x-2}$

Agar x ni o‘rniga to‘g‘ridan – to‘g‘ri limitni qo‘ysak, $\frac{0}{0}$ tipidagi aniqmaslikka ega bo‘lamiz.

$$\lim_{x\to 2}\frac{x^2-5x+6}{x-2}=\frac{4-10+6}{2-2}=\frac{0}{0} \quad \text{noaniqlik}$$

$x=2$ da kvadrat uchhad nolga aylanadi. Demak, 2 soni kvadrat uchhadning ildizi ekan. Ikkinchi ildizini Vieta teoremasini tatbiq qilib topish mumkin, bunda $x=3$ bo‘ladi.

Shunday qilib, kvadrat uchhadni chiziqli ko‘paytuvchilarga ajratish mumkin ekan. $(x-2)\cdot(x-3)$ ya’ni $x^2-5x+6=(x-2)(x-3)$. Demak,

$$\lim\frac{x^2-5x+6}{x-2}=\lim_{x\to 2}\frac{(x-2)(x-3)}{x-2}=\lim_{x\to 2}(x-3)=-1$$

2. $\lim\limits_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}=\frac{\sqrt{1+3}-2}{1-1}=\frac{0}{0} \text{ noaniqlik.}$$

Bunday noaniqlikni oldini olish uchun berilgan kasrni surat va mahrajini berilgan ifodani qo‘shmasiga (suratini) ko‘paytiriladi. So‘ngra quyidagi

$$(a+b)(a-b)=a^2-b^2$$

formuladan foydalaniladi.

$$\lim_{x\to1}\frac{\sqrt{x+3}-2}{x-1}=\lim_{x\to1}\frac{\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)}{(x-1)\left(\sqrt{x+3}+2\right)}=$$

$$=\lim_{x\to1}\frac{x+3-4}{(x-1)\left(\sqrt{x+3}+2\right)}=\lim_{x\to1}\frac{x-1}{(x-1)\left(\sqrt{x+3}+2\right)}=$$

$$=\lim_{x\to1}\frac{1}{\sqrt{x+3}+2}=\frac{1}{\sqrt{1+3}+2}=\frac{1}{2+2}=\frac{1}{4}$$

3. $\lim\limits_{x\to\infty}\frac{x^3+x}{x^2+3x}$

Agar x ning o‘rniga limit qiymatini qo‘ysak $\frac{\infty}{\infty}$ tipidagi aniqmaslikka ega bo‘lamiz, ya’ni

$$\lim_{x\to\infty}\frac{x^3+x}{x^2+3x}=\frac{\infty+\infty}{\infty+\infty}=\frac{\infty}{\infty}\quad \text{aniqmaslik.}$$

Bu aniqmaslikni ochish uchun kasrning surat va mahrajini x ning eng katta darajasiga bo‘lish kerak.

$$\lim_{x\to\infty}\frac{x^3+x}{x^2+3x}=\lim_{x\to\infty}\frac{x^3\left(1+\frac{1}{x^2}\right)}{x^2\left(1+\frac{3}{x}\right)}=\lim_{x\to\infty}\frac{x\left(1+\frac{1}{x^2}\right)}{1+\frac{3}{x}}$$

Lekin $\lim\limits_{x\to\infty}\frac{1}{x^2}=0$ ya’ni $\frac{1}{x^2}$ cheksiz katta miqdorga teskari miqdordir. Xuddi shuningdek $\lim\limits_{x\to\infty}\frac{3}{x}=0$.

Shunday qilib,

$$\lim_{x\to\infty}\frac{x^3+x}{x^2+3x}=\lim_{x\to\infty}\frac{x^3\left(1+\frac{1}{x^2}\right)}{x^2\left(1+\frac{3}{x}\right)}=\lim_{x\to\infty}\frac{x\left(1+\frac{1}{x^2}\right)}{1+\frac{3}{x}}=\lim_{x\to\infty}x\cdot\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1+\frac{3}{x}}=\infty$$

$$\frac{1+0}{1+0}=\infty\cdot1=\infty$$

Mashqlar

Limitni toping

(Qavs ichida javoblar berilgan)

1. $\lim\limits_{x\to0}\frac{3x^2+1}{x^3+2}$ ($\frac{1}{2}$)
2. $\lim\limits_{x\to2}\frac{x^2+4x+1}{x+2}$ ($3\frac{1}{4}$)
3. $\lim\limits_{x\to3}\frac{x+5}{x^2+x+1}$ ($\frac{8}{13}$)
4. $\lim\limits_{x\to4}\frac{x^2-16}{x-4}$ (8)
5. $\lim\limits_{x\to2}\frac{x^2-8x+12}{x^2-4}$ (-1)
6. $\lim\limits_{x\to2}\frac{x^2-6x+8}{x^2-5x+6}$ (2)
7. $\lim\limits_{x\to1}\frac{x^2+4x-3}{x^2-4x+3}$ (∞)
8. $\lim\limits_{x\to-1}\frac{x^2-1}{x^2+3x+2}$ (-2)
9. $\lim\limits_{x\to3}\frac{x^2-7x+12}{x^2-9}$ ($-\frac{1}{6}$)
10. $\lim\limits_{x\to-1}\frac{2x^3-2x^2+x-1}{x^3-x^2+3x-3}$ ($\frac{3}{4}$)
11. $\lim\limits_{x\to2}\frac{\sqrt{x+1}-3}{x-2}$ (-2)
12. $\lim\limits_{x\to3}\frac{\sqrt{x+1}}{x-3}$ ($\frac{1}{4}$)

13. $\lim\limits_{x\to -1}\dfrac{x-1}{\sqrt{x+1}-\sqrt{2}}$ $(\sqrt{2})$

14. $\lim\limits_{x\to 2}\dfrac{x^2-4}{\sqrt{x+3}-\sqrt{5}}$ $(8\sqrt{5})$

15. $\lim\limits_{x\to -1}\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}$ $(\dfrac{2}{\sqrt{2}})$

16. $\lim\limits_{x\to\infty}\dfrac{x^3+x}{x^4-3x+1}$ (0)

17. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+x+2}}{x^2+1}$ (1)

18. $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt[3]{x^2+4}}$ (0)

19. $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x+1}}{1+x^2}$ (1)

20. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{\sqrt{x}-1}$ (1)

21. $\lim\limits_{x\to\infty}\dfrac{+5x\sqrt{x}}{1-3x}$ $(-\dfrac{5}{3})$

22. $\lim\limits_{x\to 2}\dfrac{\sqrt{x+1}+\sqrt{3}}{x^2-3}$ $(2\sqrt{3})$

23. $\lim\limits_{x\to 1}\dfrac{\sqrt{x+3}-2}{\sqrt{x+1}-\sqrt{2}}$ $(\dfrac{1}{\sqrt{2}})$

24. $\lim\limits_{x\to\infty}\dfrac{x^3+4x^2}{x^5+1}$ (0)

25. $\lim\limits_{x\to\infty}\dfrac{x^4+1}{x^2-1}$ (∞)

Ikki ajoyib limit

Transsendent funksiyalari limitlarini hisoblashda ko‘pincha quyidagi limitlardan (ayniyatlardan) foydalanildi.

$$\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1 \qquad (1), \qquad \lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha}=1 \qquad (1')$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e \qquad (2), \qquad \lim_{x\to\infty}(1+\alpha)^{\frac{1}{\alpha}}=e \qquad (2')$$

Bu yerda e Eyler soni deyiladi. Bu irratsional son taqriban 2,7 ga teng. (1) va ($1'$) limitlar birinchi ajoyib limit deyiladi. (2) va ($2'$) limitlar ikkinchi ajoyib limit deyiladi. Quyidagi misollarni ko‘rib o‘tamiz.

Misol. $\lim\limits_{x\to 0}\frac{\sin 4x}{x}$, $4x=\alpha$ deb olamiz.

$x\to 0$ da $\alpha=4x$ ham nolga intiladi.

Shuning uchun kasrni surat va maxrajini 4 ga ko‘paytirib va (1) va ($1'$) formuladan foydalanib quyidagilarni topamiz.

$$\lim_{x\to 0}\frac{\sin 4x}{x}=\lim_{x\to 0}\frac{4\sin 4x}{4x}=4\lim\frac{\sin 4x}{4x}=4\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=4\cdot 1=4$$

Misol 2. $\lim\limits_{x\to 0}\frac{x}{tg3x}$ ni hisoblang.

$$\lim_{x\to 0}\frac{x}{tg3x}=\lim_{x\to 0}\frac{1}{\frac{\sin 3x}{\cos 3x}}=\lim_{x\to 0}\frac{\cos 3x}{\sin 3x}=\lim_{x\to 0}\frac{x}{\sin 3x}\cdot\lim_{x\to 0}\cos 3x=\frac{1}{3}\lim_{x\to 0}\frac{3x}{\sin 3x}\cdot 1=\frac{1}{3}\cdot 1=\frac{1}{3}$$

Chunki, $\lim\limits_{x\to 0}\frac{3x}{\sin 3x}=1$ va $\lim\limits_{x\to 0}\cos 3x=\cos 0=1$

3. $\lim\limits_{x\to 0}\frac{\sqrt{x+4}-2}{\sin 2x}$ ni toping.

$\lim\limits_{x\to 0}\frac{\sqrt{x+4}-2}{\sin 2x}=\frac{\sqrt{0+4}-2}{\sin 0}=\frac{0}{0}$ aniqmaslik.

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{\sin 2x}=\lim_{x\to 0}\frac{\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)}{\sin 2x\left(\sqrt{x+4}+2\right)}=\lim_{x\to 0}\frac{x+4-4}{\sin 2x\left(\sqrt{x+4}+2\right)}=$$

$$=\lim_{x\to 0}\frac{x}{\sin 2x\left(\sqrt{x+4}+2\right)}=\frac{1}{2}\cdot\lim_{x\to 0}\frac{2x}{\sin 2x\left(\sqrt{x+4}+2\right)}=\frac{1}{2}\cdot 1\cdot\frac{1}{4}=\frac{1}{8}$$

4. $\lim\limits_{x\to 0}\left(\frac{x+2}{x-2}\right)^{x}$ ni toping.

Bu yerda asosan limiti birga teng:

$$\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^x=\lim_{x\to\infty}\left[\frac{x\left(1+\frac{2}{x}\right)}{x\left(1-\frac{2}{x}\right)}\right]^x=\lim_{x\to0}\left(\frac{1+\frac{2}{x}}{1-\frac{2}{x}}\right)^x=1^\infty$$

Ko‘rsatkich cheksizlikka intilgan. Shunga asosan 1^∞ ko‘rinishdagi aniqmaslikka ega bo‘lamiz. Buni e tipdagi aniqmaslik deb ham yuritiladi.

1. l tipidagi aniqmaslikni ochish uchun asos $\frac{x+2}{x-2}$ quyidagicha o‘zgartiramiz.

$$\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^x=\lim_{x\to\infty}\left(1+\frac{x+2}{x-2}-1\right)^x=\lim_{x\to\infty}\left(1+\frac{x+2-(x-2)}{x-2}\right)^x=\lim_{x\to\infty}\left(1+\frac{4}{x-2}\right)^x$$

Endi ko‘rsatkich $x-$ni $\frac{4}{x-2}$ kasrga ko‘paytiramiz va bo‘lamiz.

$$\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)=\lim_{x\to\infty}\left(1+\frac{4}{x-2}\right)^{x\cdot\frac{4}{x-2}\cdot\frac{x-2}{4}}=\lim_{x\to\infty}\left[\left(1+\frac{4}{x-2}\right)^{\frac{x-2}{4}}\right]^{\frac{4x}{x-2}}=\left[\lim_{x\to\infty}\left(1+\frac{4}{x-2}\right)^{\frac{x-2}{4}}\right]^{\lim\limits_{x\to\infty}\frac{4x}{x-2}}=$$

$$=e^{\lim\limits_{x\to\infty}\frac{4x}{\left(1-\frac{2}{x}\right)}}=e^{\lim\limits_{x\to\infty}\frac{4}{1-\frac{2}{x}}}=e^4$$

chunki $\frac{4}{x-2}=\alpha\to0$ agar $x\to\infty$ esa $\frac{x-2}{4}=\frac{1}{2}$.

(2) formulaga asosan kvadrat qavs ichidagi ifodani limiti e ga teng. Bundan tashqari $\lim\limits_{\alpha\to\alpha}[f(x)]^{\varphi(x)}=\left[\lim\limits_{x\to a}f(x)\right]^{\lim\limits_{x\to\infty}\varphi(x)}$ teoremani qo‘lladik.

5. $\lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)^x$ ni toping.

$\lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)^x$ ham 1^∞ ko‘rinishdagi aniqmaslikni beradi. Aniqmaslikni yyechish uchun quyidagicha almashtiramiz. Demak, $-\frac{2}{x}=\alpha,\ x=-\frac{2}{\alpha},\ x\to\infty,\ \alpha\to0$. Demak,

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x=\lim_{\alpha\to0}(1+\alpha)^{-\frac{2}{\alpha}}=\left[\lim_{\alpha\to0}(1+\alpha)^{\frac{1}{\alpha}}\right]^{-2}=e^{-2}=\frac{1}{e^2}.$$

Mashqlar

Limitlarni toping
(Qavs ichida javoblari berilgan)

50. $\lim\limits_{x\to 0}\frac{\sin 5x}{2x}$ $(\frac{5}{2})$

51. $\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 7x}$ $(\frac{2}{7})$

52. $\lim\limits_{x\to 0}\frac{\sin 3x+\sin 4x}{x}$ (7)

53. $\lim\limits_{x\to 0}\frac{x}{1-\cos x}$ (2)

54. $\lim\limits_{x\to 0}\frac{tg 2x}{x}$ (2)

55. $\lim\limits_{x\to 0}\frac{x}{\sqrt{1-\cos x}}$ $(\sqrt{2})$

56. $\lim\limits_{x\to 0} x\cdot ctgx$ (1)

57. $\lim\limits_{x\to 0}\frac{\sin 4x}{tg 3x}$ $(\frac{4}{3})$

58. $\lim\limits_{x\to 0}\frac{tgx-\sin x}{\sin^3 x}$ $(\frac{1}{2})$

59. $\lim\limits_{x\to 0}\frac{\sin 4x-\sin x}{2x}$ $(\frac{3}{2})$

60. $\lim\limits_{x\to 0}\frac{\sin^2\frac{x}{2}}{x^2}$ $(\frac{1}{4})$

61. $\lim\limits_{x\to 0}\frac{\sin 2x}{\sqrt{x+4}-2}$ (8)

62. $\lim\limits_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{\sin x}$ $(\frac{1}{2\sqrt{3}})$

63. $\lim\limits_{x\to 0}\frac{x-x\sin\frac{1}{4}}{1-5x}$ $(-\frac{1}{5})$

64. $\lim\limits_{x\to 0}\frac{tg 3x}{\sqrt{x+3}-\sqrt{3}}$ $(6\sqrt{3})$

65. $\lim\limits_{\alpha\to 0}(1+5\alpha)^{\frac{3}{\alpha}}$ (e^{∞})

66. $\lim\limits_{x\to 0}\left(1-\frac{2}{x}\right)^{x}$ $\qquad$ $(\frac{1}{e^2})$

67. $\lim\limits_{x\to 0}\left(\frac{x-3}{x+3}\right)^{x}$ $\qquad$ (e^3)

68. $\lim\limits_{x\to 0}\left(\frac{x-1}{x+1}\right)^{x}$ $\qquad$ $(\frac{1}{e})$

69. $\lim\limits_{x\to 0}\left(\frac{x+2}{x-2}\right)^{3x}$ $\qquad$ (e^6)

70. $\lim\limits_{x\to 0}\left(\frac{2x-3}{2x+1}\right)^{x+1}$ $\qquad$ (e)

71. $\lim\limits_{x\to 0}\left(\frac{x^2+2}{x^2+1}\right)^{x^2}$ $\qquad$ (0)

72. $\lim\limits_{x\to 0}\left(\frac{x+1}{x-3}\right)^{x}$ $\qquad$ (-)

73. $\lim\limits_{n\to\infty} n[\ln(n-2)-\ln n]$ $\qquad$ (2)

74. $\lim\limits_{x\to 0}(1+4x)^{\frac{1}{x}}$ $\qquad$ (e^4)

75. $\lim\limits_{x\to 0}\left(\frac{x-3}{x}\right)^{\sqrt{x}}$ $\qquad$ (1)

2-§. Funksiyaning uzluksizligi

1 – ta’rif. Agar $f(x)$ funksiya $\alpha-$biror atrofida aniqlangan va

$$\lim_{x\to\alpha} f(x)=f(\alpha)$$

bo‘lsa, u $x=\alpha$ nuqtada uzluksiz deyiladi. Bu ta’rif to‘rtta uzluksiz shartini o‘z ichiga oladi.

1. $f(x)$ funksiya α ning qandaydir atrofida aniqlangan bo‘lishi kerak:

2. Chekli $\lim\limits_{x\to\alpha-0} f(x)$ va $\lim\limits_{x\to\alpha+0} f(x)$ limitlar mavjud bo‘lishi kerak.

3. Bu (chap va o‘ng) limitlar bir xil bo‘lishi kerak.

4. Bu limitlar $\lim\limits_{x\to\alpha-0} f(\alpha)$ga teng bo‘lishi kerak.

2. Funksiyaning uzilishlari. Agar funksiya α dan o‘ngda va chapda aniqlangan bo‘lsa, ammo α nuqtada uzluksizlikning to‘rtta

shartidan aqqalli bittasi bajarilmasa $f(x)$ funksiya $x=\alpha$ bo‘lganda uzilishga ega bo‘ladi. Uzilish ikki turga ega.

a) Birinchi tur uzilish chekli $\lim_{x\to\alpha-0} f(x)$ va $\lim_{x\to\alpha+0} f(x)$ limitlar mavjud, ya’ni uzluksizlik shartlaridan ikkinchisi bajariladi, qolganlari bajarilmaydi.

Masalan: $y = arctg\frac{1}{x-4}$ funksiyani $x=4$ da uzilishiga egaligini ko‘rsating:

Yyechish: Agar $x \to 4-0$ unda $\frac{1}{x-4} \to -\infty$ va $\lim_{x\to 4-0} y = -\frac{\pi}{2}$;

Demak, $x \to 4$ funksiya chap va o‘ng limitga ega ular har xildir. Shunga ko‘ra $x=4$ nuqta birinchi tur uzilish nuqtasidir.

2. Ikkinchi tur uzilish $\lim_{x\to 0} f(x)$ o‘ngdan yoki chapdan $\pm\infty$ ga teng.

Masalan: $y = \frac{x}{x-4}$ funksiyani $x=4$ da uzilishga ega ekanini ko‘rsating.

Yyechish:

$$\lim_{x\to 4-0} \frac{x}{x-4} = -\infty; \quad \lim_{x\to 4+0} \frac{x}{x-4} = +\infty$$

funksiya $x=4$ da chap tomonidan, na o‘ng tomonidan limitga ega emas.

Demak, $x=4$ ikkinchi tur uzilishiga ega nuqtadir.

Mashqlar

1. $y = \frac{x^2-16}{x-4}$ funksiyaning $x=4$ da uzilishga ega ekanligini ko‘rsating.

2.

$$y = \frac{1}{1+2^{\frac{1}{x}}}; \qquad y = arctg\frac{\alpha}{x-\alpha}; \qquad y = \frac{x^3-x^2}{2|x-1|}$$

funksiyalarning uzilish nuqtalari topilsin va grafiklari chizilsin.

3. $y = \frac{2^{\frac{1}{x-2}}-1}{2^{\frac{1}{x-2}}+1}$ funksiyaning uzilish nuqtasi topilsin.

(Javob. $x=2$ 1-chi tur uzilish nuqtasi)

4. $y=\frac{1}{(x-1)(x-5)}$ funksiyaning uzilish nuqtasi topilsin.

(Javob. $x=1$, $x=5$ 2-chi tur uzilish nuqtalari)

5. $y=\frac{x+1}{x^3+6x^2+11x+6}$ funksiyaning uzilish nuqtasi topilsin.

(Javob. $x=-2$, $x=-3$ 2-chi tur uzilishiga ega bo‘lgan nuqtalar $x=-1$)

Cheksiz kichik funksiyalarni taqqoslash

1. $x\to\alpha$, $\alpha(x)$ va $\beta(x)$ funksiyalar cheksiz kichik bo‘lsin.

1) Agar $\lim\limits_{x\to\alpha}\frac{\alpha}{\beta}=0$ bo‘lsa, α, β ga nisbatan yuqori tartibli cheksiz kichik funksiya deyiladi va $\alpha=0(\beta)$ deb yoziladi.

2) Agar $\lim\limits_{x\to\alpha}\frac{\alpha}{\beta}=m$, $m-$noldan farqli son. Bunda α va β bir xil tartibli cheksiz kichik funksiya deyiladi.

3. $\lim\limits_{x\to\alpha}\frac{\alpha}{\beta}=1$ bo‘lsa α va β ekvivalent cheksiz kichik funksiya deyiladi. Ekvivalent cheksiz kichik funksiyalar $\alpha\leftrightarrow\beta$ deb yoziladi.

4. $\lim\limits_{x\to\alpha}\frac{\alpha}{\beta^n}=A$ bo‘lsa, α, β ga nisbatan n tartibli cheksiz kichik funksiya deyiladi.

2. Ekvivalent cheksiz kichik funksiyalarning hossalari.

a) ekvivalent cheksiz kichik funksiyalarning ayirmasi ularni har biriga nisbatan ham yuqori tartibli cheksiz kichik funksiya bo‘ladi.

b) agar bir nechta har xil tartibli cheksiz kichik funksiyalar yig‘indisidan yuqori tartiblari chiqarib tashlansa, u holda qolgan qismi bosh qism deyiladi va umumiy yig‘indiga ekvivalent bo‘ladi.

1. Masalan: $X-$cheksiz kichik son bo‘lsin. $\alpha=7x^2+3x^5$ va $\beta=5x^2+3x^3$ cheksiz kichik funksiyalarni solishtiring.

$$\lim_{x\to0}\frac{\alpha}{\beta}=\lim_{x\to0}\frac{7x^2+3x^5}{5x^2+3x^3}=\lim_{x\to0}\frac{7+3x^3}{5+3x}=\frac{7}{5}$$

$\frac{\alpha}{\beta}$ nisbatning limiti noldan farqli son. Bunda α va β bir xil tartibda cheksiz kichik funksiya.

2. Masalan: $\alpha = x\sin^2 x$ va $\beta = 4x\sin x(x \to 0)$ cheksiz kichik funksiyalarni solishtiring.

$$\lim_{x\to 0}\frac{\alpha}{\beta} = \lim_{x\to 0}\frac{x\cdot\sin^2 x}{4x\sin x} = \lim_{x\to 0}\frac{\sin^2}{4\sin x} = \frac{1}{4}\lim_{x\to 0}\sin x = 0 \text{ ya'ni } \alpha = 0(\beta)$$

3. Masala. $\alpha = x\ln(1+x)$; $\beta = x\sin x(x \to 0)$ cheksiz kichik funksiyalarni solishtiring.

$$\lim_{x\to 0}\frac{\alpha}{\beta} = \lim_{x\to 0}\frac{x\ln(1-x)}{x\sin x} = \lim_{x\to 0}\frac{\ln(1+x)}{\sin x} = \lim_{x\to 0}\frac{\frac{\ln(1+x)}{x}}{\frac{\sin x}{x}} = 1 \qquad \alpha \leftrightarrow \beta$$

Mashqlar

1. $X-$nisbatan $y = \sqrt{\sin 2x}$ cheksiz kichik funksiyani tartibini aniqlang.

(Javob: 2)

2. $\alpha = x^2\sin^2 x$ va $\beta = xtg\,x,\ x \to 0$ cheksiz kichik funksiyalarni solishtiring.

(Javob: $\alpha = 0(\beta)$)

3. $\alpha = (1+t)^m - 1$ va $\beta = mx$ (agar $x \to 0$ va $m-$ratsional musbat son) cheksiz kichik funksiyalarni solishtiring.

4. $\alpha = a^x - 1$ va $\beta = x\ln a$ cheksiz kichik funksiyalarni solishtiring.

(Javob: $\alpha \leftrightarrow \beta$)

5. $X-$cheksiz kichik kattalikka solishtirib, $y = xl^x$ cheksiz kichik funksiyani tartibini aniqlang.

(Javob: $y \leftrightarrow x$)

6. $\lim_{x\to 0}\frac{\ln(1+3x\cdot\sin x)}{tgx^2}$

(Javob: 3)

7. $\lim_{x\to 0}\frac{e^{2x}-1}{\ln(1-4x)}$

(Javob: $-\frac{1}{2}$)

3-§. Funksiyaning hosilasi va differensiali

$Y=f(x)$ funksiyaning orttirmasini, argument orttirmasiga nisbatini, keyingi nolga intilgandagi limiti, agar u mavjud bo‘lsa, funksiyaning hosilasi deyiladi, ya’ni:

$$\lim_{\Delta x\to 0}\frac{\Delta Y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta)-f(x)}{\Delta x} \qquad (1)$$

Agar funksiyaning hosilasi mavjud bo‘lsa, uni differensiyalanuvchi funksiya deyiladi. Hosilani Y' yoki $f'(x)$ yoki $\frac{dY}{dx}$ yoki $\frac{df(x)}{dx}$ bilan belgilanadi.

$Y=f(x)$ funksiya hosilasining $x=x_0$ nuqtadagi $f'(x_0)$ yoki $\frac{df(x_0)}{dx_0}$ shaklida ko‘rsatiladi.

Hosila geometrik va mexanik ma’noga ega.

Mexanik ma’nosi. Mexanikada nuqtaning o‘tgan yo‘li s ning vaqt t bo‘yicha hosilasi uning tezligini beradi, ya’ni

$$v=S'(t)$$

Geometrik ma’nosi. $Y=f(x)$ funksiya hosilasining geometrik ma’nosi shundan iboratki, u shu funksiya chizmasiga o‘tkazilgan urinmaning burchak koefitsientini bildiradi, ya’ni

$$K=tg\alpha=f'(x)$$

$x=x_0$ nuqtada funksiya chizmasiga o‘tkazilgan urinmaning burchak koeffitsienti $K=f'(x_0)$ bo‘ladi.

Hosilaning ta’rifidan foydalanib berilgan funksiyaning hosilasini topish mumkin. Buning uchun quyidagi ishlar bajariladi:

1. Argumentga ixtiyoriy orttirma berib funksiyaning orttirilgan qiymati $y+\Delta Y$ topiladi.

2. Funksiyaning orttirmasi ΔY aniqlanadi:

3. $\frac{\Delta Y}{\Delta x}$ nisbat hisoblanadi.

4. Shu nisbatning $\Delta x \to 0$ dagi limiti topiladi, u mavjud bo‘lsa hosilani beradi.

$$Y' = \lim_{\Delta x \to 0} \frac{\Delta Y}{\Delta x}$$

Misol 1. Nuqta to‘g‘ri chiziq bo‘yicha $s = 3t^2 + 2t$ qonuni asosida harakatlanadi, bunda t vaqt, sekundlarda; s yo‘l, metrlarda.

Nuqtaning $t=3$ va $t=4$ momentdagi oniy tezligi topilsin.

Yyechish. Avval funksiya hosilasini topamiz.

1. $s+\Delta s = 3(t+\Delta t)^2 + 2(t+\Delta t) = 3t^2 + 6t\Delta t + 3(\Delta t)^2 + 2t + 2\Delta t$

2. $\Delta s = 3t^2 + 6t\Delta t + 3(\Delta t)^2 + 2t + 2\Delta t - \left(3t^2 + 2t\right) = 6t\Delta t + 3(\Delta t)^2 + \Delta t$

3. $v_{¢ð} = \frac{\Delta s}{\Delta t} = \frac{6t\Delta t + 3(\Delta t)^2 + 2\Delta t}{\Delta t} = 6t + 3\Delta t + 2$

4. t momentdagi nuqtaning oniy tezligi:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0}(6t + 3\Delta t + 2) = 6t + 2$$

Xususiy holda: $t=3$ bo‘lsa, $v = 20ì\ /ñåê$

$t=4$ bo‘lsa, $v = 26ì\ /ñåê$

Misol 2. Hosilaning ta’rifidan foydalanib, quyidagi funksiyalarning hosilalari topilsin:

1. $Y = 3x + 5$;

2. $f(x) = \frac{1}{x}$, $x=2$ nuqtada

3. $f(x) = \sin(2x-3)$, $x=1$ nuqtada

Yechimi: 1. Hosilaning ta’rifidan foydalanamiz:

$$Y + \Delta Y = 3(x + \Delta x) + 5 = 3x + 3\Delta x + 5$$

$$\Delta Y = 3x + 3\Delta x + 5 - (3x + 5) = 3\Delta x$$

$$\frac{\Delta Y}{\Delta x}=\frac{3\Delta x}{\Delta x}=3; \qquad Y'=\lim_{\Delta x\to 0}\frac{\Delta Y}{\Delta x}=\lim_{\Delta x\to 0}3=3$$

Demak, $Y'=3$

2. $f(x+\Delta x)=\dfrac{1}{x+\Delta x};$

$$f(x+\Delta x)-f(x)=\frac{1}{x+\Delta x}-\frac{1}{x}=\frac{x-(x+\Delta x)}{x(x+\Delta x)}=-\frac{\Delta x}{x(x+\Delta x)};$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=-\frac{\Delta x}{x(x+\Delta x)\Delta x}=-\frac{1}{x(x+\Delta x)};$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\left[-\frac{1}{x(x+\Delta x)}\right]=-\frac{1}{x(x+0)}=-\frac{1}{x^2}$$

$$f'(x)=-\frac{1}{x^2}$$

Hosilaning $x=2$ nuqtadagi qiymati esa,

$$f'(2)=-\frac{1}{2^2}=-\frac{1}{4}; \qquad f'(2)=-\frac{1}{4};$$

3. $f(x+\Delta x)=\sin(2x+2\Delta x-3)$

$$f(x+\Delta x)-f(x)=\sin(2x+2\Delta x-3)-\sin(2x-3)=$$
$$=2\cos\frac{2x+2\Delta x-3+2x-3}{2}\cdot\sin\frac{2x+2\Delta x-3-2x+3}{2}=$$
$$=2\cos(2x-3+\Delta x)\cdot\sin\Delta x$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{2\cos(2x-3+\Delta x)\cdot\sin\Delta x}{\Delta x}=2\frac{\sin\Delta x}{\Delta x}\cos(2x-3+\Delta x)$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\left[2\frac{\sin\Delta x}{\Delta x}\cos(2x-3+\Delta x)\right]=$$
$$=2\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}\cdot\lim_{\Delta x\to 0}\cos(2x-3+\Delta x)=2\cdot 1\cos(2x-3)=2\cos(2x-3);$$

$$f'(x)=2\cos(2x-3)$$

Uning $x=1$ nuqtadagi qiymati.

$$f'(x)=2\cos(2\cdot 1-3)=2\cos(-1)=2\cos 1=-2\cdot 05402=1{,}0804$$

Demak, $f'(x)=1{,}0804$

Misol 3. $f(x)=x^2-2$ parabolaga 1) $|2;\ 0|$ nuqtaga

2) $|0;\ -4|$ nuqtaga 3) $|3;\ 5|$ nuqtaga o‘tkazilgan urinmalarning burchak koeffitsientlari topilsin.

Yechimi: Avval funksiyaning x nuqtadagi hosilasini topamiz:

$$f(x+\Delta x)^2-2=x^2+2x\Delta x+(\Delta x)^2-2$$

$$f(x+\Delta x)-f(x)=2x\Delta x+(\Delta x)^2$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{2x\Delta x+(\Delta x)^2}{\Delta x}=2x+\Delta x$$

$$f'(x)=\lim_{\Delta x\to 0}(2x+\Delta x)=2x \qquad f'(x)=2x$$

$x=2$; $x=0$; $x=3$ nuqtalarda egri chiziqqa o‘tkazilgan urinmalarning burchak koeffitsientlari quyidagicha: $f'(2)=2\cdot 2=4$

$$f'(0)=2\cdot 0=0; \qquad f'(3)=2\cdot 3=6$$

Mashqlar

1. $Y=x^2-X+1$ funksiyaning $x=1$ nuqtadagi orttirmasi topilsin, agar $\Delta x=1$; $\Delta x=-0{,}5$ bo‘lsa.

Javob. $\Delta Y=3$; $\Delta Y=-0{,}75$

2. Δx orttirmani π, $\frac{\pi}{2}$ va $\frac{\pi}{4}$ ga teng hisoblab, funksiyaning $x_0=\pi$ nuqtadagi orttirmasini aniqlang.

Javob. $3\pi^2$; $\frac{5\pi^2}{4}$; $\frac{17\pi^2}{16}$

3. Quyidagi funksiyalar uchun $\frac{\Delta Y}{\Delta x}$ nisbatni berilganlar asosida hisoblang:

a) $Y=x^2$ uchun $x_0=1$ va Δx 1; 0,1 va 0,01 ga

b) $Y = 6x + 5$ uchun $x_0 = 2$ va Δx $1;\ 0{,}1$ va 0,01 ga

v) $Y = x - 3$ uchun $x_0 = 3$ va Δx $2;\ 0{,}2$ va 0,02 ga teng.

Javob:

a) 3; 2,1 va 2,01

b) 6; 6 va 6

v) 1; 1 va 1

4. Hosilaning ta'rifiga ko'ra, quyidagi funksiyalarning hosilalari topilsin:

a) $Y = 3$ b) $Y = 2x^2$ v) $Y = 2\cos x$

g) $Y = \sqrt{x}$ d) $Y = x - \sqrt{3}$ ye) $Y = tgx$

j) $Y = \dfrac{1}{x^3}$ z) $Y = 5\sin x + 3\cos x$

Javob:

a) 0 b) $Y' = 4x$ v) $Y' = -2\sin x$

g) $\dfrac{1}{2\sqrt{x}}$ d) 1 ye) $\dfrac{1}{\cos^2 x}$

j) $-\dfrac{3}{x^4}$ z) $5\cos x - 3\sin x$

5. Quyidagi funksiyalarni berilgan nuqtalarda hosilasi mavjud emasligini ko'rsating:

a) $Y = \sqrt[4]{x}$ funksiya $x = 0$ nuqtada

b) $Y = \sqrt{x - 1}$ funksiya $x = 1$ nuqtada

6. $\varphi(x) = \dfrac{2}{x}$ funksiya berilgan $\varphi'(1)$ va $\varphi'(-2)$lar hisoblansin.

Javob: -2; -0.5.

7. Agar $f(x)$ funksiya $x = 0$ nuqta atrofida (tevaragida) aniqlangan, $f(0) = 0$ bo'lib, $\lim\limits_{\Delta x \to 0} \dfrac{f(x)}{x}$ mavjud bo'lsa, oxirgi ifoda nimani bildiradi?

8. Qaysi nuqtada kubik parabola $Y = x^3$ga o'tkazilgan urinma a) $0x$ o'qiga paralell bo'ladi? b) $0x$ o'qi bilan 30^0, 45^0li burchak hosil qiladi?

Javob: a) $(0;\ 0)$, b) $\left(\pm\sqrt[4]{\frac{1}{3}};\ \pm\sqrt[4]{\frac{1}{27}}\right)$ va $\left(\pm\sqrt{\frac{1}{3}};\ \pm\frac{1}{3}\sqrt{\frac{1}{3}}\right)$.

9. $y = x^2 - 3x + 1$ parabolaga $x = 1$ nuqtadan o‘tkazilgan urinmaning tenglamasi topilsin.

Javob: $y = -x$.

4 – §. Asosiy elementar funksiyalarning hosilalari

1. Asosiy elementar funksiyalarning hosilalarini topish formulalari:

1. $\tilde{N}' = 0$ (c – o‘zgarmas son)
2. $X' = 1$ (n – o‘zgarmas son)
3. $\left(x^n\right)' = nx^{n-1}$
4. $\left(a^n\right)' = a^x \ln a$; $\left(e^x\right)' = e^x$
5. $\left(\log_a x\right)' = \frac{1}{x \ln a}$; $\left(\ln x\right)' = \frac{1}{x}$
6. $\left(\sin x\right)' = \cos x$
7. $\left(\cos x\right)' = -\sin x$
8. $\left(tgx\right)' = \frac{1}{\cos^2 x}$
9. $\left(ctgx\right)' = \frac{1}{\sin^2 x}$
10. $\left(\arcsin x\right)' = \frac{1}{\sqrt{1-x^2}}$
11. $\left(\arccos x\right)' = \frac{1}{\sqrt{1-x^2}}$
12. $\left(arctgx\right)' = \frac{1}{1+x^2}$
13. $\left(arcctgx\right)' = \frac{1}{1+x^2}$

2. Agar $u = f(x)$, $v = Y(x)$ va $w = g(x)$lar chekli hosilalarga $(u';\ v';\ w')$ ega bo‘lsa, quyidagi qoidalarga asoslanib hisoblanadi:

1^0. $(c \cdot u)' = c \cdot u'$

$2^0.\ (u+v+w)' = u'+v'+w'$

$3^0.\ (u\cdot v)' = u'\cdot v + u\cdot v'$

$4^0.\ \left(\frac{u}{v}\right)' = \frac{u'\cdot v - u\cdot v'}{v^2}$

$5^0.\ \left(\frac{c}{v}\right)' = -\frac{c\cdot v'}{v^2}$

Misol. Agar $f(x)=2x^3+4x-5$ bo‘lsa $f'(-1),\ f'(0)$ va $f'(2),\ f'(3)$ larni hisoblang.

Yechimi: Avval hosilani aniqlaymiz, (2) va so‘ngra (1) qoidalarni qo‘llasak: $f'(x)=\left(2x^3\right)'+(4x)'-5'=2\left(x^3\right)'+4x'-5'$

Hosila topshining (1), (2) va (3) formulalariga asosan

$$f'(x)=2\cdot 3x^2+4\cdot 1-0=6x^2+4$$

Endi ko‘rsatilgan nuqtalardagi hosilaning qiymatlarini aniqlaymiz:

$$f'(-1)=6\cdot(-1)^2+4=10$$
$$f'(0)=6\cdot 0+4=4$$
$$f'(2)=6\cdot 2^2+4=28$$
$$f'(3)=6\cdot 3^2+4=58$$

Eslatamiz: Bunday so‘ng qo‘llanadigan formulalarni, qoidalarni qavs ichida ko‘rsatamiz:

Misol 2. Quyidagi funksiyalarning hosilalari topilsin:

1. $y=x^{\frac{2}{3}}+3$
2. $y=\frac{2x^3}{\sqrt{x}}-\frac{3}{\sqrt[3]{x^2}}+2\sqrt[4]{x^3}$
3. $y=x^2\sqrt{x}+\frac{1}{\sqrt[3]{x}}+\frac{1}{2}x^2$
4. $y=\left(2x^3+\sqrt{3}\right)\cdot 6^x$
5. $y=\frac{x}{2-\cos x}-\frac{x^2}{\sqrt{2}}$
6. $y=\frac{3}{\sin x}+\frac{\ln x}{x^3}$

Yechimi:

1. $y' = \left(x^{\frac{2}{3}}\right)' + 3' = \frac{2}{3}x^{\frac{2}{3}-1} + 0 = \frac{2}{3}x^{-\frac{1}{3}}$ $\quad [2^0,\ (3), (1)]$

2. $y' = \left(\frac{2x^3}{\sqrt{x}}\right)' - \left(\frac{3}{\sqrt[3]{x^2}}\right)' + \left(2\sqrt[4]{x^3}\right)' = 2\left(x^{\frac{5}{2}}\right)' - 3\left(x^{-\frac{2}{3}}\right)' + 2\left(x^{\frac{3}{4}}\right) =$

$= 2\cdot\frac{5}{2}x^{\frac{3}{2}} - 3\left(-\frac{2}{3}\right)x^{-\frac{5}{3}} + 2\frac{3}{4}x^{-\frac{1}{4}} = 5x^{\frac{3}{2}} + 2x^{-\frac{5}{3}} + \frac{3}{2}x^{-\frac{1}{4}}$ $\quad [1^0, 2^0, 3]$

3. Kasr ko‘rsatkichlarga o‘tamiz.

$$y = x^{\frac{3}{2}} + x^{-\frac{1}{\sqrt{3}}} + \frac{1}{2}x^2$$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{3}x^{-\frac{4}{3}} + \frac{1}{2}2x;$ $\quad \frac{dy}{dx} = \frac{3}{2}\sqrt{x} - \frac{\sqrt[3]{x^2}}{3x^2} + x$ $\quad [2^0, 1^0, 3]$

4.

$y' = \left(2x^3 + \sqrt{3}\right)'\cdot 6^x + \left(2x^3 + \sqrt{3}\right)\cdot\left(6^x\right)' = \left(6x^2 + 0\right)\cdot 6^x + \left(2x^3 + \sqrt{3}\right)\cdot 6^x \ln 6 =$

$= 6^x\left[6x^2 + \left(2x^3 + \sqrt{3}\right)\cdot \ln 6\right]$ $\quad [3^0, 2^0, 1^0, 1, 3, 4]$

Misol 3. $y = xtgx + ctgx$ funksiyaning hosilasini toping.

Yechimi: $y' = (xtgx)' + (ctgx)' = x'tgx + x(tgx)' + (ctgx)' =$

$= tgx + \frac{x}{\cos^2 x} - \frac{1}{\sin^2 x}$ $\quad [2^0, 3^0, 2, 8, 9]$

Misol 4. Agar $f(x) = e^x \arcsin x + arctgx$ bo‘lsa, $f'(0)$ni hisoblang.

Yechimi:

$f'(x) = \left(e^x \arcsin x\right)' + \left(arctgx\right)' = \left(e^x\right)' \arcsin x + e^x\left(\arcsin x\right)' + \frac{1}{1+x^2} =$

$= e^x \arcsin x + \frac{e^x}{\sqrt{1-x^2}} + \frac{1}{1+x^2}$

$x = 0$ qiymatni qo‘yib $f'(0)$ni topamiz:

$f'(0) = e^0 \arcsin 0 + \frac{e^0}{\sqrt{1-0}} + \frac{1}{1+0} = 1\cdot 0 + \frac{1}{1} + \frac{1}{1} = 1 + 1 = 2$ $\quad f'(0) = 2$

Misol 5. $y = z^5 \cdot \log_3 z$ ning hosilasini toping.
Yechimi:
$$y' = \left(z^5\right)' \log_3 z + z^5 \left(\log_3 z\right)' = 5z^4 \log_3 z + z^4 \frac{1}{z \ln 3} = z^4 \left(5 \log_3 z + \frac{1}{\ln 3}\right)$$

Mashqlar

Quyidagi funksiyalarning hosilalarini toping
(Qavs ichida javobi berilgan)

1. $f'(x) = ax^2 + bx + c$ $\quad (2ax + b)$

2. $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x^{-3} + 1$ $\quad \left(x^2 + x + x^{-4}\right)$

3. $y = 2x^5 - 4x^3 + x^2 + \sqrt{3}$ $\quad \left(10x^4 - 12x^2 + 2x\right)$

4. $y = \left(x^2 + 3\right)(x - 1)$ $\quad \left(3x^2 - 2x + 3\right)$

5. $y = x^3\left(x^2 - 3x + 1\right)$ $\quad \left(5x^4 - 12x^3 + 3x\right)$

6. $y = (x - 1)^3$ $\quad \left(3x^2 - 6x + 3\right)$

7. $y = \frac{2}{x - 1}$ $\quad -\frac{2}{(x - 1)^2}$

8. $y = \frac{x - 1}{x + 1}$ $\quad \left(\frac{2}{(x + 1)^2}\right)$

9. $y = \frac{x^2 + 1}{x^2 - 1}$ $\quad \left(-\frac{4x}{\left(x^2 - 1\right)^2}\right)$

10. $f(x) = \frac{x}{x^3 + 1}$ bo‘lsa $f'(1)$ hisoblansin. $\left(-\frac{1}{8}\right)$

11. $y=\dfrac{3}{x^2+x+1}$ $\left(-\dfrac{3(2x+1)}{(x^2+x+1)^2}\right)$

12. $y=\dfrac{2-x^2}{2+x^2}$ $\left(-\dfrac{8x}{(2x+x^2)^2}\right)$

13. $y=\dfrac{1}{x^2+2x+3}$ bo‘lsa $y'(0)$ ni toping. $\left(-\dfrac{2}{9}\right)$

14. $v=\dfrac{z^3+1}{z^2+z+1}$ $\left[\dfrac{-3z^4+2z^3+3z^2+z-1}{(z^2+z+1)^2}\right]$

15. $s=\dfrac{t^3+3}{t+1}$ $\left[\dfrac{2t^3+3t^2-3}{(t+1)^2}\right]$

16. $y=\dfrac{x}{x^2-x^{-2}}$ $\left[-\dfrac{x^2(x^4+3)}{(x^4-1)^2}\right]$

17. $y=4\sqrt{x}+\dfrac{4}{x}-3$ $\left(\dfrac{2}{\sqrt{x}}-\dfrac{4}{x^2}\right)$

18. $y=2\sqrt{x}+3\sqrt[3]{x}$ $\left(\dfrac{1}{\sqrt{x}}+\dfrac{1}{\sqrt[3]{x^2}}\right)$

19. $y=\sqrt[3]{x^2}-x\sqrt[4]{x}$ $\left(\dfrac{2}{3\sqrt[3]{x}}-\dfrac{5}{4}\sqrt[4]{x}\right)$

20. $f(x)=\dfrac{\sqrt{x}}{\sqrt{x}-1}$ bo‘lsa $f'(4)$ ni toping $\left(-\dfrac{1}{4}\right)$

21. $y=\dfrac{\sqrt[3]{x^5}-x}{x^3}$ $\left(\dfrac{-12\sqrt[5]{x^3}+10x}{5x^4}\right)$

22. $y=\frac{1+\sqrt{x}}{1-\sqrt{x}}$ $\left(\frac{1}{\sqrt{x}\left(1-\sqrt{x}\right)^2}\right)$

23. $y=x^2\left(\sqrt{x}-2\right)$ $\left(\frac{5}{2}x\sqrt{x}-4x\right)$

24. $y=x\left(\sqrt[3]{x}-\sqrt[5]{2}\right)$ $\left(\frac{4}{3}\sqrt[3]{x}-\sqrt[5]{2}\right)$

25. $y=\sqrt[4]{x}-\frac{1}{\sqrt[3]{x}}$ $\left(\frac{1}{4\sqrt[4]{x^3}}+\frac{1}{3x\sqrt[3]{x}}\right)$

26. $y=\frac{\sqrt[3]{x}\cdot\sqrt[4]{x}}{\sqrt{x^3}}$ $\left(-\frac{11\sqrt[12]{x}}{12x^2}\right)$

27. $y=\sin x-\cos x$ $\left(\cos x+\sin x\right)$

28. $y=x\cos x$ $\left(\cos x-x\sin x\right)$

29. $y=tgx-ctgx$ $\left(4\cos ec^2\cdot 2x\right)$

30. $y=\frac{x}{\sin x}$ $\left(\frac{\sin x-x\cos x}{\sin^2 x}\right)$

31. $y=\frac{\cos x}{x}$ $\left(-\frac{x\sin x+\cos x}{x^2}\right)$

32. $y=\frac{\sin t}{1+\cos t}$ $\left(\frac{1}{1+\cos t}\right)$

33. $y=\sqrt{x}\sin x$ $\left(\frac{\sin x}{2\sqrt{x}}+\sqrt{x}\cos x\right)$

34. $y = xtgx$ $\left(tgx + \frac{x}{\cos^2 x}\right)$

35. $y = \frac{x^2 + tgx}{x^2 - tgx}$ $\left(\frac{2\left(\frac{x}{\cos^2 x} - 2xtgx\right)}{\left(x^2 - tgx^2\right)}\right)$

36. $y = \frac{1}{tgx + \sin x}$ $\left(-\frac{1 + \cos^2 x}{\cos^2 x(tgx + \sin x)^2}\right)$

37. $s = \arcsin t + \arccos t$ (0)

38. $y = x \arcsin x$ $\left(\arcsin x + \frac{x}{\sqrt{1 - x^2}}\right)$

39. $y = xarctgx$ $\left(arctgx + x\frac{1}{1 + x^2}\right)$

40. $y = \sin x \cdot \arcsin x$ $\left(\cos x \cdot \arcsin x + \frac{\sin x}{\sqrt{1 - x^2}}\right)$

5 - §. Murakkab funksiyaning hosilasi

Berilgan funksiyaning argumenti o‘z navbatida funksiyadan iborat bo‘lishi mumkin. U holda bunday funksiyalarni murakkab funksiya deyiladi.

Masalan: 1. $y = \sin x^2$ uni $y = \sin u$, $u = x^2$ shaklida ko‘rsatish mumkin. 2. $y = \left(x^3 - 1\right)^4$ uni $y = z^4$, $z = x^3 - 1$ kabi ko‘rsatish mumkin. Umumiy holda $y = f(u)$ bo‘lib $u = \varphi(x)$ bo‘lsa, berilgan funksiya $y = f(u)$ murakkab funksiya bo‘ladi, ya’ni $y = f[\varphi(x)]$.

Agar $u=\varphi(x)$ funksiya qandaydir x nuqtada $\varphi'(x)$ hosilaga, $y=f(u)$ $u-$nuqtada $y'_u=f'(u)\cdot u'$ hosilaga ega bo‘lsa, $y'_x=f'(u)\cdot\varphi'(x)$ yoki $y'_x=y'_u\cdot u'_x$ bo‘ladi.

Ya’ni berilgan hosilasini oraliq funksiya u ning hosilasiga ko‘paytiriladi. Buni hisobga olsak, murakkab funksiyalar hosilasini topish formulalari quyidagicha bo‘ladi.

1. $y=c$ $\qquad y'=0$
2. $y=u$ $\qquad y'=u'$
3. $y=u^n$ $\qquad y'=nu^{n-1}\cdot u'$
   $y=\dfrac{1}{u}$ $\qquad y'=-\dfrac{u^1}{u^2}$
   $y=\sqrt{u}$ $\qquad y'=\dfrac{u'}{2\sqrt{u}}$
4. $\begin{cases} y=a^u \\ y=e^u \end{cases}$ $\qquad y'=a^u\ln a\cdot u'$, $\quad y=e^u\cdot u'$
5. $\begin{cases} y=\log_a u \\ y=\ln u \end{cases}$ $\qquad y'=\dfrac{u'}{u\lg a}$, $\quad y=\dfrac{u'}{u}$
6. $y=\sin u$ $\qquad y'=\cos u\cdot u'$
7. $y=\cos u$ $\qquad y'=-\sin u\cdot u'$
8. $y=tgu$ $\qquad y'=\dfrac{u'}{\cos^2 u}$
9. $y=ctgu$ $\qquad y'=-\dfrac{u'}{\sin^2 u}$
10. $y=\arcsin u$ $\qquad y'=\dfrac{u'}{\sqrt{1-u^2}}$
11. $y=\arccos u$ $\qquad y'=-\dfrac{u'}{\sqrt{1-u^2}}$

12. $y = arctgu$ $\qquad y' = \frac{u'}{1+u^2}$

13. $y = arcctgu$ $\qquad y' = -\frac{u'}{1+u^2}$

Misol 1. $y = \left(\frac{x+1}{x-1}\right)^2$ $\qquad y' = ?$

Yechimi: bunda $y = u^2$ bo‘lib, $u = \frac{x+1}{x-1}$

$$y' = 2u \cdot u' = 2\frac{x+1}{x-1} \cdot \left(\frac{x+1}{x-1}\right)' = 2\frac{x+1}{x-1}$$

$$\frac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2} = 2\frac{(x+1)(x-1-x-1)}{(x-1)^3} =$$

$$= 2\frac{(x+1)(-2)}{(x-1)^3} = -\frac{4(x+1)}{(x-1)^3}; \quad y = -\frac{4(x+1)}{(x-1)^3}; \qquad [3,\ 4^0,\ 2^0,\ 1^0,\ 2]$$

Misol 2. $y = (1+2x)^{50}$; $\qquad y' = ?$

Yechimi: bunda $y = 50$ bo‘lib, $u = 1-2x$

$$y' = 50u^{49} \cdot u' = 50(1-2x)^{49} \cdot (1-2x)' =$$

$$= 50(1-2x)^{49}(-2) = -100(1-2x)^{49}; \qquad [3,\ 2^0,\ 1^0,\ 1,\ 2]$$

Misol 3. $y = 2^{x^2+1}$ ya’ni $y = 2u$ bo‘lib, $u = x^2 + 1$ bunda y' topilsin.

Yechimi:

$$y' = 2^u \ln 2 \cdot u' = 2^{x^2+1} \cdot \ln 2\left(x^2+1\right)' =$$

$$= 2^{x^2+1} \ln 2 \cdot 2x = 2^{2^x+1} x \ln 2 \qquad [3,\ 2^0,\ 1^0,\ 1,\ 2]$$

Misol 4. $y = \log_3\left(4+x^2\right)$ bunda $y = \log_3 u$ bo‘lib, $u = 4 + x^2$; y' topilishi kerak.

Yechimi:

$$y'=\frac{u'}{u\ln 3}=\frac{\left(4+x^2\right)'}{\left(4+x^2\right)\cdot\ln_3}=\frac{2x}{\left(4+x^2\right)\ln_3};\quad [3,\ 2^0]$$

Misol 5. $y=\sin 3^x$ ya’ni $y=\sin u$ bo‘lib, $u=3^x$; $y'=?$
Yechimi:

$$y'=\cos u\cdot u'=\cos 3^x\cdot\left(3^x\right)'=3^x\cos 3^x\ln 3;\quad [6,\ 4]$$

Misol 6. $y=tg2x$ ya’ni $y=tgu$ bo‘lib, $u=2x$; $y'=?$
Yechimi:

$$y'=\frac{u'}{\cos^2 u}=\frac{(2x)'}{\cos^2 2x}=\frac{2}{\cos^2 2x};$$

$$y'=2\sec^2 2x$$

Misol 7. $y=\arcsin\sqrt{x}$ ya’ni $y=\arcsin u$ va $u=\sqrt{x}$; $y'=?$
Yechimi:

$$y'=\frac{u'}{\sqrt{1-u^2}}=\frac{\left(\sqrt{x}\right)}{\sqrt{1-\left(\sqrt{x}\right)^2}}=\frac{1}{2\sqrt{x}\sqrt{1-x}};$$

$$y'=\frac{1}{2\sqrt{x(1-x)}};$$

Misol 8. $y=\arccos\frac{2x-1}{\sqrt{3}}$ ya’ni $y=\arccos u$ va $u=\frac{2x-1}{\sqrt{3}}$;

$$y'=-\frac{u'}{\sqrt{1-u^2}}=-\frac{\left(\frac{2x-1}{\sqrt{3}}\right)'}{\sqrt{1-\left(\frac{2x-1}{\sqrt{3}}\right)^2}}=-\frac{\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)'}{\sqrt{1-\frac{(2x-1)^2}{3}}}=$$

$$=-\frac{\frac{2}{\sqrt{3}}}{\frac{\sqrt{3-(2x-1)^2}}{\sqrt{3}}}=-\frac{2}{\sqrt{3-(2x-1)^2}}$$

Misol 9. $y=\cos^2(3x-4)$ $\qquad y'=?$

Yechimi: bunda $y=u^2$, $u=\cos t$ va $t=3x-4$ dan iborat. Shunga ko‘ra:

$$y'=2u\cdot u'=2\cos t\cdot(\cos t)'=2\cos t(-\sin t)\cdot t'=$$

$$=2\cos(3x-4)[-\sin(3x-4)](3x-4)=$$

$$=-2\cos(3x-4)\cdot\sin(3x-4)\cdot 3=-3\sin(6x-8)$$

Misol 10. $y=\ln^5\left(\frac{1}{5}x+1\right)$; $\qquad y'=?$

Yechimi: $y=u^5$ $\quad u=\ln t$, $\quad t=\frac{1}{5}x+1$

$$y'=5u^4\cdot u'=5\ln^4 t(\ln t)'=5\ln^4 t\cdot\frac{t'}{t}=$$

$$=5\ln^4\left(\frac{1}{5}x+1\right)\frac{\left(\frac{1}{5}x+1\right)'}{\frac{1}{5}x+1}=5\ln^4\left(\frac{1}{5}x+1\right)\cdot\frac{\frac{1}{5}}{\frac{x+5}{5}}=$$

$$=5\ln^4\left(\frac{1}{5}x+1\right)\cdot\frac{1}{x+5}=\frac{5}{x+5}\ln^4\left(\frac{1}{5}x+1\right)$$

Misol 11. $f(t)=\ln\frac{\sqrt{1-\sin t}}{1+\sin t}$ da $f'\left(\frac{\pi}{4}\right)$ topilsin.

Yechimi: Avval berilgan funksiyani logarifmlab olish maqsadga muvofiqdir:

$$f'(t)=\frac{1}{2}[\ln(1-\sin t)-\ln(1+\sin t)]$$

$$f'(t)=\frac{1}{2}\left(\frac{-\cos t}{1-\sin t}-\frac{\cos t}{1+\sin t}\right)=-\frac{\cos t}{2}\cdot\frac{1+\sin t+1-\sin t}{1-\sin^2 t}=$$

$$=-\frac{\cos t}{2}\cdot\frac{2}{\cos^2 t}=-\frac{1}{\cos t};\qquad f'\left(\frac{\pi}{4}\right)=-\frac{1}{\cos\frac{\pi}{4}}=-\sqrt{2}$$

Mashqlar

Funksiyalarning hosilalari topilsin:

1. $y=\sin(2x-1)$ bunda $y=\sin u$ va $u=2x-1$.

Javob: $y'=2\cos(2x-1)$

2. $y=\cos(1-x)$ bunda $y=\cos u$ va $u=1-x$

Javob: $y'=\sin(1-x)$

3. $y=\sin at$ bunda $y=\sin u$ va $u=at$

Javob: $y'=a\cos at$

4. $y=\cos\left(\frac{\pi}{5}+x^2\right)$ bunda $y=\cos u$ va $u=\frac{\pi}{5}+x^2$

Javob: $y'=-2x\sin\left(\frac{\pi}{5}+x^2\right)$

5. $y=(1-2x)^7$ bunda $y=u^7$ va $u=(1-2x)$

Javob: $y'=-14(1-2x)^6$

6. $y=\lg(ax^2+bx+c)$ bunda $y=\lg u$ va $u=ax^2+bx+c$

Javob: $y'=\dfrac{2ax+b}{(ax^2+bx+c)\ln 10}$

7. $y=\ln(1+2x-x^2)$ bunda $y=\ln u$ va $u=1+2x-x^2$

Javob: $y'=\dfrac{2(1-x)}{1+2x-x^2}$

8. $y=\left(\dfrac{x^2}{2x-1}\right)^{10}$ bunda $y=u^{10}$ va $u=\dfrac{x^2}{2x-1}$

Javob: $y'=\dfrac{20x^{19}(x-1)}{(2x-1)^{11}}$

9. $y = \sin x^2$

Javob: $y' = 2x\cos x^2$

10. $y = tg(\sin x)$

Javob: $y' = \dfrac{\cos x}{\cos^2(\sin x)}$

11. $y = \arcsin\sqrt[4]{x}$

Javob: $y' = \dfrac{1}{4\sqrt[4]{x^3}\sqrt{1-\sqrt{x}}}$

12. $y = \cos^2 2x$

Javob: $y' = -2\sin 4x$

13. $y = \dfrac{1}{5}tg^3 x^3$

Javob: $y' = \dfrac{3x^2 tg^2 x^3}{\cos^2 x^3}$

14. $y = \ln^3 x$

Javob: $y' = \dfrac{3\ln^2 x}{x}$

15. $y = e^{-x^2}$

Javob: $y' = -2xe^{-x^2}$

16. $y = arctg(tgx)$

Javob: $y' = 1$

17. $y = \arcsin(\sin x)$

Javob: $y' = 1$

18. $y = u^{100}$ bunda $u = 2 + 5x$

Javob: $y' = 500(2 + 5x)^{99}$

19. $y = u^8$ bunda $u = \dfrac{x+1}{x-1}$

Javob: $y' = -\dfrac{16(x+1)^7}{(x-1)^9}$

20. $y = u^{10}$ bunda $u = 2x + 1$

Javob: $y' = 2\ln 10 \cdot 10^{2x+1}$

21. $y = \log_3 u$ bunda $u = x^5 + 1$

Javob: $\left(\dfrac{5x^4}{(x^5+1)\ln 3}\right)$

22. $y = \sin^2 x + \cos^2 x$

Javob: 0

23. $y = \arcsin x + \sqrt{1 - x^2}$

Javob: $\left(\dfrac{1-2x}{\sqrt{1-x^2}}\right)$

24. $y = (\arcsin x)^3$

Javob: $\left(\dfrac{2\arcsin x}{\sqrt{1-x^2}}\right)$

25. $y = \sin(x + \sin x)$

Javob: $(1 + \cos x)\cos(x + \sin x)$

26. $y = \cos(3^x + 3^{-x})$

Javob: $(\ln 3(3^{-x} - 3^x) \cdot \sin(3^x + 3^{-x}))$

27. $y = 5\sin(2-3x)$
Javob: $[-15\cos(2-3x)]$

28. $y = \cos\left(6x - \frac{1}{x}\right)$
Javob: $\left[-\left(6x + \frac{1}{x^2}\right)\sin\left(6x - \frac{1}{x}\right)\right]$

29. $y = \sin(x^2 - 2^x)$
Javob: $(2x - 2^x \ln 2)\cos(x^2 - 2^x)$

30. $y = tg(3x+1)^3$
Javob: $\left(\frac{9(3x+1)^2}{\cos^2(3x+1)}\right)$

31. $y = ctg(x\cos x)$
Javob: $\left(\frac{x\sin x - \cos x}{\sin^2(x\cos x)}\right)$

32. $y = 10^{x^2+x+1}$
Javob: $\left[10^{x^2+x+1}\ln 10(2x+1)\right]$

33. $y = 6^{\arcsin x}$
Javob: $\left(\frac{6^{\arcsin x}\ln 6}{\sqrt{1-x^2}}\right)$

34. $y = e^{ax}\cos bx$
Javob: $\left[e^{ax}(a\cos bx - b\sin bx)\right]$

35. $z=(2a+3bu)^4$

Javob: $12b(2a+3bu)^3$

36. $y=7^{\frac{x\sin x}{1+x}}$

Javob: $\left[7^{\frac{x\sin x}{1+x}}\ln 7\frac{\sin x+x\cos x+x^2\cos x}{(1+x)^2}\right]$

37. $y=\frac{\cos x}{3\sin^2 x}$

Javob: $\left(-\frac{1+\cos^2 x}{3\sin^2 x}\right)$

38. $y=\frac{\cos x}{3\sin^2 x}$

Javob: $\left(-\frac{1+\cos^2 x}{3\sin^2 x}\right)$

39. $y=\ln\frac{a^2+x^2}{a^2-x^2}$

Javob: $\left(\frac{4a^2x}{a^4-x^4}\right)$

40. $z=\ln\sqrt{\frac{e^{2t}}{1+e^{2t}}}$ Javob: $\left(\frac{1}{1+e^{2t}}\right)$

Oshkormas funksiyaning hosilasi

Funksiya x va y oralig‘idagi munosabat $F(x, y)=0$ shaklida ko‘rsatilgan bo‘lsa, (funksiya y ga nisbatan yechilmagan) berilgan funksiya oshkormas deyiladi.

Masalan:

$$x^2 + y^2 = R^2$$

$$x^3 + y3 - xy + 5 = 0$$

$$x^2 + \sin xy - 3 = 0$$

Bunday funksiyalarni differensiyalashda y, x ning murakkab funksiyasi deb hisoblanadi va y' ni topiladi.

Misol 1. Quyidagi funksiyalarning hosilalari topilsin.

a) $y^3 - 3y + 2ax = 0$ b) $x^2 + 3xy + y^2 + 1 = 0$ funksiyada $y-$ning $(2; -1)$ nuqtadagi qiymati hisoblansin, v) $\sin\varphi + r\varphi - 5r = 0$ $\frac{dr}{d\varphi}$ topilsin, g) $e^y + xy + 0 = 0$ funksiyada y' ning $(0; 1)$ nuqtadagi qiymati hisoblansin.

Yechimi: a) tenglikning har ikki qismidan x ga nisbatan hosila olamiz:

$$3y^2 \cdot y' - 3y' + 2a = 0$$

$$3y'(y^2 - 1) = -2a \qquad y' = \frac{2a}{3(1 - y^2)}$$

b) x ga nisbatan hosila olsak:

$$2x + 3y + 3xy' + 2y \cdot y' = 0$$

y' ga nisbatan tenglama yechamiz:

$$y' = -\frac{2x + 3y}{3x + 2y}$$

$x = 2$ va $y = -1$larni o‘rniga qo‘ysak

$$y' = -\frac{2 \cdot 2 + 3(-1)}{3 \cdot 2 + 2(-1)} = -\frac{1}{4}$$

v) φ ga nisbatan hosila olamiz:

$$\cos\varphi + \varphi \cdot \frac{dr}{d\varphi} + r - 5\frac{dr}{d\varphi} = 0$$

$$\frac{dr}{d\varphi}(\varphi - 5) = -(r + \cos\varphi)$$

$$\frac{dr}{d\varphi} = \frac{r + \cos\varphi}{5 - \varphi}$$

g) x ga nisbatan differensiallasak:

$$e^{y} \cdot y' + y + xy' = 0 \text{ bundan}$$

$$y' = -\frac{y}{e^{y} + x}$$ x va y larni berilgan qiymatlarini o‘rniga qo‘ysak:

$$y' = -\frac{1}{\acute{o} + 0} \quad -\frac{1}{\mathring{a}}.$$

Mashqlar

(Javobi qavs ichida berilgan)

Quyidagi funksiyalarning hosilalari topilsin:

1. $2x + 3y + 1 = 0$ $\left(-\frac{2}{3}\right)$
2. $x^2 + y^2 = 5e^x$ $\left(\frac{5e^e - 2x}{2y}\right)$
3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\left(-\frac{b^2 x}{a^2 y}\right)$
4. $x^2 - 5y^2 + 4xy - 1 = 0$ $\left(\frac{x + 2y}{5y - 2x}\right)$
5. $x^3 + y^3 - 3xy + a^2 = 0$ $\left(\frac{x^2 - y}{x - y^2}\right)$
6. $y = \sin(x + 2y)$ $\left(\frac{\cos(x + 2y)}{1 - 2\cos(x - 2y)}\right)$

7. $x^4-6x^2y^2+9y^4-5x^2+15y^2-100=0$ $\quad\left(y'=\dfrac{x}{3y}\right)$

8. $\sin\left(y-x^2\right)-\ln\left(y-x^2\right)+2\sqrt{y-x^2}=3$ $\quad\left(y'=2x\right)$

Quyidagi oshkormas funksiyalarning hosilalarini ko‘rsatilgan nuqtadagi qiymati hisoblansin:

(Javobi qavs ichida berilgan)

9. $x^2+y^2=1$ $\quad\left(-\dfrac{\sqrt{2}}{2};\ \dfrac{\sqrt{2}}{2}\right)$ nuqtada

10. $y^2=2px$ $\quad\left(\dfrac{p}{2},\ p\right)$ nuqtada

11. $x=y+\sin y$ $\quad(0,\ 0)$ nuqtada $\left(\dfrac{1}{2}\right)$

12. $x^2+xy+y^2=3$ $\quad\left(0,\ -\sqrt{3}\right)$ nuqtada $\left(-\dfrac{1}{2}\right)$

13. $ye^y-xe^x=y(x-1)$ $\quad(1,\ 1)$ nuqtada $\left(\dfrac{2e+1}{2e}\right)$

14. $e^y+xy=e$ $\quad(0,\ 1)$ nuqtada $\left(-\dfrac{1}{e}\right)$

15. $e^{xy}+x^2+y^2=2$ $\quad(1,\ 0)$ nuqtada (-2)

Logarifmik funksiyalarning hosilasi

Berilgan funksiyaning $y=f(x)$ logarifmik hosilasi, uning logarifmidan olingan hosilasidir, ya’ni

$$(\ln y)'=\frac{y'}{y} \quad (\text{bunda } y>0)$$

Logarifmlash mumkin bo‘lgan funksiyalar hosilasini topishdan avval, logarifmlab olish hosila topish ishini anchagina soddalashtiradi.

Misol 1. Ko‘rsatkichli – darajali funksiya $y = u^v$ ning hosilasini toping.

Yechimi: Berilgan funksiyani dastlab logarifmlaymiz, so‘ngra murakkab funksiya hosilasini topamiz.

$$\ln y = v \ln u$$

$$\frac{y'}{y} = v' \ln u + v\frac{u'}{u}$$

$$y' = y\left(v' \ln u + v\frac{u'}{u}\right)$$

Misol 2. $y = x^x$ ning hosilasini toping.

Yechimi: $\ln y = x \ln x$ bu tengliknin har ikki qismidan hosila olamiz:

$$\frac{y'}{y} = \ln x + x \cdot \frac{1}{x}$$

$$\frac{y'}{y} = \ln x + 1, \quad y' = y(\ln x + 1) \quad \text{yoki}$$

$$y' = x^x(\ln x + 1)$$

Misol 3. $y = (x-1)\sqrt[3]{(5x+1)^2(x+1)}$; $\quad y' = ?$

Yechimi: Funksiyani logarifmlaymiz:

$$\ln y = \ln(x-1) + \frac{1}{3}\left[2\ln(5x+1) + \ln(x+1)\right]$$

$$\frac{y'}{y} = \frac{1}{x-1} + \frac{1}{3}\left[2\frac{5}{5x+1} + 1\frac{1}{x+1}\right] = \frac{2(15x^2+7x-4)}{3(x^2-1)(5x+1)}$$

undan

$$y' = \frac{2(15x^2+7x-4)}{3(x^2-1)(5x+1)}(x-1)\sqrt[3]{(5x+1)^2(x+1)}$$

Soddalashtirsak:

$$y' = \frac{2(15x^2+7x-4)}{3\sqrt[3]{(x+1)^2(5x+1)}}$$

Misol 4. Agar $S=(\sin t)^{\cos 2t}$ bo‘lsa, s' ni toping.

Yechimi: $\ln s=\cos 2t \ln\sin t$

$$\frac{s'}{s}=(\cos 2t)'\ln\sin t+\cos 2t(\ln\sin t)'=$$

$$=-2\sin 2t\ln\sin t+\cos 2t\cdot\frac{\cos t}{\sin t}=$$

$$=-2\sin 2t\cdot\ln\sin t+\cos 2t\cdot ctg\ t$$

$$s'=s(\cos 2t\cdot ctg\, t-2\sin 2t\ln\sin t)$$

yoki

$$s'=(\sin t)^{\cos 2t}(\cos 2t ctg\, t-2\sin 2t\ln\sin t)$$

Mashqlar

Funksiyalarning hosilasi topilsin.
(Javobi qavs ichida berilgan)

1. $y=x^{\cos x}$ $\quad\left[x^{\cos x-1}(\cos x-x\sin x\ln x)\right]$

2. $y=(\cos x)^x$ $\quad\left[(\cos x)^x(\ln\cos x-xtgx)\right]$

3. $y=\sqrt[x]{x}$ $\quad\left[\frac{\sqrt[x]{x}}{x^2}(1-\ln x)\right]$

4. $y=x^{x^3}$ $\quad\left[x^{x^3+2}(3\ln x+1)\right]$

5. $y=\frac{(x-1)^3\sqrt{x+2}}{\sqrt[3]{(x+1)^2}}$ $\quad\left[\frac{(17x^2+62x+21)(x-1)^2}{\sqrt[3]{(x+1)^5}\sqrt{x+2}}\right]$

6. $s=\frac{\sqrt{1-t^2}}{3t+1}$ $\left[\frac{3-t-6t^2}{\sqrt{1-t^2}(3t+1)^2}\right]$

7. $s=(t+1)^3(t-1)^2\sqrt[3]{(t+2)^2}$

$\left[\frac{1}{3\sqrt[3]{t+2}}(t+1)^2(t-1)(17t^2+27t-8)\right]$

8. $y=(\sin x)^{\ln x}$ $\left[(\sin x)^{\ln x}\left(\frac{1}{x}\ln\sin x+ctg\,x\ln x\right)\right]$

9. $y=\sqrt[4]{\frac{x(x^3+1)}{(x^3-1)^3}}$ $\frac{-5x^4-12x^3-1}{4\sqrt[4]{(x^3-1)^7(x^3+1)^3}}$

Yuqori tartibli hosilalar

$y=f(x)$ funksiyaning hosilasi y' ni birinchi tartibli hosila deb yuritamiz. Birinchi tartibli hosilaning hosilasiga ikkinchi tartibli hosila deb yuritiladi va uni y'' yoki $f''(x)$ bilan belgilanadi. Xuddi shu tartibda uchinchi, to‘rtinchi va hokazo tartibli hosilalarni topish mumkin.

Misol 1. Quyidagi funksiyalarning ko‘rsatilgan tartibdagi hosilalarni toping.

a) $y=x^3+2x^2-x-3$ $y'''=?$

b) $s=\ln t$ $s''''=?$

v) $s=t^3-t-3$ $s''(0)=?$

g) $f(x)=\sin 2x$ $f'''''\left(\frac{\pi}{2}\right)=?$

Yechimi: a) Birinchi tartibli hosilani olamiz:

$$y'=3x^2+2\cdot 2x-1$$

undan yana hosila olamiz:

$$y'' = 6x + 4$$

yana bir marta hosila olsak, uchinchi tartibli hosila kelib chiqadi, demak,

$$y''' = 6$$

b) Berilgan funksiyadan ketma – ket to‘rt marta hosila olamiz:

$$s' = (\ln t)' = \frac{1}{t};$$

$$s'' = (s')' = \left(\frac{1}{t}\right)' = -\frac{1}{t^2};$$

$$s''' = (s'')' = \left(-\frac{1}{t^2}\right)' = \frac{1}{t^3};$$

$$s'''' = (s''')' = \left(\frac{1}{t^3}\right)' = (2t^{-3}) = 2 \cdot (-3)t^{-4} = -\frac{6}{t^4}$$

demak, $s'''' = -\frac{6}{t^4}$.

v) $s = t^3 - t + 3$

$s' = (t^3 - t + 3)' = 3t^2 - 1$

$s'' = (s')' = (3t^2 - 1)' = 6t$. $t = 0$ da

$s''(0) = 6 \cdot 0 = 0$

demak,

$$s''(0) = 0$$

g) $f(x) = \sin 2x$

$f'(x) = 2\cos 2x$

$f''(x) = (f'(x))' = (2\cos 2x)' = -4\sin 2x$

$f'''(x) = (f''(x))' = (-4\sin 2x)' = -8\cos 2x$

$f''''(x) = (f'''(x))' = (-8\cos 2x)' = 16\sin 2x$

$f'''''(x) = (f''''(x))' = (16\sin 2x)' = 32\sin 2x$

$$f''''\left(\frac{\pi}{2}\right)=32\cos\left(2\frac{\pi}{2}\right)=32\cos\pi=-32$$

Demak, $f''''\left(\frac{\pi}{2}\right)=-32$

Misol 2. $y=\cos 2x$ funksiyaning $y''+4y=0$ tenglamani qanoatlantirishini isbotlang.

Yechimi:

$$y'=-2\sin 2x;\qquad y''=\left(-2\sin 2x\right)'=-4\cos 2x$$

o‘rniga qo‘ysak

$$-4\cos 2x+4\cdot\cos 2x=0;\quad 0=0$$

Mashqlar

Quyidagi funksiyalarni ko‘rsatilgan tartibdagi hosilalarini toping: (Javobi qavs ichida berilgan)

1. $y=x^3+4x^2-7x+1;$ $\qquad y''''=?$ $\qquad$ (0)
2. $f(x)=x^8;$ $\qquad f'''(1)=?$ $\qquad$ (336)
3. $y=x^5+4x^3-x;$ $\qquad y''''=?$ $\qquad$ (120)
4. $y=\cos x;$ $\qquad y''''=?$ $\qquad$ $(\cos x)$

Aralash misollar

(Javobi qavs ichida berilgan)

1. $y=\dfrac{ax+b}{cx+d}$ $\qquad \left(\dfrac{at-bc}{(cx+d)^2}\right)$
2. $u=\left(\dfrac{v}{1-v}\right)^n$ $\qquad \left(\dfrac{n\cdot v^{n-1}}{(1-v)^{n+1}}\right)$
3. $s=\dfrac{1-t^5}{\sqrt{3}}$ $\qquad \left(-\dfrac{5}{\sqrt{3}}t^4\right)$
4. $y=\dfrac{2}{\sin x+\cos x}$ $\qquad \dfrac{2(\sin x-\cos x)}{(\sin x+\cos x)^2}$

5. $\varphi(\alpha)=\dfrac{a-\alpha}{1+\alpha}$, $\varphi'(1)$ va $\varphi'(0)$ ni hisoblang. $\left[-\dfrac{1+\alpha}{4};\ -(1+a)\right]$

6. $y=\sqrt{(a+x)(b+x)}$ $\left(\dfrac{2x+a+b}{2\sqrt{(a+x)(b+x)}}\right)$

7. $s=a\sin wt+b\cos wt$ $(aw\cos wt-bw\sin wt)$

8. $f(x)=\ln(1+x)++\arccos\dfrac{x}{2}$, $f'(1)$ va $f'(0)$ ni hisoblang. $\left(\dfrac{3-2\sqrt{3}}{6};\ \dfrac{1}{2}\right)$

9. $y=\ln\left(x+\sqrt{a^2-x^2}\right)$; $\left(\dfrac{a^2-2x\sqrt{a^2-x^2}}{(2x^2-a^2)\sqrt{a^2-x^2}}\right)$

10. $y=\ln\sin 3^x$ $\left(3^x\ln 3ctg3^x\right)$

11. $y=tg\left(7^x+x^7\right)$ $\left(\dfrac{7^x\ln 7+7x^6}{\cos^2\left(7^x+x^7\right)}\right)$

12. $y=arctg\ln x$ $\left(\dfrac{1}{x\left(1+\ln^2 x\right)}\right)$

13. $z = \left(1+\sqrt{v}\right)^5$ $\left(\dfrac{5\left(1+\sqrt{v}\right)^4}{2\sqrt{v}}\right)$

14. $y=\arcsin x+\sqrt{1-x^2}$ $\left(\sqrt{\dfrac{1-x}{1+x}}\right)$

15. $f(x)=x\sqrt{1-x^2}+\arcsin x$ $\left(2\sqrt{1-x^2}\right)$

16. $y=arctg\dfrac{x}{1-x}$ $\left(\dfrac{1}{1-2x+2x^2}\right)$

17. $f(x)=x\arccos x$ $\left(\arccos x-\dfrac{x}{\sqrt{1-x^2}}\right)$

18. $y=\arcsin\sqrt{x}$ $\left(\dfrac{1}{2\sqrt{x(1-x)}}\right)$

19. $f(x) = arctg\sqrt[4]{e^x + 2}$ $\left[\dfrac{e^x}{4\sqrt[4]{(e^x+2)^3}\left(1+\sqrt[4]{e^x+2}\right)}\right]$

20. $y = \arcsin\sqrt[3]{\ln x}$ $\left(\dfrac{1}{3x\sqrt[3]{\ln^2 x}\sqrt{1-\sqrt[3]{\ln^2 x}}}\right)$

21. $f(x) = \arcsin\ln x$ $\left(\dfrac{1}{x\sqrt{1-\ln^2 x}}\right)$

22. $y = \ln\arcsin x$ $\left(\dfrac{1}{\arcsin x\cdot\sqrt{1-x^2}}\right)$

23. $y = a^{\arcsin x}$ $\left(\dfrac{a^{\arcsin x}\ln a}{\sqrt{1-x^2}}\right)$

24. $y = a^x\sin\ln x$ $a^x\left(\dfrac{\cos\ln x}{x} + \ln a\sin\ln x\right)$

25. $y = (\arcsin x)^3$ $\left(\dfrac{3(\arcsin x)^2}{\sqrt{1-x^2}}\right)$

9-§. Hosilaning qo‘llanishi

Urinma va normal tenglamasi

Berilgan nuqtadan $y = f(x)$ funksiya chizmasiga o‘tkazilgan urinmaning tenglamasini tuzishda hosilaning geometrik ma’nosidan foydalanamiz. Hosila, egri chiziqqa o‘tkazilgan urinmaning burchak koeffitsienti edi. Binobarin $M(x_0;\ y_0)$ nuqtadan o‘tkazilgan to‘g‘ri chiziqlar dastasi $y - y_0 = k(x - x_0)$ dan egri chiziqqa o‘tkazilgan urinmaning tenglamasini ajratish kerak. Buning uchun $k = (x_0)$ ni tenglamaga qo‘yamiz.

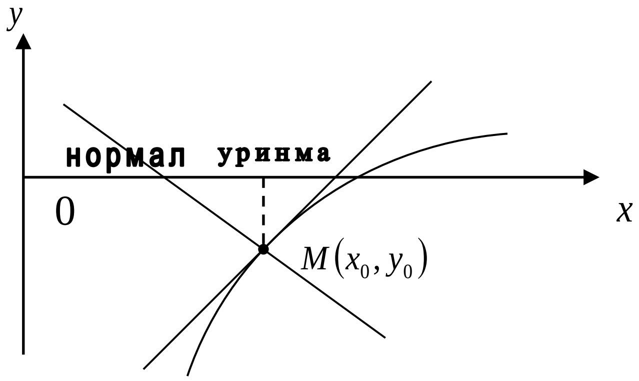
U holda

$$y - y_0 = f'(x_0)(x - x_0) \qquad (1)$$

(1) egri chiziqqa $M(x_0;\ y_0)$ nuqtadan o‘tkazilgan urinmaning tenglamasi bo‘ladi. Urinish nuqtasida urinmaga perpendikulyar bo‘lgan chiziq normal deb aytiladi. Uning tenglamasi:

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \qquad (2)$$

Urinma va normal quyidagi shaklda ko‘rsatilgan:

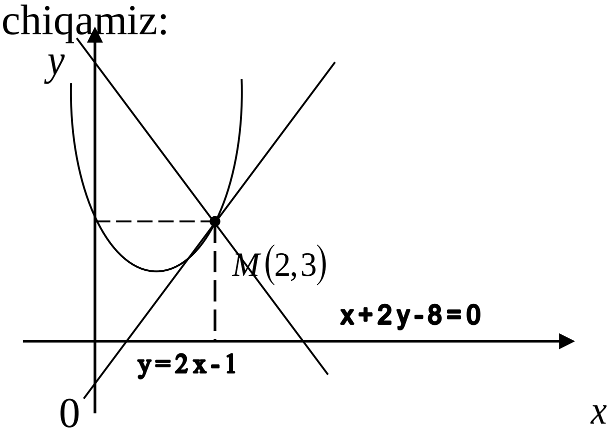


Misol 1. $f(x) = x^2 - 2x + 3$ parabolaga abssissasi $x = 2$ bo‘lgan nuqtadan o‘tkazilgan urinma va normal tenglamasi tuzilsin.

Yechimi: Shartga ko‘ra $x_0 = 2$. Bu qiymatni parabola tenglamasiga qo‘yib, nuqtaning ordinatasini aniqlaymiz:

$$f(2) = 2^2 - 2\cdot 2 + 3,\ \ y_0 = 3, \qquad f'(2) = 2\cdot 2 - 2 = 2$$

Bularni (1) va (2) formulalarga qo‘yib urinma, normal tenglamasini keltirib chiqamiz:



Urinma tenglamasi: $y - 3 = 2(x - 2)$,

$$y = 2x - 1$$

Normal tenglamasi: $y - 3 = -\frac{1}{2}(x - 2)$,

$$x + 2y = 8$$

Misol 2. $x^2 - 2xy + 3y^2 - 2y - 16 = 0$

Egri chiziqqa (1; 3) nuqtada o‘tkazilgan urinmaning tenglamasi tuzilsin:

Yechimi: nuqta koordinatalirini tenglamaga qo‘yib, uning shu egri chiziqda yotishiga ishonch hosil qilamiz:

$$1^2 - 2 \cdot 1 \cdot 3 + 3 \cdot 3^2 - 2 \cdot 3 - 16 = 0$$

yoki

$$0 = 0$$

Oshkormas funksiyaning x bo‘yicha hosilasini topamiz:

$$2x - 2y - 2xy' + 6yy' - 2x' = 0$$

$$y'(3y - x - 1) = y - x; \qquad y' = \frac{y - x}{3y - x - 1};$$

$$k = y'_{\substack{x=1 \\ y=3}} = \frac{3 - 1}{3 \cdot 3 - 1 - 1} = \frac{2}{7}; \qquad k = \frac{2}{7}.$$

Berilganlarga ko‘ra urinma va normal tenglamalarni tuzamiz.

Urinma tenglamasi:

$$y - 3 = \frac{2}{7}(x - 1) \text{ yoki } 2x - 7y + 19 = 0$$

Normal tenglamasi:

$$y - 3 = -\frac{7}{2}(x - 1) \text{ yoki } 7x + 2y - 13 = 0$$

Mashqlar

1. Berilgan egri chiziqlarga ko‘rsatilgan nutalarda o‘tkazilgan urinma va normal tenglamalari tuzilsin.

1. $y = x^2 - 3x + 4$ va (1; 2) nuqtada
2. $y = -x^2 + 3x$ ga, abssissasi 2 bo‘lgan nuqtada
3. $f(x) = x^2 + 3$ ga, ordinatasi 4 bo‘lgan nuqtada
4. $\varphi(x) = \frac{1}{x}$ ga, abssissasi 1 bo‘lgan nuqtada
5. $y = tgx$ ga ordinatasi 1 bo‘lgan nuqtada

6. $y=\ln x$ ning abssissa o‘qi bilan kesishgan nuqtada

Javoblar:

1. $y=-x+3$ va $y=x+1$
2. $y=-x+4$ va $y=x$
3. $y=2x+2$ va $x+2y-y=0$ yoki $y=-2x+6$ va $x-2y+7=0$
5. $2x-y-\frac{\pi}{2}-2n\pi+1=0$ va $x+2y-\frac{\pi}{4}-n\pi-2=0$
6. $y=x-1$ va $y+y=1$

2. $y=\sqrt[3]{3x^4+2xy}$ egri chiziqqa (-1; 1) nuqtada o‘tkazilgan urinma tenglamasi tuzilsin.

Javob: $10x+7y+3=0$

3. $y=\frac{1}{x^2+1}$ egri chiziqda shunday nuqta topilsinki, u nuqta orqali o‘tkazilgan urinma abssissa o‘qiga parallel bo‘lsin:

Javob: (0; 1)

4. $y=\frac{x-4}{x-2}$ giperbolaning koordinata o‘qlari bilan kesishgan nuqtalaridan o‘tkazilgan urinmalarning o‘zaro parallel ekanligi ko‘rsatilgan.

5. $y=x^3$ da shunday nuqta topilsinki, u nuqtadan o‘tkazilgan urinma, birinchi koordinata burchagining bissektrisasiga parallel bo‘lsin.

Javob: $\left(\pm\sqrt{\frac{1}{3}};\ \pm\frac{1}{3}\sqrt{\frac{1}{3}}\right)$

10-§. Funksiya differensiali va defferensial hisobning asosiy teoremalari

$y = f(x)$ funksiya x nuqtada hosilaga ega bo‘lsin, ya’ni

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = y' \qquad (1)$$

mavjud bo‘lsin, u holda limit ta’rifiga ko‘ra $\frac{\Delta y}{\Delta x} = y' + \alpha$ bunda α cheksiz kichik miqdor bo‘lib, $\alpha \to 0$ da $\Delta x \to 0$ (1) formuladan:

$$\Delta y = y'\Delta x + \alpha\Delta x \qquad (2)$$

Funksiya orttirmasining bosh qismiga funksiyaning differensiali deyiladi va dy bilan belgilanadi. Ta’rifga ko‘ra:

$$dy = y'\Delta x$$

Agar funksiya argumenti orttirmasining argument differensialiga teng $dx = \Delta x$ ekanligini e’tiborga olsak:

$$dy = y'dx \qquad (3)$$

ya’ni funksiyaning differensiali, funksiya hosilasi bilan argument differensialining ko‘paytmasiga teng.

Misol 1. a) Agar $\Delta x = 0.3$ bo‘lsa, $y = x^3$ funksiyaning $x = 0$ nuqtadagi differensialini hisoblang.

b) Agar $\Delta x = 0.1$ bo‘lsa, $y = \ln(x^2+1) + arctg\sqrt{x}$ funksiyaning $x = 1$ nuqtadagi differensialini hisoblang.

v) $y = x^3 - x^2 + 3x - 1$ ning differensialini toping.

g) $r = y^4 + 2^{\sin 3y}$ ning differensialini toping.

Yechimi: a) $y = x^3$ funksiyaning orttirmasi.

$$\Delta y = (x+\Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

bo‘lib, uning bosh qismi $3x^2\Delta x$ dir. Binobarin uning differensiali $dy = 3x^2\Delta x,\ x = 0,\ \Delta x = 0.3$ ekanligini hisobga olsak: $dy = 3\cdot 0^2\cdot 0.3 = 0$. Demak, $dy = 0$.

b) Qolgan misollarni yechishda (3) formuladan foydalanamiz:

$$dy = \left[\ln(x^2+1) + arctg\sqrt{x}\right]dx = \left(\frac{2x}{x^2+1} + \frac{1}{2(1+x)\sqrt{x}}\right)\cdot dx$$

$x = 1$ va $\Delta x = 0.1 = dx$ ekanligining nazarda tutsak

$$dy = \left(\frac{2\cdot 1}{1^2+1} + \frac{1}{2(1+1)\sqrt{1}}\right)\cdot 0.1 = 0.125, \quad dy = 0.125$$

v) $dy = \left(x^3 - x^2 + 3x - 1\right)' dx = \left(3x^2 - 2x + 3\right)dx$

g) $dr = \left(\varphi^4 + 2^{\sin 3\varphi}\right)' d\varphi = \left(4\varphi^3 + 3\cos 3\varphi \cdot 2^{\sin 3\varphi} \ln 2\right) \cdot d\varphi$

Mashqlar

1. $y = \sin x$ funksiyaning $\Delta x = \frac{\pi}{2}$ bo‘lganda $x = 0$ nuqtadagi differensialini toping:

Javob: $\Delta y = \frac{\pi}{2}$

2. $y = tgx$ funksiyaning $\Delta x = 0.5;\ 0.1$ va 0.001 bo‘lganda $x = \frac{\pi}{4}$ nuqtadagi differensiallari topilsin:

Javob: $dy = 1,\quad dy = 0.2$ va $dy = 0.002$

3. $y = x^2$ funksiyaning $\Delta x = 2;\ 1;\ -0.1$ va -0.01 bo‘lganda $x = 1$ nuqtada differensiallari topilsin:

Javob: 4; 2; -0.2; -0.02

Funksiyalarning differensiallari topilsin: (Javob qavs ichida berilgan).

4. $y = 2\sin x$ $\qquad\qquad$ $(2\cos x dx)$

5. $y = 3x^2 + 1$ $\qquad\qquad$ $(6xdx)$

6. $s = \frac{at^2}{2}$ $\qquad\qquad$ $(at\,dt)$

7. $y = ctg\frac{x}{2}$ $\qquad\qquad$ $\left(-\frac{dx}{2\sin^2\frac{x}{2}}\right)$

8. $s = a\sin(t-1)$ $\qquad\qquad$ $(a\cos(t-1)dt)$

9. $y=\frac{x+1}{x-1}$ $\left(-\frac{2dx}{(x-1)^2}\right)$

10. $s=a\sin(wt+\varphi_0)$ $[aw\cos(wt+\varphi_0)dt]$

11. $p=\frac{1}{v}$ $\left(-\frac{dv}{p^2}\right)$

12. $y=\sqrt[3]{x^2-1}$ $\left(\frac{2x\,dx}{3\sqrt[3]{(x^2-1)^2}}\right)$

13. $y=x\cdot 2^x$ $(2^x+x\cdot 2^x\ln 2)dx$

14. $y=\arccos\sqrt[3]{x}$ $\left(\frac{dx}{3\sqrt[3]{x^2}\sqrt{1-\sqrt[3]{x^2}}}\right)$

15. $y=tgx^2$ $\left(\frac{2x\,dx}{\cos x^2}\right)$

16. $y=\sin(x^2+x+1)$ $[\cos(x^2+x+1)(2x+1)dx]$

17. $y = 3^{\arccos\ x}$ $\left(-\frac{3^{\arccos x}\ln 3dx}{\sqrt{1-x^2}}\right)$

18. $y=\frac{1}{\sqrt{1-x^2}}$ $\left(\frac{xdx}{\sqrt{(1-x^2)^3}}\right)$

19. $y=\ln\sin(x+1)$ $[ctg(x+1)dx]$

20. $v=e^{\sin t}$ $(e^{\sin t}\cos tdt)$

21. $y=\sin(\cos x)$ $(-\sin x\cos(\cos x)dx)$

22. $y=(xtgx)^2$ $\left(2xtgx\left(tgx+\frac{x}{\cos^2 x}\right)dx\right)$

23. $y=2^{arctg(2x-1)}$ $\left(-\frac{\ln 2\cdot 2^{arctg(2x-1)}}{2x^2-2x+1}dx\right)$

24. $y=\sin\ln\left(x^2-1\right)$ $\qquad\left(\dfrac{2x\cos\ln\left(x^2-1\right)}{x^2-1}dx\right)$

25. $y=\dfrac{1}{2}\ln tg\dfrac{x}{2}-\dfrac{\cos^2 x}{\sin x}$

Differensialning taqribiy hisoblashda qo‘llanishi

$y=f(x)$ funksiyaning hosilasi, ta’rifiga ko‘ra quyidagicha edi:

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

Limitning ta’rifiga ko‘ra:

$$\frac{\Delta y}{\Delta x}=y'+\alpha \qquad \text{yoki} \qquad \Delta y=y'\Delta x+\Delta x\cdot\alpha$$

$\alpha\Delta x$ yuqori tartibli cheksiz kichik miqdor bo‘lgani uchun funksiyaning orttirmasining qiymati uchun $y'\Delta x$ ni olish mumkin:

$$\Delta y\approx y'\Delta x \quad (1) \qquad \Delta y\approx dy$$

$y=f(x)$ funksiyaning orttirilgan qiymati $y+\Delta y=f(x+\Delta x)$ yoki $y=f(x+\Delta x)=f(x)+\Delta y$ bundan (1) formulaga ko‘ra

$$f(x+\Delta x)\approx f(x)+y'\Delta x$$

$$f(x+\Delta x)\approx f(x)+f'(x)\Delta x \qquad (2)$$

yuqoridagi formulaga asosan funksiya orttirmasining va orttirilgan qiymatini taqribiy qiymatlarini hisoblash mumkin.

Misol 1. Qirrasi 20 sm. bo‘lgan kubning qirrasi 0.1 sm orttirilsa, uning hajmi qanday o‘zgaradi?

Yechimi: Kubning qirrasini x bilan, uning o‘zgarishini esa Δx bilan belgilaymiz. U holda $v=x^3$ bo‘lib Δy ni topish kerak bo‘ladi.

Agar yuqorida keltirilgan (1) formuladan foydalansak:

$dv = v'dx = 3x^2 dx$ va

$dx = \Delta x$ ekanligini hisobga olsak:

$\Delta v \approx 3x^2 \Delta x = 3 \cdot 20^2 \cdot 0.1 = 120; \quad \Delta v = 120$ ñì ^{3}

Funksiya orttirmasining haqiqiy qiymatini esa

$$\Delta v = 3 \cdot 20^2 \cdot 0.1 + 3 \cdot 20 \cdot (0.1)^2 + (0.1)^3 = 120 + 0.60 + 0.001 = 120.601$$

Bundagi xato 0.601 ga teng bo‘lib, juda kichik qiymatga ega ekanligi ko‘rinib turibdi.

Misol 2 $\cos 40^\circ = 0.766$ ekanligini bilgan holda jadvaldan foydalanmay $\cos 41^\circ$ ni toping.

Yechimi: $\cos(x + \Delta x) \approx \cos x + d\cos x$ va

$d\cos x \approx -\sin x dx$ yoki $d\cos x = -\sin x \Delta x$

ekanligini hisobga olsak, quyidagilarni hisobga olish kerak bo‘ladi:

$$\sin 40^0 = \sqrt{1 - \cos^2 40^0} = \sqrt{1 - (0.766)^2} =$$

$$= \sqrt{1 - 0.587} = \sqrt{0.413} \approx 0.6426$$

$$\Delta x = 1^0 \approx \frac{3.14}{180} \approx 0.0175$$

O‘rniga qo‘ysak:
$\cos 41^0 \approx 0.766 - 0.6426 \cdot 0.0175 = 0.766 - 0.0112 = 0.7548, \ \cos 41^0 \approx 0.7548$

Bu esa funksiyaning qiymatini jadvaldan topilgan qiymati $\cos 41^0 \approx 0.7548$ orasidagi farqni juda kichik ekanligini ko‘rsatadi.

Misol 3. Differensial yordamida $\sqrt[3]{25}$ ning taqribiy qiymatini hisoblang.

Yechimi: Bu misol yechimiga $y = \sqrt[3]{x}$ funksiya mos keladi. Izlangan ildiz $x = 27$ ga yaqin bo‘lgani uchun $\Delta x = -2$ bo‘ladi.

Bundan

$$\sqrt[3]{x + \Delta x} \approx \sqrt[3]{x} + \left(\sqrt[3]{x}\right)' \Delta x$$

yoki

$$\sqrt[3]{x + \Delta x} \approx \sqrt[3]{x} + \frac{\Delta x}{3\sqrt[3]{x^2}}$$

o‘rniga qo‘ysak:

$$\sqrt[3]{25} \approx \sqrt[3]{27} + \frac{-2}{3\sqrt[3]{27^2}} = 3 - \frac{2}{27} \approx 2.926$$

Demak, $\sqrt[3]{25} \approx 2.926$

Mashqlar

1. Taqribiy qiymatini topilsin:

1. $\sqrt{26}$ 2. $\sqrt{35}$ 3. $\sqrt{148}$ 4. $\sqrt[3]{31}$

5. $\sqrt[4]{80}$ 6. $\sqrt[3]{28}$ 7. $\sqrt[3]{65}$ 8. $\sqrt[5]{33}$

9. $\cos 46^0$ 10. $tg46^0$ 11. $ctg29^0 30'$ 12. $\sin 29^0 30'$

13. $\ln(e+0.1)$ 14. $(3.03)^3$ 15. $\ln 0.97$ 16. $(9.09)^3$

Javoblar: 1) 5.1 2) 5.917 3) 12.167 4) 1.988

5) 2.991 6) 3.037 7) 4.021 8) 2.012

9) 0.695 10) 1.035 11) 1.765 12) 0.492

13) 1.037 14) 255.15 15) -0.03 16) 997

Funksiyalarning taqribiy qiymatlari hisoblansin:
(Javobi qavs ichida berilgan)

1. $y = x^3 + x^2$ $x = 2.01$ bo‘lganda (1216)

2. $y = \dfrac{x}{\sqrt{x^2+16}}$ $x = 2.9$ bo‘lganda (0.587)

3. $y = \sqrt{\dfrac{4-2}{1+x}}$ $x = 3.02$ bo‘lganda (0.494)

4. $y = x^3 - 2x$ $x = 0.02$ bo‘lganda (0.04)

5. $y = x^5 - x^3 + x$ $\quad x = 1.01$ bo‘lganda (1.03)

3. $\ln 85 = 4.4427$ ekanligini bilgan holda jadvalsiz $\ln 86$ ni toping.
Javob: 4.4544

4. To‘g‘ri burchakli uchburchakning katetlari $a = 12$ ñì, $b = 5$ ñì bo‘lib, kichik kateti 0.5 sm orttirilsa, gipotenuza qancha ortadi?
Javob: 0.19 sm

5. Radiusi 20 sm bo‘lgan sharning radiusi 0.0024 sm ga ortsa, uning hajmi qancha ortadi?
Javob: 12.06 sm^3 .

Aniqmasliklarni ochishda Lopital qoidasining qo‘llanilishi

(a, b) oralig‘ida uzluksiz $f(x)$ va $\varphi(x)$ funksiyalar nisbati $\dfrac{f(x)}{\varphi(x)}$ berilgan va shu oraliqda ularning chekli hosilalari $f'(x)$ va $\varphi'(x)$ mavjud bo‘lib, $\varphi'(x) \neq 0$ bo‘lsin.

Agar $x \to a$ da har ikki funksiya cheksiz kichik yoki cheksiz katta miqdordan iborat bo‘lsa, ya’ni $x \to a$ da $\dfrac{f(x)}{\varphi(x)}$ nisbat $\dfrac{0}{0}$ yoki $\dfrac{\infty}{\infty}$ ko‘rinishidagi aniqmaslikdan iborat bo‘lsa,

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \lim_{x \to a} \frac{f'(x)}{\varphi'(x)}$$

bo‘ladi.

(agar hosilalari nisbatining limiti (chekli yoki cheksiz) mavjud bo‘lsa)

Misol 1.

1. $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x^2 - 5x + 4}$

2. $\lim\limits_{x\to 1}\dfrac{x^3-3x+2}{x^3-x^2-x+1}$

3. $\lim\limits_{x\to 0}\dfrac{x^3-6x+6\sin x}{x^5}$

4. $\lim\limits_{x\to 0}\dfrac{tgx-x}{x-\sin x}$

5. $\lim\limits_{x\to \infty}\dfrac{ax^2+b}{cx^2+d}$

6. $\lim\limits_{x\to \infty}\dfrac{x+\ln x}{x}$

Yechimi: 1 – 4 misollarda $\dfrac{0}{0}$ ko‘rinishdagi 5 – 6 misollarda $\dfrac{\infty}{\infty}$ ko‘rinishdagi aniqmaslik mavjud.

Lopital qoidasidan foydalansak:

1. $\lim\limits_{x\to 4}\dfrac{x^2-16}{x^2-5x+4}=\lim\limits_{x\to 4}\dfrac{2x}{2x-5}=\dfrac{2\cdot 4}{2\cdot 4-5}=\dfrac{8}{3}$

2. $\lim\limits_{x\to 1}=\dfrac{x^3-3x+2}{x^3-x^2-x+1}=\lim\limits_{x\to 1}\dfrac{3x^2-3}{3x^2-2x-1}=\lim\limits_{x\to 1}\dfrac{6x}{6x-2}=\dfrac{6}{4}=1.5$

Bu misolni yechishda Lopital qoidasi ikki marta ketma – ket qo‘llanadi.

3. $\lim\limits_{x\to 0}\dfrac{x^3-6x+6\sin x}{x^5}=\lim\limits_{x\to 0}\dfrac{3x^2-6x+6\cos x}{5x^4}=$

$=\lim\limits_{x\to 0}\dfrac{6x-6\sin x}{20x^3}=\lim\limits_{x\to 0}\dfrac{6-6\cos x}{60x^2}=$

$=\lim\limits_{x\to 0}\dfrac{6\sin x}{120x}=\lim\limits_{x\to 0}\dfrac{\cos x}{20x}=\dfrac{\cos 0}{20}=\dfrac{1}{20}$

4. $\lim\limits_{x\to 0}\dfrac{tg-x}{x-\sin x}=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{\cos^2 x}-1}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{1-\cos^2 x}{\cos^2 x(1-\cos x)}=$

$$=\lim_{x\to 0}\frac{1+\cos x}{\cos^2 x}=\frac{1+\cos 0}{\cos^2 0}=\frac{1+1}{1}=2$$

5. $\lim_{x\to\infty}\frac{ax^2+b}{cx^2+d}=\lim_{x\to\infty}\frac{2ax}{2cx}=\lim_{x\to\infty}\frac{2a}{2c}=\frac{a}{c}$

6. $\lim_{x\to\infty}\frac{x+\ln x}{x}=\lim_{x\to\infty}\frac{1+\frac{1}{x}}{1}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1+0=1$

Mashqlar

Limitlarni hisoblang

(Qavs ichida javobi berilgan)

1. $\lim_{x\to 1}\frac{x^3-3x^2+2}{x^3-4x^2+3}$ $\left(\frac{3}{5}\right)$

2. $\lim_{x\to 0}\frac{e^x-e^{-x}}{\ln(1+x)}$ (2)

3. $\lim_{x\to\infty}\frac{\pi-2arctg}{-e^{\frac{3}{x}}-1}$ $\left(\frac{2}{3}\right)$

4. $\lim_{x\to 0}\frac{2-\left(e^x+e^{-x}\right)\cos x}{x^4}$ $\left(\frac{1}{3}\right)$

5. $\lim_{x\to 0}\frac{e^{3x}-3x-1}{\sin^2 5x}$ (0.18)

6. $\lim_{x\to 0}\frac{\sin 3x-3xe^x+3x^2}{arctgx-\sin x-\frac{x^3}{6}}$ (18)

7. $\lim\limits_{x\to 1}\dfrac{x^5+1}{x^3+1}$ $\left(\dfrac{5}{3}\right)$

8. $\lim\limits_{x\to 0}\dfrac{\sin 7x}{\sin 3x}$ $\left(\dfrac{7}{3}\right)$

9. $\lim\limits_{x\to 0}\dfrac{tg2x-2x}{x^3}$ (∞)

10. $\lim\limits_{x\to 0}\dfrac{e^{-3x}-e^{\sin x}}{x}$ (-4)

11. $\lim\limits_{x\to a}\dfrac{\ln(x-a)}{\ln(e^x-e^a)}$ (1)

12. $\lim\limits_{x\to \infty}\dfrac{\ln x}{x^n},\quad (n>0)$ (0)

13. $\lim\limits_{x\to 0}\dfrac{\ln x}{1+2\ln\sin x}$ $\left(\dfrac{1}{2}\right)$

14. $\lim\limits_{x\to 1}\dfrac{tg\frac{\pi}{2}x}{\ln(1-x)}$ (∞)

15. $\lim\limits_{x\to 1}\dfrac{\ln(x-1)}{ctg\pi x}$ (0)

16. $\lim\limits_{x\to 0}\dfrac{\ln x}{x}$ (0)

17. $\lim\limits_{x\to 1}\left(\dfrac{x}{\ln x}-\dfrac{1}{\ln x}\right)$ (1)

18. $\lim\limits_{x\to 0}\dfrac{ctgx}{\ln 2x}$ $(-\infty)$

19. $\lim_{x\to 0}\left(\frac{1}{x}\right)^{tgx}$ (1)

20. $\lim_{x\to 0}\frac{\sin x - x\cos x}{\sin^3 x}$ $\left(\frac{1}{3}\right)$

Funksiyaning o‘sishi va kamayishi oraliqlari

Berilgan $y = f(x)$ funksiyaning (a, b) oraliqda o‘suvchi bo‘lishi uchun uning hosilasini shu oraliqda nomanfiy bo‘lishi yetarli va zarur. Ya’ni $y = f(x)$ funksiyaning hosilasi (a, b) oraliqda musbat bo‘lsa, u shu oraliqda o‘suvchi bo‘ladi. Masalan: $y = x^2 - 3x + 1$ funksiya berilgan bo‘lsin. Uning hosilasi $y' = 2x - 3;$ $x > 2$ bo‘lsa, ya’ni $(2, \infty)$ oraliqda hosila musbat bo‘ladi, shuningdek $(-\infty, 1)$ oraliqda esa funksiya kamayuvchidir.

Funksiyaning o‘sishi va kamayishi oraliqlarini aniqlash uchun quyidagilar bajariladi:

1. Berilgan funksiyaning hosilasi topiladi.
2. Shu hosilani nolga tenglab, uning haqiqiy ildizlari topiladi, (aks holda, haqiqiy ildizlar mavjud bo‘lmasa, funksiya faqat o‘suvchi yoki kamayuvchi bo‘ladi).
3. Topilgan ildizlar yordamida funksiyaning aniqlanish sohasi oraliqlarga ajratiladi. Oraliqlardagi hosilaning ishorasiga qarab uning o‘sishi yoki kamayishi aniqlanadi.

Misol 1. $y = x^3 + 2x - 5$

1. $y' = 3x^2 + 2$
2. $y' = 0$ $\quad 3x^2 + 2 = 0$

$$3x^2 = -2; \quad x^2 = -\frac{2}{3}$$

bu tenglama haqiqiy yechimga ega emas. Haqiqatan ham bu funksiyaning hosilasi x ning istalgan haqiqiy qiymatida musbat bo‘ladi. Binobarin bu funksiyaning aniqlanish sohasi hamma vaqt o‘suvchidir.

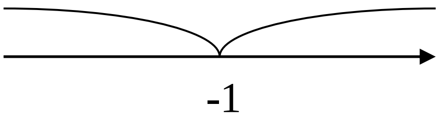
Misol 2. $y = \ln(x^2 + 2x + 3)$

1. $y' = \frac{(x^2 + 2x + 3)'}{x^2 + 2x + 3} = \frac{2x + 2}{x^2 + 2x + 3}$

2. $y' = 0, \quad 2x + 2 = 0 \quad x = -1$

-1 nuqta haqiqiy son o‘qini ikki qismga ajratadi:

$(-\infty,\ 1)$ va $(-1,\ \infty)$



$(-\infty,\ 1)$ oraliqda $y' < 0$ va $(-1,\ \infty)$ oraliqda bo‘lgani uchun berilgan funksiya $(-\infty,\ -1)$ oraliqda kamayuvchi va $(-1,\ \infty)$ oraliqda o‘suvchidir.

Mashqlar

Quyidagi funksiyalarning o‘sish va kamayish oralig‘i aniqlansin:

1. $y = x^2 + x + 1$ — $\left(-\infty,\ -\frac{1}{2}\right)$ da kamayuvchi

   $\left(-\frac{1}{2},\ \infty\right)$ da o‘suvchi

2. $y = 3x - x^2$ — $\left(-\infty,\ \frac{1}{2}\right)$ da o‘suvchi

   $\left(\frac{1}{2},\ \infty\right)$ da kamayuvchi

3. $y = x^3 + 3x^2 + 3x + 1$ — $(-\infty,\ \infty)$ da o‘suvchi

4. $y = 1 - x + 2x^4$ — $\left(-\infty,\ \frac{1}{2}\right)$ da kamayuvchi

   $\left(\frac{1}{2},\ \infty\right)$ da o‘suvchi

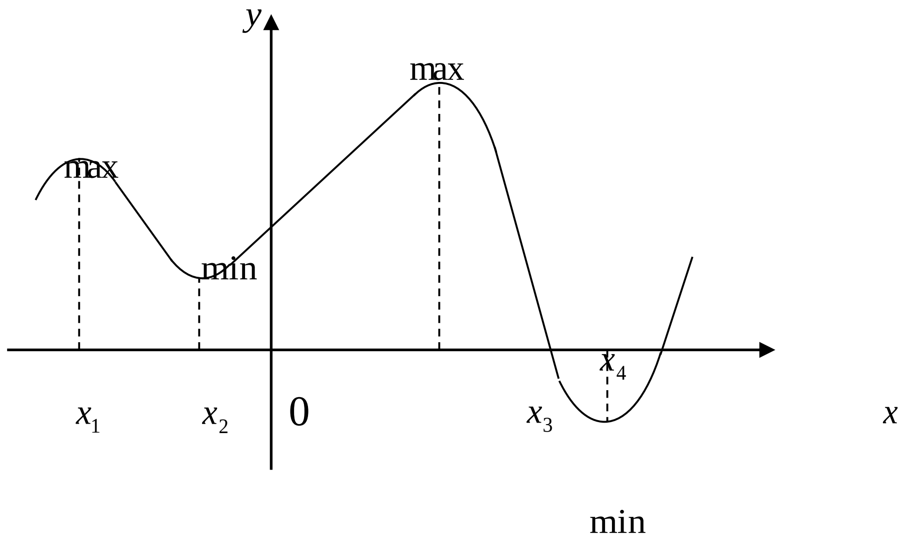
5. $y = \frac{x}{1 + x^2}$ — $(-\infty,\ -1)$ va $(1,\ \infty)$ oraliqda kamayuvchi, $(-1,\ 1)$ da o‘suvchi

6. $y = x^2 e^{-x}$	$(-\infty,\ 0)$ va $(2,\ \infty)$ oraliqlarda kamayuvchi $(0,\ 2)$ da o‘suvchi
7. $y = x \ln x$	$(0,\ e^{-1})$ da kamayuvchi $(e^{-1},\ \infty)$ da o‘suvchi
8. $y = x - e^{x}$	$(-\infty,\ 0)$ da o‘suvchi $(0,\ \infty)$ da kamayuvchi
9. $y = e^{-x^2}$	$(-\infty,\ 0)$ da o‘suvchi $(0,\ \infty)$ da kamayuvchi
10. $y = x + \cos x$	$(-\infty,\ \infty)$ da o‘suvchi

Funksiyaning ekstremumi va uning aniqlanish usullari

$y = f(x)$ funksiya uchun x_0 nuqtaga yetarli yaqin yotuvchi (ammo $x \neq x_0$) barcha nuqtalar uchun $f(x) < f(x_0)$ (yoki $f(x) > f(x_0)$) tengsizligi bajarilsa, shu x nuqtada funksiya maksimumga (yoki minimumga) ega deyiladi.

Funksiyaning maksimum yoki minimumi uning ekstremumi deyiladi.



Agar $y = f(x)$ funksiya x_0 nuqtada ekstremumga ega bo‘lsa, uning shu nuqtadagi hosilasi nolga teng yoki mavjud emas. (funksiyaning ekstremumga ega bo‘lishi mumkin bo‘lgan nuqtalar kritik nuqtalar deyiladi).

Funksiyaning ekstremumga ega bo‘lishini aniqlashda ikki qoida mavjud.

Qoida 1. Agar funksiyaning hosilasi $f'(x)$ kritik nuqtadan o‘tishda, chapdan o‘ngga, o‘z ishorasini musbatdan manfiyga o‘zgartirsa, $y = f(x)$ minimumga ega bo‘ladi; agar ishorasini o‘zgartirmasa ekstremumi yo‘q.

Shunday qilib, berilgan funksiyaning ekstremumini aniqlash uchun quyidagilar bajariladi:

1. Funksiyaning hosilasini nolga tenglab haqiqiy yechim topiladi, ular kritik nuqtalar bo‘ladi. Kritik nuqtalar yordamida funksiyaning aniqlanish sohasi bir necha intervallarga (oraliqlarga) ajraladi. Har bir intervalda funksiya hosilasining ishorasi aniqlanadi.

2. Kritik nuqtadan o‘tish hosilasi ishorasining o‘zgarishiga qarab, uning ekstremumi aniqlanadi.

Qoida 2. $y = f(x)$ funksiyaning x_0 nuqta atrofida chekli hosilasi mavjud va $f'(x_0) = 0$ bo‘lib, shu nuqtaning o‘zida $f''(x)$ mavjud bo‘lsin, u holda funksiya: 1) agar $f''(x_0) > 0$ bo‘lsa, x_0 nuqtada minimum; 2) agar $f''(x_0) < 0$ bo‘lsa, maksimumga ega bo‘ladi; 3) agar $f'(x_0) = 0$ bo‘lganda bu qoida ekstremumni aniqlay olmaydi, shuning uchun 1 – qoidaga murojaat qilinadi.

Misol 1. Quyidagi funksiyalarning ekstremumlari aniqlansin:

1. $f(x) = x^3$ 2. $f(x) = \sqrt[3]{x}$ 3. $y = |x|$

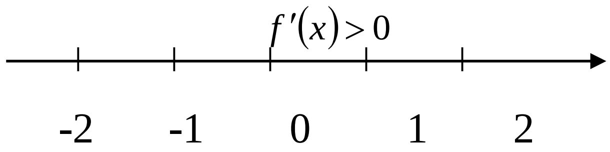
4. $f(x) = \frac{1}{4}x^4 - 2x^2 + 3$ 5. $f(x) = x^3 - 6x^2 + 9x - 4$

6. $y = 3\sqrt[3]{x^2} - x^2$ 7. $\varphi(x) = x\sqrt{2 - x^2}$

Yechimi: 1. Funksiya $(-\infty, \infty)$ oraliqda aniqlansin. Uning hosilasi $f'(x) = 3x^2$; kritik nuqtani aniqlaymiz. Buning uchun $3x^2 = 0$ tenglamani yechamiz: $3x^2=0$; $x_1=0$, $x_2=0$ nuqtadan o‘tishda hosila ishorasini o‘zgartirmaydi.

Masalan: $x=-1$ bo‘lsa, $f'(-1)=3$; $x=0$ va agar $x=1$ bo‘lsa, $f'(1)=3$.

Binobarin funksiya ekstremumga ega emas.



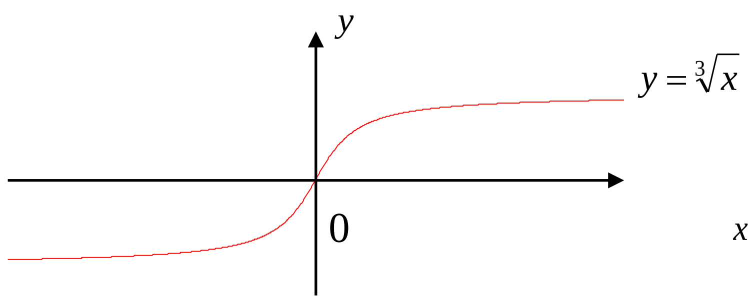
funksiya hamma vaqt o‘suvchidir.

2. Funksiyaning aniqlanish sohasi $(-\infty, \infty)$

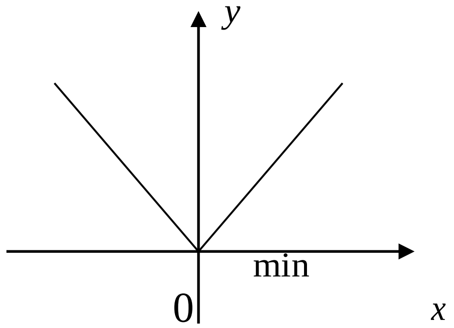
$$f'(x)=\frac{1}{3\sqrt[3]{x^2}}; \qquad f'(x)=0$$

$$\frac{1}{3\sqrt[3]{x^2}}=0$$

Bu tenglama haqiqiy yechimga ega emas. Shunga ko‘ra funksiyaning ekstremumi mavjud emas.



3. Funksiya $(-\infty, \infty)$ oralig‘ida aniqlangan. Funksiya $x=0$ nuqtada funksiyaning hosilasi mavjud bo‘lmasa ham, bu nuqtada funksiya minimumga ega.



4. Funksiya haqiqiy sonlar to‘plamida aniqlangan. Uning hosilasini nolga tenglab, kritik nuqtalarni topamiz:

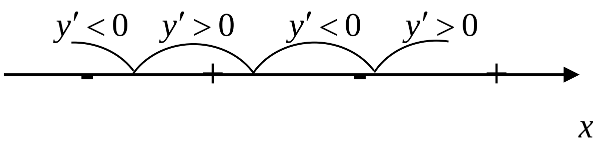
$$f(x)=\frac{1}{4}x^4-2x^2+3$$

$$f'(x)=x^3-4x \qquad x^3-4x=0$$

$$x(x^2-4)=0 \qquad (x+2)\times(x-2)=0$$

Bundan $x=-2;\ x=0;\ x=2$

Uch nuqta bilan son o‘qi to‘rt oraliqqa bo‘linadi:



Har bir oraliqda funksiya hosilasining ifodasini aniqlaymiz:

$(-\infty, -2)$ oraliqda $f'(-3)=(-3)^3-4(-3)=-15<0$

$(-2, 0)$ oraliqda $f'(-1)=(-1)^3-4(-1)=3>0$

$(0, 2)$ oraliqda $f'(1)=1^3-4\cdot 1=-3<0$

$(2, 4)$ oraliqda $f'(3)=3^3-4\cdot 3=15>0$

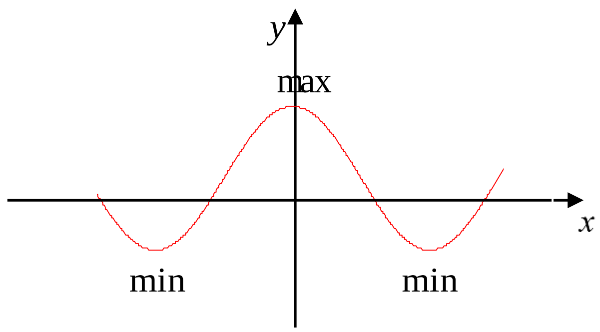
(Oraliqdagi ixtiyoriy haqiqiy sonni tanlab olish mumkin, masalan $(-\infty, -2)$ oraliqda $x<-2$ tanlanishi mumkin).

Kritik nuqtalar: ± 2 dan o‘tish vaqtida hosilaning ishorasi manfiydan musbatga o‘zgargani uchun bu nuqtalarda funksiya minimumga ega. $x=0$ nuqtada esa funksiya maksimumga ega, chunki undan o‘tishda hosila ishorasi musbatdan manfiyga o‘zgaradi. Funksiyaning minimum va maksimum qiymatlarini aniqlaymiz.

$$f(-2)=\frac{1}{4}(-2)^4-2(-2)^2+3=-1=f(2)$$

$$f(0)=\frac{1}{4}\cdot 0-2\cdot 0+3=3$$

Demak, funksiya maksimum (0; 3) nuqtada bo‘lib, minimumlari esa (-2; -1) va (+2; -1) nuqtalarda



5. Funksiyaning aniqlanish sohasi barcha haqiqiy sonlar to‘plamidan iborat. Kritik nuqtalarni topamiz.

$$f'(x)=3x^2-12x+9; \qquad 3x^2-12x+9=0$$

$$x^2-4x+3=0; \qquad x_1=1; \quad x_2=3$$

Ikkinchi qonundan foydalanish uchun ikkinchi tartibli hosilani topamiz:

$$f''(x)=6x-12=6(x-2)$$

$$f''(1)=6(1-2)<0 \text{ maksimum } f(1)=0$$

$$f''(3)=6(3-2)>0 \text{ minimum } f(3)=-4$$

Demak, funksiya (1; 0) nuqtada maksimum (3; -4) nuqtada minimumga ega.

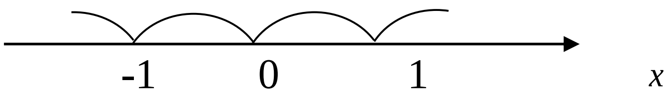
6. Funksiya $(-\infty, \infty)$ oralig‘ida aniqlangan.

$$y'=2\left(\frac{1}{\sqrt[3]{x^2}}-x\right); \qquad y'=0$$

$$\frac{1}{\sqrt[3]{x^2}}-x=0; \qquad 1=x\sqrt[3]{x^2}; \quad 1=x^3$$

$$x_1=-1; \qquad x_2=1$$

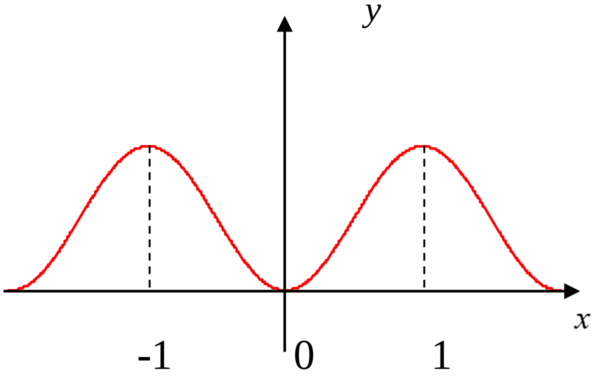
Funksiya hosilasi $x=0$ nuqtada ma’noga эga эmas. Bundan uchinchi kritik nuqtaning 0 эkanligi kelib chiqadi: $x_3=0$. Uchta kritik nuqta atrofida hosilaning ishorasini aniqlab, jadval tuzamiz:



x	-1	0	1
y	2	0	2
y	0		0
Xulosa	+ − max	− + min	+ − max

Demak, funksiya ikki maksimumga va bir minimumga эga:

(-1; 2) va (1; 2) nuqtada maksimum (0; 0) nuqtada minimumga эga:



7. Funksiyaning aniqlanish sohasi $\left[-\sqrt{2};\ \sqrt{2}\right]$ kesmadan iborat,

$$\varphi'(x) = \frac{2\left(1 - x^2\right)}{\sqrt{2 - x^2}}$$

Kritik nuqtalarni topamiz:

$$\frac{2\left(1 - x^2\right)}{\sqrt{2 - x^2}} = 0; \qquad 1 - x^2 = 0; \quad x_1 = -1; \qquad x_2 = 1$$

Funksiya hosilasining mahrajini nolga tenglab yana ikki qiymat topamiz:

$$\sqrt{2 - x^2} = 0 \qquad x^2 = 2$$

$$x = \pm\sqrt{2}$$

Ammo bu nuqtalar funksiyaning aniqlanish sohasidan tashqari bo‘lgani uchun, kritik nuqta bo‘la olmaydi,

$$\varphi''(x)=\frac{2\left[\left(1-x^2\right)'\sqrt{2-x^2}-\left(\sqrt{2-x^2}\right)'\left(1-x^2\right)\right]}{2-x^2}=$$

$$=\frac{21-2x\sqrt{2-x^2}-\dfrac{-2x\left(1-x^2\right)}{2\sqrt{2-x^2}}}{2-x^2}=\frac{2\left(-4x+2x^3+x-x\right)}{\sqrt{2-x^2}\left(2-x^2\right)}=$$

$$=\frac{2\left(x^3-3x\right)}{\left(2-x^2\right)\sqrt{2-x^2}}$$

Ikkinchi qoidadan foydalanamiz:

$$\varphi''(-1)=\frac{2\left[(-1)^3-3(-1)\right]}{\left(2-(-1)^2\right)\sqrt{2-(-1)}}>0 \qquad \text{min}$$

$$\varphi''(1)=\frac{2\left(1^3-3\cdot1\right)}{\left(2-x^2\right)\sqrt{2-1^2}}<0 \qquad \text{max}$$

$$\varphi(-1)=-1, \qquad \varphi(1)=1$$

bo‘lgani uchun funksiya (-1; 1) nuqtada minimumga va (1; 1) nuqta maksimumga эga.

Mashqlar

Quyidagi funksiyalarning эkstremumlari aniqlansin.
(Javobi qavs ichida berilgan)

1. $y=x^2+x+1$ $\left(-\frac{1}{2};\ \frac{3}{4}\right)$ nuqtada min

2. $y=2x^3-3x^2$ $(1;\ -1)$ nuqtada min

$(0;\ 0)$ nuqtada max

3. $y = x^2 + ax + a^2$ — $\left(-\frac{a}{2};\ \frac{3}{4}a^2\right)$ nuqtada min

4. $y = 2x^3 - 6x^2 - 18x + 7$ — $(3;\ -47)$ nuqtada min
$(-1;\ 17)$ nuqtada max

5. $y = \frac{x}{x^2 + x + 1}$ — $(-1;\ 1)$ nuqtada min
$\left(1;\ \frac{1}{3}\right)$ nuqtada max

6. $y = 4x - x^6$ — $(1;\ 3)$ nuqtada max

7. $y = \frac{2}{3}x^2\sqrt[3]{6x-7}$ — $(0;\ 0)$ nuqtada max
$\left(1;\ -\frac{2}{3}\right)$ nuqtada min

8. $y = x^4 + 4x^3 - 2x^2 - 12x + 5$ — $(-1;\ 12)$ nuqtada max
$(1;\ -4)$ va $(-3;\ -4)$ min

9. $y = x^2\sqrt{x^2 + 2}$ — $(0;\ 0)$ nuqtada max

10. $y = 2^{\sin x}$ — $\left(\frac{\pi}{2} + 2k\pi;\ 2\right)$ nuqtada max
$\left(-\frac{\pi}{2} + 2k\pi;\ \frac{1}{2}\right)$ nuqtada min

11. $y = x^3 - 6x^2 + 12x$ — эkstremumga эga эmas.

12. $y = e^x + e^{-x}$ — $(0;\ 2)$ nuqtada min

13. $y = x + \sqrt{x^2 + a^2}$ — экstremumga эga эmas.

14. $y = x\log_7 x$ — $\left(\frac{1}{e};\ -\frac{1}{e}\log_7 x\right)$ nuqtada $\min$

15. $y = \sqrt[3]{x^2} - x$ — $(0;\ 0)$ nuqtada $\min$
$\left(\frac{8}{27};\ \frac{4}{27}\right)$ nuqtada $\max$

16. $y = \frac{\ln^2 x}{x}$ — $(1;\ 0)$ nuqtada $\min$
$\left(e^2;\ e^{-2}\right)$ nuqtada $\max$

17. $y = \sqrt{5 - 4x}$ — экstremumga эga эmas.

18. $y = \frac{x^2}{2} + \frac{1}{x}$ — $\left(1;\ \frac{5}{3}\right)$ nuqtada $\min$

19. $y = \sqrt{x^2 - 5x - 5}$ — экstremumga эga эmas.

20. $f(x) = x - arctgx$ — экstremumga эga эmas.

21. $y = 2x^3 - 3x^2$ — $(0;\ 0)$ nuqtada $\max$
$(1;\ -1)$ nuqtada $\min$

22. $y = 2x^3 - 6x^2 - 18x + 7$ — $(-1;\ 17)$ nuqtada $\max$
$(3;\ -47)$ nuqtada $\min$

23. $y = (x^2 + x + 2)(x^2 + x - 2)$ — $(-1;\ -4)$, $(0;\ -4)$ nuqtada $\min$
$\left(-\frac{1}{2};\ -\frac{63}{16}\right)$ nuqtada $\max$

24. $y = 2x^3 - 3x^2 + 1$ — $(0;\ 1)$ nuqtada $\max$
$(1;\ 0)$ nuqtada $\min$

25. $f(x)=\ln(1+x)-x+\dfrac{x^2}{2}$ экstremumga эga эmas.

26. $f(x)=\dfrac{e^x}{(x+3)^2}$ $\left(-1;\ \dfrac{1}{4e}\right)$ nuqtada min

27. $f(x)=\dfrac{x^3}{(x-2)(x+3)}$ $(\approx 14;\ \approx 5.5)$

$\left(1;\ -\sqrt{19};\ \approx 14\right)$ nuqtada max

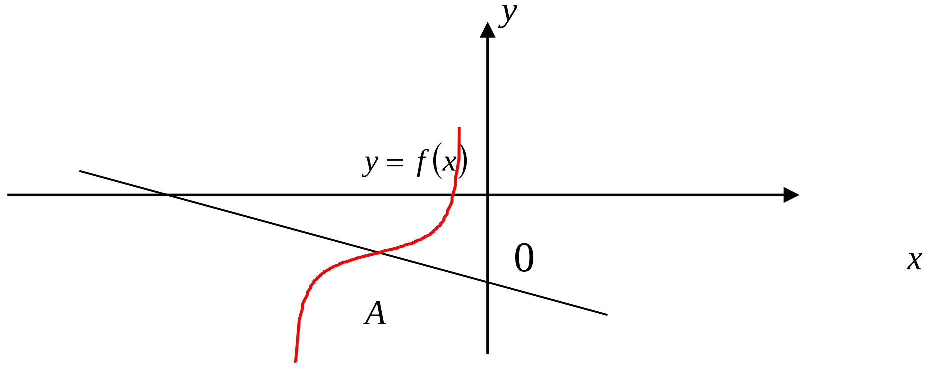
$\left(1;\ +\sqrt{19};\ \approx 5.5\right)$ nuqtada min

Funksiyalar chizmasining botiq va qavariqligi. Burilish nuqtasi.

$(a;b)$ oraliqda funksiya chizmasi o‘ngga o‘tkazilgan urinmalarga nisbatan quyida (ёki юqorida) bo‘lsa, $y=f(x)$ funksiyaning chizmasini shu oraliqda qavariq (botiq) deyiladi.

Agar $(a;b)$ oraliqda funksiyaning ikkinchi tartibli hosilasi mavjud bo‘lsa, uning botiq ёki qavariqligini $f''(x)$ uning ishorasidan bilsak bo‘ladi. Agar $f''(x)<0$ (ёki $f''(x)>0$) bo‘lsa, funksiya chizmasi shu oraliqda qavariq (botiq) bo‘ladi.

Funksiyaning botiqligi qavariqlik bilan almashadigan (ёki aksincha) nuqta burilishi nuqtasi deyiladi.



A – burilish nuqtasi

Funksiyaning burilish nuqtasini aniqlash uchun $f''(x)=0$ tenglamani qanoatlantiradigan x ning qiymatlarini va funksiyaning aniqlanish sohasidan $f''(x)$ funksiya mavjud bo'lganlari aniqlanadi. Bu nuqtalar ikkinchi hil kritik nuqtalar bo'ladi. Kritik nuqtadan o'tish vaqtida ikkinchi tartibli hosila ishorasini almashtirsa, kritik nuqta – burilish nuqtasi bo'ladi.

Misol. Quyidagi funksiyalarning qavariqlikdagi (botiqlik) oraliqlari, burilish nuqtasi topilsin:

1. $y=x^4-6x^2+5$ 2. $y=\sqrt[3]{x^4}$ 3. $y=\sqrt[3]{x}$

Echimi 1: Ikkinchi tartibli hosilani topamiz: $y'=4x^3-12x$

$y''=12x^2-12$ tenglamani echamiz: $y''=0$ $12x^2-12=0$

$$x^2-1=0; \quad x_{1,2}=\pm1; \quad x_1=-1, \quad x_2=+1$$

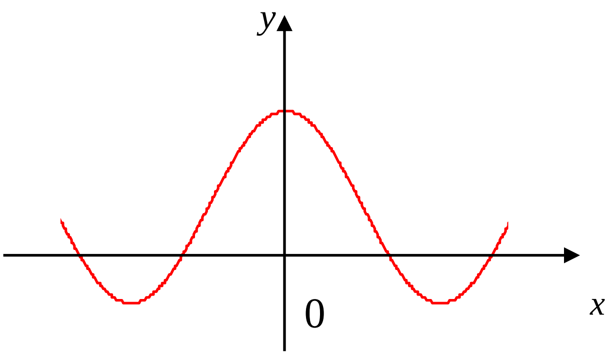
Shu nuqtalar tevaragida y'' ning ishorasini tekshiramiz:

$x<-1$ bo'lganda $y''>0$; $-1<x<1$ bo'lganda $y''<0$ va $x>1$ bo'lganda $y''>0$. Binobarin $(-1;\,0)$ va $(1;\,0)$ nuqtalar burilish nuqtasidir.

$(-\infty;\,-1)$ oraliqda $y''>0$ bo'lgani uchun chizmasi botiq.

$(-1;\,1)$ oraliqda $y''<0$ bo'lgani uchun chizmasi qavariq va

$(1;\,\infty)$ oraliqda $y''>0$ bo'lgani uchun chizmasi botiqdir.



2. $y'=\frac{4}{3}x^{\frac{1}{3}}$; $y''=\frac{4}{9}x^{-\frac{2}{3}}=\frac{4}{9\sqrt[3]{x^2}}$

y'', x ning biror qiymatida nolga aylanmaydi, ammo $x=0$ bo‘lganda y'' ma’noga эga эmas. Shuning uchun $x=0$ nuqta tevaragi y'' ishorasining o‘zgarishini tekshiramiz: agar $x<0$ bo‘lsa, $y''>0$, $x>0$ bo‘lganda $y''>0$. Binobarin funksiyada burilish nuqtasi mavjud эmas, uning chizmasi botiqdir.

3. $y'=\frac{1}{3}x^{-\frac{2}{3}}$; $y''=-\frac{4}{9}x^{-\frac{5}{3}}=\frac{2}{9\sqrt[3]{x^5}}$

Xuddi oldingi misolga o‘xshash $x=0$ nuqta kritik nuqtadir. Ammo $x<0$ bo‘lganda $y''>0$ va $x>0$ bo‘lganda $y''<0$ bo‘lgani uchun $x=0$ nuqta $y=\sqrt[3]{x}$ funksiyaning burilish nuqtasidir. Funksiya $(-\infty;\,0)$ oraliqda qavariq, $(0;\,\infty)$ oraliqda botiq bo‘ladi.

Mashqlar

Quyidagi funksiyaning burilish nuqtalarini chizmadan botiqlik va qavariqlik oraliqlarini aniqlang. (Javobi qavs ichida berilgan)

1. $y=xe^x$ — $(-\infty;\,-2)$ da qavariq va $(-2;\,\infty)$ da botiq $\left(-2;\,-\frac{2}{e^2}\right)$ burilish nuqtasi

2. $y=(x-4)^5+4x+4$ — $(-\infty;\,4)$ da qavariq va $(4;\,\infty)$ da botiq $(4;\,20)$ burilish nuqtasi

3. $y=(x-1)\sqrt[7]{(x-1)^6}$ — $(1;\,0)$ da burilish nuqtasi $(-\infty;\,1)$ da qavariq va $(1;\,\infty)$ da botiq

4. $y=x^3-3x^2+2x$ — $(1;\,0)$ da burilish nuqtasi $(-\infty;\,1)$ da qavariq va $(1;\,\infty)$ da botiq

5. $y=x^4-6x^2$ — $(1;\,-5)$ va $(-1;\,-5)$ burilish nuqtalari bo‘lib, $(-\infty;\,-1)$ va $(1;\,\infty)$ oraliqlarda botiq.

$(-1;1)$ da qavariq

6. $y = x^3 + 1$ — $(0;1)$ burilish nuqtasi
$(-\infty;0)$ da qavariq va $(0;\infty)$ da botiq

7. $y = \ln(1 + x^2)$ — $(1;\ln 2)$ va $(-1;\ln 2)$ burilish nuqtalari
$(-1;1)$ da botiq va $(-\infty;1)$, $(1;\infty)$ da qavariq

8. $y = \dfrac{a^3}{a^2 + x^2}$ — $\left(\dfrac{a}{\sqrt{3}};\dfrac{3a}{4}\right)$, $\left(-\dfrac{a}{\sqrt{3}};\dfrac{3a}{4}\right)$ burilish nuqtalari
$\left(-\dfrac{a}{\sqrt{3}};\dfrac{a}{\sqrt{3}}\right)$ da qavariq

9. $y = \sqrt[3]{x^2}$ — $\left(-\infty;-\dfrac{a}{\sqrt{3}}\right)$, $\left(\dfrac{a}{\sqrt{3}},\infty\right)$ larda botiq
$(-\infty;\infty)$ da qavariq

10. $y = e^{-x^2}$ — $\left(\dfrac{\sqrt{2}}{2},\dfrac{1}{\sqrt{e}}\right)$ va $\left(-\dfrac{\sqrt{2}}{2},\dfrac{1}{\sqrt{e}}\right)$ burilish nuqtalari bo‘lib,
$\left(-\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$ da qavariq
$\left(-\infty;-\dfrac{\sqrt{2}}{2}\right)$; $\left(\dfrac{\sqrt{2}}{2};\infty\right)$ larda botiq.

Asimptotalar

Agar $y = f(x)$ funksiya chizmasidagi $M(x,y)$ nuqtadan biror to‘g‘ri chiziqdagi ℓ nuqtagacha bo‘lgan masofa, shu nuqta эgri chiziq bo‘yicha koordinata boshidan cheksiz uzoqlashish bilan, nolga intilsa bu to‘g‘ri chiziqni эgri chiziq $y = f(x)$ning asimptotasi deyiladi.

Agar $\lim\limits_{x\to\alpha} f(x) = +\infty$ ёki $\lim\limits_{x\to\alpha} f(x) = -\infty$ bo‘lsa, $x = \alpha$ to‘g‘ri chizig‘i $y = f(x)$ funksiyaning vertikal asimptotasi bo‘ladi.

Agar $\lim\limits_{x\to\infty} f(x)=b$ ёki $\lim\limits_{x\to\infty} f(x)=-b$ limit mavjud bo‘lsa, $y=b$ to‘g‘ri chizig‘ini $y=f(x)$ funksiyaning gorizontal asimptotasi deyiladi.

Agar $y=f(x)$ funksiya uchun $k=\lim\limits_{x\to\infty}\frac{f(x)}{x}$; $b=\lim\limits_{x\to\infty}[f(x)-kx]$ ёki $k=\lim\limits_{x\to-\infty}\frac{f(x)}{x}$ va $\lim\limits_{x\to-\infty}[f(x)-kx]$ limitlar mavjud bo‘lsa, $y=kx+b$ funksiyaning og‘ma asimptotasi bo‘ladi.

Misol 1. $y=\sqrt{\frac{x^3}{x-2}}$ эgri chiziqning asimptotalari topilsin.

Echimi: funksiya $(-\infty, 0)$ va $(2, +\infty)$ oraliqlarda aniqlangan. $\lim\limits_{x\to 2}\sqrt{\frac{x^3}{x-2}}=\infty$ bo‘lgani uchun $x=2$ funksiya uchun vertikal asimptota bo‘ladi. $\lim\limits_{x\to+\infty}\sqrt{\frac{x^3}{x-2}}$ chekli qiymatga эga bo‘lmagani uchun gorizontal asimptota mavjud эmas.

Og‘ma asimptotalar borligini tekshiramiz:

$$1.\ \hat{E}=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\sqrt{\frac{x^3}{x-2}}}{x}=\lim_{x\to\infty}\sqrt{\frac{x}{x-2}}=\lim_{x\to\infty}\sqrt{\frac{1}{1-\frac{2}{x}}}=1.$$

$$b_1\lim_{x\to\infty}[f(x)-kx]=\lim_{x\to\infty}\left[\sqrt{\frac{x3}{x-2}}-x\right]=\lim_{x\to\infty}\frac{x(\sqrt{x}-\sqrt{x-2})}{\sqrt{x-2}}=$$

$$=\lim_{x\to\infty}\frac{x(x-x+2)}{\sqrt{x-2}(\sqrt{x}+\sqrt{x-2})}=\lim_{x\to\infty}\frac{2}{\sqrt{1-\frac{2}{x}}\left(1+\sqrt{1-\frac{2}{x}}\right)}=1.$$

Binobarin $y=x+1$ og‘ma asimptota.

2. Xuddi oldingiga o‘xshash

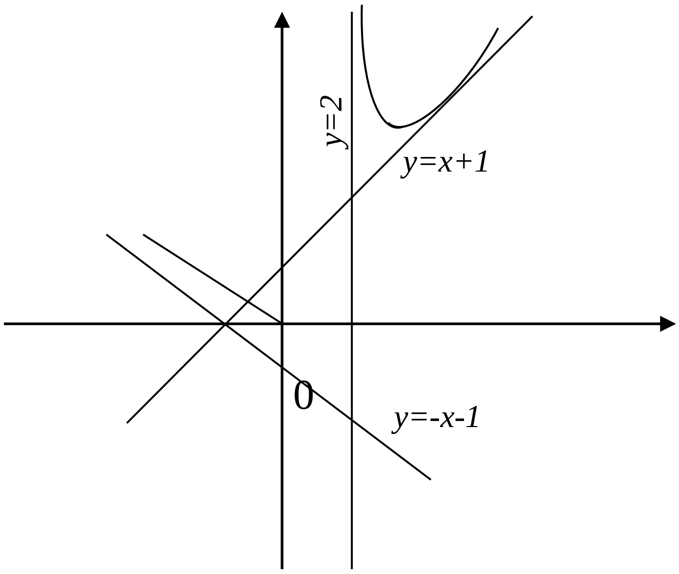
$$\hat{E}_2=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\sqrt{\frac{x^3}{x-2}}}{x}=\lim_{x\to\infty}\frac{\sqrt{\frac{x}{x-2}}}{-1}=-1$$

(Kasrning surat va mahrajini musbat son “x” ga bo‘ldik).

$$\hat{A}_2 = \lim_{x\to\infty}[f(x)-kx] = \lim_{x\to\infty}\left[\sqrt{\frac{x^3}{x-2}}+x\right] = \lim_{x\to\infty}\left[\sqrt{\frac{x^3}{2-x}}+x\right] = -1$$

demak, $y = x-1$ ikkinchi og‘ma asimptota bo‘ladi.

Yuqoridagilarga asosan эgri chiziq chizmasini chizamiz.



Misol 2. Quyidagi эgri chiziqlarning asimptotalari topilsin.

1. $y = \frac{1}{\tilde{o}-2}$; 2. $y = arctgx$;

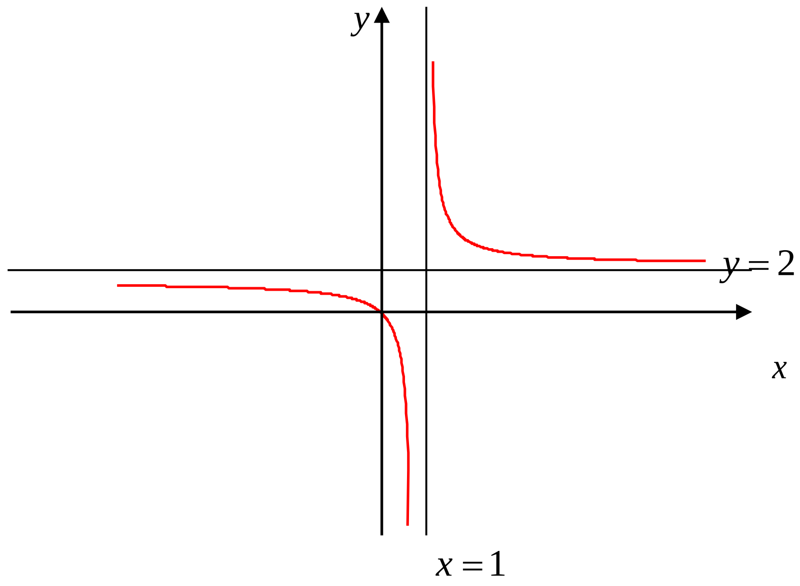
3. $f(x) = \frac{2x}{x-1}$; 4. $y = -e^{\frac{1}{x}}$.

Echimi: 1. $\lim_{x\to\infty}\frac{1}{x-2} = 0$ bo‘lgani uchun эgri chiziq $y = 0$ ya’ni Ox o‘qidan iborat gorizontal asiptotaga эga.

$\lim_{x\to 2}\frac{1}{x-2} = \infty$ bo‘lgani uchun $x = 2$ vertikal asimptota bo‘ladi.

2. Эgri chiziq ikki gorizontal asimtotaga эga: $y=\frac{\pi}{2}$; va $y=-\frac{\pi}{2}$ chunki $\lim\limits_{x\to\infty} arctgx=\frac{\pi}{2}$ va $\lim\limits_{x\to-\infty} arctgx=-\frac{\pi}{2}$ vertikal asimptota mavjud эmas.

3. $\lim\limits_{x\to 1-0}\frac{2x}{x-1}=-\infty$ va $\lim\limits_{x\to 1+0}\frac{2x}{x-1}=\infty$ bo‘lgani uchun $x=1$ vertikal asimptota bo‘ladi, $\lim\limits_{x\to\infty}\frac{2x}{x-1}=2$ bo‘lgani uchun $y=2$ gorizontal bo‘ladi.



4. Gorizontal asimptotani topamiz.

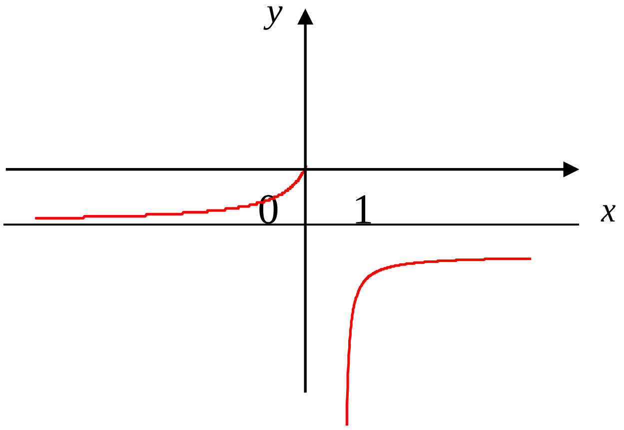
$\lim\limits_{x\to\infty}\left(-e^{\frac{1}{x}}\right)=-1$ ya’ni $y=-1$

Vertikal asimptotani topamiz:

$$\lim_{x\to 0}\left(-e^{\frac{1}{x}}\right)=-\lim_{t\to\infty}e^{t}=-\infty$$

$$\lim_{x\to 0}\left(-e^{\frac{1}{x}}\right)=-\lim_{t\to-\infty}e^{t}=0$$

Binobarin $x=0$ chiziq, ya’ni kordinata o‘qi vertikal asimptotadir



Mashqlar

Quyidagi funksiyaning asimptotalari topilsin:
(Javob qavs ichida berilgan.)

1. $y=\dfrac{3\tilde{o}}{\tilde{o}-2}$; $y=3$ gorizontal asimptota

2. $y=\dfrac{\tilde{o}}{2\tilde{o}-1}+\tilde{o}$; $y=2$ vertikal asimptota

$x=\dfrac{1}{2}$ vertikal asimptota

3. $y=\dfrac{1-\tilde{o}^2}{\tilde{o}^2}$; $y=-1$ gorizontal asimptota

$x=0$ vertikal asimptota

4. $y=\tilde{o}+\dfrac{1}{\tilde{o}^2}$; $x=0$ vertikal asimptota

5. $\dfrac{\tilde{o}^2}{16}-\dfrac{y^2}{25}=1$; asimptotalarga эga эmas.

6. $y=\ln(x-1)$. $x=1$ vertikal asimptota

7. $f(x)=\dfrac{x^2+1}{x^2-4}$ $y=1$ gorizontal asimptota

$x=2$ va $x=-2$ vertikal asimptotalar

8. $y = \sqrt{\dfrac{x^3}{x+2}}$; $\quad x = -2$ vertikal asimptota

9. $y = x^2 e^{-x}$ $\quad y = 0$ gorizontal asimptota

10. $y = \dfrac{\tilde{o}^2 - 2x + 3}{\tilde{o} + 2}$. $\quad x = -2$ vertikal asimptota.

Funksiya chizmasini chizish

Funksiya chizmasini chizishda quyidagilarni aniqlash kerak:

1. Funksiyani aniqlanish sohasi.
2. Funksiyaning juftligi (toqligi), davriyligi.
3. Uzilish nuqtalari.
4. Chizmani kordinata o‘qlari bilan kesishish nuqtalari.
5. Funksiyaning экstremumini aniqlash.
6. Funksiyaning burilish nuqtasi, botiqligi.
7. Funksiyaning o‘sish va kamayish oraliqlari.
8. Asimptotalarning mavjudligi.
9. Funksiyaning aniqlanish soha chegaralariga yaqinlashgandagi limitning qiymati.

Bu ishlarni istalgan tartibda bajarish mumkin. Yana bir necha qo‘shimcha nuqtalar topish bilan funksiya chizmasini chizish mumkin.

Misol: Quiyidagi funksiyaning chizmasini chizing:

$$y = \left(\frac{\tilde{o} + 1}{\tilde{o} - 1} \right)^2$$

Echimi: 1. Funksiya $x = 1$ nuqtadan boshqa barcha haqiqiy sonlar to‘plamida aniqlangan. Binobarin bu nuqta uzilish nuqtasi va $x = 1$ to‘g‘ri chizig‘i vertikal asimptotadan iborat.

2. Funksiya juft ham, toq ham эmas.

3. Gorizontal asimptota borligini tekshiramiz.

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)=\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^2}{\left(1-\frac{1}{x}\right)^2}=1$$

Demak, $y=1$ gorizontal asimptota экan.

4. $\left(\frac{x+1}{x-1}\right)^2=0$ ning echimini topamiz: $x=-1$

Demak, chizma Ox o‘qiga $x=-1$ nuqtada urinadi, chunki funksiyani aniqlash sohasida musbat bo‘lgani uchun, chizmasi Ox o‘qining юqori qismiga joylashadi.

5. Funksiya экstremumini aniqlaymiz:

$$y'=-\frac{4(\tilde{o}+1)}{(\tilde{o}-1)^2}; \qquad y''=\frac{8(\tilde{o}+2)}{(\tilde{o}-1)^4}$$

$y'=0$ desak, ya’ni $-\frac{4(\tilde{o}+1)}{(\tilde{o}-1)^2}=0, \quad x=-1$

Bitta kritik nuqta mavjud экan.

$y''_{x=-1}=\frac{8(-1+2)}{(-1-1)^4}=0$ (-1; 0) bo‘lgani uchun bu nuqtada funksiya minimumga эga.

$$y_{\min}=\left(\frac{-1+1}{-1-1}\right)^2=0 \qquad (-1,\ 0)$$

Funksiyaning o‘sish va kamayish oraliqlarini aniqlash uchun birinchi tartibli hosilaning ishorasini belgilaymiz: $(-\infty,-1)$ va $(1,\infty)$ oraliqlarda $\acute{o}'<0$ binobarin kamaюvchi, (-1; 1) oralig‘ida эsa $\acute{o}'>0$ bo‘lgani uchun funksiya o‘suvchi.

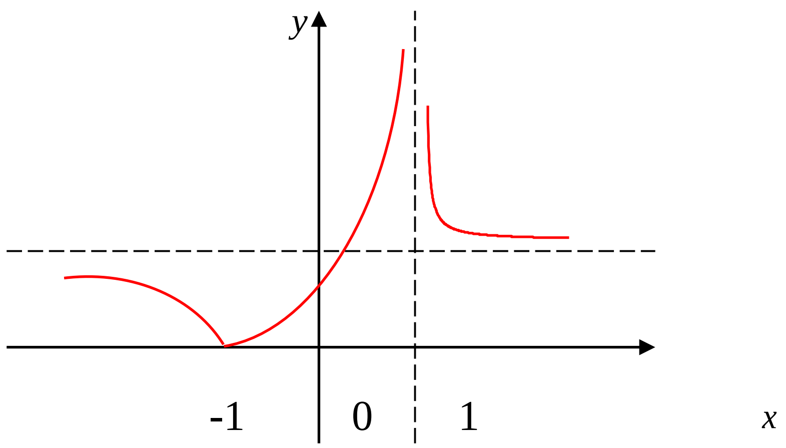
6. Chizmaning botiqlik va qavariqligini, burilish nuqtasini topamiz. $y''=0$ bo‘lsin. U vaqtda $\frac{8(x+2)}{(x-1)^4}=0,\ x=-2$

$x<-2$ bo‘lganda $y''<0$

$1>x>-2$ bo‘lganda $y''>0$

$x>1$ bo‘lganda $y''<0$ bo‘lgani uchun

$(-\infty,-2)$ oralig‘ida chizma qavariq va (-2, 1) va (1, ∞) oraliqlarida botiq bo‘lib $x=2$ burilish nuqtasidir. Yuqoridagilarga asosan funksiya chizmasini chizamiz:



Mashqlar

Quyidagi funksiyalar chizmasini chizing.

1. $y=3\tilde{o}^4+2\tilde{o}^2-5$

2. $y=\tilde{o}^5$

3. $y=\tilde{o}+\dfrac{1}{\tilde{o}}$

4. $y=\dfrac{1}{1-\tilde{o}^2}$

5. $y=\dfrac{3\tilde{o}^2-4}{\tilde{o}^2+2\tilde{o}}$

6. $y=\dfrac{\tilde{o}}{\tilde{o}^2+1}$

7. $y=\dfrac{1}{\tilde{o}}e^x$

8. $y = \tilde{o}^3 e^{-\tilde{o}}$

9. $y = e^{2x - x^2}$

10. $y = \ln\left(x + \sqrt{\alpha^2 + x^2}\right)$, $\quad (a > 0)$

Funksiyaning эng katta va kichik qiymatlari

Berilgan $y = f(x)$ fukцiyaning (a, b) oralig‘idagi эng katta qiymatini topish uchun shu oraliqda funksiyaning maksimum qiymatini va oraliq chegaralaridagi qiymatlarni xisoblab, ulardan эng kattasini tanlanadi, shu qiymat funksiyaning эng katta qiymati bo‘ladi.

Misol: Funksiyalarning ko‘rsatilgan oraliqdagi эng katta va эng kichik qiymatlari topilsin.

1. $y = \tilde{o}^4 - 8\tilde{o}^2 + 3$ $\qquad [-2, 2]$ kesmada

2. $y = tgx - x$ $\qquad \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ kesmada

Echim: 1. Kritik nuqtalarni topamiz.

$y' = 4\tilde{o}^3 - 16\tilde{o} \qquad 4x\left(x^2 - 4\right) = 0$

$x = 0, \quad x_2 = 2, \quad x_3 = -2.$

$\acute{o}'' = 12\tilde{o}^2 - 16, \; y''_{x=\pm 2} = 12(\pm 2)^2 - 16 = 32 > 0$ min

$y''_{x=0} = 12 \cdot 0 - 16 = -16 < 0$ max

$y_{\min} = (\pm 2)^4 - 8(\pm 2)^2 + 3 = -13$

$y_{\max} = 0 - 8 \cdot 0 + 3 = 3.$

Demak $[-2, 2]$ kesmada funksiyaning эng katta qiymati 3 va эng kichik qiymati – 13 эkan.

2. $y' = \frac{1}{\cos^2 x} - 1 = \frac{1 - \cos^2 x}{\cos^2 x}, \quad \frac{\sin^2 x}{\cos^2 x} = tg^2 x.$

$y' = 0, \; tg^2 x = 0, \; tgx = 0, \; x = n\pi$

$y'' = 2tgx \cdot \frac{1}{\cos^2 x} = 0$ aniq эmas. Ammo funksiya $y' > 0$ bo‘lgani uchun o‘suvchi. $y_{x=\frac{\pi}{4}} = tg\left(-\frac{\pi}{4}\right) + \frac{\pi}{4} = \frac{\pi}{4} - 1$

$$y_{\delta=\frac{\pi}{4}} = tg\frac{\pi}{4} - \frac{\pi}{4} = 1 - \frac{\pi}{4}$$

Javob: 10 va 10

3. Цilindrlarning radiusi R va balandligi H orasidagi munosabat qanday bo‘lganda, uning to‘liq sirti эng kichik bo‘ladi?

Javob: $H = 2R$

4. Berilgan perimetrga эga bo‘lgan to‘g‘ri to‘rtburchaklar orasida, qandayining diagonali kichik bo‘ladi?

Javob: kvadrat

5. Radiusi R bo‘lgan shar ichiga chizilgan цilindrning hajmi эng katta bo‘lishi uchun uning balandligi qanday bo‘lishi kerak?

Javob: $\frac{2R\sqrt{3}}{3}$

6. $x^2 + px + q$ uchxadning $x = 2$ bo‘lganda 1ga teng minimumga эga bo‘lishi uchun p va q lar qanday son bo‘lishi kerak?

7. Asosi ℓ va balandligi h bo‘lgan uchburchakka ichki chizilgan to‘g‘ri to‘rtburchaklardan эng katta юzaligini aniqlang.

Javob: To‘g‘ri to‘rtburchakning balandligi $\frac{h}{2}$

8. Radiusi R bo‘lgan doiradan qanday sektor qirqib tashlanganda, qolgan qismidan эng katta hajmli voronka yasab bo‘ladi?

Javob: $\alpha = 2\pi\left(1 - \sqrt{\frac{1}{8}}\right)$

9. To‘g‘ri doiraviy цilindr shaklidagi jism юqori tomondan yarim shar bilan cheklangan bo‘lib, hajmi V ga teng. Uning o‘lchamlari qanday bo‘lganda, uning to‘la sirti эng kichik bo‘ladi?

10. Berilgan hajmga эga bo‘lgankonus shaklidagi chaylaga, asosining radiusidan balandligi 2 marta katta bo‘lganda эng kam material ketishini isbotlang.

11. Deraza юqori tomoni yarim doiradan iborat to‘g‘ri to‘rtburchakdan iborat. Uning perimetri P ga teng. Deraza o‘lchamlari qanday bo‘lganda u эng ko‘p ёrug‘lik o‘tkazadi?

Javob: doiraning radiusi bilan deraza balandligi o‘zaro teng bo‘lganda.

12. Temir yo‘ldan 60 km masofada A punkti bor. Temir yo‘ldagi A punktiga эng yaqin joylashgan C stanцiyadan B stanцiyagacha эng qisqa muddatda etib kelish uchun stanцiyani C dan qancha uzoqlikka qurish kerak. Harakat tezligi quruqlikda 20 km soat, temir yo‘l bo‘yicha эsa 52 km soat.

Javob: 25 km.

13. $\frac{x^2}{25} + \frac{y^2}{9} = 1$ эllipsga ichki chizilgan эng katta юzada to‘g‘ri to‘rtburchakning tomonlari topilsin.

Javob: $5\sqrt{2}$ va $3\sqrt{2}$

14. Shar radiusi R ga teng. Unga ichki chizilgan цilindrlar ichida эng katta ёn sirtga эga bo‘lganini aniqlang.

Javob: $H = R\sqrt{2}$

(Цilindrning o‘q kesimi - kvadrat)

15. R radiusli sharga ichki chizilgan barcha konuslar ichida hajmi эng katta bo‘lganini toping.

Javob: Í $= \frac{4R}{3}$

IV – BOB

BIR NECHA O‘ZGARUVCHI FUNKSIYANING HOSILASI VA DIFFERENSIALI

1 - §. Birinchi tartibli xususiy hosila

$y = f(x,t)$ funksiyada t ni o‘zgarmas deb qarab, undan x bo‘yicha olingan hosila $y-$ni $x-$bo‘yicha xususiy hosilasi deyiladi, va $\frac{\partial y}{\partial x}$ ëki $f'_x(x,\, t)$ ko‘rinishda belgilanadi.

$$\frac{\partial y}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,t) - f(x,t)}{\Delta x} = f'_x(x,\, t)$$

$y-$ni t bo‘yicha xususiy hosilasi ham shunday ta’riflanadi va belgilanadi.

$$\frac{\partial y}{\partial t} = f'_t(x,\, t)$$

$$\frac{\partial y}{\partial t} = \lim_{\Delta t \to 0} \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t} = f'_t(x,\, t)$$

xususiy hosilaga aytiladi. Bu ta’rifni uchta, to‘rtta v.h. o‘zgaruvchi funksiyasiga ham tadbiq qilsak bo‘ladi.

Xususiy hosilalar uchun differensiallashning oddiy qoida va formulalari o‘rinlidir.

1. $y = x^3 - 5xt + 6t^2 + 3t + 3$; $\frac{\partial y}{\partial x}$, $\frac{\partial y}{\partial t}$ toping.

Echimi: t ni doimiy kattalik deb, $\frac{\partial y}{\partial x} = 3x^2 - 5t$ topamiz.

x - ni doimiy kattalik deb, $\frac{\partial y}{\partial t} = 5x + 12t + 3$ topamiz.

2. $y = e^{x^3} + t^6$; $\frac{\partial y}{\partial x}$, $\frac{\partial y}{\partial t}$ toping.

Echimi: t ni doimiy kattalik deb, $\frac{\partial y}{\partial x} = e^{x^3}\left(x^3\right)' = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$ topamiz.

x - ni doimiy kattalik deb, $\frac{\partial y}{\partial t}=6t^5$ topamiz.

3. $\varphi=u^4\cos^2 x$, $\frac{\partial\varphi}{\partial\varphi}$, $\frac{\partial\varphi}{\partial x}$ ni toping.

Echimi: u - ni doimiy kattalik deb, $\frac{\partial\varphi}{\partial x}=u^4 2\cos x(-\sin x)=-u^4\sin 2x$ topamiz.

x - ni doimiy kattalik deb, $\frac{\partial\varphi}{\partial u}=4u^3\cos^2 x$ topamiz.

4. $\varphi=(x^3+1)\cdot\ln^2 3t^2$; $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial t}$ ni toping.

Echimi: $\frac{\partial\varphi}{\partial x}=3x^2\ln^2 3t^2$;

$$\frac{\partial\varphi}{\partial t}=(x^3+1)2\ln 3t^2\cdot\frac{6t}{3t^2}=4(x^3+1)\frac{\ln 3t^2}{t}$$

5. $\varphi=\arcsin 3x\cdot\sqrt{u}$; $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial u}$ ni toping.

Echimi: $\frac{\partial\varphi}{\partial x}=\frac{3}{\sqrt{1-9x^2}}\sqrt{u}$; $\frac{\partial\varphi}{\partial u}=\arcsin 3x\cdot\frac{u'}{2\sqrt{u}}$

6. $\varphi=u^6 tgx$; $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial u}$ ni toping.

$\frac{\partial\varphi}{\partial x}=u^6\frac{1}{\cos^2 x}$; $\frac{\partial\varphi}{\partial u}=6u^5 tgx$

7. $u=x^3+3x\cdot y^4+y^5+3$, $\frac{\partial u}{\partial u}$, $\frac{\partial\varphi}{\partial y}$ ni toping.

8. $u=\sqrt{x}+2e^{xy}\ln x+2$, $\frac{\partial u}{\partial u}$, $\frac{\partial\varphi}{\partial y}$ ni toping.

9. $\varphi=\rho^6\sin^3 x$, $\frac{\partial\varphi}{\partial\rho}$, $\frac{\partial\varphi}{\partial x}$ ni toping.

10. $\varphi=e^{\sin^3 x}\arccos 2\alpha$, $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial\alpha}$ ni toping.

11. $z=a^{2x^2y}$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ ni toping.

12. $z=\left(x^4+2y\right)\sqrt{x^3}$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ ni toping.

13. $z=3y^2\sqrt{x}+5t^3\sqrt[5]{y^2}$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial z}{\partial t}$ ni toping.

14. $u=e^{\frac{1}{y^2}}+e^{\frac{1}{x^2}}$, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial z}$ ni toping.

15. $z=\arcsin\frac{x^2}{1+y^3}$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ ni toping.

16. $z=\left(x^5+1\right)^2\ln^3\left(y^2+1\right)$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ ni toping.

17. $z=a^{\left(x^4+y^3\right)^2}$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ ni toping.

18. $u=\rho^3\cos^2 3x$, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial\rho}$ ni toping.

19. $u=e^{\sin^2 x}\ln\left(y^3+1\right)$, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ ni toping.

20. $u=a^3\sin^4\varphi$, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial\varphi}$ ni toping.

21. $\varphi = e^{tgx} \arccos 3\alpha$, $\frac{\partial \varphi}{\partial x}, \ \frac{\partial \varphi}{\partial \alpha}$ ni toping.

22. $\varphi = z^3 arctg(x^3 + y^4)$, $\frac{\partial \varphi}{\partial x}, \ \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}$ ni toping.

23. $\varphi = (x^4 + 1)^3 \cos^2 5\alpha$, $\frac{\partial \varphi}{\partial x}, \ \frac{\partial \varphi}{\partial \alpha}$ ni toping.

24. $\varphi = 2x^5 \sin(x^3 + y^2)$, $\frac{\partial \varphi}{\partial x}, \ \frac{\partial \varphi}{\partial y}$ ni toping.

25. $\varphi = (x^3 + y^2 + 3) \cdot \sqrt[5]{z^3}$, $\frac{\partial \varphi}{\partial x}, \ \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}$ ni toping.

26. $\varphi = e^{x^3 \sin\alpha} \ln\left(x^3 + y\right)$, $\frac{\partial \varphi}{\partial x}, \ \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial \alpha}$ ni toping.

27. $\varphi = 2x^3 y + 3y^3 + 6$, $\frac{\partial \varphi}{\partial x}, \ \frac{\partial \varphi}{\partial y}$ ni toping.

28. $\varphi = \frac{x^2 + y}{\sin 3x}$, $\frac{\partial \varphi}{\partial x}, \ \frac{\partial \varphi}{\partial y}$ ni toping.

29. $z = \rho^3 \sin^3 x + y^4 \cos\left(x^3 + y^2\right)$, $\frac{\partial z}{\partial x}, \ \frac{\partial z}{\partial y}, \frac{\partial z}{\partial \rho}$ ni toping.

30. $z = \frac{a^{x^2 + y^2} \ln x}{\sin^3 \alpha}$, $\frac{\partial z}{\partial x}, \ \frac{\partial z}{\partial y}, \frac{\partial z}{\partial \alpha}$ ni toping.

2-§. To‘la differensial

Berilgan $y = f(x,\ y)$ funksiyaning $A(x,\ y)$ nuqtadagi to‘la orttirmasi deb quyidagi $\Delta\varphi = f(x+\Delta x,\ y+\Delta y) - f(x,\ y)$ayirmaga aytiladi. Bu erdagi Δx, Δy funksiya argumentining (o‘zgaruvchisining) ixtiёriy orttirmasidir. $\varphi = f(x,\ y)$ funksiya $(x,\ y)$ nuqtada differensiallanuvchi deyiladi, agar nuqtada uning to‘la orttirmasini quyidagicha $\Delta\varphi = A\Delta x + B\Delta y + O(\rho)$ (bu erda $\rho = \sqrt{\Delta x^2 + \Delta y^2}$) kabi tasavvur qila olsak.

To‘la differensial deb, to‘la orttirmaning bosh qismi bo‘lgan va $\Delta x\ \Delta y$ argument orttirmalariga chiziqli bo‘lgan $dz = A\Delta x + B\Delta y$ tenglikka aytiladi.

Bir – biriga bog‘liq bo‘lmagan o‘zgaruvchilarning differensiali, ularning orttirmalari bilan mos keladi, ya’ni:

$$dx = \Delta x,\ \ dy = \Delta y$$

$\varphi = f(x,\ y)$ funksiyaning to‘la differensiali

$$d\varphi = \frac{\partial\varphi}{dx}dx + \frac{\partial\varphi}{dy}dy$$

formula ёrdamida hisoblanadi. Shunga o‘xshash uch o‘zgaruvchiga (argumentga) эga funksiyalarning ham $\varphi = f(x,\ y,\ z)$ to‘la differensiali

$$d\varphi = \frac{\partial\varphi}{dx}dx + \frac{\partial\varphi}{dy}dy + \frac{\partial\varphi}{dz}dz$$

formula orqali hisoblanadi.

$\rho = \sqrt{\Delta x^2 + \Delta y^2}$ ni kichik qiymatlarida, $\varphi = f(x,\ y)$ differensiallashuvchi funksiyalar uchun quyidagi tahminiy tenglik mos keladi.

$$\Delta\varphi \approx d\varphi \qquad f(x+\Delta x,\ y+\Delta y) \approx f(x,\ y) + dz$$

1. $z = arctg(x^3 + y)$ $\qquad$ $dz -$ni toping

$$\frac{\partial z}{\partial x} = \frac{(x^3+y)'_x}{1+(x^3+y)^2} = \frac{3x^2}{1+(x^3+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x^3+y)'_y}{1+(x^3+y)^2} = \frac{1}{1+(x^3+y)^2}$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{3x^2}{1+\left(x^3+y\right)^2}dx+\frac{1}{1+\left(x^3+y\right)}dy=$$

$$=\frac{3x^2dx+dy}{1+\left(x^3+y\right)^2}$$

2. $\varphi=x^{2y^3z^2}$ $\qquad$ $d\varphi-$ni toping.

Yechish:

$$d\varphi=\frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy+\frac{\partial\varphi}{\partial z}dz$$

$$\frac{\partial\varphi}{\partial x}=2y^3z^2x^{2y^3z^2-1}$$

$$\frac{\partial\varphi}{\partial y}=x^{2y^3z^2}\ln x\cdot 6y^2z^2$$

$$\frac{\partial\varphi}{\partial z}=x^{2y^3z^2}\ln x\cdot 4y^3z$$

$$d\varphi=2y^3z^2\cdot x^{2y^3z^2-1}dx+x^{2y^3z^2}\ln x\left(6y^2zdy+4y^3zdz\right)$$

3. $arctg(1,02/0.95)$ bu funksiyani, $z=arctgu$ $x=1,$ $y=1$ qiymatlarida taxminiy hisoblang.

Yechish: $x=1,$ $y=1$ qiymatlarida funksiyani qiymati

$$z=arctg(1/1)=\frac{\pi}{4}\cong 0.785 \qquad \text{teng} \qquad \Delta x=-0.05$$

$\Delta y=-0.02$ da funksiyaning orttirmasi:

$$\Delta z\cong dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y=-\frac{y\Delta x}{x^2+y^2}+\frac{x\Delta y}{x^2+y^2}=$$

$$=\frac{x\Delta y+y\Delta x}{x^2+y^2}=\frac{1\cdot 0.02+1\cdot 0.05}{2}=0.035$$

$$arctg(1.02/0.95)=z+\Delta z=0.785+0.035=0.082$$

Mashqlar

1. $\varphi = x^3 + xy^2 + y^2z^3 + 3$ $d\varphi -$ ni toping.
2. $\varphi = e^{x^2 + y^3}$ $d\varphi -$ ni toping.
3. $\varphi = \frac{x^4 + y^5}{\sqrt{x}}$ $d\varphi -$ ni toping.
4. $\varphi = \ln\left(x^3 + y^6 + z^4\right)$ $d\varphi -$ ni toping.
5. $\varphi = \ln ctg\left(x^2 \cdot y\right)$ $d\varphi -$ ni toping.
6. $\varphi = \cos\left(x^3 + 2y^2\right)$ $d\varphi -$ ni toping.
7. $\varphi = \sin\frac{3x^2}{y^3}$ $d\varphi -$ ni toping.
8. $\varphi = x^{y^2}$ $d\varphi -$ ni toping.
9. $\varphi = \ln\frac{x^3 + 1}{\sin y}$ $d\varphi -$ ni toping.
10. $z = \left(x^3 + 2y^3\right)\cdot e^{u^2}$ $dz -$ ni toping.
11. $z = e^x\left(\cos y + tgx + ctgu\right)$ $dz -$ ni toping.
12. $z = e^{2x^2 + y^3}\left(x\sin y + yctgx\right)$ $dz -$ ni toping.
13. $z = arctg3x^2y^4$ $dz -$ ni toping.
14. $z = \arcsin\left(x^5 + y^4\right)$ $dz -$ ni toping.
15. $u = \sqrt[5]{x^3} \cdot e^{4y^2}$ $du -$ ni toping.
16. $u = \frac{1}{e^{x^2}} + \frac{1}{e^{3y^5}}$ $du -$ ni toping.
17. $u = e^{xy^2} + e^{z^2} + 5$ $du -$ ni toping.

V-BOB
ANIQMAS INTEGRAL
§ 1. Boshlang‘ich funksiya va aniqmas integral

1. Boshlang‘ich funksiya

Agar $f(x)$ funksiyaning aniqlanish sohasini hamma nuqtalarida $F'(x)=f(x)$ tenglik o‘rinli bo‘lsa, (yoki $dF(x)=F'(x)dx=f(x)dx$), u holda $F(x)$ funksiya $f(x)$ funksiyani boshlang‘ich funksiyasi deyiladi. Aniqmas integral $\int$ - simvol bilan belgilanadi. Bunda: $\int f(x)dx=F(x)+c$.

$f(x)$ - integral ostidagi funksiya

$f(x)dx$ - integral ostidagi ifoda

$\int$ - integral belgisi

Shunday qilib ta’rifga ko‘ra $\int f(x)dx=F(x)+c$ berilgan funksiyani boshlang‘ich funksiyasini topish, $f(x)$ funksiyani integrallash deb ataladi. Aniqmas integral bir necha xossaga ega. Asosiysi 2 ta xossa hisoblanadi.

1. O‘zgarmas ko‘paytuvchini aniqmas integral ishorasidan tashqariga chiqarish mumkin, ya’ni: $\int kf(x)dx=k\int f(x)dx$ bunda k - o‘zgarmas.

2. Bir necha funksiyalar algebraik yig‘indisining aniqmas integrali shu funksiyalar aniqmas integrallarining algebraik yig‘indisiga teng, ya’ni:

$$\int[f(x)\pm\varphi(x)]dx=\int f(x)dx\pm\int\varphi(x)dx$$

Integrallarni hisoblashda integrallar jadvali quyidagicha:

1. $\int x^{\alpha}dx=\dfrac{x^{\alpha+1}}{\alpha+1}+c \qquad (\alpha\neq-1)$

2. $\int\dfrac{dx}{x^2}=-\dfrac{1}{x}+c$

3. $\int\dfrac{dx}{\sqrt{x}}=2\sqrt{x}+c$

4. $\int\dfrac{dx}{x}=\ln|x|+c$

5. $\int e^x dx=e^x+c$

6. $\int a^x dx=\dfrac{a^x}{\ln a}+c$

7. $\int\cos dx=\sin x+c$

8. $\int \sin x dx = -\cos x + c$

9. $\int \frac{dx}{\cos^2 x} = tgx + c$

10. $\int \frac{dx}{\sin^2 x} = -ctgx + c$

11. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} arctg \frac{x}{a} + c$

12. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$

13. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$

14. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + c$

2 va 3 formulalar 1 formulaning xususiy ko‘rinishi hisoblanadi.

$2 =$da $\alpha = -2$ $3 =$da $\alpha = -\frac{1}{2}$ deb olish mumkin. Shunday qilib, integrallash amali differensiallash amaligi teskari amal hisoblanadi. Jadvaldan har bir formulani differensialash yo‘li bilan to‘g‘riligini tekshirish mumkin, ya’ni o‘ng tomonda turgan ifodaning hosilasini topsak, u integral ostidagi funksiyaga teng bo‘ladi.

Endi integrallash metodlarini ko‘rib o‘tamiz.

1. To‘g‘ridan – to‘g‘ri integrallash metodi.

Bu metodda berilgan integralni, aniqmas integralning xossalaridan, asosiy integral jadvalidan va boshlang‘ich funksiyalarning ta’rifidan foydalanib topiladi. Quyidagi misollarni ko‘rib o‘tamiz:

1. $\int dx$ ni toping.

Bu integralni quyidagi ko‘rinishda $\int x^0 dx$ yozib olamiz. Bunga jadvaldagi 1 formulani tatbiq qilamiz: $\alpha = 0$ bo‘lganda

$$\int dx = \int x^0 dx = \frac{x^0 + 1}{0 + 1} + c = x + c$$

Shunday qilib, $\int dx = x + c$ teng ekan. Bundan quyidagini topish mumkin. $\int$ va d ishorasi ketma – ket kelganda bir – birini yo‘qotar ekan. Buning natijasida x va o‘zgarmas son C ni ($x + C$ - ning) yig‘indisiga ega bo‘lar ekanmiz. Bu sonda integral murakkab integralni topishda ham tez-tez uchrab turadi. Shuning uchun $\int dx = x + c$ ekanligini doimo yodda tutish zarur.

2. $\int (5x^3 - 3x^2 + 4x + 6) dx$ ni toping.

Bu integralni topish uchun yuqoridagi aniqmas integralni 1 va 2 xossalaridan foydalanamiz. Berilgan integralning 4 ta funksiyasini aniqmas integrallarini algebraik yig‘indiga tenglab yozamiz va o‘zgarmas ko‘paytuvchini aniqmas integral ishorasidan tashqariga chiqarib yozamiz, ya’ni:

$$\int(5x^3-3x^2+4x+6)dx=\int 5x^3dx-\int 3x^2dx+\int 4xdx+\int 6dx=5\int x^3dx-3\int x^2dx+4\int xdx+6\int dx$$

Dastlabki 3 ta integralni har biriga 1 formulani tatbiq qilamiz. Jadvaldagi α - ning o‘rniga $\alpha=3,\ \alpha=2,\ \alpha=1$ mos keladi, ya’ni

$$\int x^3dx=\frac{x^{3+1}}{3+1}+c_1=\frac{x^4}{4}+c_1$$

$$\int x^2dx=\frac{x^{2+1}}{2+1}+c_2=\frac{x^3}{3}+c_2$$

$$\int xdx=\frac{x^{1+1}}{1+1}+c_3=\frac{x^4}{2}+c_3$$

va

$$\int dx=x+c_4 \qquad c=c_1+c_2+c_1+c_4$$

Shunday qilib,

$$\int(5x^3-3x^2+4x+6)dx=5\int x^3dx-3\int x^2dx+4\int xdx+6\int dx=5\frac{x^4}{4}-3\frac{x^3}{3}+4\frac{x^2}{2}+6x+c=$$

$$\frac{5}{4}x^4-x^3-2x^2+6x+c$$

Bunda 4 ta integralga qo‘shilgan c o‘zgarmas ixtiyoriy son. O‘zgarmas sonlarning yig‘indisi ham o‘zgarmas sondir. Shuning uchun algebraik yig‘indining integrallarini topishda chiqqan natijaga bitta o‘zgamas son yozilsa, ham bo‘ladi.

3. $\int\frac{dx}{x^4}$ ni toping.

Bu integralni topish uchun o‘rta maktab kursida bizga ma’lum bo‘lgan quyidagi $x^{-n}=\frac{1}{x^n}$ formuladan foydalanamiz. Integral ostidagi ifodani $\frac{1}{x^4}$ ni x^{-4} deb yozib olamiz. Ya’ni $\frac{1}{x^4}=x^{-4}$. So‘ngra integralga 1 formulani tatbiq qilamiz $\alpha-4$deb.

Shunday qilib, $\int\frac{dx}{x^4}=\int x^{-4}dx=\frac{x^{-4+1}}{-4+1}+c=\frac{x^{-3}}{-3}+c=-\frac{1}{3x^3}+c$

4. $\int\sqrt[3]{x^2dx}$ ni toping.

Bu berilgan integralni topinsh uchun o‘rta maktab kursidan bizga ma’lum bo‘lgan quyidagi $\sqrt[n]{x^m}=x^{\frac{m}{n}}$ formuladan foydalanamiz. Buning uchun integral ostidagi funksiyani quyidagi ko‘rinishda yozib olamiz: $\sqrt[3]{x^2} = x^{\frac{2}{3}}$ so‘ngra, jadvaldagi 1 formuladan foydalanmiz.

$$\alpha = \frac{2}{3} \text{ deb}$$

$$\int \sqrt[3]{x^2 dx} = \int x^{\frac{2}{3}} dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + c = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + c = \frac{3}{5}\sqrt[3]{x^5} + c = \frac{3}{5} x\sqrt[3]{x^2} + c$$

5. $\int \frac{3x^3 + x - \sqrt{x}}{\sqrt{x}} dx$ ni toping.

Avval integral ostidagi kasrning suratini mahrajiga bo‘lib, shaklni o‘zgartirib olamiz.

$$\frac{3x^3 + x - \sqrt{x}}{\sqrt{x}} = \frac{3x^3}{\sqrt{x}} + \frac{x}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}} = \frac{3x^2}{x^{\frac{1}{2}}} + \frac{x}{x^{\frac{1}{2}}} - 1 = 3x^{3-\frac{1}{2}} + x^{1-\frac{1}{2}} - 1 = 3x^{\frac{5}{2}} + x^{\frac{1}{2}} - 1$$

Buni olib borib, berilgan integralga qo‘yamiz va aniqmas integralning xossalaridan va integral jadvaldagi 1 formuladan foydalanib topamiz.

$$\int \frac{3x^3 + x - \sqrt{x}}{\sqrt{x}} dx = \int \left(3x^{\frac{5}{2}} + x^{\frac{1}{2}} - 1\right) dx = 3\int x^{\frac{5}{2}} dx + \int x^{\frac{1}{2}} dx - \int dx = \frac{6}{7}\sqrt{x^7} + \frac{2}{3}\sqrt{x^3} - x + c$$

6. $\int \frac{dx}{x^4 + x^2}$ ni toping.

Integral ostidagi kasrning suratiga x^2 - ni qo‘shamiz va ayiramiz hamda berilgan integralni quyidagi ko‘rinishda yozib olamiz:

$$\int \frac{dx}{x^4 + x^2} = \int \frac{dx}{x^2(x^2+1)} = \int \frac{1 + x^2 - x^2}{x^2(x^2+1)} dx = \int \frac{(1 + x^2) - x^2}{x^2(x^2+1)} dx =$$

$$\int \frac{1 + x^2}{x^2(x^2+1)} dx - \int \frac{x^2}{x^2(x^2+1)} dx = \int \frac{dx}{x^2} - \int \frac{dx}{x^2+1} = -\frac{1}{x} - arctg + c$$

Bunda $\int \frac{dx}{x^2} = -\frac{1}{x} + c$ (jadvaldagi 2 integral)

$\int \frac{dx}{x^2+1} = arctgx + c$ (Bunda jadvaldagi 11 integral)

7. $\int \frac{dx}{\sqrt{5 - x^2}}$ ni toping.

Har qanday musbat a sonini quyidagi ko‘rinishda yozib olish mumkin $\left(\sqrt{a}\right)^2$ ya’ni $a = \left(\sqrt{a}\right)^2$. Shunga asosan $5 = \left(\sqrt{5}\right)^2$ deb yozib olamiz va jadvaldagi formuladan foydalanib berilgan integralni topamiz, bunda $a=\sqrt{5}$.

$$\int \frac{dx}{\sqrt{5-x^2}} = \int \frac{dx}{\sqrt{\left(\sqrt{5}\right)^2 - x^2}} = \arcsin\frac{x}{\sqrt{5}} + c$$

$$\left(\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c\right)$$

Mashqlar

Mustaqil yechish uchun misollar

Quyidagi integrallarni toping.

1. $\int \left(3x^4 - 2x^3 + 5x - 7\right)dx$ Javob: $\frac{3}{5}x^5 - \frac{2^4}{2} + \frac{5}{2}x^2 - 7x + c$
2. $\int \left(4x^5 - 6x^2 + 1\right)dx$ Javob: $\frac{2}{3}x^6 - 2x^3 + x + c$
3. $\int \left(2x^6 - 4x + 5\right)dx$ Javob: $\frac{2}{7}x^7 - 2x^2 + 5x + c$
4. $\int \left(3\sqrt{x} - 2x + 3\right)dx$ Javob: $2x\sqrt{x} - x^2 + 3x + c$
5. $\int \left(\sqrt[3]{x} - 2\sqrt[4]{x} + 5\right)dx$ Javob: $\frac{3}{5}x\sqrt[3]{x} - \frac{8}{5}x\sqrt[4]{x} + 5x + c$
6. $\int \left(\frac{3}{x^2} + 7\right)dx$ Javob: $7x - \frac{3}{x} + c$
7. $\int \left(\frac{1}{x} - 2x + 4\right)dx$ Javob: $\ln|x| - x^2 + 4x + c$
8. $\int \frac{dx}{\sqrt[4]{x}}$ Javob: $\frac{4}{3}\sqrt[4]{x^3} + c$
9. $\int \frac{x+1}{x^2}dx$ Javob: $\ln|x| - \frac{1}{x} + c$
10. $\int \frac{\sqrt{x}+5}{x}dx$ Javob: $2\sqrt{x} - 5\ln|x| + c$

11. $\int \frac{4-2x+x^2}{\sqrt{x}}dx$ Javob: $8\sqrt{x}-\frac{4}{3}x\sqrt{x}+\frac{2}{5}x^2\sqrt{x}+c$

12. $\int \frac{x^2dx}{x^2+2}$ Javob: $x-\sqrt{2}arctg\frac{x}{\sqrt{2}}+c$

13. $\int \frac{dx}{x^2+5}$ Javob: $\frac{1}{\sqrt{5}}arctg\frac{x}{\sqrt{5}}+c$

14. $\int \frac{dx}{x^2-4}$ Javob: $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+c$

15. $\int \frac{dx}{\sqrt{2-x^2}}$ Javob: $\arcsin\frac{x}{\sqrt{2}}+c$

16. $\int \frac{dx}{x^2+9}$ Javob: $\frac{1}{3}arctg\frac{x}{3}+c$

17. $\int \frac{x^2+4}{1+x^2}dx$ Javob: $x+3arctgx+c$

18. $\int \frac{2+3x^2}{x^4+x^2}dx$ Javob: $arctgx-\frac{2}{x}+c$

19. $\int \frac{dx}{x^4-x^2}$ Javob: $\left(\frac{1}{x}+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+c\right)$

20. $\int (2\sin x-5\cos\alpha)dx$ Javob: $-2\cos\alpha-5\sin x+c$

21. $\int \frac{2x\sin^2 x-1}{\sin^2 x}$ Javob: $x^2+ctg+c$

22. $\int \frac{dx}{\sin^2 x\cdot\cos^2 x}$ Javob: $tgx-ctgx+c$

23. $\int \frac{1+\cos^2 x}{1+\cos 2x}$ Javob: $\frac{1}{2}tg+\frac{1}{2}x+c$

24. $\int \left(\frac{2}{x}+\frac{3}{x^2}+\frac{4}{x^3}\right)dx$ Javob: $2\ln|x|-\frac{3}{x}-\frac{2}{x^2}+c$

25. $\int \left(\frac{2}{x^2}+3x^3+6\frac{1}{x^4}\right)dx$ Javob: $-\frac{2}{x}+\frac{3}{4}x^4-\frac{2}{x^3}+c$

Ko‘paytuvchini differensial ishorasi ostiga kiritish

Funksiya differensialining ta’rifiga ko‘ra $df(x) = f'(x)dx$. Shu formulani o‘ngdan chapga qo‘llash $f'(x)$ ko‘paytuvchini differensial ishorasi ostiga kiritish deyiladi.

Masalan, $\cos x dx = d\sin x$ ya’ni ko‘paytuvchi $f'(x) = \cos x$ differensial ishorasi ostiga kiritilgan, bunda $f(x) = \sin x$ yoki $xdx = \frac{dx^2}{2}$ ya’ni ko‘paytuvchi $f'(x) = x$ differensial ishorasi ostiga kiritilgan va $\frac{1}{2}dx^2$ hosil qilgan.

Integral ostida $f'(x)$ ni differensial ishorasi ostida kiritish natijasida ba’zida berilgan integralni jadvaldagi integral ko‘rinishiga keltirish mumkin.

Quyidagi misollarni ko‘rib o‘tamiz.

1. $\int e^{4x}dx$ integralni toping.

Ma’lumki, $4dx = d4x$ hamda integral ostidagi ifodani 4 ga bo‘lib va ko‘paytirib, quyidagini hosil qilamiz:

$$\int e^{4x}dx = \int e^{4x}\frac{4dx}{4} = \frac{1}{4}\int e^{4x}d(4x) = \frac{1}{4}e^{4x} + c$$

Bu yerda $\int e^{4x}d4x = e^{4x} + c$ jadvaldagi integrallardir.

2. $\int \sin(2x+3)dx$ integralni toping.

Bizga ma’lum bo‘lgan (8) formula $\int \sin x dx = -\cos x + c$. Berilgan misolda $2x+3$ asosiy rol o‘ynaydi. Shuning uchun dxni 2 – ga ko‘aytirib va bo‘lib quyidagicha shakl o‘zgartiramiz $\frac{dx \cdot 2}{2} = \frac{1}{2}d2x$ so‘ngra quyidagicha yozish mumkin:

$\frac{1}{2}d(2x+3)$ chunki $\frac{1}{2}d(2x+3) = \frac{1}{2}(d2x + d3) = \frac{1}{2}d2x$

Demak, $dx = \frac{dx \cdot 2}{2} = \frac{1}{2}d2x = \frac{1}{2}d(2x+3)$

Shunday qilib quyidagiga ega bo‘lamiz:

$$\int \sin(2x+3)dx = \int \sin(2x+3)\frac{d2x}{2} = \frac{1}{2}\int \sin(2x+3)d(2x+3) = \frac{1}{2}\cos(2x+3) + c$$

Bunda $\int \sin x dx = -\cos x + c$ formuladan foydalaniladi.

3. $\int \frac{x\,dx}{x^2+1}$ ni toping.

x ko‘paytuvchini differensial ishorasi ostiga kiritib, quyidagini hosil qilamiz:

$$xdx = \frac{dx^2}{2}; \quad \left(\frac{dx^2}{2} = \frac{\left(x^2\right)' dx}{2} = \frac{2xdx}{2} = xdx \right)$$

$$\text{Demak, } \int \frac{xdx}{x^2+1} = \int \frac{\frac{dx^2}{2}}{x^2+1} = \frac{1}{2}\int \frac{dx^2}{x^2+1}$$

Differensial ishorasi ostiga x^2-ga ixtiyoriy o‘zgarmas sonni qo‘shish mumkin va bundan differensial o‘zgarmaydi, ya’ni $d\left(x^2+c\right) = dx^2 + dc$ ammo o‘zgarmas sonni differensial 0 ga teng. Shunday qilib

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{dx^2}{x^2+1} = \frac{1}{2}\int \frac{d\left(x^2+1\right)}{x^2+1} = \frac{1}{2}\ln\left(x^2+1\right) + c$$

Bunda $\int \frac{dx}{x} = \ln|x| + c$ formuladan foydalanildi. Misolda absolyut qiymatning ishorasi tushib qoldirildi, chunki x ning har qanday qiymatida x^2+1 ifoda musbat.

4. $\int e^{\sqrt{x}} \frac{dx}{\sqrt{x}}$ integralni toping.

Ma’lumki, $d\sqrt{x} = \left(\sqrt{x}\right)' dx = \frac{1}{2\sqrt{x}}dx$ ammo $\frac{dx}{\sqrt{x}} = 2d\sqrt{x}$.

Demak, $\int e^{\sqrt{x}} \frac{dx}{\sqrt{x}} = \int e^{\sqrt{x}}\, 2d\sqrt{x} = 2\int e^{\sqrt{x}} d\sqrt{x} = 2e^{\sqrt{x}} + c$

Bunda $\int e^x dx = e^x + c$ formuladan foydalaniladi. x o‘rnida $\sqrt{x}$ kelyapti.

5. $\int \frac{\ln^3 x}{x} dx$ ni toping.

Berilgan integralni quyidagi ko‘rinishda yozib olamiz. Ma’lumki,

$\int \ln^3 x \frac{dx}{x}$. Demak $d\ln x = \frac{1}{x}dx$ yoki $\frac{dx}{x} = d\ln x$.

Shunday qilib, $\int \ln^3 x \frac{dx}{x} = \int \ln^3 x d(\ln x) = \int (\ln x)^3 d(\ln x) = \frac{(\ln x)^4}{4} + c = \frac{1}{4}\ln^4 x + c$

Bunda $\int x^{\alpha}dx=\frac{x^{\alpha+1}}{\alpha+1}+c$ formuladan foydalaniladi.
Bu misolda x o‘rnida $\ln x$ kelyapti.

Mashqlar

Mustaqil ish uchun misollar

1. $\int e^{5x}dx$ J: $\left(\frac{1}{5}e^{5x}+c\right)$

2. $\int \sin(2x+1)dx$ J: $\left(-\frac{1}{2}\cos(2x+1)+c\right)$

3. $\int \frac{dx}{\cos^2 6x}$ J: $\left(\frac{1}{6}tg6x+c\right)$

4. $\int e^{-3x}dx$ J: $\left(-\frac{1}{3}e^{-3x}+c\right)$

5. $\int \cos(3x-2)dx$ J: $\left(\frac{1}{3}\sin(3x-2)+c\right)$

6. $\int (x+4)^5 dx$ J: $\left(\frac{1}{6}(x+4)^6+c\right)$

7. $\int \sqrt{x+7}\, dx$ J: $\left[\frac{2(x+7)\sqrt{x+7}}{3}+c\right]$

8. $\int \frac{d}{\sqrt{3x-1}}$ J: $\left(\frac{2\sqrt{3x-1}}{3}+c\right)$

9. $\int (4x-1)^7\, dx$ J: $\left[\frac{(4x-1)^8}{32}+c\right]$

10. $\int \ell^{2x-1}dx$ J: $\left(\frac{1}{2}e^{2x-1}+c\right)$

11. $\int e^{\frac{x}{4}}\,dx$ J: $\left(4e^{\frac{x}{4}}+c\right)$

12. $\int \sin\frac{x}{5}\,dx$ J: $\left(-5\cos\frac{x}{5}+c\right)$

13. $\int \frac{2x\,dx}{x^2+5}$ J: $\left(\ln\left(x^2+5\right)+c\right)$

14. $\int \sin x\cos x dx$ J: $\left(\frac{\sin^2 x}{2}+c\right)$

15. $\int e^{x^2}x\,dx$ J: $\left(\frac{1}{2}e^{x^2}+c\right)$

16. $\int \ln\frac{dx}{x}$ J: $\left(\frac{\ln^2 x}{2}+c\right)$

17. $\int \cos^3 x\sin x\,dx$ J: $\left(-\frac{1}{4}\cos^4 x+c\right)$

18. $\int \frac{x^2\,dx}{x^3-1}$ J: $\left(\frac{1}{3}\ln\left|x^3-1\right|+c\right)$

19. $\int e^{\sqrt{x}}\frac{dx}{\sqrt{x}}$ J: $\left(e^{\sqrt{x}}+c\right)$

20. $\int \cos\sqrt{x}\frac{dx}{\sqrt{x}}$ J: $\left(2\sin\sqrt{x}+c\right)$

§2. O‘zgaruvchini almashtirish usuli (yoki o‘rniga qo‘yish usuli)

Ayrim hollarda integral ostidagi o‘zgaruvchini yangi o‘zgaruvchiga almashtirish berilgan integralni jadvaldagi integral ko‘rinishiga olib keladi. Bu usul o‘zgaruvchini almashtirish usuli yoki o‘rniga qo‘yish usuli (metodi) deb ataladi.

Bizga quyidagi $\int f(x)dx$, integralni topish kerak bo‘lsin. Ammo to‘g‘ridan – to‘g‘ri $f(x)$ funksiyani boshlang‘ich funksiyasini topish murakkab. Shuning uchun $x=\varphi(t)$ deb o‘zgaruvchini almashtiramiz, bunda $\varphi(t)$ uzluksiz, monoton va differensiallanuvchi funksiyadir. Differensialning ta’rifiga asosan quyidagini topamiz:

$dx = d\varphi(t) = \varphi'(t)dt$ ya’ni $dx = \varphi'(t)dt$

U holda quyidagi formula o‘rinli bo‘ladi.

$$\int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt \qquad (1)$$

(1) formula aniqmas integralda o‘zgaruvchini almashtirish formulasi deyiladi.

Integral $\int f[\varphi(t)]\varphi'(t)dt$ ni topgandan so‘ng, oldingi o‘zgaruvchiga qaytish kerak, ya’ni t ni o‘rniga uni x bilan aniqlangan ifodasini qo‘yish kerak.

Quyidagi misollarni ko‘rib o‘tamiz:

Misol 1. $\int \frac{x\,dx}{x^2+4}$ ni toping.

O‘zgaruvchini almashtiramiz $x^2+4=t$ deb, bu tenglikni ikkala tomonini differensiallaymiz va quyidagiga ega bo‘lamiz. $d(x^2+4)dt$ yoki $2xdx=dt$ ikkala tomonini 2 ga bo‘lib, quyidagiga ega bo‘lamiz: $xdx=\frac{1}{2}dt$. Ammo xdx ifoda berilgan integral ostidagi kasrning suratiga teng. Demak integral ostidagi x^2+4 o‘rniga t ni hamda xdx ifodani o‘rniga $\frac{dt}{2}$ ni qo‘yib quyidagini hosil qilamiz:

$$\int \frac{x\,dx}{x^2+4} = \frac{1}{2}\int \frac{dt}{t}$$

$\int \frac{dt}{t}$ integral jadvaldagi integral, agar quyidagi $\int \frac{dx}{x} = \ln|x| + c$ formulani esga olsak, bunda x o‘zgaruvchini o‘rniga t o‘zgaruvchi kelayapti. Demak, $\int \frac{dt}{t} = \ln|t| + c$.

Shunday qilib, $\int \frac{x\,dx}{x^2+4} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + c = \frac{1}{2}\ln(x^2+4) + c$.

Bunda t - o‘rniga uning x bilan aniqlangan ifodasi qo‘yiladi, ya’ni $t=x^2+4$ va absolyut miqdorning ishorasi tushirib qoldirildi, chunki x^2+4 ifoda x-ning har qanday qiymatida musbat.

Misol 2. $\int \frac{x\,dx}{(x+1)^2}$ ni toping.

$x+1=t$ deb o‘zgaruvchini almashtiramiz, u holda $x=t-1$ bo‘ladi. bundan $dx=dt$. So‘ngra bu topilganlarni olib borib berilgan integralga qo‘yamiz. Quyidagi ko‘rinishda yozib olsak qulayroq bo‘ladi:

$$\int \frac{x\,dx}{(x+1)^2}=\left|\begin{matrix} x+1=t \\ x=t-1 \\ dx=dt \end{matrix}\right|=\int \frac{(t-1)dt}{t^2}=\int \frac{tdt}{t^2}-\int \frac{dt}{t^2}=\int \frac{dt}{t}-\int \frac{dt}{t^2}=\ln|t|-\left(-\frac{1}{t}\right)+c=$$

$$\ln|x+1|+c$$

Bunda $\int \frac{dt}{t}=\ln|t|+c$ (jadvaldagi 4 formulaga asosan)

$\int \frac{dt}{t^2}=-\frac{1}{t}+c$ (jadvaldagi 3 formulaga asosan)

Misol 3. Integral $\int \sqrt{e^x+1}e^x dx$ ni toping.

$e^x+1=t^2$ deb o‘zgaruvchini almashtiramiz, so‘ngra differensiallab, $(e^x+1)'dx=2tdt$ ni topamiz. Yoki $e^x dx=2tdt$ almashtirishdan quyidagini olish mumkin.

$$\sqrt{e^x+1}=\sqrt{t^3}=t$$

Shunday qilib,

$$\int \sqrt{e^x+1}e^x dx=\left|\begin{matrix} e^x+1=t^2 \\ e^x dx=(t^2)'dt= \\ =2tdt.\ \ \sqrt{e^x+1}=t \end{matrix}\right|=\int t2tdt=2\int t^2dt=\frac{2}{3}t^3+c=$$

$$=\frac{2}{3}\left(\sqrt{e^x+1}\right)^3+c$$

Bunda $\int t^2 dt=\frac{t^3}{3}+c$ jadvaldagi 1 formulaga asosan, bunda $\alpha=2$.

Misol 4. $\int x^2\cos(x^3+2)dx$ ni toping.

$x^3+2=t$ deb, o‘zgaruvchini almashtiramiz. Unda

$$\left(x^3+2\right)'=dx=t'dt$$

Bundan $3x^2dx=dt$ yoki $x^2dx=\frac{dt}{3}$

Demak,

$$\int x^2\cos\left(x^3+2\right)dx=\left|\begin{matrix}x^3+2=t\\ 3x^2dx=dt\\ x^2dx=\frac{1}{3}dt\end{matrix}\right|=\int\cos t\frac{1}{3}dt=\frac{1}{3}\int\cos tdt=\frac{1}{3}\sin t+c=$$

$$=\frac{1}{3}\sin\left(x^3+2\right)+c$$

Misol 5. $\int\frac{e^xdx}{\sqrt{3-e^{2x}}}$ ni toping.

Ma’lumki, $e^{2x}=\left(e^x\right)^2$ va $e^x=t$ deb o‘zgaruvchini almashtiramiz. Differensiallab quyidagini topamiz: $e^xdx=dt$.

Shunday qilib,

$$\int\frac{e^xdx}{\sqrt{3-e^{2x}}}=\int\frac{e^xdx}{\sqrt{\left(\sqrt{3}\right)^2-\left(e^x\right)^2}}=\left|\begin{matrix}e^x=t\\ e^xdx=dt\end{matrix}\right|=\int\frac{dt}{\sqrt{\left(\sqrt{3}\right)^2-t^2}}=\arcsin\frac{t}{\sqrt{3}}+c=$$

$$=\arcsin\frac{e^x}{\sqrt{3}}+c$$

Bunda $\int\frac{dx}{\sqrt{\left(\sqrt{3}\right)^2-t^2}}=\arcsin\frac{t}{\sqrt{3}}+c$. Bu jadvaldagi 12 formulaga asoslanib topildi. Bunda $a=\sqrt{3}$ formuladagi x o‘rnida t kelayapti.

Mashqlar

Mustaqil yechish uchun misollar

1. $\int\frac{2xdx}{x^2+3}$ J: $\left[\ln\left(x^2+3\right)+c\right]$

2. $\int\left(e^x+1\right)^2e^xdx$ J: $\left[\frac{\left(e^x+1\right)^3}{3}+c\right]$

3. $\int \frac{x^2 dx}{(x^3+5)^2}$ J: $\left[-\frac{1}{3(x^3+5)}+c\right]$

4. $\int \frac{xdx}{\sqrt{x^2+1}}$ J: $\left(\sqrt{x^2+1}+c\right)$

5. $\int \frac{xdx}{x+2}$ J: $(x-2\ln(x+2)+c)$

6. $\int \frac{xdx}{\sqrt{x+4}}$ J: $\left(\frac{2}{3}\sqrt{x+4}(x+8)+c\right)$

7. $\int \frac{\cos xdx}{\sin x}$ J: $(-\ln|\sin|+c)$

8. $\int \frac{\sin xdx}{\cos^2 x}$ J: $\left(-\frac{1}{\cos x}+c\right)$

9. $\int \frac{\cos xdx}{2+\sin x}$ J: $(\ln|2+\sin x|+c)$

10. $\int tgxdx$ J: $(-\ln|\cos x|+c)$

11. $\int \frac{xdx}{(x-3)^2}$ J: $\ln|x-3|-\frac{3}{x-3}+c$

12. $\int \frac{1+\ln x}{x}dx$ J: $\left(\frac{(1+\ln)^2}{2}+c\right)$

13. $\int \frac{dx}{x\ln x}$ J: $(\ln|\ln x|+c)$

14. $\int \frac{\sqrt{1+\ln x}}{x}dx$ J: $\left(\frac{2}{3}(1+\ln x)\sqrt{1+\ln x}+c\right)$

15. $\int \frac{dx}{(2-\ln x)^2 x}$ J: $\left(-\frac{1}{2+\ln x}+c\right)$

16. $\int \frac{\sqrt[3]{arctgx}}{1+x^2}dx$ J: $\left(\frac{3}{4}arctgx\right)^{\frac{4}{3}}+c$

17. $\int \frac{(2x+1)}{x^2+x+1}dx$ J: $\left(\ln\left|x^2+x-1\right|+c\right)$

18. $\int \frac{xdx}{x^4+1}$ J: $\left(\frac{1}{2}arctgx^2+c\right)$

19. $\int e^{tgx}\frac{dx}{\cos^2 x}$ J: $\left(e^{tgx}+c\right)$

20. $\int \frac{\sin 2xdx}{1+\sin^2 x}$ J: $\left(\ln\left(1+\sin^2 x\right)+c\right)$

§ 3. Bo‘laklab integrallash usuli

Bo‘laklab integrallash formulasi deb, quyidagi tenglikka aytiladi:

$$\int udv = uv - \int vdu \qquad (1)$$

Bu formulani qo‘llashdan maqsad, o‘ng tomonda turgan integralni chap tomondagi berilgan integraldan sodda qo‘rinishga keltirishdir. Bu usul quyidagi asosiy hollarda qo‘llaniladi.

1. Agar (1) formula chap tomonida turgan integralda integral ostidagi $f(x)$ ko‘phad bilan quyidagi funksiyalarni birini ko‘paytmasidan iborat bo‘lsa:

$$e^{ax},\ \sin ax,\ \cos ax,\ \ln x,\ \arcsin x,\ arctgx$$

$P(x)$ ko‘phad x^n ko‘rinishdagi darajali funksiyadan iborat bo‘lsa,

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

Agar integral ostidagi funksiya quyidagi ko‘rinishdagi funksiyalardan biri bo‘lsa, $\sin x$, $\arcsin x$, $\arccos x$, $arctgx$, $arcctgx$ yoki $e^x \cos x$ yoki $e^x \sin x$.

Bir necha misollar ko‘rib o‘tamiz:

Misol 1. $\int x\sin xdx$ni toping.

$x = u,\quad \sin xdx = dv$ deb olamiz.

Bunda birinchi tenglikni differensiallab, ikkinchi tenglikni integrallab, o‘zgarmas son c ni qo‘shmasdan quyidagini topamiz:

$dx = du,\quad \int \sin xdx = \int dv$ yoki $-\cos x = v$

Shunday qilib,

$$\int x\sin x dx = \begin{vmatrix} x = u & \sin x dx = dv \\ dx = du & -\cos x = v \end{vmatrix} = uv - \int v du = x(-\cos x) -$$

$$\int(-\cos x)dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + c$$

Misol 2. $\int xe^{-5x}\,dx$ ni toping.

$x = u,\ e^{-5x}dx = dv$ deb olamiz. Bunda

$$\int xe^{-5x}\,dx = \begin{vmatrix} x = u, \ \ e^{-5x}dx = dv, & \\ dx = du, \ \int e^{-5x}dx = \int dv, \ -\frac{1}{5}e^{-5x} = v \end{vmatrix} = uv - \int v du =$$

$$x\left(-\frac{1}{5}e^{-5x}\right) - \int\left(-\frac{1}{5}e^{-5x}\right)dx = -\frac{x}{5}e^{-5x} + \frac{1}{5}\int e^{-5x}dx = -\frac{x}{5}e^{-5x} + \frac{1}{5}\int e^{-5x}\,\frac{d(-5x)}{-5} =$$

$$= -\frac{x}{5}e^{-5x} - \frac{1}{25}\int e^{-5x}d(-5x) = -\frac{x}{5}e^{-5x} - \frac{1}{25}e^{-5x} + c$$

Misol 3. $\int x^3\ln x\,dx$ ni toping.

$\ln x = u,\ x^3dx = dv$ deb olamiz. Birinchi tenglikni differensiallab, ikkinchisini integrallab quyidagini hosil qilamiz:

$$d\ln x = du$$

$$(\ln x)'\,dx = \frac{dx}{x} = du$$

$$x^3dx = dv$$

$$\int x^3dx = \int dv$$

$$\frac{x^4}{4} = v$$

Shunday qilib,

$$\int x^3\ln x\,dx = \begin{vmatrix} \ln x = u, & x^3dx = dv \\ \frac{dx}{x} = du, & \frac{x^4}{4} = v \end{vmatrix} = uv - \int v du = \frac{x^4}{4}\ln x - \int\frac{x^4}{4}\cdot\frac{dx}{x} =$$

$$\frac{x^4}{4}\ln x - \frac{1}{4}\int x^3dx = \frac{x^4}{4}\ln x - \frac{1}{4}\cdot\frac{x^4}{4} + c = \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + c$$

Mashqlar

Mustaqil yechish uchun misollar

1. $\int x\cos xdx$ J: $(x\sin x+\cos x+c)$

2. $\int xe^{x}dx$ J: $(e^{x}(x-1)+c)$

3. $\int x\ln xdx$ J: $\left(\frac{x^2}{2}\left(\ln x-\frac{1}{2}\right)+c\right)$

4. $\int xe^{3x}dx$ J: $\left(\frac{x}{3}e^{3x}-\frac{1}{9}e^{3x}+c\right)$

5. $\int arctgxdx$ J: $\left(xarctgx-\frac{1}{2}\ln(1+x)^2+c\right)$

6. $\int x\cdot\sin 4xdx$ J: $\left(\frac{1}{16}\sin 4x-\frac{x}{4}\cdot\cos 4x+c\right)$

7. $\int \arcsin xdx$ J: $\left(x\arcsin x+\sqrt{1-x^2}+c\right)$

8. $\int xarctgxdx$ J: $\left(\frac{x^2+1}{2}arctgx-\frac{x}{2}+c\right)$

9. $\int\frac{xdx}{\sin^2 x}$ J: $(-xctgx+\ln|\sin x|+c)$

10. $\int x^6\ln xdx$ J: $\left[\frac{x^7}{7}\left(\ln|x|-\frac{1}{7}\right)+c\right]$

11. $\int\frac{x\cos xdx}{\sin^3 x}$ J: $\left[-\frac{1}{2}\left(\frac{x}{\sin^2 x}+ctgx+c\right)\right]$

12. $\int x^2\cos xdx$ J: $(x^2\sin x+2x\cos x-\sin x+c)$

13. $\int x^2e^{-x}dx$ J: $\left[-e^{-x}(x^2+2x+2)+c\right]$

14. $\int x^2 \sin x dx$ J: $\left(-x^2 \cos x + 2x \sin x + 2\cos x + c\right)$

15. $\int x \ln^2 x dx$ J: $\left(\frac{1}{2}x^2 \ln^2|x| - \frac{1}{2}x^2 \ln|x| + \frac{1}{4}x^2 + c\right)$

16. $\int e^x \cos x dx$ J: $\frac{e^x}{2}(\cos x + \sin x) + c$

17. $\int e^x \sin x dx$ J: $\frac{e^x}{2}(\sin x - \cos x) + c$

18. $\int \sqrt{1+x^2}dx$ J: $\left[\frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left(x + \sqrt{1+x^2} + c\right)\right)\right]$

19. $\int \cos(\ln x)dx$ J: $\left\{\frac{x}{2}[\sin(\ln x) + \cos(\ln x)] + c\right\}$

20. $\int arctg\sqrt{x}dx$ J: $xarctg\sqrt{x}2\sqrt{x} + 2arctg\sqrt{x} + c$

§ 4. Eng sodda kasrlarni integrallash

Eng sodda kasrlar deb, quyidagi ko‘rinishdagi kasrlarga aytiladi:

$$\frac{1}{x+a} \quad (1) \qquad \frac{1}{(x+a)^n} \quad (2)$$

$$\frac{1}{x^2+px+q} \quad (3) \qquad \frac{Mx+N}{x^2+px+q} \quad (4)$$

Bu yerda x^2+px+q kvadrat uchhad haqiqiy ildizlarga ega emas.

(1) va (2) ko‘rinishdagi kasrlarni o‘zgaruvchini almashtirish usulidan foydalanib integrallash mumkin. Bunda $x+a=t$ deb almashtiriladi.

Masalan: $\int \frac{dx}{(x+2)^2} = \left|\begin{matrix} x+2=t \\ dx=dt \end{matrix}\right| = \int \frac{dt}{t^2} = -\frac{1}{t} + c = -\frac{1}{x+2} + c$

(3) va (4) ko‘rinishdagi sodda kasrlarni integrallash uchun eng avval x^2+px+q kvadrat uchhaddan to‘la kvadrat ajratiladi, ya’ni bu kvadrat uchhadni $(x \pm a)^2 = x^2 \pm 2ax + a^2$ ko‘rinishga keltiriladi. So‘ngra

o‘zgaruvchini almashtiriladi. Buni quyidagi misollarda ko‘rish mumkin:

Misol 1. $\int \frac{dx}{x^2+3x+7}$ ni toping.

x^2+3x+7 kvadrat uchhaddan to‘la kvadrat ajratamiz.

Ya’ni

$$x^2+3x+7=x^2+2\frac{3}{2}x+7=x^2+2\frac{3}{2}x+7=x^2+2\frac{3}{2}x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+7$$

$$\left[\left(x^2+2\frac{2}{3}x+\frac{9}{4}\right)-\frac{9}{4}+7=\left(x+\frac{3}{2}\right)^2-\frac{9}{4}+\frac{28}{4}=\left(x+\frac{3}{2}\right)^2+\frac{19}{4}\right.$$

Shunday qilib,

$$\int\frac{dx}{x^2+3x+7}=\int\frac{dx}{\left(x+\frac{3}{2}\right)^2+\frac{19}{4}}=\left|\begin{matrix}x+\frac{3}{2}=t\\ dx=dt\end{matrix}\right|=\int\frac{dt}{t^2+\frac{19}{4}}=\int\frac{dt}{t^2+\left(\sqrt{\frac{19}{4}}\right)^2}=$$

$$\frac{1}{\frac{\sqrt{19}}{2}}arctg\frac{t}{\frac{\sqrt{19}}{2}}+c=\frac{2}{\sqrt{19}}arctg\frac{2t}{\sqrt{19}}+c=\frac{2}{\sqrt{19}}arctg\frac{2\left(x+\frac{3}{2}\right)}{\sqrt{19}}+c=$$

$$\frac{2}{\sqrt{19}}arctg\frac{2x+3}{\sqrt{19}}+c$$

Misol 2. $\int\frac{2x+1}{3x^2-x+3}dx$ ni toping.

$3x^2-x+3=3\left(x^2-\frac{1}{3}x+1\right)$ demak $\int\frac{(2x+1)dx}{3x^2-x+3}=\frac{1}{3}\int\frac{2x+1}{x^2-\frac{1}{3}x+1}dx$

$x^2-\frac{1}{3}x+1$ kvadrat uchhaddan to‘la kvadrat ajratamiz.

$(x\pm a)^2=x^2\pm 2ax+a^2$ yoki $(x-a)^2=x^2-2ax+a^2$

Shunday qilib,

$$x^2-\frac{1}{3}x+1=x^2-2\cdot\frac{1}{6}x+1=x^2-2\cdot\frac{1}{6}x+\left(\frac{1}{6}\right)^2-\left(\frac{1}{6}\right)^2+1=\left(x^2-2\cdot\frac{1}{6}x+\frac{1}{36}\right)-\frac{1}{36}+1=$$

$$=\left(x-\frac{1}{6}\right)^2+\frac{35}{36}$$

Demak,

$$\int\frac{(2x+1)dx}{x^2-x+3}=\frac{1}{3}\int\frac{(2x+1)dx}{x^2-\frac{1}{3}x+1}=\int\frac{(2x+1)dx}{\left(x-\frac{1}{6}\right)+\frac{35}{36}}=\left|\begin{array}{l} x-\frac{1}{16}=t \\ x=t+\frac{1}{6} \\ dx=dt \end{array}\right|=\frac{1}{3}\int\frac{\left[2\left(t+\frac{1}{6}\right)+1\right]dx}{t^2+\frac{35}{36}}=$$

$$\frac{1}{3}\int\frac{2t+\frac{1}{3}+1}{t^2+\frac{35}{36}}dt=\frac{1}{3}\int\frac{2t+\frac{4}{3}}{t^2+\frac{35}{36}}dt=\frac{1}{3}\int\frac{2tdt}{t^2+\frac{35}{36}}+\frac{1}{3}\int\frac{\frac{4}{3}dt}{t^2+\frac{35}{36}}$$

Lekin bizga ma’lumki $dt^2=2tdt$. Shuning uchun $2tdt$ ifodani dt^2 bilan almashtiramiz. Birinchi integraldan va ikkinchi integraldan o‘zgarmas sonlarni integraldan tashqariga chiqarib, quyidagiga ega bo‘lamiz.

$$\int \frac{2x+1}{3x^2-x+3}dx = \frac{1}{3}\int \frac{(2x+1)dx}{x^2-\frac{1}{3}x+1} = \frac{1}{3}\int \frac{(2x+1)dx}{\left(x-\frac{1}{6}\right)^2+\frac{35}{36}} = \left| \begin{array}{l} x-\frac{1}{16}=t \\ x=t+\frac{1}{6} \\ dx=dt \end{array} \right| = \frac{1}{3}\int \frac{2tdx}{t^2+\frac{35}{36}} =$$

$$\frac{1}{3}\int \frac{\frac{4}{3}dt}{t^2+\frac{35}{36}} = \frac{1}{3}\int \frac{dt^2}{t^2+\frac{35}{36}} = \frac{4}{9}\int \frac{dt}{t^2+\frac{35}{36}} = \frac{1}{3}\int \frac{d\left(t^2+\frac{35}{36}\right)}{t^2+\frac{35}{36}} + \frac{4}{9}\int \frac{dt}{t^2+\left(\sqrt{\frac{35}{36}}\right)^2} =$$

$$= \frac{1}{3}\ln\left(t^2+\frac{35}{36}\right) + \frac{4}{9}\frac{1}{\frac{\sqrt{35}}{36}} arctg \frac{t}{\frac{\sqrt{35}}{36}} + c$$

t - o‘rniga uning qiymati $\left|x-\frac{1}{6}\right|$ ni qo‘yib ixchamlaymiz.

Mashqlar

Mustaqil yechish uchun misollar

1. $\int \frac{dx}{x^2-6x+11}$ J: $\left(\frac{1}{\sqrt{2}} arctgx\frac{x-3}{\sqrt{2}}+c\right)$

2. $\int \frac{dx}{x^2+5x+4}$ J: $\left(\frac{1}{3}\ln\left|\frac{x+1}{x+4}\right|+c\right)$

3. $\int \frac{dx}{2x^2+x-1}$ J: $\left(\frac{2}{3}\ln\left|\frac{2x-1}{2(x+1)}\right|+c\right)$

4. $\int \frac{xdx}{x^2+7x+13}$ J: $\left(\frac{1}{2}\ln\left|x^2+7x+13\right| - \frac{7}{\sqrt{13}} arctg\frac{2x+7}{\sqrt{13}}+c\right)$

5. $\int \frac{(x+5)dx}{2x^2+2x+3}$ J: $\left(\frac{1}{4}\ln\left|2x^2+2x+3\right| + \frac{9}{2\sqrt{5}} arctg\frac{2x+1}{\sqrt{5}}+c\right)$

6. $\int \frac{dx}{4x^2+6x+5}$ J: $\left(\frac{1}{\sqrt{11}} arctg \frac{2x+1}{\sqrt{11}} + c \right)$

§ 5. Ratsional kasrlarni integrallash

Ratsional kasr deb ikki ko‘phadning nisbatiga aytiladi, ya’ni quyidagi ko‘rinishdagi kasrga aytiladi.

$$\frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m} = \frac{P(x)}{Q(x)} \qquad (1)$$

Agar kasrni suratini darajasi mahrajining darajasidan katta yoki teng bo‘lsa, ya’ni $n \geq m$ bo‘lsa, u holda (1) kasr noto‘g‘ri kasr deyiladi. Agar kasrning suratining darajasi mahrajining darajasidan kichik bo‘lsa, ya’ni $n < m$ u holda bunday kasr to‘g‘ri kasr deyiladi.

Masalan: $\frac{x^2+1}{x^3+3x+5}$ to‘g‘ri kasr $(n < m)$.

$\frac{x^4 - x^3 + 1}{x^2 + x + 2}$ kasr esa noto‘g‘ri kasr $(n > m)$.

Har qanday noto‘g‘ri kasrni butun qismi bilan to‘g‘ri kasrning yig‘indisi ko‘rinishida tasvirlash mumkin. Quyidagi kasrni $\frac{x^4 - x^3 + 1}{x^2 + x - 2} = (x^2 - 2x + 4) + \frac{-8x + 9}{x^2 + x - 2}$ ko‘rinishda yozish mumkin ya’ni ko‘phadni ko‘phadga bo‘lish qoidasi bilan bo‘lib quyidagiga ega bo‘lamiz:

$$\begin{array}{l|l} x^4 - x^3 + 1 & x^2 + x - 2 \\ \cline{2-2} & x^2 - 2x + 4 \\ \underline{x^4 + x^3 - 2x^2} & \\ -2x^3 + 2x + 1 & \\ \underline{-2x^3 - 2x^2 + 4x} & \\ 4x^2 - 4x + 1 & \\ \underline{4x^2 + 4x - 8} & \\ -8x + 9 & \end{array}$$

Bunda $x_2 - 2x + 4$, $\dfrac{x^4 - x^3 + 1}{x^2 + 2x - 2}$ noto‘g‘ri kasrning butun qismi $\dfrac{-8x + 9}{x^2 + x - 2}$ esa to‘g‘ri kasr.

Shuning uchun noto‘g‘ri kasrlarni integrallash uchun eng avval to‘g‘ri kasrni integrallashni bilish zarur.

Masalan:

$$\int \frac{x^4 - x^3 + 1}{x^2 + x + 2} dx = \int \left(x^2 - 2x + 4\right) dx + \int \frac{-8x + 9}{x^2 + x + 2} dx = \int x^2 dx - 2\int x dx + 4\int dx +$$
$$+ \int \frac{-8x + 9}{x^2 + x - 2} dx$$

$x^2 + x - 2$ - kvadrat uchhad haqiqiy ildizga ega, ya’ni $x = -2,\ x = 1$ demak, buni $x^2 + x - 2 = (x - 1)(x + 2)$ ko‘rinishda yozish mumkin.

Har qanday ko‘rinishdagi $\dfrac{P_1(x)}{Q_1(x)}$ to‘g‘ri kasrni mahraji $(x + a)$ ko‘rinishdagi bir necha ko‘paytuvchilardan iborat bo‘lsa, bunday kasrni yig‘indisi ko‘rinishida yozish mumkin.

Masalan: $\dfrac{-8x + 9}{x^2 + x - 1} = \dfrac{-8x + 9}{(x - 1)(x + 2)}$

to‘g‘ri kasrni ikkita elementar kasrning yig‘indisi shaklida yozish mumkin.

Masalan: $\dfrac{-8x + 9}{x^2 + x - 2} = \dfrac{\frac{1}{3}}{x - 1} - \dfrac{\frac{25}{3}}{x + 2}$

Agar o‘ng tomondagi kasrni umumiy mahrajiga keltirsak, ya’ni $\dfrac{-8x + 9}{(x - 1)(x + 2)}$ yoki $\dfrac{-8x + 9}{x^2 + x - 2}$ kasr kelib chiqadi. Albatta berilgan to‘g‘ri kasrni yuqoridagi ko‘rinishdagi elementar kasrlarni yig‘indisi shaklida yozish oson emas. Buning uchun avval $\dfrac{-8x + 9}{x^2 + x - 2}$ kasrni quyidagi kasrning yig‘indisi ko‘rinishida ya’ni $\dfrac{A}{x - 1} + \dfrac{B}{x + 2}$ yozib olamiz. Bunda A, B hozircha noma’lum miqdorlardir.

A va B larni noma’lum koeffitsientlarni topish qoidasi bo‘yicha topib olinadi.

Demak: $\frac{-8x+9}{x^2+x-2}=\frac{A}{x-1}+\frac{B}{x+2}$

Buning o‘ng tomonini umumiy mahrajga keltiramiz,

$\frac{-8x+9}{x^2+x-2}=\frac{A(x+2)+B(x-1)}{(x+2)(x-1)}$

Agar teng kasrlarni mahrajlari bir – biriga teng bo‘lsa, u holda suratlari ham teng bo‘ladi, ya’ni

$-8x+9=A(x+2)+B(x-1)$ (2)

Bunda $x=1$ desak, unda (2) tenglikdan quyidagini hosil qilamiz:

$-8+9=A(1+2)+B$ yoki $1=3A$ bundan $A=\frac{1}{3}$

$x=-2$ desak u holda (2) tenglikdan quyidagini hosil qilamiz:

$-8(-2)+9=A(-2+2)+B(-2-1)$

yoki $25=-3B$ $B=-\frac{25}{3}$

Umuman x uchun har qanday son qiymat olsak ham bo‘ladi. Ammo x ni shunday tanlab olish kerakki, (2) ayniyatdagi qo‘shiluvchilardan bittasi nolga teng bo‘lsin. Shuning uchun $x=-2,\ x=1$ deb, $A=\frac{1}{3},\ B=-\frac{25}{3}$ larga ega bo‘ldik. Demak,

$$\frac{-8x+9}{x^2+x-2}=\frac{A}{x-1}+\frac{B}{x+2}=\frac{\frac{1}{3}}{x-1}+\frac{-\frac{25}{3}}{x+2}$$

Shunday qilib,

$$\int\frac{-8x+9}{x^2+x-2}dx=\int\left(\frac{\frac{1}{3}}{x-1}+\frac{-\frac{25}{3}}{x+2}\right)dx=\frac{1}{3}\int\frac{dx}{x-1}-\frac{25}{3}\int\frac{dx}{x+1}=\frac{1}{3}\int\frac{d(x-1)}{x-1}-\frac{25}{3}\int\frac{d(x+2)}{x+2}=$$

$$\frac{1}{3}\ln|x-1|-\frac{25}{3}\ln|x+2|+c$$

Berilgan integral esa quyidagiga teng bo‘ladi:

$$\int\frac{x^4-x^3+1}{x^2+x+2}dx=\int\left(x^2-2x+4\right)dx+\int\frac{-8x+9}{x^2+x+2}dx=\int x^2dx-2\int xdx+4\int dx+$$

$$+\frac{1}{3}\int\frac{dx}{x-1}-25\int\frac{dx}{x+2}=\frac{x^3}{3}-2\frac{x^2}{2}+4x+\frac{1}{3}\ln(x-1)-\frac{25}{3}\ln(x+2)+c=$$

$$\frac{1}{3}x^3+x^2+4x+\frac{1}{3}\ln|x-1|-\frac{25}{3}\ln|x+2|+c$$

Umumiy holda agar $\frac{P_0(x)}{Q_0(x)}$ to‘g‘ri kasrning mahraji $(x+a)^{m_1}\cdot\left(x^2+px+q\right)^{n_1}$ ko‘rinishidagi ko‘paytuvchilardan iborat bo‘lsa, hamda x^2+px+q kvadrat uchhad haqiqiy ildizga ega bo‘lmasa, u holda quyidagi teorema o‘rinli bo‘ladi.

Teorema. $\frac{P_0(x)}{Q_0(x)}$ to‘g‘ri kasrni mahrajini quyidagi $(x+a)^{m_1}\cdot\left(x^2+px+q\right)^{n_1}$ ko‘rinishdagi ko‘paytuvchilar shaklida yozish mumkin bo‘lsa, u holda bu kasrni quyidagi shaklda yozish mumkin:

$$\frac{P_0(x)}{Q_0(x)}=\frac{P_0(x)}{(x+a)^{m_1}\left(x^2+px+q\right)^{n_1}}=\frac{A_1}{x+a}+\frac{A_2}{(x+a)^2}+\ldots+\frac{Am_1}{(x+a)^{m_1}}+\frac{M_1x+N_1}{x^2+px+q}+$$

$$\ldots+\frac{M_{n_1}x+N_m}{\left(x^2+px+q\right)^{n_1}}$$

Bunda $A_1,A_2,\ldots+Am,M_1,N_2,\ldots,M_{n_1},N_{n_1}$ noaniq (noma’lum koeffitsientlardir).

Misol. $\int\frac{dx}{x^4-1}$

Bizga ma’lumki $x^4-1=\left(x^2+1\right)\left(x^2-1\right)=\left(x^2+1\right)(x-1)(x+1)$

U holda $\frac{1}{x^4-1}=\frac{1}{(x+1)(x-1)\left(x^2+1\right)}$ bu to‘g‘ri kasrni uchta elementlar kasrning yig‘indisi shaklida yozib olamiz.

$$\frac{1}{x^4-1}=\frac{1}{(x+1)(x-1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{B}{x-1}=\frac{M_x+N}{x^2+1}$$

O‘ng tomonni umumiy mahrajga keltirib, so‘ngra chap va o‘ng tomonlarining suratlarini tenglab, quyidagini hosil qilamiz:

$$1=A(x-1)\left(x^2+1\right)+B(x+1)\left(x^2+1\right)+(x-1)(x+1)(M_x+N)$$

Bunda $x=1,\ x=-1,\ x=0,\ x=2$ deb quyidagilarni topamiz.

Agar $x=1$ bo‘lsa $1=A\cdot 0+B(1+1)(1+1)+M\cdot 0$ ga ega bo‘lamiz, bundan $1=B4,\ B=\frac{1}{4}$

Agar $x=-1$ desak $1=A(-1-1)2+B\cdot 0+M\cdot 0$ yoki $1=-4A$ bundan $A=-\frac{1}{4}$

Agar $x=0$ desak $1=A(-1)\cdot 1+B\cdot 1\cdot 1+N(-1)\cdot 1$ yoki $1=-A+B-N$. Ammo $A=-\frac{1}{4},\ B=\frac{1}{4}$ edi. Demak: $1=\frac{1}{4}+\frac{1}{4}-N$ bundan $N=-\frac{1}{2}$

Agar $x=2$ desak: $1=A|5|B35+(M2+N)$ yoki $1=5A+15B+6M+3N$

$$1=5\left(-\frac{1}{4}\right)+15\left(\frac{1}{4}\right)+6M+3\left(-\frac{1}{2}\right)$$

$$1=-\frac{5}{4}+\frac{15}{4}+6M-\frac{3}{2},\ 1=6M+1$$

bundan $M=0$. Shunday qilib,

$$\frac{1}{(x+1)(x-1)(x^2+1)}=\frac{-\frac{1}{4}}{x+1}=\frac{\frac{1}{4}}{x-1}+\frac{-\frac{1}{2}}{x^2+1}$$

Demak, $\int\frac{dx}{x^4-1}=\int\frac{dx}{(x+1)(x-1)(x^2+1)}=\int\left[\frac{-\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-1}+\frac{-\frac{1}{2}}{x^2+1}\right]dx=$

$$-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{2}\int\frac{dx}{x^2+1}=-\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-1|-\frac{1}{2}arctgx+c=$$

$$\frac{1}{4}[\ln|x-1|-\ln|x+1|]-\frac{1}{2}arctgx+c=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}arctgx+c$$

Mashqlar

Mustaqil yechish uchun misollar

1. $\int \frac{x-4}{(x-2)(x-3)}dx$ J: $\left(\ln\frac{c(x-2)^2}{x-3}\right)$

2. $\int \frac{2x+7}{(x+2)(x-1)}dx$ J: $\left(+\ln\frac{(x-1)^3}{x+2}+c\right)$

3. $\int \frac{5x^3+9x^2-22x-8}{x^3-4x}dx$ J: $\left(5x+2\ln|x|+3\ln|x-2|+4\ln|x+2|+c\right)$

4. $\int \frac{3x+2}{x(x+1)}dx$ J: $2\ln|x|+\ln|x+1|+c$

5. $\int \frac{xdx}{(x+1)(2x+1)}dx$ J: $\left(\ln\left|\frac{x+1}{\sqrt{2x+1}}\right|+c\right)$

6. $\int \frac{dx}{x^4-x^3}$ J: $\frac{1}{x}+\frac{1}{2x}+\ln|x|+\ln|x-1|+c$

7. $\int \frac{dx}{(x+3)(x-4)}$ J: $\ln\left|\frac{x+3}{x+4}\right|+c$

6-§. Trigonometrik ifodalarni integrallash

1. $\int c(\sin x)\cdot\cos xdx$ va $\int R(\cos x)\cdot\sin xdx$ ko‘rinishdagi integrallar o‘rniga qo‘yish usuli bilan topiladi. Bunda $\sin x=t$ va $\cos x=t$ deb, o‘zgaruvchini almashtiriladi. Integral ostidagi $R(\sin x)$ funksiya $\sin x$ ni ratsional funksiyasi.

Masalan: $\int\left(\sin^3 x+\sin^2 x-3\sin x\right)\cdot\cos xdx$ integral $\int R(\sin x)\cdot\cos xdx$ ko‘rinishidagi integraldir.

Bunda $R(\sin x)=\sin^3 x+\sin^2 x-3\sin x$ deb o‘zgaruvchini almashtiramiz. $\sin x=t$ deb olamiz, bundan $\cos xdx=dt$ bo‘ladi. Shunday qilib,

$$\int\left(\sin^3 x+\sin^2 x-3\sin x\right)\cos xdx=\left|\begin{matrix}\sin x=t\\ \cos xdx=dt\end{matrix}\right|=\int\left(t^3+t^2-3t\right)dt=\int t^3dt+$$

$$\int t^2dt-3\int tdt=\frac{t^4}{4}+\frac{t^3}{3}-\frac{3t^2}{2}+c=\frac{1}{4}\sin^4 x+\frac{\sin^3 x}{3}-\frac{3}{2}\sin x+c$$

2. $\int \sin^{2n+1} xdx$ $\int \cos^{2n+1} xdx$

ko'rinishdagi integrallar quyidagicha topiladi (bunda $2n+1$ natural son). Avval integral ostidagi funksiya quyidagi ko'rinishda yozib olinadi. $\sin^{2n+1} x = \sin^{2n} x \cdot \sin x = \left(\sin^2 x\right)^n \cdot \sin x$.

Shuningdek $\cos^{2n+1} x = \cos^{2n} x \cdot \cos x = \left(\cos^2 x\right)^n \cdot \cos x$. So'ngra $\sin^2 x + \cos^2 x = 1$ formuladan foydalanilsa berilgan integral holga keladi.

Masalan: $\int \sin^5 xdx = \int \sin^4 x \cdot \sin xdx = \int \left(\sin^2 x\right)^2 \sin xdx$

Ammo $\sin^2 x = 1 - \cos^2 x$ ga teng.

Demak

$$\int \sin^5 xdx = \int \left(\sin^2 x\right)^2 \sin xdx = \int \left(1 - \cos^2 x\right)\sin xdx = \begin{vmatrix} \cos x = t \\ -\sin xdx = dt \\ \sin xdx = -dt \end{vmatrix} =$$

$$-\int \left(1 - t^2\right)^2 dt = -\int \left(1 - 2t^2 + t^4\right)dt = -\int dt + 2\int t^2 dt - \int t^4 dt =$$

$$-t + \frac{2}{3}t^3 - \frac{1}{5}t^5 + c = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

3. $\int \sin^{2n} xdx, \int \cos^{2n} xdx$ ko'rinishidagi integrallar quyidagi trigonometrik formulalar yordami bilan topiladi:

$$\sin^2 x = \frac{1 - \cos 2x}{2}; \ \cos^2 x = \frac{1 + \cos 2x}{2}$$

Masalan,

$$\int \cos^4 xdx = \int \left(\cos^2 x\right)^2 dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx = \int \frac{1 + 2\cos 2x + \cos^2 2x}{4} dx =$$

$$\frac{1}{4}\int dx + \frac{1}{2}\int \cos 2xdx + \frac{1}{4}\int \cos^2 2xdx$$

Lekin $\cos^2 2x$ ni (2) formulaga asosan quyidagicha yozish mumkin: $\frac{1 + \cos 4x}{2}$ ya'ni $\cos^2 2x = \frac{1 + \cos 4x}{2}$.

Demak:

$$\int \cos^2 2xdx = \int \frac{1+\cos 4x}{2}dx = \int \frac{1}{2}dx + \int \frac{1}{2}\cos 4xdx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 4xdx =$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\cdot\frac{1}{4}\int \cos 4xd4x = \frac{1}{2}x + \frac{1}{8}\sin 4x + c$$

Shunday qilib,

$$\int \cos^4 xdx = \int \left(\cos^2 x\right)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \int \frac{1+2\cos 2x+\cos^2 2x}{4}dx = \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2xdx +$$

$$+ \frac{1}{4}\int \cos^2 2xdx = \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2x\frac{d2x}{2} + \frac{1}{4}\int \frac{1+\cos 4x}{2}dx = \frac{1}{4}\int dx + \frac{1}{4}\int \cos 2xd2x + \frac{1}{8}\int dx +$$

$$+ \frac{1}{8}\int \cos 4xdx = \frac{1}{4}x + \frac{1}{8}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + c$$

4. $tg^m xdx$ ko‘rinishdagi integral quyidagicha topiladi. Agar $m \neq 0$ va m toq son bo‘lsa, u holda $tgx = t$ deb o‘zgaruvchini almashtiriladi, unda $x = arctgt$ bo‘ladi, bundan $dx = (arctdx)' dt = \dfrac{dt}{1+t^2}$.

Agar $m<0$ va m toq son bo‘lsa, u holda $\sin x = t$ deb o‘zgaruvchi almashtiriladi.

Quyidagi misollarni ko‘rib o‘tamiz: $m<0,\ m = 2n+1\ (n = 1,2,3,\ldots)$.

a)

$$\int \frac{dx}{tg^3 x} = \int \frac{dx}{\dfrac{\sin^3 x}{\cos^3 x}} = \int \frac{\cos^3 x}{\sin^3 x}dx = \int \frac{\cos^2 x\cdot\cos xdx}{\sin^3 x} = \int \frac{\left(1-\sin^2 x\right)\cos xdx}{\sin^3 x} = \begin{vmatrix}\sin x = t\\ \cos xdx = dt\end{vmatrix} =$$

$$= \int \frac{\left(1-t^2\right)dt}{t^3} = \int \frac{dt}{t^3} - \int \frac{dt}{t} = -\frac{1}{2t^2} + c = -\frac{1}{2\sin^2 x} - \ln\sin x + c$$

b) $m>0$

$$\int tg^3 xdx = \int \frac{\sin^3 x}{\cos^3 x}dx = \int \frac{\sin^2 x\sin xdx}{\cos^3 x} = \int \frac{\left(1-\cos^2 x\right)\cdot\sin xdx}{\cos^3 x} = \begin{vmatrix}\cos x = t\\ -\sin xdx = dt\\ \sin xdx = -dt\end{vmatrix} =$$

$$= -\int \frac{\left(1-t^2\right)dt}{t^3} = -\int \frac{dt}{t^3} - \int \frac{dt}{t} = \frac{1}{2t^2} - \int \frac{dt}{t} + c = -\frac{1}{2\cos^2 x} - \ln\left|\cos x\right| + c$$

v) $m>0$

$$\int tg^3 x dx = \left| \begin{aligned} & tgx = t \\ & x = arctgt \\ & dx = \frac{dt}{1+t^2} \end{aligned} \right| = \int \frac{t^3 dt}{1+t^2} = \int \frac{t^2 t dt}{1+t^2} = \int \frac{(t^2+1-1)t dt}{1+t^2} = \int \frac{(t^2+1)t dt}{1+t^2} - \int \frac{t dt}{1+t^2} =$$

$$\int t dt - \frac{1}{2}\int \frac{dt^2}{t^2+1} = \frac{t^2}{2} - \frac{1}{2}\ln(t^2+1) + c = \frac{tg^2 x}{2} - \frac{1}{2}\ln(tg^2 x + 1) + c = \frac{1-\cos^2 x}{2\cos^2 x} + \ln\frac{1}{(\cos^2 x)} + c$$

$$\left(\cos^2 x = \frac{1}{1+tg^2\alpha}; \ \frac{1}{\cos^2 x} = 1 + tg^2 x \right)$$

$m \neq 0$ va m toq son bo‘lsin: $(m = 2n, \ n = 1,2,3,\dots)$

$$\int tg^4 x dx = \left| \begin{aligned} & tgx = t \\ & x = arctgt \\ & dx = \frac{dt}{1+t^2} \end{aligned} \right| = \int \frac{t^4 dt}{1+t^2} = \int \frac{t^2 \cdot t^2 dt}{1+t^2} = \int \frac{(t^2+1-1)t^2 dt}{1+t^2} = \int \frac{(t^2+1)t^2 dt}{1+t^2} - \int \frac{t^2 dt}{1+t^2} =$$

$$\int t^2 dt - \int \frac{t^2+1-1}{1+t^2} dt = \int t^2 dt - \int \frac{t^2+1}{1+t^2} dt + \int \frac{dt}{1+t^2} = \int t^2 dt - \int dt + \int \frac{dt}{1+t^2} =$$

$$\frac{t^3}{3} - t + arctgt + c = \frac{(tgx)^3}{3} - tgx + x + c$$

Universal almashtirish: $tg\frac{x}{2} = t$

Agar trigonometrik ifodalarni integrali berilsa va unga 1, 2, 3 hollarni qo‘llab bo‘lmasa, u holda universal almashtirishdan foydalaniladi.

Universal almashtirish quyidagi ko‘rinishdagi integrallarni topishda qo‘llaniladi.

$$\int \frac{dx}{\sin x - \cos x}, \quad \int \frac{dx}{4 - \cos x}, \quad \int \frac{dx}{1 + \cos x - \sin x} \quad \text{va xokazo.}$$

Universal almashtirish integral ostidagi trigonometrik funksiyani darajasi yuqori bo‘lganda qo‘llansa, u holda qiyin hollarga olib kelishi mumkin.

Quyidagi misollarni ko‘rib o‘tamiz:

$$\int \frac{dx}{2\sin x - \cos x}$$

$tg\frac{x}{2}=t$ deb universal almashtirishdan foydalanamiz. bundan $\frac{x}{2}=arctgx,\quad x=2arctgx,\quad dx=\frac{2dt}{1+t^2}$ hamda $\sin x$ va $\cos x$ ni $tg\frac{x}{2}$ orqali ifoda qilib olamiz.

Trigonometriyadan bizga ma'lumki

$$\sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}},\quad \cos x=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}},$$

u holda $tg\frac{x}{2}=t$ edi.

Demak, $\sin x=\frac{2t}{1+t^2},\quad \cos x=\frac{1-t^2}{1+t^2}$

$$\int\frac{dx}{2\sin x-\cos x}=\left|\begin{matrix}tg\frac{x}{2}=t, & \sin x=\frac{2t}{1+t^2}\\ dx=\frac{2dt}{1+t^2}, & \cos x=\frac{1-t^2}{1+t^2}\end{matrix}\right|=\int\frac{\frac{2dt}{1+t^2}}{2\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}=2\int\frac{dt}{\left(1+t^2\right)\frac{4t-1+t^2}{1+t^2}}=$$

$$=2\int\frac{dt}{(t+2)^2-5}=\left|\begin{matrix}t+2=z\\ dt=dz\end{matrix}\right|=2\int\frac{dz}{z^2-5}=2\int\frac{dz}{z^2-\left(\sqrt{5}\right)^2}=2\cdot\frac{1}{2\sqrt{5}}\ln\left|\frac{z-\sqrt{5}}{z+\sqrt{5}}\right|+c=$$

$$=\frac{1}{\sqrt{5}}\ln\left|\frac{t+2-\sqrt{5}}{t+2+\sqrt{5}}\right|+c=\frac{1}{\sqrt{5}}\ln\left|\frac{tg\frac{x}{2}+2-\sqrt{5}}{tg\frac{x}{2}+2+\sqrt{5}}\right|+c$$

Mashqlar

Mustaqil yechish uchun misollar

1. $\int\sin^2 2xdx$ J: $\left(\frac{1}{x}x-\frac{1}{8}\sin 4x+c\right)$

2. $\int\left(1+2\cos x\right)^2dx$ J: $\left(3x+4\sin x+\sin 2x+c\right)$

3. $\int\cos^4 xdx$ J: $\left(\frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}\right)+c$

4. $\int\sin^2 x\cdot\cos^2 xdx$ J: $\left(\frac{x}{8}-\frac{\sin 4x}{32}+c\right)$

5. $\int \sin^2 x \cdot \cos^4 x dx$ J: $\left(\frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + c\right)$

6. $\int \sin^3 x dx$ J: $\left(\frac{\cos^3 x}{3} - \cos x + c\right)$

7. $\int \frac{\cos^3 x dx}{\sin^2 x}$ J: $-\frac{1}{\sin x} - \sin x + c$

8. $\int \frac{\sin^3 x}{\cos^2 x} dx$ J: $-\frac{1}{\cos x} + \cos x + c$

9. $\int tg^5 x \cdot dx$ J: $\left(\frac{tg^4 x}{4} - \frac{tg^2 x}{2} - \ln|\cos x| + c\right)$

10. $\int \cos^3 x dx$ J: $\left(\sin x - \frac{\sin^3 x}{3} + c\right)$

11. $\int \frac{\sin^3 x}{\cos x} dx$ J: $\left(\frac{\cos^2 x}{2} - \ln|\cos x| + c\right)$

12. $\int tg^4 x dx$ J: $\left(\frac{tg^3 x}{3} - tgx + x + c\right)$

13. $\int (5\sin^2 x - 3\sin x)\cos x dx$ J: $\left(\frac{5\sin^3 x}{3} - \frac{3}{2}\sin^2 x + c\right)$

14. $\int \sin^3 x \cdot \cos^2 x dx$ J: $\left[\frac{1}{15}\cos^3 x(3\cos^2 x - 5) + c\right]$

15. $\int \cos^5 x \sin x dx$ J: $\left(-\frac{\cos^6 x}{6} + c\right)$

16. $\int \frac{dx}{\sin x + \cos x}$ J: $\frac{\sqrt{2}}{2} \ln\left|\frac{tg\frac{x}{2} + 1 + \sqrt{2}}{tg\frac{x}{2} + 1 - \sqrt{2}}\right| + c$

17. $\int \frac{dx}{5-3\cos x}$ J: $\frac{1}{2}arctg\left(2tg\frac{x}{2}\right)+c$

18. $\int \frac{dx}{5+4\sin x}$ J: $\frac{2}{3}arctg\frac{5tg\frac{x}{2}+4}{3}+c$

19. $\int \frac{dx}{5-4\sin x+3\cos x}$ J: $\left(\frac{1}{2-tg\frac{x}{2}}+c\right)$

20. $\int \frac{dx}{\cos x}$ J: $\left(2n\ln\left|\frac{tg\frac{x}{2}}{tg\frac{x}{2}}\right|+c\right)$

7-§. Sodda funksional ifodalarni integrallash

Quyidagi integral $\left(R\left(x\sqrt[n]{x}\right)\right)dx$ berilgan bo‘lsin. Bunda (n - natural son) $R\left(x\sqrt[n]{x}\right)$ esa x va $\sqrt[n]{x}$ ning ratsional funksiyadir.

Masalan: $\frac{1+\sqrt[4]{x+1}}{\sqrt[4]{x+1}}$ funksiya $R\left(x, \sqrt[4]{x+1}\right)$ ko‘rinishidagi funksiyadir. $\frac{x^2+\sqrt{x}}{1-\sqrt[3]{x}}$ esa $R\left(x, \sqrt[6]{x}\right)$ funksiyadir.

Agar $\int R\left(x\sqrt[n]{x}\right)dx$ ko‘rinishdagi integralda $x=t^n$ deb o‘zgaruvchini almashtiramiz. Bunda n ildizlarni hammasining darajasi uchun kichik umumiy bo‘linuvchi sondir. Masalan: $\int \frac{\sqrt{x}dx}{1+\sqrt[3]{x}}$ integralni topish uchun $x=t^6$ deb, o‘zgaruvchini almashtiramiz, ya’ni $6=$soni ildizlarni ko‘rsatkichi 2 va 3 uchun eng kichik umumiy bo‘linuvchi sondir.

Shunday qilib,

$$\int \frac{\sqrt{x}dx}{1+\sqrt[3]{x}} = \left| \begin{array}{l} x = t^6 \\ dx = 6t^5dt \\ \sqrt{x} = \sqrt{t^6} = t^3 \end{array} \right| = \int \frac{t^3 6t^5 dt}{1+t^2} = 6\int \frac{t^8 dt}{t^2+1} = \left| \begin{array}{l} \begin{array}{l|l} t^8 & t^2+1 \\ \hline & t^6 - t^4 + t^2 - 1 \end{array} \\ \underline{t^8 + t^6} \\ \quad -t^6 \\ \underline{-t^6 - t^4} \\ \quad\quad t^4 \\ \underline{-t^4 + t^2} \\ \quad\quad -t^2 \\ \underline{-t^2 - 1} \\ \quad\quad 1 \end{array} \right| =$$

$$= 6\int \left(t^6 - t^4 + t^2 - 1 + \frac{+1}{t^2+1} \right) dt = 6\int t^6 dt - 6\int t^4 dt + 6\int t^2 dt - 6\int dt +$$

$$6\int \frac{dt}{1+t^2} = \frac{6}{7}t^2 - \frac{6}{5}t^5 - \frac{6}{3}t^3 - 6t + 6arctgt + c = \frac{6}{7}\left(\sqrt[6]{x}\right)^7 - \frac{6}{5}\left(\sqrt[6]{x}\right)^5 -$$

$$2\left(\sqrt[6]{x}\right)^3 - 6\sqrt[6]{x} + arctg\sqrt[6]{x} + c$$

Quyidagi $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ ko‘rinishdagi integralni topish uchun mahrajidan to‘la kvadratlar ajratiladi, so‘ngra jadvaldagi $\int \frac{dx}{\sqrt{a-x^2}}$ (f.12 $a<0$ bo‘lganda) integral ko‘rinishiga olib kelinadi.

Quyidagi $\int \frac{dx}{x\sqrt{ax^2+bx+c}}$ ko‘rinishidagi integral uchun $\frac{1}{x} = t$ almashtirish bajariladi va $x = \frac{1}{t}$ bundan $dx = -\frac{dt}{t^2}$.

Masalan:

$$\int \frac{dx}{x\sqrt{ax^2-bx+1}} = \left| \begin{matrix} \frac{1}{x}=t \\ x=\frac{1}{t} \\ dx=-\frac{dt}{t^2} \end{matrix} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{10}{t^2}-\frac{6}{t}+1}} = \int \frac{-dt}{10-6t+t^2} =$$

$$= -\int \frac{dt}{\sqrt{10-6t+t^2}} = -\int \frac{dt}{\sqrt{(t-3)^2+1}} = -\int \frac{d(t-3)}{\sqrt{(t-3)^2+1}}$$

$$\left\{\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x+\sqrt{x^2+a}\right|\right\} = -\ln\left|(t+3)+\sqrt{(t-3)^2+1}\right|+c=$$

$$= -\ln\left|(t+3)+\sqrt{t^2-6t+10}\right|+c = -\ln\left|\frac{1-3x+\sqrt{10x^2-6x+1}}{x}\right|+c$$

3. $\int R\left(x,\sqrt{Q^2-x^2}\right)dx$ ko‘rinishdagi integralni hisoblash uchun $x=a\sin t$ deb o‘zgaruvchini almashtiriladi (yoki $x=a\cos t$).

$\int R\left(x,\sqrt{x^2+a^2}\right)dx$ ko‘rinishdagi integralni hisoblash uchun $x=atgt$
(yoki $x=arctgt$ deb o‘zgaruvchini almashtiriladi.

$\int R\left(x,\sqrt{x^2-a^2}\right)dx$ ko‘rinishdagi integralni hisoblash uchun $x=\frac{a}{\cos t}$ (yoki $x=\frac{a}{\sin t}$) deb, o‘zgaruvchini almashtiriladi.

Quyidagi misollarni ko‘rib o‘tamiz.

1.

$$\int \frac{dx}{x^2\sqrt{9+x^2}} = \left| \begin{matrix} 9=3^2=a^2,\ a=3 \\ x=3tgt \\ dx=\dfrac{3dt}{\cos^2 t} \end{matrix} \right| = \int \frac{\dfrac{3dt}{\cos^2 t}}{9tg^2t\sqrt{9+9tg^2t}} =$$

$$= \int \frac{3dt}{\cos^2 t\cdot \dfrac{9\sin^2 x}{\cos^2 t}\sqrt{9\left(1+tg^2 t\right)}} = \left(1+tg^2x=\frac{1}{\cos^2 x}=\sec^2 x\right) =$$

$$= \int \frac{dt}{3\sin^2 t\sqrt{9\dfrac{1}{\cos^2 t}}} = \int \frac{dt}{3\sin^2 t\cdot 3\dfrac{1}{\cos t}} = \frac{1}{9}\int \frac{\cos t dt}{\sin^2 x} = \frac{1}{9}\int \frac{d\sin t}{\sin^2 t} =$$

$$\frac{1}{9}\cdot\frac{1}{\sin t}+c = -\frac{1}{9\sin(arctgx)}+c$$

$(\cos t dt = d\sin t)$,

$$\left(\int \frac{dx}{x^2} = -\frac{1}{x}+c,\quad t=arctg\frac{x}{3};\ x=3tgt\right)$$

2. $\displaystyle\int \frac{x^2dx}{\sqrt{4-x^2}}$ integralni topish talab qilinsin.

Bizga ma’lumki, $4=2^2$ hamda berilgan integral $R\left(x\sqrt{2^2-x^2}\right)$ ko‘rinishdagi integraldir. Demak berilgan integral uchun $x=2\sin t$ deb o‘zgaruvchini almashtiramiz.

Bundan $dx=2\cos t dt$

Shunday qilib,

$$\int \frac{x^2dx}{\sqrt{4-x^2}} = \left| \begin{matrix} x=2\sin t \\ dx=2\cos t dt \end{matrix} \right| = \int \frac{4\sin^2 t 2\cos t dt}{\sqrt{4-4\sin^2 t}} = 8\int \frac{\sin^2 t\cdot \cos t dt}{\sqrt{4\left(1-\sin^2 t\right)}} =$$

$$= 8\int \frac{\sin^2 t\cdot\cos t dt}{2\sqrt{\cos^2 t}} = 4\int \frac{\sin^2 t\cdot \cos t dt}{\cos t} = 4\int \sin^2 t dt = 4\int \frac{1-\cdot\cos 2t}{2}dt$$

$$= 2\int dt - \int \cos 2t d2t = 2t - \sin 2t + c$$

Bizga ma'lumki, $x = 2\sin t$. Demak $\frac{x}{2} = \sin t \quad t = \arcsin\frac{x}{2}$ topilgan javobini quyidagi formulalarni qo'llab, soddallashtirish mumkin.

Shunday qilib, quyidagiga ega bo'lamiz:

$$\int \frac{x^2dx}{\sqrt{4-x^2}} = 2t - \sin 2t + c = 2t - 2\sin t \cdot \cos t + c = 2\arcsin\frac{x}{2} -$$

$$-2\sin\left(arc \cdot \sin\frac{x}{2}\right) \cdot \cos\left(arc \cdot \sin\frac{x}{2}\right) + c = 2\arcsin\frac{x}{2} -$$

$$2 \cdot \frac{x}{2}\cos\left(\arcsin\frac{x}{2}\right) + c = 2\arcsin\frac{x}{2} - x \cdot \sqrt{1 - \sin^2\left(\arcsin\frac{x}{2}\right) + c} =$$

$$2\arcsin\frac{x}{2} - x \cdot \sqrt{1 - \left(\frac{x}{2}\right)^2} + c = 2\arcsin\frac{x}{2} - x \cdot \frac{1}{2}\sqrt{4 - x^2} + c$$

$$(\sin(\arcsin x) = x)$$

Mashqlar

Mustaqil yechish uchun misollar

1. $\int \frac{dx}{\sqrt{2x^2 - x - 3}}$ J: $\frac{1}{\sqrt{2}}\ln\left|x - \frac{1}{4} + \frac{1}{\sqrt{2}}\sqrt{2x^2 - x \pm 3}\right|$

2. $\int \frac{dx}{\sqrt{5 - 2x - 3x^2}}$ J: $\left(\frac{1}{\sqrt{3}}\arcsin\frac{3x+1}{4} + c\right)$

3. $\int \frac{dx}{\sqrt{5x^2 + 3x - 2}}$ J: $\frac{1}{\sqrt{5}}\ln\left|10x + 3 + 2\sqrt{5}\sqrt{5x^2 + 3x + 2}\right| + c$

4. $\int \frac{xdx}{\sqrt{x^2 + 4x + 5}}$ J: $\sqrt{x^2 + 4x + 5} - 2\ln\left|x + 2 + \sqrt{x^2 + 4x + 5}\right| + c$

5. $\int \frac{dx}{x\sqrt{2x^2 - 5x + 3}}$ J: $\left(-\frac{1}{\sqrt{5}}\ln\left|\frac{6 - 5x}{6x} + \sqrt{\frac{3 - 5x + 2x^2}{3x^2}}\right| + c\right)$

6. $\int \frac{dx}{x\sqrt{7x^2-2x+5}}$ J: $\left(-\frac{1}{\sqrt{5}}\ln\left|\frac{-x+5+\sqrt[5]{7x^2-2x+5}}{5x}\right|\right)$

7. $\int \frac{\sqrt{x}dx}{1+\sqrt{x}}$ J: $x-2\sqrt{x}+2\ln\left|\sqrt{x+1}\right|+c$

8. $\int \frac{\sqrt{x}dx}{1-\sqrt[3]{x^2}}$ J: $-\frac{6}{5}\sqrt[6]{x^5}-6\sqrt[6]{x}+3arctg\sqrt[6]{x}-\frac{3}{2}\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt{x}+1}\right|+c$

9. $\int \frac{dx}{\left(1+\sqrt[3]{x}\right)\sqrt{x}}$ J: $\left(6\sqrt[6]{x}-6arctg\sqrt[6]{x}+c\right)$

10. $\int \sqrt{4-x^2}dx$ J: $2\arcsin\frac{x}{2}+x\cos\arcsin\frac{x}{2}+c$

11. $\int \frac{x^2dx}{\sqrt{16-x^2}}$ J: $2\arcsin\frac{x}{2}-\frac{x}{2}\sqrt{16-x^2}+c$

12. $\int \frac{dx}{x^2\sqrt{1+x^2}}$ J: $\left(c-\frac{\sqrt{x^2+1}}{x}\right)$

13. $\int \frac{dx}{x\sqrt{x^2-4}}$ J: $\frac{1}{2}\arccos\frac{2}{x}+c$

14. $\int \frac{dx}{x\sqrt{1+x^2}}$ J: $\left(\ln\frac{|x|}{1+\sqrt{x^2+1}}+c\right)$

15. $\int \frac{\sqrt{1-x^2}}{x^2}dx$ J: $\left(c-\frac{\sqrt{1-x^2}}{x^2}-\arcsin x\right)$

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